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Erratum on the Paper Helv.Phys.Acta 65 (1992) 748: Semiclassical expansions of the thermodynamic limit for a Schrödinger equation II. The double well case

By Bernard Helffer

DMI-ENS
45 rue d'Ulm
F-75230 Paris Cédex

and Johannes Sjöstrand

Département de Mathématiques
Université Paris-Sud
F-91405 Orsay
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In our paper there is a gap in the proof of (3.7) p.761. It is indeed not always possible to construct κ with the property that $\kappa(x^{\Omega}) = x^{\Omega_+}$ is the center of the ℓ^{∞}-ball containing one well and satisfying (3.7). We overlooked the rôle of $\Sigma_k(x_k - x_{k+1})^2$ in the comparison of V and $V\circ\kappa$. The text starting from p.760 line -8 to p.761 line -9 has to be modified as follows.

Let $\kappa : \Omega \to \Omega_+ = (I_+)^m$ be the translation with $\kappa(x^{\Omega}) = \tilde{x}^{\Omega_+}$. Here $\tilde{x}_j^{\Omega_+} = 0$, if I_j contains t_0 and in the other case $\tilde{x}_j^{\Omega_+} = \pm\rho\delta_0$ for a suitable ρ independent of Ω and δ satisfying: $0 < \rho < 1$ where we take the sign $+$ if I_j is on the right of t_0 and the sign $-$ if I_j

is on the left of t_0.

The choice of ρ:

In order to precise the choice of ρ, we observe that the problems occur for the boxes which are in the vicinity of one well and we consider a simplified function by taking the quadratic approximation of v at t_0. This permits us to approximate the function $(s,t) \to W(t,s) = (1/16)(t-s)^2 + v(t_0 + (t+s)/2)$ by

$$(s,t) \to W_0(t,s) = C(t-s)^2 + D(t+s)^2$$

with $C > 0$ and $D > 0$. We consider the evolution of this potential restricted to an interval $I(x,\delta)$ centered at the point $(t = x, s = 0)$ and of size $2\delta_0$.

We observe that after dilation we can reduce our study to the case $\delta_0 = 1$, and we consider consequently the function:

$$]0, +\infty[\ni x \to W_0(x,s) \text{ for } s \in [-1,1]\ .$$

We have

$$(\partial_t W_0)(x,s) = 2C(x-s) + 2D(x+s) = (2C+2D)x + (2D-2C)s\ .$$

We first observe that if $C = D$, the function is a monotone increasing function of $x \geq 0$ and we could have taken $\rho = 0$ in this case but this is not the case here (in other papers we do a change of coordinates in order to be in this situation but it is not convenient here); we recall that $C = 1/16$, $D = v"(t_0)/8$ in our case.

Let us assume in order to fix the ideas that $C > D$. Then the derivative as a function of s is minimal at $s = 1$ and $(\partial_t W_0)(x,s)$ is positive for $x \geq (C-D)/(C+D)$.

Our ρ is chosen such that $|(C-D)|/(C+D) < \rho < 1$.

Using (3.4) and our choice of ρ, we see that, chosing ϵ_0 so small such that $\epsilon_0 << \delta_0$ with $\delta_0 \leq 1$, we get for $x \in \Omega$:

(3.7) $\quad V(x) - V(\kappa(x)) \geq C\delta_0^2\beta(\Omega)$

where $\beta(\Omega)$ is the number of intervals I_j which do not contain t_0 (or $-t_0$). Notice that $\beta(\Omega)$ is unchanged by the first "κ".

Actually in this discussion, we have different cases to consider depending on the vanishing $\tilde{x}_j^{\Omega_+}\tilde{x}_{j+1}^{\Omega_+} = 0$ or not. We have only discussed above the most difficult case when this product is 0 with $|\tilde{x}_j^{\Omega_+}| + |\tilde{x}_{j+1}^{\Omega_+}| \neq 0$.

We now compose our two maps, we notice that $\alpha_0(\Omega) \leq \beta(\Omega)$ and we then get a new map $x \to \kappa(x)$, being the composition of reflexions in 0 in some of the coordinates and of a translation, such that

$$\kappa : \Omega \to \tilde{\Omega}_+ \subset 2.\Omega_+,$$

(where Ω is the original box)

$$\kappa(x^\Omega) = \tilde{x}^{\Omega_+}$$

and

(3.8) $\quad V(x) - V(\kappa(x)) \geq (1/C)(\alpha_+(\Omega) + \beta(\Omega)), x \in \Omega.$

C is here a strictly positive constant, independent of Ω, once we have fixed ϵ_0 and δ_0 conveniently as explained before.
Let P_Ω denote the Dirichlet realization of $-h^2\Delta + V$ in Ω and let μ_+ denote the lowest eigenvalue of $P_{\tilde{\Omega}_+}$. Let μ_0 denote the lowest eigenvalue of $-h^2\Delta + V$ on $I\!R^n$. By the minimax principle, we have
(3.9) $\mu_0 \leq \mu_+$
and recall from Section 1 that, under the assumption $m = \mathcal{O}(h^{-N_0})$, we have a good knowledge of the asymptotics for μ_+ deduced from WKB constructions.
Formula (3.8) shows that:
(3.10) $P_\Omega - \mu_+ \geq (1/C)(\alpha_+(\Omega) + \beta(\Omega))$.
Using κ we will also get, **under the assumption** $m = \mathcal{O}(h^{-N_0})$,
(3.11) $P_\Omega - \sum_1^m (x_j - x_j^\Omega)^{2M} - \mu_+^M(h) \geq (1/C)(\alpha_+(\Omega) + \beta(\Omega))$, if $\Omega \neq \Omega_\pm$,
with a new constant $C > 0$, where
(3.12) $\mu_+^M(h) - \mu_+(h) = \mathcal{O}(h^{N(M)})$,
with $N(M) \to \infty$ as $M \to \infty$ and $h > 0$ sufficiently small. We observe indeed that:
$\sum_1^m (\tilde{x}_j - \tilde{x}_j^\Omega)^{2M} = \sum' (\tilde{x}_j)^{2M} + \sum'' (\tilde{x}_j - \tilde{x}_j^\Omega)^{2M}$,
where $\sum'$ corresponds to the sum over the j such that $t_0 \in I_j$. We now obtain the majoration:

$$\sum_1^m (\tilde{x}_j - \tilde{x}_j^\Omega)^{2M} \leq \sum_1^m \tilde{x}_j^{2M} + \beta(\Omega)\epsilon_0^{2M} \; ; \; \forall \tilde{x} \in \kappa(\Omega)$$

and then get (3.11) as in $[Sj]_1$ (Section 6), $[Sj]_2$ or [He-Sj].