

**Zeitschrift:** Helvetica Physica Acta

**Band:** 67 (1994)

**Heft:** 1

**Erratum:** Erratum on the paper Helv. Phys. Acta 65 (1992) 748 : semiclassical expansions of the thermodynamic limit for a Schrödinger equation. II, The double well case

**Autor:** Helffer, Bernard

#### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 02.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## Erratum on the Paper

**Helv.Phys.Acta 65 (1992) 748:**

**Semiclassical expansions of the thermodynamic limit  
for a Schrödinger equation**

**II. The double well case**

By Bernard Helffer

DMI-ENS  
45 rue d'Ulm  
F-75230 Paris Cédex

and Johannes Sjöstrand

Département de Mathématiques  
Université Paris-Sud  
F-91405 Orsay  
(2. II. 1994)

In our paper there is a gap in the proof of (3.7) p.761. It is indeed not always possible to construct  $\kappa$  with the property that  $\kappa(x^\Omega) = x^{\Omega_+}$  is the center of the  $\ell^\infty$ -ball containing one well and satisfying (3.7). We overlooked the rôle of  $\Sigma_k(x_k - x_{k+1})^2$  in the comparison of  $V$  and  $V \circ \kappa$ . The text starting from p.760 line -8 to p.761 line -9 has to be modified as follows.

Let  $\kappa : \Omega \rightarrow \Omega_+ = (I_+)^m$  be the translation with  $\kappa(x^\Omega) = \tilde{x}^{\Omega_+}$ . Here  $\tilde{x}_j^{\Omega_+} = 0$ , if  $I_j$  contains  $t_0$  and in the other case  $\tilde{x}_j^{\Omega_+} = \pm \rho \delta_0$  for a suitable  $\rho$  independent of  $\Omega$  and  $\delta$  satisfying:  $0 < \rho < 1$  where we take the sign + if  $I_j$  is on the right of  $t_0$  and the sign - if  $I_j$

is on the left of  $t_0$ .

**The choice of  $\rho$ :**

In order to precise the choice of  $\rho$ , we observe that the problems occur for the boxes which are in the vicinity of one well and we consider a simplified function by taking the quadratic approximation of  $v$  at  $t_0$ . This permits us to approximate the function  $(s, t) \rightarrow W(t, s) = (1/16)(t - s)^2 + v(t_0 + (t + s)/2)$  by

$$(s, t) \rightarrow W_0(t, s) = C(t - s)^2 + D(t + s)^2$$

with  $C > 0$  and  $D > 0$ . We consider the evolution of this potential restricted to an interval  $I(x, \delta)$  centered at the point  $(t = x, s = 0)$  and of size  $2\delta_0$ .

We observe that after dilation we can reduce our study to the case  $\delta_0 = 1$ , and we consider consequently the function:

$$]0, +\infty[ \ni x \rightarrow W_0(x, s) \text{ for } s \in [-1, 1].$$

We have

$$(\partial_t W_0)(x, s) = 2C(x - s) + 2D(x + s) = (2C + 2D)x + (2D - 2C)s.$$

We first observe that if  $C = D$ , the function is a monotone increasing function of  $x \geq 0$  and we could have taken  $\rho = 0$  in this case but this is not the case here (in other papers we do a change of coordinates in order to be in this situation but it is not convenient here); we recall that  $C = 1/16$ ,  $D = v''(t_0)/8$  in our case.

Let us assume in order to fix the ideas that  $C > D$ . Then the derivative as a function of  $s$  is minimal at  $s = 1$  and  $(\partial_t W_0)(x, s)$  is positive for  $x \geq (C - D)/(C + D)$ .

Our  $\rho$  is chosen such that  $|(C - D)|/(C + D) < \rho < 1$ .

Using (3.4) and our choice of  $\rho$ , we see that, choosing  $\epsilon_0$  so small such that  $\epsilon_0 \ll \delta_0$  with  $\delta_0 \leq 1$ , we get for  $x \in \Omega$ :

$$(3.7) \quad V(x) - V(\kappa(x)) \geq C\delta_0^2\beta(\Omega)$$

where  $\beta(\Omega)$  is the number of intervals  $I_j$  which do not contain  $t_0$  (or  $-t_0$ ). Notice that  $\beta(\Omega)$  is unchanged by the first " $\kappa$ ".

Actually in this discussion, we have different cases to consider depending on the vanishing  $\tilde{x}_j^{\Omega+}\tilde{x}_{j+1}^{\Omega+} = 0$  or not. We have only discussed above the most difficult case when this product is 0 with  $|\tilde{x}_j^{\Omega+}| + |\tilde{x}_{j+1}^{\Omega+}| \neq 0$ .

We now compose our two maps, we notice that  $\alpha_0(\Omega) \leq \beta(\Omega)$  and we then get a new map  $x \rightarrow \kappa(x)$ , being the composition of reflexions in 0 in some of the coordinates and of a translation, such that

$$\kappa : \Omega \rightarrow \tilde{\Omega}_+ \subset 2\Omega_+,$$

(where  $\Omega$  is the original box)

$$\kappa(x^\Omega) = \tilde{x}^{\Omega+}$$

and

$$(3.8) \quad V(x) - V(\kappa(x)) \geq (1/C)(\alpha_+(\Omega) + \beta(\Omega)), x \in \Omega.$$

$C$  is here a strictly positive constant, independent of  $\Omega$ , once we have fixed  $\epsilon_0$  and  $\delta_0$  conveniently as explained before.

Let  $P_\Omega$  denote the Dirichlet realization of  $-h^2\Delta + V$  in  $\Omega$  and let  $\mu_+$  denote the lowest eigenvalue of  $P_{\tilde{\Omega}_+}$ . Let  $\mu_0$  denote the lowest eigenvalue of  $-h^2\Delta + V$  on  $\mathbb{R}^n$ . By the minimax principle, we have

$$(3.9) \quad \mu_0 \leq \mu_+$$

and recall from Section 1 that, under the assumption  $m = \mathcal{O}(h^{-N_0})$ , we have a good knowledge of the asymptotics for  $\mu_+$  deduced from WKB constructions.

Formula (3.8) shows that:

$$(3.10) \quad P_\Omega - \mu_+ \geq (1/C)(\alpha_+(\Omega) + \beta(\Omega)).$$

Using  $\kappa$  we will also get, **under the assumption**  $m = \mathcal{O}(h^{-N_0})$ ,

$$(3.11) \quad P_\Omega - \sum_1^m (x_j - x_j^\Omega)^{2M} - \mu_+^M(h) \geq (1/C)(\alpha_+(\Omega) + \beta(\Omega)), \text{ if } \Omega \neq \Omega_\pm,$$

with a new constant  $C > 0$ , where

$$(3.12) \quad \mu_+^M(h) - \mu_+(h) = \mathcal{O}(h^{N(M)}),$$

with  $N(M) \rightarrow \infty$  as  $M \rightarrow \infty$  and  $h > 0$  sufficiently small. We observe indeed that:

$$\sum_1^m (\tilde{x}_j - \tilde{x}_j^\Omega)^{2M} = \sum' (\tilde{x}_j)^{2M} + \sum'' (\tilde{x}_j - \tilde{x}_j^\Omega)^{2M},$$

where  $\sum'$  corresponds to the sum over the  $j$  such that  $t_0 \in I_j$ . We now obtain the majoration:

$$\sum_1^m (\tilde{x}_j - \tilde{x}_j^\Omega)^{2M} \leq \sum_1^m \tilde{x}_j^{2M} + \beta(\Omega) \epsilon_0^{2M}; \quad \forall \tilde{x} \in \kappa(\Omega)$$

and then get (3.11) as in [Sj]1 (Section 6), [Sj]2 or [He-Sj].