Localization for semiclassical continuous random Schrödinger operators. II, The random displacement model

Autor(en): **Klopp, Frédéric**

Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **66 (1993)**

Heft 7-8

PDF erstellt am: **26.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-116591

Helv Phys Acta
Vol. 66 (1993)

0018-0238/93/080810-32$1.50+0.20/0

Localization for Semiclassical Continuous Random Schrödinger Operators II: the Random Displacement Model.

Frédéric Klopp

U.R.A 760 C.N.R.S
Département de Mathématique, Bât. 425,
Université de Paris-Sud, Centre d'Orsay,
91405 Orsay Cédex, France
e-mail: klopp@lanor.matups.fr

16.IX.1993

Abstract. In $L^2(\mathbf{R}^\mathbf{d})$, we study a perturbation of a semi-classical periodic Schrödinger operator where each of the wells has been randomly deviated from its initial position. For this operator, we show exponential localization for energies close to the bottom of the spectrum. Then we study the effect of random perturbation of the bottom of each of the wells of the previous model. We show that the summing up of the two types of disorder leads to a stronger localization than if we would only consider one kind of disorder alone.

Résumé. Dans $L^2(\mathbf{R}^\mathbf{d})$, nous étudions une perturbation d'un opérateur de Schrödinger semi-classique à potentiel périodique dont chacun des puits a été aléatoirement déplacé. Pour cet opérateur, nous démontrons un résultat de localisation exponentielle pour des énergies voisines de la borne inférieure du spectre. Puis nous étudions l'effet d'une perturbation aléatoire des fonds de puits du modèle précédemment étudié. Nous constatons que l'effet conjugué des deux types de désordre conduit à une localisation plus forte que celle due à l'un de ces désordres pris tout seul.

0) Introduction.

In [Kl 2], we studied the following class of random perturbations of a periodic Schrödinger operator: in each of its wells, the periodic potential was perturbated by some compactly supported function which size was given by a random variable; all these random variables were assumed to be independently identically distributed. We proved localization results for such random operators. In this work, we study other classes of random perturbations of a periodic Schrödinger operator. In our first model, we will randomly distort the lattice that is defined by the periodic potential, at each of its sites and study the random operator thus obtained. Then as a second model, we will consider the sum of both disorders that is, each of the lattice sites will be randomly distorted and the bottom of the wells will be randomly perturbed.

The hamiltonians we will study are of the following form:

$$P_{q,\omega} = -h^2\Delta + \sum_{\gamma\in\mathbb{Z}^d} q_\gamma\theta(x-\gamma-\omega_\gamma) = -h^2\Delta + V_{q,\omega}, \tag{0.1}$$

where

(*) θ is a $\mathcal{C}^\infty$ function supported in a sufficiently small compact,

(**) $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ is a collection of i.i.d random variables valued in G, some sufficiently small neighborhood of 0 in $\mathbb{R}^d$,

(* * *)

(1) either $\forall\gamma\in\mathbb{Z}^d$, $q_\gamma = 1$; in this case, we will speak of model 1.

(2) or $\forall\gamma\in\mathbb{Z}^d$, $q_\gamma = 1 + t_\gamma$, $(t_\gamma)_{\gamma\in\mathbb{Z}^d}$ is a collection of i.i.d random variables valued in some neighborhood of 0 in $\mathbb{R}$; in this case we speak of model 2.

Such hamiltonians are models used in solid state physics (see, for example, the works of B. Halperin [Ha], E. Lieb and D. Mattis [Li-Mat]). They describe the behaviour of an electron in a pure crystal (model 1) (or in an alloy (model 2)) which lattice structure was disturbed.

Mathematicians also have been interested in this class of random Schrödinger operators (see, for example [Ki 2]). In [Ki-Ma 1], W. Kirsch and F. Martinelli characterized the set supporting the spectrum of such hamiltonians (see also [Ki 1]). In [Ki-Ma 2], they defined and studied the integrated density of states of these operators, and gave asymptotics for this quantity in the regimes of very large positive or negative energy.

In this paper, we are interested in studying the spectrum of $P_{q,\omega}$, more precisely the nature of this spectrum. Using the ergodicity of the random field defining the potentials seen, we know that the nature of the spectrum of $P_{q,\omega}$ is the same for almost every realization of the potential (see for example [C-L], [Ki 2] or [P-Fi]). In [Co-Hi], J-M. Combes and P. Hislop study ,among others, an operator similar to $P_{q,\omega}$. Nevertheless, in their study, the presence of the random variables $(t_\gamma)_{\gamma\in\mathbb{Z}^d}$ is essential as it is the main argument to prove the Wegner estimate.

Let us first discuss model 1. Let Q be the following operator: $Q = -h^2\Delta + \theta$. Let $\mu(h)$ be the ground state of Q. The semi-classical method we use (i.e we study $P_{1,\omega}$ in the limit h tends to 0) permits us only to study $P_{1,\omega}$ in some fixed energy interval where we know precise information on the spectral data for Q; in the present case we choose this interval to be a neighborhood of $\mu(h)$, the minimum of the spectrum of Q. First, as in [Kl 2], we will restrict $P_{1,\omega}$ to some suitable energy interval, neighborhood of $\mu(h)$, and show that it is then unitarily equivalent to some matrix which coefficients we control. To the opposite of what happens in [Kl 2], $P_{1;\omega}$ is not a small perturbation of $P_{1,0}$, the periodic operator where all the $(q_\gamma)_{\gamma\in\mathbb{Z}^d}$ are set to 1 and the $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ to 0. Indeed, one notices if G is much larger than supp(θ), for some configuration $\omega = (\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$,

$$\| P_{1,\omega} - P_{1,0} \| = | \theta |_\infty .$$

So one can not use simple perturbation theory to reduce the operator $P_{1,\omega}$ from the reduction known for $P_{1,0}$ by Floquet theory. Instead of this, we use a ω-uniform version of the results of U. Carlsson [Ca]. The main assumption for this to work is that, uniformly in ω, the Agmon distance relevant for our problem does not degenerate. This permits us to reduce, uniformly in ω, the operator $P_{1,\omega}$ restricted to an energy interval neighborhood of $\mu(h)$, to a matrix acting on $\ell^2(\mathbb{Z}^d)$.

Under suitable geometric assumptions on supp(θ) and G, as all the wells of $V_{1,\omega}$ are at the same energy level, the main term in the reduced matrix will be the interaction matrix (see [Ca] or [He-Sj 1] for the case of finitely many wells). This matrix has only non zero terms on the first off-diagonals (i.e the terms indexed by (α,β) with $|\alpha-\beta|=1$). Using the works of B. Helffer and J. Sjöstrand [He-Sj 1] and [He-Sj 2], we study these terms carefully. Under suitable geometric assumptions on θ and the support of the distribution of the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$, the interaction coefficient between the wells α and β is only a function of the Agmon distance between $\alpha+\omega_\alpha$ and $\beta+\omega_\beta$.

Using this study, we are able to show a Wegner estimate for this matrix in the high energy regime that is, in this case, for energies outside some arbitrarily small neighborhood of $\mu(h)$ when h is small enough. Then using a rewritten version of Theorem 1.7 of [Kl 2], we are able to prove exponential localization (when h is small enough) for $P_{1,\omega}$.

Let us mention that, if in the reduced matrix, we only keep the interaction matrix, then the operator we get is close to the "off-diagonal disorder" model studied by W. Faris (see [Fa 1] and [Fa 2]) for which he proved localization at high energy (see also the section 5 of the work of M. Aizenman and S. Molchanov [Ai-Mo]).

For the second model, as the random variables $(t_\gamma)_{\gamma\in\mathbb{Z}^d}$ are supposed to be small, we reduce $P_{q,\omega}$ (restricted to a suitable energy interval) to a well controled infinite matrix using the reduction known for $P_{1,\omega}$ and perturbation theory. We then prove localization results for the second model. This shows how both disorder add up to produce stronger localization. Indeed, we can separate 3 regimes depending on the relative strength of the disorders due to the $(t_\gamma)_{\gamma\in\mathbb{Z}^d}$ and the $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$.

We are in the first regime when the order of magnitude of the random variables $(t_\gamma)_{\gamma\in\mathbb{Z}^d}$ is smaller than the order of magnitude of the interaction coefficient depending on the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$. In this case, we get localization in the same conditions as for model 1. In this regime, one can forget the disorder due to the $(t_\gamma)_{\gamma\in\mathbb{Z}^d}$.

In the second regime, when the order of magnitude of the interaction coefficient depending on the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ is smaller than the order of magnitude of the random variables $(t_\gamma)_{\gamma\in\mathbb{Z}^d}$, we get localization in the same conditions as in the model studied in [Kl 2] that is the whole energy band gets localized. In this regime we can forget the disorder due to the $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$.

In the third and last regime, when both order of magnitudes are equal, we are able to prove localization in the whole energy band though none of the disorders alone would be strong enough to ensure this.

The paper is organized as follows: in section I, we give a precise description of the models we study and state the main results. Section II is devoted to the study of the interaction coefficients relevant to our problem. In section III, we state a generalization of the main localization result of [Kl 2], which we apply to model 1 having proved the Wegner estimate in section IV. We study the model 2 in section V.

Acknowledgement: the author would like to thank the Mittag-Leffler Institute for its hospitality as well as B. Helffer, A. Jensen et J. Sjöstrand, the organizers of the program "Spectral Problems in Mathematical Physics" for their kind invitation to participate.

I) Definitions and Results.

A) The first model.

1) The semi-classical reduction. Let $|\cdot|$ be the supremum norm on $\mathbb{Z}^d$. Let Ω be a relatively compact open subset of $\mathbb{R}^d$, and $\theta \in C_0^\infty\left((-\frac{1}{2},\frac{1}{2})^d\right)$ such that

(H.1)

1) $\overline{\Omega} + \text{supp}\theta \subset (-\frac{1}{2},\frac{1}{2})^d$,

2) $-1 \leq \theta \leq 0$, $\theta^{-1}(-1) = \{0\}$ and -1 is a non degenerate minimum of θ.

Consider, on $L^2(\mathbb{R}^d)$,

$$Q = -h^2\Delta + \theta. \tag{1.4}$$

By standard semi-classical analysis (see, for example [He] and references therein), we know that there exists $h_0 > 0$, $C_0 > 0$ such that, for $h \in (0, h_0)$, there exists $\mu(h) = \inf \sigma(Q)$, a simple eigenvalue of Q, verifying:

* $\mu(h) \to -1$ when $h \to 0$,

** $\sigma(Q) \cap [\mu(h) - 2C_0h, \mu(h) + 2C_0h] = \{\mu(h)\}$

(here $\sigma(Q)$ denotes the spectrum of Q).

Let us denote by $\psi(h)$, the normalized positive eigenfunction of Q associated to $\mu(h)$.

Pick $\omega = (\omega_\alpha)_{\alpha\in\mathbb{Z}^d} \in \Omega^{\mathbb{Z}^d}$. For $\alpha \in \mathbb{Z}^d$, define the functions

$$\theta_{\omega,\alpha}(x) = \theta(x - \alpha - \omega_\alpha) \text{ and } V_\omega = \sum_{\alpha\in\mathbb{Z}^d} \theta_{\omega,\alpha}.$$

and, the transformations

$$\begin{aligned} \tau_\alpha : \Omega^{\mathbb{Z}^d} &\to \Omega^{\mathbb{Z}^d} \\ \omega &\mapsto \tau_\alpha\omega = (\omega_{\alpha+\beta})_{\beta\in\mathbb{Z}^d} \end{aligned}$$

and

$$\begin{aligned} \tau_\alpha : \mathcal{F}(\mathbb{R}^d, \mathbb{C}) &\to \mathcal{F}(\mathbb{R}^d, \mathbb{C}). \\ u &\mapsto \tau_\alpha(u) : \mathbb{R}^d \to \mathbb{C} \\ & \qquad x \mapsto \tau_\alpha(u)(x) = u(x-\alpha) \end{aligned}$$

where $\mathcal{F}(\mathbb{R}^d, \mathbb{C})$ is the space of $\mathbb{C}$-valued functions on $\mathbb{R}^d$.

Then one has, for any $\alpha \in \mathbb{Z}^d$ and $\omega \in \Omega^{\mathbb{Z}^d}$,

$$\tau_\alpha(V_\omega) = V_{\tau_\alpha \omega}. \tag{1.2}$$

Let us consider the following Schrödinger operator acting on $L^2(\mathbb{R}^d)$,

$$P_\omega = -h^2 \Delta + V_\omega. \tag{1.3}$$

As V_ω is bounded for $\omega \in \Omega^{\mathbb{Z}^d}$, P_ω is self-adjoint, semi-bounded from below with domain $H^2(\mathbb{R}^d)$.

For $\omega \in \Omega^{\mathbb{Z}^d}$, we define d_ω the Agmon distance defined by the metric $(V_\omega + 1)dx$. Notice that, by assumption (H.1) 1), for α and β in $\mathbb{Z}^d$ such that $\alpha \neq \beta$,

$$\text{supp}(\theta) + \overline{\Omega} + \alpha \bigcap \text{supp}(\theta) + \overline{\Omega} + \beta = \emptyset. \tag{1.1}$$

This implies that the Agmon distance d_ω does not degenerate when ω varies in $\Omega^{\mathbb{Z}^d}$.

We define $I(h) = [\mu(h) - C_0 h, \mu(h) + C_0 h]$. Let Π_ω be the orthogonal projection on F_ω, the spectral space associated to $I(h)$ and P_ω. We get

Theorem 1.1. *Let P_ω be defined as above and assumption (H.1) hold. Then, there exists $\epsilon_0 > 0$ such that, for $\epsilon \in (0, \epsilon_0)$, there exists $h_\epsilon > 0$ such that, for any $h \in (0, h_\epsilon)$,*

1) there exists $(\varphi_{\alpha,\omega})_{\alpha \in \mathbb{Z}^d}$, a Hilbert basis of F_ω, and a constant $C(h) > 0$ such that,

$$\| e^{\frac{1-\epsilon}{h} d_\omega(\alpha + \omega_\alpha, \cdot)} \varphi_{\alpha,\omega} \| + h \| e^{\frac{1-\epsilon}{h} d_\omega(\alpha+\omega_\alpha, \cdot)} \nabla \varphi_{\alpha,\omega} \| \leq C(h), \tag{1}$$

2) the matrix of $P_{\omega|F_\omega} = P_\omega \Pi_\omega$ expressed in this basis, is

$$H(\omega) = \mu(h) + W(\omega) + M(\omega),$$

where

a) $\mu(h)$ is the first eigenvalue of Q,

b) the two self-adjoint matrix valued mappings $\omega \mapsto H(\omega)$ and $\omega \mapsto M(\omega)$ are in $C^1(\Omega^{\mathbb{Z}^d}, \mathcal{B}(\ell^2(\mathbb{Z}^d))),$.

c) if we write $W(\omega) = ((w(\omega;\alpha,\beta)))_{(\alpha,\beta)\in\mathbb{Z}^d\times\mathbb{Z}^d}$, *one has,*

$$w(\omega;\alpha,\beta) = \begin{cases} \langle \psi_{\alpha,\omega}, \dfrac{\theta_{\alpha,\omega}+\theta_{\beta,\omega}}{2}\psi_{\beta,\omega}\rangle & \text{if } \mid \alpha-\beta \mid = 1 \\ 0 & \text{if not} \end{cases}$$

d) if we write $M(\omega) = ((m(\omega;\alpha,\beta)))_{(\alpha,\beta)\in\mathbb{Z}^d\times\mathbb{Z}^d}$, *then we get*

$$\mid m(\omega;\alpha,\beta) \mid \leq \exp\Big(-\frac{1-\epsilon}{h}\inf_{\substack{\gamma\neq\alpha\\ \gamma'\neq\beta}}(d_\omega(\alpha+\omega_\alpha,\gamma+\omega_\gamma+supp(\theta))+$$

$$+d_\omega(\gamma+\omega_\gamma+supp(\theta),\gamma'+\omega_{\gamma'}+supp(\theta))+d_\omega(\gamma'+\omega_{\gamma'}+supp(\theta),\beta+\omega_\beta))\Big)$$

and

$$\mid \partial_{\omega_\gamma} m(\omega;\alpha,\beta) \mid \leq \begin{cases} e^{-\frac{1-\epsilon}{h}(d_\omega(\alpha+\omega_\alpha,\gamma+\omega_\gamma+\text{supp}(\theta))+d_\omega(\gamma+\omega_\gamma+\text{supp}(\theta),\beta+\omega_\beta))} \\ \quad \text{if } \alpha\neq\gamma \text{ and } \beta\neq\gamma \\ e^{-\frac{1-\epsilon}{h}(d_\omega(\mu+\omega_\mu,\gamma+\omega_\gamma+\text{supp}(\theta))+\inf_{\nu\neq\gamma} d_\omega(\gamma+\omega_\gamma+\text{supp}(\theta),\nu+\omega_\nu))} \\ \quad \text{if } \alpha\neq\beta \text{ and } (\mu,\gamma)=(\alpha,\beta) \text{ or } (\beta,\alpha) \\ e^{-2\frac{1-\epsilon}{h}\inf_{\nu\neq\gamma} d_\omega(\gamma+\omega_\gamma+\text{supp}(\theta),\nu+\omega_\nu)} \\ \quad \text{if } \alpha=\gamma=\beta \end{cases}$$

Remark. In fact, one may weaken assumption (H.1) as the only point that is really needed to get Theorem 1.1 is ω-uniform non degeneracy of the Agmon metric defined by $(V_\omega+1)dx$.

We will not give a detailed proof of this result. To get the reduction to the matrix form, one merely rewrites the proof of the Main Theorem of [Ca] taking into account the uniform nondegeneracy of the relevant Agmon distance. Then to get the estimates on the terms of the matrix of follows the strategy given in [Kl1].

In the next proposition, under additionnal assumptions on the Agmon distance induced by $V_\omega+1$, we will give more precise informations about the matrix $W(\omega)$. This analysis mainly relies on the analysis of the interaction coefficients that was developped be B. Helffer and J. Sjöstrand (see, for example, [He-Sj1], [He-Sj2] or [He]).

Let us assume that

(H.2)

a) if $B_A(0)$ is the smallest closed Agmon ball containing supp(θ) (here the Agmon distance we consider is the one defined by $\theta+1$). Then $B_A(0)$ is strictly convex.

b) if, for $\alpha\in\mathbb{Z}^d$, we define $B_{A,\Omega}(\alpha)=\alpha+\overline{\Omega}+B_A(0)$, then, for $\mid\alpha-\beta\mid=1$ and any $\gamma\neq\alpha,\beta$, one has

$$d_e(B_{A,\Omega}(\alpha),B_{A,\Omega}(\gamma))+d_e(B_{A,\Omega}(\gamma),B_{A,\Omega}(\beta))>d_e(B_{A,\Omega}(\alpha),B_{A,\Omega}(\beta))$$

(here d_e denotes the euclidian distance in $\mathbb{R}^d$.)

Remark.

It is easily seen that this assumption holds if, for example, θ is spherically symmetric (to get (a)) and the euclidian diameters of Ω and supp(θ) are small enough (to get (b)).

Let us define,

$$d_\omega(\alpha,\beta) = d_\omega(\alpha+\omega_\alpha, \beta+\omega_\beta).$$

One shows the

Proposition 1.2. *Under assumption (H.1) and (H.2), there exists $h_0 > 0$ and $C_0 > 0$ such that, for $h \in (0, h_0)$, one has*

$$(1) \qquad w_{\alpha,\beta}(\omega) = h^{1-\frac{d}{2}} a(\omega_\alpha,\omega_\beta) e^{-\frac{d_\omega(\alpha,\beta)}{h}} + \mathcal{O}\left(h^{2-\frac{d}{2}}\right) e^{-\frac{d_\omega(\alpha,\beta)}{h}},$$

where:

$*$ *$a(\omega_\alpha,\omega_\beta)$ is a C^∞ function of $(\omega_\alpha,\omega_\beta)$ in $\Omega\times\Omega$ and continuous in $\overline{\Omega}\times\overline{\Omega}$,*

$**$ *for any $(\omega_\alpha,\omega_\beta) \in \overline{\Omega}\times\overline{\Omega}$,*

$$\frac{1}{C_0} \leq a(\omega_\alpha,\omega_\beta) \leq C_0.$$

and

$$(2) \qquad -h\nabla_{\omega_\alpha} w_{\alpha,\beta}(\omega) = w_{\alpha,\beta}(\omega)\nabla_{\omega_\alpha} d_\omega(\alpha,\beta) + \mathcal{O}\left(h^{2-\frac{d}{2}}\right) e^{-\frac{d_\omega(\alpha,\beta)}{h}},$$

In both formula (1) and (2), the $\mathcal{O}$ is uniform in $(\alpha,\beta) \in \mathbb{Z}^d \times \mathbb{Z}^d$ such that $\mid \alpha-\beta \mid = 1$.

2) The random structure.

Let g be a distribution density in $\mathbb{R}^d$ (i.e $g \geq 0$ and $\int_{\mathbb{R}^d} g = 1$) and G be its closed support, that is

$$G = \{x \in \mathbb{R}^d;\ \forall U, \text{ neighborhood of } x, \text{ one has } \int_U g dx > 0\}.$$

Without restriction, we may suppose that $0 \in G$. Suppose that G and θ satisfy (H.1) and (H.2). Moreover assume that

(H.3)

a)

$$S_0 = \inf_{\substack{\omega\in G^{\mathbb{Z}^d}\\ \alpha\neq\beta}} d_\omega(\alpha,\beta) < \inf_{\substack{\omega\in G^{\mathbb{Z}^d}\\ |\alpha-\beta|\geq 2}} d_\omega(\alpha+\omega_\alpha, \beta+\omega_\beta+\text{supp}\theta),$$

and

$$S_0 < 2 \inf_{\substack{\omega \in G^{\mathbb{Z}^d} \\ |\alpha-\beta|\geq 1}} d_\omega(\alpha + \omega_\alpha, \beta + \omega_\beta + \text{supp}\theta),$$

b) $\exists \delta_0 > 0$ and ϵ_0 such that, if $\Omega = G + B(0, \epsilon_0)$, then $\forall \omega \in \Omega^{\mathbb{Z}^d}$ and $\forall (\alpha, \beta) \in \mathbb{Z}^d \times \mathbb{Z}^d$ such that $| \alpha - \beta |= 1$, one has

$$d_\omega(\alpha, \beta) \leq S_0 + \delta_0 \Rightarrow \forall \gamma \neq \alpha,\ \gamma \neq \beta \begin{cases} d_\omega(\alpha, \gamma) \geq S_0 + 2\delta_0 \\ d_\omega(\beta, \gamma) \geq S_0 + 2\delta_0 \end{cases}$$

c) $\exists \rho_0 > 0$, $\eta_0 > 0$ such that, $\forall \eta \in [0, \eta_0]$,

$$\int_{\mathbb{R}^d} \sup_{\nu \in B(0,1)} | g(x + \eta\nu) - g(x) | \, dx \leq \left(\frac{\eta}{\eta_0}\right)^{\rho_0}.$$

Remark.

Point a) of assumption (H.3) ensures that the leading order term in $H(\omega) - \mu(h)$ will be $W(\omega)$. It may be obtained by assuming that G and supp(θ) are small enough.

Point b) is a geometric assumption on G. We assume that a well can be nearest neighbor (in the Agmon distance sense) of at most one other well. So a well may interact significantly with at most one of its $| \cdot |_\infty$-nearest neighbors. This assumption is merely technical. It restricts the choice for G (e.g G can not be a square). However, it may easily be relaxed (though its formulation gets more complicated) so as to include a larger class of admissible G's.

Point d) is a regularity assumption on g. One can notice that it implies that g is bounded. Point d) will be essential in our proof of the Wegner estimate. Moreover it will allow us to use a high energy regime at the edges of the band of spectrum we study. Indeed we will only be able to prove localization near the edges of the part of the spectrum we consider. The fact that the random variables $(\omega_\alpha)_{\alpha \in \mathbb{Z}^d}$ admit a density will tell us that the probability that a finite rank restriction of our reduced operator is of this size is very small

For $0 < h < h_0$, we define the interval

$$I(h_0, h) = \mu(h) + [-C_0 h, -e^{-\frac{S_0+h_0}{h}}] \cup [e^{-\frac{S_0+h_0}{h}}, C_0 h]$$

As the random variables $(\omega_\alpha)_{\alpha \in \mathbb{Z}^d}$ are supposed to be independent identically distributed, we may apply Theorem 2 of [Ki-Ma 1], and so get that $\sigma(P_\omega)$, the spectrum of P_ω, its pure point part, its absolutely continuous part and its singular continuous part are non-random sets.

We now state our main result concerning model 1 that is,

Theorem 1.3.

Let G and θ satisfy (H.1) and (H.2). Let $(\omega_\alpha)_{\alpha\in\mathbb{Z}^d}$ be a family of independent identically distributed random variables with common distribution density g (with closed support G) satisfying (H.3). Let P_ω be defined by (1.1). Then there exists $h_0 > h_0' > 0$ such that, for $h \in (0, h_0')$, with probability 1, one has,

a)

$$\sigma(P_\omega) \cap I(h_0, h) \neq \emptyset,$$

b) the spectrum of P_ω in $I(h_0, h)$ is pure point,

c) if φ is an eigenfunction of P_ω associated to an eigenvalue in $I(h_0, h)$, then there exists $C(h, \varphi) > 0$ such that, for $x \in \mathbb{R}^d$,

$$| \varphi(x) | \leq C(h, \varphi) e^{-\frac{h_0'}{h}|x|}.$$

Remark. Using the ergodicity of our family of operators P_ω, one shows that, there exists some constant $C > 0$ such that, for ω in some set of probability 1, for h small enough,

$$\| P_{\omega|F_\omega} - \mu(h) \| \geq \frac{1}{C} h^{1-\frac{d}{2}} e^{-\frac{S_0}{h}},$$

(see, for example, the proof of point a) given in section 3.)

Looking at sublattices of $\mathbb{Z}^d$, using Theorem 4 of [Ki-Ma 2] and semi-classical techniques conjuguated to Floquet theory, one may prove that, for any $\epsilon_0 > 0$ small enough and h small enough, there exists an interval of width $2e^{-\frac{S_0+\epsilon_0}{h}}$ around $\mu(h)$ that is in the spectrum of P_ω for almost every ω.

So, by point b) and c), we see that, when h goes to 0, most of the spectrum becomes localised.

We see that we are not able to prove that the spectrum of P_ω is localized for energies close to $\mu(h)$. This is mainly due to the fact that for energies to close to $\mu(h)$, the Wegner estimate breaks down (see also [Fa 2] and [Ai-Mo]). The reason for this breakdown is the same as for the break down in the "off-diagonal" disorder.

B) The second model.

1) The reduction theorem.

Using the same notations as in part A) of this section, we consider the operator

$$P_{t,\omega} = -h^2\Delta + \sum_{\gamma\in\mathbb{Z}^d} (1 + t_\gamma)\theta(x - \gamma - \omega_\gamma), \tag{1.5}$$

where $(t_\gamma)_{\gamma\in\mathbb{Z}^d}$ are real parameters in $[-\frac{1}{2}C_0 h, \frac{1}{2}C_0 h]$. We suppose that Ω and θ satisfy (H.1) and that, for $\gamma \in \mathbb{Z}^d$, $\omega_\gamma \in \Omega$. Then, for every (t, ω), $P_{t,\omega}$ is a semi-bounded self-adjoint operator with domain H^2.

Let $\Pi_{t,\omega}$ be the orthogonal projection on $F_{t,\omega}$, the spectral space associated to $I(h) = [\mu(h) - C_0 h, \mu(h) + C_0 h]$ and $P_{t,\omega}$. We show the

Theorem 1.4.

Let $P_{t,\omega}$ be defined as above, and suppose that Ω and θ satisfy (H.1). Then there exists $\epsilon_0 > 0$ such that, for $\epsilon \in (0, \epsilon_0)$, there exist $h_\epsilon > 0$ such that, for $h \in (0, h_\epsilon)$, $\omega \in \Omega^{\mathbb{Z}^d}$ and $t \in [-\frac{1}{2}C_0 h, \frac{1}{2}C_0 h]^{\mathbb{Z}^d}$,

1) there exists $(\varphi_{\alpha,t,\omega})_{\alpha\in\mathbb{Z}^d}$, a Hilbert basis of $F_{t,\omega}$ and a constant $C(h) > 0$ such that,

$$(1) \qquad \| e^{\frac{1-\epsilon}{h} d_\omega(\alpha+\omega_\alpha,\cdot)}\varphi_{\alpha,t,\omega} \| + h \| e^{\frac{1-\epsilon}{h} d_\omega(\alpha+\omega_\alpha,\cdot)}\nabla\varphi_{\alpha,t,\omega} \| \leq C(h),$$

2) the matrix of $P_{t,\omega|F_{t,\omega}} = P_{t,\omega}\Pi_{t,\omega}$ expressed in this basis, is

$$H(t,\omega) = \mu(h) + W(\omega) + D(t) + M(t,\omega),$$

where

a) $\mu(h)$ is the first eigenvalue of Q,

b) the self-adjoint matrix valued mapping $(t,\omega) \mapsto H(t,\omega)$ is analytic in t and C^1 in ω, the mapping $t \mapsto D(t)$ is analytic and the mapping $\omega \mapsto W(\omega)$ is C^1,

c) point c) of Theorem 1.1 is valid for $W(\omega)$,

d) $D(t)$ is the diagonal matrix $((b(t_\alpha\delta_{\alpha,\beta}))_{(\alpha,\beta)\in\mathbb{Z}^d\times\mathbb{Z}^d}$ where b is an analytic bijection between two neighborhoods of 0,

e) if we write $M(t,\omega) = ((m(t,\omega;\alpha,\beta)))_{(\alpha,\beta)\in\mathbb{Z}^d\times\mathbb{Z}^d}$, then, for $m(\omega;\alpha,\beta)$ and $\nabla_{t_\gamma,\omega_\gamma} m(t,\omega;\alpha,\beta)$, we get the estimates given in point d) of Theorem 1.1.

Remark.

a) We will not give a detailed proof of this result. Using Theorem 1.1, one may prove it by regular perturbation theory (using the same method as in [Kl 1]), as for $\omega \in \Omega^{\mathbb{Z}^d}$ and $t \in [-\frac{1}{2}C_0 h, \frac{1}{2}C_0 h]^{\mathbb{Z}^d}$,

$$\| P_{t,\omega} - P_\omega \| \leq \frac{1}{2}C_0 h \text{ and } \sigma(P_\omega\Pi_\omega) \subset [\mu(h) - e^{-\frac{c_0}{h}}, \mu(h) + e^{-\frac{c_0}{h}}]$$

for some $c_0 > 0$ and h small enough.

b) By section 3 of [Kl 1], we know that the bijection b satisfies, for u in some complex neighbohood of 0,

$$b(u) = \mu(u,h) - \mu(h) = \rho(h)t(1 + tq(t))$$

where $\mu(u,h)$ is the infimum of the spectrum of $Q + u\theta$ (Q is defined in part A) of this section), where q is analytic and, where $\frac{1}{C} < \rho(h) < C$ for h small enough and some $C > 0$ independent of h.

2) Localization for the second model.

Let g_ω be a distribution density on $\mathbb{R}^d$ satisfying (H.1)-(H.3). Let g_T be a distribution density on $\mathbb{R}$ supported in $[-1,1]$ such that, there exists $\rho_0 > 0$ and $\epsilon_0 > 0$ such that for $\epsilon \in [0, \epsilon_0]$,

$$\int_{\mathbb{R}^d} \sup_{|u|\leq\epsilon} \mid g_T(x+u) - g(x) \mid dx \leq \left(\frac{\epsilon}{\epsilon_0}\right)^{\rho_0}.$$

Pick a function $a(h)$ such that, for $h \in (0, h_0)$, $0 < a(h) < \frac{1}{2}C_0 h$. Suppose that the limit of $h \log a(h)$ exists when h tends to 0. Define $g_{a,T}(u) = \frac{1}{a(h)} g(\frac{1}{a(h)}u)$ and $S_0 = \inf\limits_{\omega\in(G_\Omega)^{\mathbb{Z}^d}} \inf\limits_{\alpha\neq\beta} d_\omega(\alpha,\beta)$ (here G_Ω denotes the closed support of g_Ω).

We will now define two different asymptotic regimes in ω and t depending on $a(h)$. We will say that we are in regime (1) if

$$\lim_{h\to 0} h \log a(h) < -S_0,$$

and then, for $0 < h < h_0$, define

$$I_1(h_0, h) = \mu(h) + [-C_0 h, -e^{-\frac{S_0+h_0}{h}}] \cup [e^{-\frac{S_0+h_0}{h}}, C_0 h].$$

We will be in regime (2) if

$$\lim_{h\to 0} h \log a(h) \geq -S_0,$$

and then, for $0 < h < h_0$, define

$$I_2(h_0, h) = \mu(h) + [-C_0 h, C_0 h].$$

Then the following result holds

Theorem 1.5.

Let $g_{a,T}$ and g_Ω be constructed as above. Let $(t_\alpha, \omega_\alpha)_{\alpha\in\mathbb{Z}^d}$ be independent identically distributed random variables having $g_{a,T} \otimes g_\Omega$ as common distribution density. Let $P_{t,\omega}$ be the operator defined by (1.4).

Then, in regime (k) for $k = 1$ or 2, there exists $h_0 > h_0' > 0$ such that for $h \in (0, h_0')$, with probability 1,

a) $\sigma(P_{t,\omega}) \cap I_k(h_0, h) \neq \emptyset$,

b) the spectrum of $P_{t,\omega}$ in $I_k(h_0, h)$ is pure point,

c) if φ is an eigenfunction associated to E, an eigenvalue of $P_{t,\omega}$ in $I_k(h_0, h)$ then there exists $C(h, \varphi) > 0$ such that, for $x \in (R)^d$,

$$\mid \varphi(x) \mid \leq C(h, \varphi) e^{-\frac{h_0'}{h}|x|}.$$

Remark. A priori in the way the result is stated one sees only two different asymptotic regimes. In fact these should be separated into three regimes. The regimes are:

(1) when $\lim_{h\to 0} h\log a(h) < -S_0$,

(2) when $\lim_{h\to 0} h\log a(h) = -S_0$,

(3) when $\lim_{h\to 0} h\log a(h) > -S_0$.

In regime (1), the effect due to the random variables $(\omega_\alpha)_{\alpha\in\mathbb{Z}^d}$ dominates the one created by $(t_\alpha)_{\alpha\in\mathbb{Z}^d}$. So the $(\omega_\alpha)_{\alpha\in\mathbb{Z}^d}$ are alone responsible for localization. Hence only a part of the band gets localized.

In regime (2), it is the effect due to the random variables $(t_\alpha)_{\alpha\in\mathbb{Z}^d}$ that dominates; the whole band gets localised.

In regime (3), both of the effects are of the same order of magnitude but none alone would be strong enough to localize the whole band. It is the summing up of both effects that localizes the band.

II) The interaction coefficients.

a) An auxiliary lemma.

Let $B_{A,\omega}(\alpha) = \alpha + \omega_\alpha + B_A(0)$. One proves

Lemma 2.1. *Under assumptions (H.1)-(H.2), for $\mid \alpha - \beta \mid = 1$, the following assertions hold*

a) For any $\omega \in \Omega^{\mathbb{Z}^d}$, there exists a unique minimal Agmon geodesic going from $\alpha+\omega_\alpha$ to $\beta+\omega_\beta$; this geodesic depends only on $(\omega_\alpha,\omega_\beta)$.

b) $d_\omega(\alpha,\beta)$ depends only on $(\omega_\alpha,\omega_\beta)$ and is a C^∞ function on $\Omega\times\Omega$.

c) Let $x_0(\omega_\alpha,\omega_\beta)$ be the intersection point of the unique minimal Agmon geodesic going from $\alpha+\omega_\alpha$ to $\beta+\omega_\beta$ and the hyperplane $\Gamma_{\alpha,\beta} = \{x \in \mathbb{R}^n;\ \mid x-\alpha \mid = \mid x-\beta \mid\}$. Then $x_0(\omega_\alpha,\omega_\beta)$ is a C^∞ function of $(\omega_\alpha,\omega_\beta)\in\Omega\times\Omega$.

d) $\nabla_{\omega_\alpha} d_\omega(\alpha,\beta) = -\nabla_{\omega_\beta} d_\omega(\alpha,\beta)$.

e) There exists $C>1$ such that, for h small enough and $(\omega_\alpha,\omega_\beta)\in\Omega\times\Omega$,

$$\frac{1}{C} \leq \mid \nabla_{\omega_\alpha} d_\omega(\alpha,\beta) \mid \leq C.$$

Proof. To prove Lemma 2.1, we will use the following elementary geometric lemma, the proof of which is left to the reader:

Lemma 2.2. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two strictly convex compact subsets of $\mathbb{R}^d$ with C^∞ boundary. Then, for $z\in\mathbb{R}^d$ such that $(z+\mathcal{C})\cap\mathcal{C}' = \emptyset$,*

1) there exists a unique $(x(z),x'(z))\in(z+\mathcal{C})\times\mathcal{C}'$ such that $\mid x(z)-x'(z)\mid = d_e(z+\mathcal{C},\mathcal{C}')$.

2) $z\mapsto x(z)$ and $z\mapsto x'(z)$ are C^∞ functions; hence $z\mapsto d_e(z+\mathcal{C},\mathcal{C}')$ is C^∞.

3) For any compact K such that $(K+\mathcal{C})\cap\mathcal{C}'=\emptyset$, there exists $C_K>0$ such that

$$\frac{1}{C_K}\leq\mid\nabla_z d_e(z+\mathcal{C},\mathcal{C}')\mid\leq C_K.$$

By Proposition 6.5 of [He-Sj 1] (see also [He] Proposition 4.4.2), we know, that, for any $\alpha\in\mathbb{Z}^d$, for $x\in\partial B_{A,\omega}(\alpha)$ (here $\partial B_{A,\omega}(\alpha)$ is the boundary of $B_{A,\omega}(\alpha)$), there exists a unique minimal Agmon geodesic from $\alpha+\omega_\alpha$ to x. We also know that $\partial B_{A,\omega}(\alpha)$ is C^∞ (see [He-Sj 1] and [He]).

Let $\mathcal{C}=B_{A,\omega}(\alpha)$, $\mathcal{C}'=B_{A,\omega}(\beta)$ and $z(\omega)=z(\omega_\alpha,\omega_\beta)=\omega_\alpha-\omega_\beta$. Then, by assumption (H.1), we know that $(\overline{z(\Omega\times\Omega)}+\mathcal{C})\cap\mathcal{C}'=\emptyset$. So we may apply Lemma 2.2. For x and x' defined by Lemma 2.2, let $x_\alpha(\omega)=x(z(\omega))+\omega_\alpha\in\partial B_{A,\omega}(\alpha)$ and $x_\beta(\omega)=x'(z(\omega))+\omega_\alpha\in\partial B_{A,\omega}(\beta)$. Let $\gamma_{\omega_\alpha,\omega_\beta}$, be the path sum of the minimal Agmon geodesic from $\alpha+\omega_\alpha$ to $x_\alpha(\omega)$, the straight line from $x_\alpha(\omega)$ to $x_\beta(\omega)$ and the minimal Agmon geodesic from $x_\beta(\omega)$ to $\beta+\omega_\beta$. Then, one sees easily that $\gamma_{\omega_\alpha,\omega_\beta}$ is the unique minimal Agmon geodesic (for the metric defined by $(V_\omega+1)$) from $\alpha+\omega_\alpha$ to $\beta+\omega_\beta$, which proves point a) of Lemma 2.1.

Points c), d) and e) of Lemma 2.1 are immediate consequences of points 2) and 3) of Lemma 2.2. We know that $x_\alpha(\omega)$ and $x_\beta(\omega)$ are C^∞ in $(\omega_\alpha,\omega_\beta)$. Moreover, by assumption (H.1), we know that

$$B_{A,\Omega}(\alpha)\subset\left\{x\in\mathbb{R}^d;\ \mid x-\alpha\mid<\mid x-\beta\mid\right\}$$
$$B_{A,\Omega}(\beta)\subset\left\{x\in\mathbb{R}^d;\ \mid x-\beta\mid<\mid x-\alpha\mid\right\}.$$

So, for $(\omega_\alpha,\omega_\beta)\in\overline{\Omega\times\Omega}$,

$$(x_\alpha(\omega)-x_\beta(\omega))\cdot(\alpha-\beta)\neq 0.$$

This in turn implies

$$\Gamma_{\alpha,\beta}\cap\gamma_{\omega_\alpha,\omega_\beta}=\{x_0(\omega_\alpha,\omega_\beta)\}$$

where

$$x_0(\omega_\alpha,\omega_\beta)=x_\alpha(\omega)+\frac{\beta^2-\alpha^2}{(x_\alpha(\omega)-x_\beta(\omega))\cdot(\alpha-\beta)}(x_\alpha(\omega)-x_\beta(\omega)).$$

So one derives point b) of Lemma 2.1 from point 2) of Lemma 2.2. This ends the proof of Lemma 2.1.

■

b) Proof of Proposition 1.2.

We recall that ψ denotes the normalized positive eigenvector associated to $\mu(h)$, the ground state of the operator $Q=-h^2\Delta+\theta$, and that $\psi_{\alpha,\omega}(x)=\psi(x-\alpha-\omega_\alpha)$. By definition, for $\mid\alpha-\beta\mid=1$,

$$w_{\alpha,\beta}(\omega)=\langle\psi_{\alpha,\omega},\left(\frac{\theta_{\alpha,\omega}+\theta_{\beta,\omega}}{2}\right)\psi_{\beta,\omega}\rangle. \tag{2.1}$$

Following the appendix of [Kl 1] (see also [He-Sj 1]), we define $\Gamma^+_{\alpha,\beta} = \{x \in \mathbb{R}^n;\ | x-\alpha | \leq | x-\beta |\}$ and $\Gamma^-_{\alpha,\beta} = \{x \in \mathbb{R}^n;\ | x-\alpha | \geq | x-\beta |\}$, and compute

$$\begin{aligned}
w_{\alpha,\beta}(\omega) &= \int_{\Gamma^+_{\alpha,\beta}} \psi_{\alpha,\omega} \cdot \left(\frac{\theta_{\alpha,\omega}+\theta_{\beta,\omega}}{2}\right) \psi_{\beta,\omega} + \int_{\Gamma^-_{\alpha,\beta}} \psi_{\alpha,\omega} \cdot \left(\frac{\theta_{\alpha,\omega}+\theta_{\beta,\omega}}{2}\right) \psi_{\beta,\omega} \\
&= \frac{1}{2}\left(\int_{\Gamma^+_{\alpha,\beta}} \psi_{\alpha,\omega} \cdot \theta_{\beta,\omega}\psi_{\beta,\omega} + \int_{\Gamma^-_{\alpha,\beta}} \psi_{\alpha,\omega} \cdot \theta_{\alpha,\omega}\psi_{\beta,\omega}\right),
\end{aligned} \tag{2.2}$$

as, for any w, $\mathrm{supp}(\theta_{\alpha,\omega}) \cap \Gamma^+_{\alpha,\beta} = \mathrm{supp}(\theta_{\beta,\omega}) \cap \Gamma^-_{\alpha,\beta} = \emptyset$.

Using the fact that $Q\psi = \mu(h)\psi$ and Green's formula, we get

$$\begin{aligned}
w_{\alpha,\beta}(\omega) &= \frac{1}{2}\left(\int_{\Gamma^+_{\alpha,\beta}} \psi_{\alpha,\omega} \cdot \left(h^2\Delta + \mu(h)\right) \psi_{\beta,\omega} + \int_{\Gamma^-_{\alpha,\beta}} \psi_{\beta,\omega} \cdot \left(h^2\Delta + \mu(h)\right) \psi_{\alpha,\omega}\right) \\
&= \frac{1}{2}\left(h^2 \int_{\Gamma_{\alpha,\beta}} \left(\psi_{\beta,\omega}\nabla\psi_{\alpha,\omega} - \psi_{\alpha,\omega}\nabla\psi_{\beta,\omega}\right) \cdot \vec{n}\, d\sigma + \right. \\
&\qquad\qquad \left. + \int_{\Gamma^+_{\alpha,\beta}} \psi_{\beta,\omega} \cdot \left(h^2\Delta + \mu(h)\right) \psi_{\alpha,\omega}\right) \\
&\quad + \frac{1}{2}\left(h^2 \int_{\Gamma_{\alpha,\beta}} \left(\psi_{\beta,\omega}\nabla\psi_{\alpha,\omega} - \psi_{\alpha,\omega}\nabla\psi_{\beta,\omega}\right) \cdot \vec{n}\, d\sigma + \right. \\
&\qquad\qquad \left. + \int_{\Gamma^-_{\alpha,\beta}} \psi_{\alpha,\omega} \cdot \left(h^2\Delta + \mu(h)\right) \psi_{\beta,\omega}\right)
\end{aligned} \tag{2.3}$$

where $\vec{n}$ is the normal to $\Gamma_{\alpha,\beta}$ oriented toward α.

Hence, because $Q\psi = \mu(h)\psi$ and $\mathrm{supp}(\theta_{\alpha,\omega}) \cap \Gamma^+_{\alpha,\beta} = \mathrm{supp}(\theta_{\beta,\omega}) \cap \Gamma^-_{\alpha,\beta} = \emptyset$, we get

$$w_{\alpha,\beta}(\omega) = h^2 \int_{\Gamma_{\alpha,\beta}} \left(\psi_{\beta,\omega}\nabla\psi_{\alpha,\omega} - \psi_{\alpha,\omega}\nabla\psi_{\beta,\omega}\right) \cdot \vec{n}\, d\sigma. \tag{2.4}$$

Now, following [He-Sj 1] pp 397-398 (see also [He-Sj 2] Theorem 6.1.1), using the W.K.B expansions known for $\psi_{\alpha,\omega}$ and $\psi_{\beta,\omega}$ along $\gamma_{\alpha,\beta}(\omega)$ (see [He-Sj 1] Theorem 5.8 and Remark 3.10), and a stationnary phase method, by Lemma 2.1, we get

$$w_{\alpha,\beta}(\omega) = h^{1-\frac{n}{2}} a(x_0(\omega_\alpha,\omega_\beta)) e^{-\frac{1}{h}d_\omega(\alpha,\beta)} + \mathcal{O}(h^{2-\frac{n}{2}} e^{-\frac{1}{h}d_\omega(\alpha,\beta)}), \tag{2.5}$$

where $x \mapsto a(x)$ is a C^∞ and $\frac{1}{C} > a(x) > C$ for some $C > 0$, for x in a neighborhood of $x_0(\omega_\alpha, \omega_\beta)$ and h small enough.

To estimate $\nabla_{\omega_\alpha} w_{\alpha,\beta}(\omega)$, we just derivate (2.4) with respect to ω_α to get

$$\nabla_{\omega_\alpha} w_{\alpha,\beta}(\omega) = h^2 \int_{\Gamma_{\alpha,\beta}} \psi_{\beta,\omega} \nabla_{\omega_\alpha} (\nabla\psi_{\alpha,\omega} \cdot \vec{n}) - \nabla_{\omega_\alpha}\psi_{\alpha,\omega} (\nabla\psi_{\beta,\omega} \cdot \vec{n})\, d\sigma. \tag{2.6}$$

Computing $\nabla_{\omega_\alpha}(\nabla\psi_{\alpha,\omega}\cdot\overrightarrow{n})$ and $\nabla_{\omega_\alpha}\psi_{\alpha,\omega}$ using the W.K.B expansions known for ψ near the point $x_0(\omega_\alpha,\omega_\beta)$, and using a stationnary phase method as above, by Lemma 2.1, we get

$$\nabla_{\omega_\alpha}w_{\alpha,\beta}(\omega)=-h^{-\frac{n}{2}}a(x_0(\omega_\alpha,\omega_\beta))e^{-\frac{1}{h}d_\omega(\alpha,\beta)}\nabla_{\omega_\alpha}d_\omega(\alpha,\beta)+\mathcal{O}(h^{1-\frac{n}{2}}e^{-\frac{1}{h}d_\omega(\alpha,\beta)}),$$

that is, using the h-uniform boundedness of $\frac{1}{a(x_0(\omega_\alpha,\omega_\beta))}$,

$$-h\nabla_{\omega_\alpha}w_{\alpha,\beta}(\omega)=w_{\alpha,\beta}(\omega)\nabla_{\omega_\alpha}d_\omega(\alpha,\beta)+\mathcal{O}(h^{2-\frac{n}{2}}e^{-\frac{1}{h}d_\omega(\alpha,\beta)}).$$

This end the proof of Proposition 1.2.

III) The proof of Theorem 1.3.

The idea of the proof is to reduce our initial operator $P_\omega\Pi_\omega$ via Theorem 1.1, then to rescale this reduced operator so as to be able to use a slightly modified version of the localization theorem, Theorem 1.7, of [Kl 2].

1) A uniform version of the localization theorem.

Let $m>0$ and $d>0$. Let D be some fixed set. Let $H(m,d):D^{\mathbb{Z}^d}\to\mathcal{M}at(\mathbb{Z}^d)$, a matrix valued application satisfying

(H".1)

a) $\forall t\in D^{\mathbb{Z}^d}$, $H(m,d,t)=((H(m,d,t;x,y)))_{(x,y)\in\mathbb{Z}^d\times\mathbb{Z}^d}$ is formally self-adjoint,

b) $\exists K>0$ such that, for $(x,y)\in\mathbb{Z}^d\times\mathbb{Z}^d$ such that $x\neq y$,

$$\sup_{t\in D^{\mathbb{Z}^d}}|H(h,t;x,y)|\leq e^{m(K-|x-y|)}.$$

Under assumption (H".1), $H(m,d,t)$ defines a self-adjoint operators on $\ell^2(\mathbb{Z}^d)$. One shows

Theorem 3.1. *(Theorem 1.7 [Kl 2])*

Let $m_0>0$ and $d_0>0$. Let $K_0>0$ and $K_0'>0$. Let $\epsilon_0>0$, $\rho_0>0$ and $\eta_0>0$. Pick $p>\sup(4,d)$, $\beta>0$ such that $\beta p<\inf(4,d)$. Pick $\epsilon\in(0,\frac{1}{2})$. Let $I\subset\mathbb{R}$.

Assume that, for $m\geq m_0$ and $\frac{d}{d_0}\geq\frac{m}{m_0}$, $H(m,d,t)$ satisfies (H".1) for (m,K_0) and a decoupling property of order $(d,e^{K_0'd})$. Suppose that the $(t_x)_{x\in\mathbb{Z}^d}$ are i.i.d random variables such that, for $m\geq m_0$ and $\frac{d}{d_0}\geq\frac{m}{m_0}$, $H(m,d,t)$ satisfies in I a Wegner estimate of type $(\epsilon_0,\rho_0,\eta_0)$.

Then, there exists $L_0>0$(depending on m_0, d_0, K_0, K_0', η_0, ρ_0, ϵ_0, p, β, ϵ) such that, for $m\geq m_0$, if there exists $l\geq L_0$ such that $\forall(x,y)\in\mathbb{Z}^d\times\mathbb{Z}^d$ satisfying $|x-y|>l(1+\epsilon)$,

$$P(\{\forall E\in I,\ \Lambda_l(x)\ or.\Lambda_l(y)\text{ is }(E,m(1-\epsilon),\beta,\epsilon)-\text{regular}\})>1-l^{-p}$$

then, with probability 1, the spectrum of $H(m,d,t)$ in I is pure point; and if φ is an eigenvector associated to E, an eigenvalue of $H(m,d,t)$ in I, then

$$\limsup_{|x|\to\infty} \frac{\log |\varphi(x)|}{|x|} \leq -m(1-2\epsilon)$$

Remark. The terminology used here (e.g decoupling property of order (d,k), Wegner estimate of type $(\epsilon_0,\rho_0,\eta_0)$, $(E, m_0(1-\epsilon),\beta,\epsilon)$-regularity) is defined in section I of [Kl 2].

We will not give a detailed proof of Theorem 3.1. It is obtained from the proof of Theorem 1.7 of [Kl 2] by merely following what happens to the constants $e^{K_0 m}$ and $e^{K_0' d}$ through this proof. Without too many difficulties, one gets an (m,d)-uniform version of Lemma 2.2, 2.3 and 2.6 of [Kl 2] (for $m \geq m_0$ and $\frac{d}{d_0} \geq \frac{m}{m_0}$); then, following the arguments of [Kl 2], one gets a (m,d)-uniform version of Theorem 1.7 of [Kl 2] that is Theorem 3.1.

2) The rescaling and the proof Theorem 1.3.

Let δ_0 be given as in assumption (H.3)b). By assumption (H.1),

$$c_0 = \inf_{\omega\in\Omega^{\mathbb{Z}^d}} \inf_{\alpha\neq\beta} \frac{d_\omega(\alpha+\mathrm{supp}(\theta),\beta+\mathrm{supp}(\theta))}{|\alpha-\beta|} > 0.$$

By assumptions (H.1) and (H.3)a), we may pick $c_0' > 0$ small enough such that, if $\Omega = G + B(0,c_0')$, then (H.1) holds for $\overline{\Omega}$ and θ and

$$\delta_0' = \inf_{\omega\in\Omega^{\mathbb{Z}^d}} \inf_{|\alpha-\beta|\geq 2} d_\omega(\alpha+\mathrm{supp}(\theta),\beta) - S_0 > 0.$$

Fix $h_0 = \inf(\delta_0,\delta_0',\frac{c_0}{2}, h_{\inf(\frac{1}{4},\frac{c_0}{2})})$ (here $h_{\inf(\frac{1}{4},\frac{c_0}{2})}$ is given by Theorem 1.1).

For $0 < h < h' \leq h_0$, we rescale

$$\begin{aligned}\tilde{H}(\omega) &= e^{\frac{1}{h}(S_0+h')}(H(\omega)-\mu(h))\\ &= \tilde{M}(\omega)+\tilde{W}(\omega)\\ &= e^{\frac{1}{h}(S_0+h')}M(\omega) + e^{\frac{1}{h}(S_0+h')}W(\omega)\end{aligned}$$

(where A is given in Proposition 1.2).

In the sequel, for the sake of simplicity, we omit to write the h and h' dependence of $\tilde{H}(\omega)$. One should keep in mind that the only semi-classical parameter is h. h' only appears as a renormalisation parameter.

To get point a) of Theorem 1.3, we just have to prove

Lemma 3.2. *There exists $0 < h_0' < \frac{h_0}{2}$ such that $\forall 0 < h < h_0'$ and $h' = h_0$, with probability 1,*

$$\| \tilde{H}(\omega) \| > 1.$$

Proof. By the estimates known for $M(\omega)$ (see Theorem 1.1), we get that there exists $C>0$ and $h_0'>0$, such that for any $0<h<h'<h_0'$,

$$\| \tilde{M}(\omega) \| \leq e^{-\frac{1}{Ch}}.$$

Moreover,if $\tilde{w}_{\alpha,\beta}(\omega)$ is the generic coefficient of $\tilde{W}(\omega)$, by Proposition 1.2, we know that, for some $A>0$ and for h small enough,

$$| \tilde{w}_{\alpha,\beta}(\omega) | \geq \frac{A}{2} h^{1-\frac{d}{2}} e^{-\frac{1}{h}(d_\omega(\alpha,\beta)-S_0-h')}. \tag{3.1}$$

By the ergodicity of the family $(\omega_\alpha)_{\alpha\in\mathbb{Z}^d}$, we know that there exists E a subset of $G^{\mathbb{Z}^d}$ of probability 1 such that, for any $\omega\in E$, there exists $(\alpha,\beta)\in\mathbb{Z}^d\times\mathbb{Z}^d$ such that $|\alpha-\beta|=1$ and such that

$$d_\omega(\alpha,\beta)=S_0.$$

Pick $0<h_0'<h_0$ such that

$$\frac{A}{2}(h_0')^{1-\frac{d}{2}} e^{\frac{h_0}{h_0'}} \geq \frac{3}{2}.$$

So, for any $\omega\in E$, for $0<h<h_0'$ and $h'=h_0$, we get

$$| \tilde{w}_{\alpha,\beta}(\omega) | \geq \frac{3}{2}.$$

Consequently, $\| \tilde{H}(\omega) \| > 1$, which ends the proof of Lemme 3.2.

■

Now to get points c) and d), we will apply Theorem 3.1. Let us fix $h'=h_0$ and write $\tilde{H}(\omega)=((\tilde{H}(\omega;\alpha,\beta)))_{(\alpha,\beta)\in\mathbb{Z}^d\times\mathbb{Z}^d}$. Using the estimates given Theorem 1.1, reducing h_0' if necessary, we get

$$| \tilde{H}(\omega;\alpha,\beta) | \leq e^{\frac{h_0'}{h}(2-|\alpha-\beta|)} \text{ if } \alpha\neq\beta, \tag{1}$$

and

$$| \nabla_\gamma \tilde{H}(\omega;\alpha,\beta) | \leq e^{\frac{h_0'}{h}(3-|\alpha-\gamma|-|\gamma-\beta|)}. \tag{2}$$

Using (2) in the same way we did in [Kl 2] Section IV)B), one proves that, for some $K_0'>0$, $\tilde{H}(\omega)$ satisfies a decoupling estimate of order $(\frac{h_0'}{h}, e^{K_0'\frac{h_0'}{h}})$, for $0<h<h_0'$.

Now, if we prove

Lemma 3.3 (The Wegner estimate). *There exists $h_0''>0$, $\epsilon_0>0$ such that, for $0<h<h'<h_0''$, $\tilde{H}(\omega)$ satisfies a Wegner estimate of type $(1+\sup(1,\frac{1}{\rho_0}),\inf(1,\rho_0),1)$ in $(-\infty,-1]\cup[1,+\infty)$ where ρ_0 is given in assumption (H.3)d).*

and

Lemma 3.4. *For any $l < 1$, there exists $h_l > 0$ such that, for $h \in (0, h_l)$ and $(\alpha, \beta) \in \mathbb{Z}^d \times \mathbb{Z}^d$ such that $| \alpha - \beta |> \frac{5}{4} l$,*

$$P(\{\forall \in (-\infty, -1] \cup [1, +\infty),\ \Lambda_l(\alpha) \text{ or } \Lambda_l(\beta) \text{ is } (E, \frac{2h_l}{h}, \beta, \frac{1}{4})\text{-regular}\}) > 1 - l^{-p}.$$

then, fixing $\epsilon = \frac{1}{4}$, we can apply Theorem 3.1 to $\tilde{H}(\omega)$ to get that, there exists $h_0 > h_0' > 0$ such that, for $h \in (0, h_0')$, with probability 1,

a) $\sigma(\tilde{H}(\omega)) \cap ((-\infty, -1] \cup [1, +\infty)) \neq \emptyset$,

b) the spectrum of $\tilde{H}(\omega)$ in $(-\infty, -1] \cup [1, +\infty)$ is purely punctual,

c) if φ is an eigenvector associated to E, an eigenvalue of $\tilde{H}(\omega)$ in $(-\infty, -1] \cup [1, +\infty)$, then

$$\limsup_{|x| \to +\infty} \frac{\log | \varphi(x) |}{| x |} \leq -\frac{h_0'}{h}.$$

Then, rescaling $\tilde{H}(\omega)$ to $H(\omega)$ and using the fact that the vectors of our basis of F_ω are uniformly exponentially decreasing, we get Theorem 1.3 (see the end of the proof of Theorem 1.9 in [Kl 2]).

The proofs of Lemma 3.3 and 3.4 are given the next section, section V.

IV) The proof of Lemma 3.3 and 3.4.

A) The proof of the Wegner estimate (Lemma 3.3).

To prove this, we will apply the strategy we applied to prove the Wegner estimate in [Kl 2]. We will construct a vector field in the ω variables such that when you derivate the operator $\tilde{H}_\Lambda(\omega)$ (i.e. the operator $\tilde{H}(\omega)$ restricted to Λ some cube of $\mathbb{Z}^d$ (see section I of [Kl 2])) along this vector field, you obtain an operator that is of constant sign when restricted to the spectral space associated to $\tilde{H}_\Lambda(\omega)$ and the energy interval you consider. In our present case, we will give a vector field such that the derivative of $\tilde{H}_\Lambda(\omega)$ along it is approximately $2\tilde{H}_\Lambda(\omega)$. This will be enough to conclude a Wegner estimate.

For $\alpha \in \mathbb{Z}^d$ such that $| \alpha |= 1$, define

$$W_{0,\alpha} = \{\omega_0 \in G;\ \exists \omega_\alpha \in G \text{ such that } d_\omega(0, \alpha) = S_0\},$$

and

$$\Xi = \{\alpha \in \mathbb{Z}^d;\ | \alpha |= 1 \text{ and } W_{0,\alpha} \neq \emptyset\}.$$

Then, by definition of S_0, continuity of d_ω and compacity of G, $\Xi \neq \emptyset$. Moreover, there exists $\delta_0' > 0$ such that, for $\alpha \in \Xi$ and $\omega \in G^{\mathbb{Z}^d}$,

$$d_\omega(0, \alpha) > S_0 + \delta_0'.$$

For $\delta > 0$ and $\alpha \in \Xi$, let $W_{0,\alpha}(\delta)$ be the following open neighborhood of $W_{0,\alpha}$,

$$W_{0,\alpha} \subset W_{0,\alpha}(\delta) = \{\omega_0 \in \Omega;\ \exists \omega_\alpha \in \Omega \text{ such that } d_\omega(0, \alpha) < S_0 + \delta\},$$

(here Ω is the neighborhood of G defined in section IV)2)).

Let $\chi_{0,\alpha}$ be a C_0^∞ function such that

$$\begin{cases} \chi_{0,\alpha} \equiv 1 \text{ on } W_{0,\alpha}(\frac{1}{2}\delta_0) \\ \chi_{0,\alpha} \equiv 0 \text{ outside of } W_{0,\alpha}(\frac{3}{4}\delta_0) \end{cases}$$

where $\delta_0 \leq \inf(\delta_0, \delta_0')$ (here δ_0 given by assumption (H.3) b)) and δ_0 is such that $W_{0,\alpha}(\frac{1}{2}\delta_0) \cap W_{0,\beta}(\frac{3}{4}\delta_0) = \emptyset$ if $\alpha \neq \beta$. For $\alpha \notin \Xi$, let $\chi_{0,\alpha} \equiv 0$.

Then define, for $\mid \alpha - \beta \mid = 1$ and $\omega \in \mathbb{R}^{\mathbb{Z}^d}$,

$$\chi_{\alpha,\beta}(\omega) = \chi_{0,\alpha-\beta}(\omega_\beta) \cdot \chi_{0,\beta-\alpha}(\omega_\alpha).$$

Let $\Lambda \subset \mathbb{Z}^d$. Let $V_\Lambda(h)$ be the following vector field,

$$\begin{aligned} V_\Lambda(h;\omega) &= -h \sum_{\substack{\alpha\in\Lambda \\ \beta\in\Lambda;\ |\beta-\alpha|=1}} \chi_{\alpha,\beta}(\omega) \frac{\nabla_{\omega_\alpha} d_\omega(\alpha,\beta)}{\mid \nabla_{\omega_\alpha} d_\omega(\alpha,\beta) \mid^2} \cdot \nabla_{\omega_\alpha} \\ &= -h \left(\sum_{\beta\in\Lambda;\ |\beta-\alpha|=1} \chi_{\alpha,\beta}(\omega) \frac{\nabla_{\omega_\alpha} d_\omega(\alpha,\beta)}{\mid \nabla_{\omega_\alpha} d_\omega(\alpha,\beta) \mid^2} \right)_{\alpha\in\Lambda}. \end{aligned} \tag{4.1}$$

Remark. If one changes assumption (H.3), one must change the definition of the vector field $V_\Lambda(h,\omega)$ so as to take into account the fact that a well may interact significantly with more than one of its $|\cdot|_\infty$-nearest neighbor wells.

By Proposition 1.3 and the localization of $\mathrm{supp}(\chi_{0,\alpha})$, $V_\Lambda(h)$ is defined and C^∞ on $\mathbb{R}^{d\cdot|\Lambda|}$ (here $\mid \Lambda \mid$ denotes the cardinal of Λ). Moreover, there exists $C > 0$ (independent of Λ),

$$\sup_{\omega\in\mathbb{R}^{d\cdot|\Lambda|}} \sup_{\alpha\in\Lambda} \mid (V_\Lambda(h;\omega))_\alpha \mid \leq C \cdot h, \tag{4.2}$$

and

$$\sup_{\omega\in\mathbb{R}^{d\cdot|\Lambda|}} \parallel \nabla_\omega V_\Lambda(h;\omega) \parallel_{\mathcal{B}(\mathbb{R}^{d\cdot|\Lambda|})} \leq C \cdot h, \tag{4.3}$$

where $\nabla_\omega V_\Lambda(h;\omega)$ denotes the Jacobian matrix of $V_\Lambda(h;\omega)$, $\parallel \cdot \parallel_E$, the operator norm for bounded linear operators from E to E, a Hilbert space ((4.3) holds because, by construction, $\nabla_{\omega_\beta}[(V_\Lambda(h,\omega))_\alpha] = 0$ for $\mid \alpha - \beta \mid > 1$).

So, by the ordinary differential equation theory (see, for example, [A]), we know that, there exists $h_0 > 0$, such that, for $h \in (0, h_0)$ and any $\Lambda \subset \mathbb{Z}^d$, there exists a C^∞ semigroup $SG_\Lambda(t) : [-2,2] \times \mathbb{R}^{d\cdot|\Lambda|} \mapsto \mathbb{R}^{d\cdot|\Lambda|}$ such tha , for $\mid t \mid \leq 2$,

$$\frac{d}{dt} SG_\Lambda(t) = V_\Lambda\left(h; SG_\Lambda(t)\right) \text{ and } SG_\Lambda(0) = \mathrm{Id}, \tag{4.4}$$

and

$$\frac{d}{dt}\nabla_\omega SG_\Lambda(t) = [\nabla_\omega V_\Lambda(h; SG_\Lambda(t))] \cdot \nabla_\omega SG_\Lambda(t) \text{ and } \nabla_\omega SG_\Lambda(0) = \mathrm{Id}. \tag{4.5}$$

Moreover, for some $C > 0$ (independent of Λ and h), one has

$$\sup_{\omega\in\mathbb{R}^{d\cdot|\Lambda|}} | SG_\Lambda(t)(\omega) - \omega | \leq Ch \mid t \mid, \tag{4.6}$$

and

$$\sup_{\omega\in\mathbb{R}^{d\cdot|\Lambda|}} \| \nabla_\omega SG_\Lambda(t) - \mathrm{Id} \|_{\mathcal{B}(\mathbb{R}^{d\cdot|\Lambda|})} \leq Ch \mid t \mid . \tag{4.7}$$

Now choose $h_0 > 0$ small enough such that, for $h \in (0, h_0)$ and $\mid t \mid \leq 2$, $SG_\Lambda(t)$ maps G^Λ into Ω^Λ; and define the mapping $\tilde{SG}_\Lambda(t) : (\mathbb{R}^d)^{\mathbb{Z}^d} \mapsto (\mathbb{R}^d)^{\mathbb{Z}^d}$ in the following way: for $\alpha \in \mathbb{Z}^d$,

$$\left(\tilde{SG}_\Lambda(t)(\omega)\right)_\alpha = \begin{cases} (SG_\Lambda(t)\omega)_\alpha \text{ if } \alpha \in \Lambda \\ \omega_\alpha \text{ if } \alpha \notin \Lambda \end{cases} \tag{4.8}$$

At last, for $h \in (0, h_0)$, $\mid t \mid \leq 2$ and $\omega \in G^{\mathbb{Z}^d}$, we define

$$\begin{aligned} \tilde{H}_\Lambda(t,\omega) &= \tilde{H}_\Lambda(\tilde{SG}_\Lambda(t)(\omega)) \\ &= \tilde{W}_\Lambda(\tilde{SG}_\Lambda(t)(\omega)) + \tilde{M}_\Lambda(\tilde{SG}_\Lambda(t)(\omega)) \\ &= \tilde{W}_\Lambda(t,\omega) + \tilde{M}_\Lambda(t,\omega). \end{aligned}$$

We show

Lemma 4.1. *There exists $0 < h_0' < h_0$ such that, for $0 < h < h' < h_0'$, $-2 \leq t \leq 2$, $\omega \in G^{\mathbb{Z}^d}$ and any $\Lambda \subset \mathbb{Z}^d$, one has*

$$\frac{d}{dt}\tilde{H}_\Lambda(t,\omega) = (1 + A(t,\omega))\tilde{H}_\Lambda(t,\omega) + \tilde{H}_\Lambda(t,\omega)(1 + A(t,\omega)) + \mathcal{O}\left(e^{-\frac{h_0}{h}}\right),$$

where:

(*) $A(t,\omega)$ is a diagonal matrix satisfying, for some $C > 0$ (independent of ω, Λ and t),

$$\| A(t,\omega) \| \leq Ch. \tag{1}$$

(**) $\mathcal{O}$ is uniform in ω, t and Λ.

Proof. To compute $\frac{d}{dt}\tilde{H}_\Lambda(t,\omega)$, we will compute separately $\frac{d}{dt}\tilde{W}_\Lambda(t,\omega)$ and $\frac{d}{dt}\tilde{M}_\Lambda(t,\omega)$. Let us begin with $\frac{d}{dt}\tilde{M}_\Lambda(t,\omega)$. Write $\tilde{M}_\Lambda(t,\omega) = ((\tilde{m}(t,\omega;\alpha,\beta)))_{(\alpha,\beta)\in\mathbb{Z}^d\times\mathbb{Z}^d}$. So, by (4.1), (4.4) and (4.8), for $(\alpha,\beta) \in \mathbb{Z}^d \times \mathbb{Z}^d$,

$$\begin{aligned} \frac{d}{dt}\tilde{m}(t,\omega;\alpha,\beta) &= V_\Lambda(h)(SG_\Lambda(t)\omega) \cdot \nabla_{\omega_\alpha}\tilde{m}(\tilde{SG}_\Lambda(t)\omega;\alpha,\beta) \\ &= -h \sum_{\substack{\gamma\in\Lambda \\ |\mu-\gamma|=1}} \chi_{\gamma,\mu}(SG_\Lambda(t)\omega)\frac{\nabla_{\omega_\gamma} d_\omega(\gamma,\mu)}{| \nabla_{\omega_\gamma} d_\omega(\gamma,\mu) |^2}\nabla_{\omega_\gamma}\tilde{m}(SG_\Lambda(t)\omega;\alpha,\beta). \end{aligned}$$

So, by Lemma 2.1, for some $C > 0$ (independent of Λ and t),

$$| \frac{d}{dt}\tilde{m}(t,\omega;\alpha,\beta) | \leq Ch \sum_{\gamma\in\Lambda} | \nabla_{\omega_\gamma} \tilde{m}(SG_\Lambda(t)\omega;\alpha,\beta) | .$$

By estimate d) of Theorem 1.1, assumption (H.3) a) and the definition of $\tilde{M}$, we get that, there exists $0 < h_0' < h_0$ and $\tilde{C} > 0$ (independent of Λ) such that, for $0 < h < h' < h_0'$, $-2 \leq t \leq 2$ and $\omega \in G^{\mathbb{Z}^d}$,

$$\begin{aligned} | \frac{d}{dt}\tilde{m}(t,\omega;\alpha,\beta) | &\leq Che^{-\frac{h_0}{h}} \sum_{\gamma\in\Lambda} e^{-\frac{h_0}{h}(|\alpha-\gamma|+|\gamma-\beta|)} \\ &\leq Che^{-\frac{h_0}{h}} \sum_{\gamma\in\mathbb{Z}^d} e^{-\frac{h_0}{h}(|\alpha-\gamma|+|\gamma-\beta|)} \\ &\leq \tilde{C}he^{-\frac{h_0}{h}} e^{-\frac{h_0}{h}|\alpha-\beta|}. \end{aligned}$$

Hence, using Schur's Lemma, we get, for h small enough, for any admissible t, ω and Λ,

$$\| \frac{d}{dt}\tilde{M}_\Lambda(t,\omega) \|_{\ell^2(\Lambda)} \leq e^{-\frac{h_0}{h}}. \tag{4.9}$$

We already knew that, for h small enough, for any admissible t, ω and Λ,

$$\| \tilde{M}_\Lambda(t,\omega) \|_{\ell^2(\Lambda)} \leq e^{-\frac{h_0}{h}}. \tag{4.10}$$

Let us now estimate $\frac{d}{dt}\tilde{W}_\Lambda(t,\omega)$. First, notice that, by (4.6) and by Proposition 1.3, there exists $C > 0$ (independent of (α,β)) such that, for $| \alpha - \beta | = 1$ and any $\omega \in G^{\mathbb{Z}^d}$

$$\sup_{|t|\leq 2} | d_{\tilde{SG}_\Lambda(t)\omega}(\alpha,\beta) - d_\omega(\alpha,\beta) | \leq Ch,$$

so, for $-2 \leq t \leq 2$,

$$e^{-C} e^{-\frac{1}{h}(d_\omega(\alpha,\beta)-S_0)} \leq e^{-\frac{1}{h}\left(d_{\tilde{SG}_\Lambda(t)\omega}(\alpha,\beta)-S_0\right)} \leq e^{C} e^{-\frac{1}{h}(d_\omega(\alpha,\beta)-S_0)}. \tag{4.11}$$

By assumption (H.3) b), defintion of Ξ and localization of $\mathrm{supp}(\chi_{0,\alpha-\beta})$, for $(\alpha,\beta,\gamma) \in \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{Z}^d$ such that $\alpha \neq \beta$ and $| \alpha - \gamma | = | \beta - \gamma | = 1$, and $\omega \in \Omega^{\mathbb{Z}^d}$, we have

$$| e^{\frac{1}{h}d_\omega(\alpha,\beta)} \chi_{\alpha,\gamma}(\omega) | \leq e^{-\frac{S_0+\frac{\delta_0}{2}}{h}} \text{ and } | e^{\frac{1}{h}d_\omega(\alpha,\beta)} (1 - \chi_{\alpha,\beta}(\omega)) | \leq e^{-\frac{S_0+\frac{\delta_0}{2}}{h}}. \tag{4.12}$$

By Proposition 1.2, we know that, for h small enough and $\omega \in G^{\mathbb{Z}^d}$ and $-2 \leq t \leq 2$,

$$-h\nabla_{\omega_\alpha} \tilde{w}_\alpha(\tilde{SG}_\Lambda(t)\omega) = \left(\nabla_{\omega_\alpha} d_{\tilde{SG}_\Lambda(t)\omega}(\alpha,\beta) + v_\alpha(\alpha,\beta)\right) \tilde{w}_\alpha(\tilde{SG}_\Lambda(t)\omega), \tag{4.13}$$

where:

$(*)$ $v_\alpha(\alpha,\beta) = v_\alpha(\beta,\alpha) = -v_\beta(\alpha,\beta)$ (using the results of Theorem 1.1),

$(**)$ there exists $C>0$ (independent of (α,β)) such that

$$| v_\alpha(\alpha,\beta) | \leq Ch. \tag{4.14}$$

Now, we compute

$$\begin{aligned}
&\frac{d}{dt}\tilde{w}_{\alpha,\beta}(\tilde{SG}_\Lambda(t)\omega) = \\
&= \sum_{\substack{\gamma\in\Lambda \\ |\mu-\gamma|=1;\ \mu\in\Lambda}} \chi_{\gamma,\mu}\left(\tilde{SG}_\Lambda(t)\omega\right) \frac{\nabla_{\omega_\gamma} d_{\tilde{SG}_\Lambda(t)\omega}(\gamma,\mu)\cdot\left(-h\nabla_{\omega_\gamma}\tilde{w}_{\alpha,\beta}(\tilde{SG}_\Lambda(t)\omega)\right)}{| \nabla_{\omega_\gamma} d_{\tilde{SG}_\Lambda(t)\omega}(\gamma,\mu) |^2} \\
&= \sum_{|\mu-\alpha|=1} \left(\chi_{\alpha,\mu}\left(\tilde{SG}_\Lambda(t)\omega\right)\tilde{w}_{\alpha,\beta}(\tilde{SG}_\Lambda(t)\omega)\right)\cdot \\
&\qquad\cdot\frac{\nabla_{\omega_\alpha} d_{\tilde{SG}_\Lambda(t)\omega}(\alpha,\mu)\cdot\left(\nabla_{\omega_\alpha} d_{\tilde{SG}_\Lambda(t)\omega}(\alpha,\beta)+v_\alpha(\alpha,\beta)\right)}{| \nabla_{\omega_\alpha} d_{\tilde{SG}_\Lambda(t)\omega}(\alpha,\mu) |^2} \\
&+ \sum_{|\mu-\beta|=1} \left(\chi_{\beta,\mu}\left(\tilde{SG}_\Lambda(t)\omega\right)\tilde{w}_{\alpha,\beta}(\tilde{SG}_\Lambda(t)\omega)\right)\cdot \\
&\qquad\cdot\frac{\nabla_{\omega_\beta} d_{\tilde{SG}_\Lambda(t)\omega}(\beta,\mu)\cdot\left(\nabla_{\omega_\beta} d_{\tilde{SG}_\Lambda(t)\omega}(\alpha,\beta)+v_\beta(\alpha,\beta)\right)}{| \nabla_{\omega_\beta} d_{\tilde{SG}_\Lambda(t)\omega}(\beta,\mu) |^2},
\end{aligned}$$

using (4.13) and the fact that $\tilde{w}_{\alpha,\beta}$ only depends on $(\omega_\alpha,\omega_\beta)$.

Using (4.11), (4.12) and the localization of $\mathrm{supp}(\chi_{0,\alpha})$, we get

(4.15)

$$\begin{aligned}
&\frac{d}{dt}\tilde{w}_{\alpha,\beta}(\tilde{SG}_\Lambda(t)\omega) = \\
&= \tilde{w}_{\alpha,\beta}(\tilde{SG}_\Lambda(t)\omega)\left(2+\nabla_{\omega_\alpha} d_{\tilde{SG}_\Lambda(t)\omega}(\alpha,\beta)\cdot v_\alpha(\alpha,\beta)+\nabla_{\omega_\beta} d_{\tilde{SG}_\Lambda(t)\omega}(\alpha,\beta)\cdot v_\beta(\alpha,\beta)\right) \\
&+\left(\sum_{|\mu-\alpha|=1;\ \mu\in\Lambda}\chi_{\alpha,\mu}\left(\tilde{SG}_\Lambda(t)\omega\right)\frac{\nabla_{\omega_\alpha} d_{\tilde{SG}_\Lambda(t)\omega}(\alpha,\mu)}{| \nabla_{\omega_\alpha} d_{\tilde{SG}_\Lambda(t)\omega}(\alpha,\mu) |^2}\right. \\
&+\sum_{|\mu-\beta|=1;\ \mu\in\Lambda}\chi_{\beta,\mu}\left(\tilde{SG}_\Lambda(t)\omega\right)\frac{\nabla_{\omega_\beta} d_{\tilde{SG}_\Lambda(t)\omega}(\beta,\mu)}{| \nabla_{\omega_\beta} d_{\tilde{SG}_\Lambda(t)\omega}(\beta,\mu) |^2} \\
&+\left(\left(\chi_{\alpha,\beta}(\tilde{SG}_\Lambda(t)\omega)-1\right)+\left(\chi_{\beta,\alpha}(\tilde{SG}_\Lambda(t)\omega)-1\right)\right)\cdot\mathcal{O}\left(e^{-\frac{1}{h}\left(d_{\tilde{SG}_\Lambda(t)\omega}(\alpha,\beta)-(S_0+h')\right)}\right) \\
&= \tilde{w}_{\alpha,\beta}(\tilde{SG}_\Lambda(t)\omega)\left(2+\nabla_{\omega_\alpha} d_{\tilde{SG}_\Lambda(t)\omega}(\alpha,\beta)\cdot v_\alpha(\alpha,\beta)+\nabla_{\omega_\beta} d_{\tilde{SG}_\Lambda(t)\omega}(\alpha,\beta)\cdot v_\beta(\alpha,\beta)\right) \\
&\qquad\qquad+\mathcal{O}\left(e^{-\frac{1}{h}(\frac{\delta_0}{2}-h')}\right),
\end{aligned}$$

where $\mathcal{O}$ is uniform in α and β.

We define the matrix $A(t,\omega) = ((A_\alpha(t,\omega)\delta_{\alpha,\beta}))_{(\alpha,\beta)\in\Lambda\times\Lambda}$ where

$$A_\alpha(t,\omega) = \sum_{|\mu-\alpha|=1;\ \mu\in\Lambda} \chi_{\alpha,\mu}\left((\tilde{SG}_\Lambda(t)\omega\right)\nabla_{\omega_\alpha} d_{\tilde{SG}_\Lambda(t)\omega}(\alpha,\mu)\cdot v_\alpha(\alpha,\mu).$$

Estimate (1) of Lemma 4.1 follows from (4.14) and the localization of the supports of the functions $\chi_{0,\alpha}$.

Using (4.11), we compute the coefficients of the matrix $\tilde{W}(t,\omega)A(t,\omega) = ((wa_{\alpha,\beta}))_{(\alpha,\beta)\in\Lambda\times\Lambda}$ to get

$$\begin{aligned} wa_{\alpha,\beta} &= \tilde{w}_{\alpha,\beta}(\tilde{SG}_\Lambda(t)\omega)A_\beta(t,\omega) \\ &= \tilde{w}_{\alpha,\beta}(\tilde{SG}_\Lambda(t)\omega)\nabla_{\omega_\beta} d_{\tilde{SG}_\Lambda(t)\omega}(\beta,\alpha)\cdot v_\beta(\beta,\alpha) \\ &\quad + \sum_{\substack{|\mu-\alpha|=1;\ \mu\in\Lambda \\ \mu\neq\alpha}} \chi_{\beta,\mu}\left(\tilde{SG}_\Lambda(t)\omega\right) e^{-\frac{1}{h}\left(d_\omega(\alpha,\beta)-(S_0+h')\right)}\cdot\mathcal{O}(1) \\ &\quad + (\left(1-\chi_{\beta,\alpha}(\tilde{SG}_\Lambda(t)\omega)\right) e^{-\frac{1}{h}\left(d_\omega(\alpha,\beta)-(S_0+h')\right)}\cdot\mathcal{O}(1) \end{aligned}$$

then, according to (4.12), we get

$$wa_{\alpha,\beta} = \tilde{w}_{\alpha,\beta}(\tilde{SG}_\Lambda(t)\omega)\nabla_{\omega_\beta} d_{\tilde{SG}_\Lambda(t)\omega}(\beta,\alpha)\cdot v_\beta(\beta,\alpha) + \mathcal{O}\left(e^{-\frac{1}{h}(\frac{\delta_0}{2}-h')}\right), \tag{4.16}$$

where $\mathcal{O}$ is uniform in α and β.

At last, combining (4.15) and (4.16), we showed that, there exists $0 < h_0' < h_0$ such that, for $0 < h < h' < h_0'$ and for any Λ and any admissible t and ω,

$$\frac{d}{dt}\tilde{W}_\Lambda(t,\omega) = (1+A(t,\omega))\tilde{W}_\Lambda(t,\omega) + \tilde{W}_\Lambda(t,\omega)(1+A(t,\omega)) + \mathcal{O}\left(e^{-\frac{h_0}{h}}\right).$$

Lemma 4.1 then follows from (4.9) (4.10) and (4.13).

■

Now define the following semi-group of $|\Lambda|\times|\Lambda|$-matrices,

$$U(t,\omega) = \exp\left(\int_0^t (1+A(u,\omega))du\right).$$

Notice that, as $A(t,\omega)$ is diagonal so is $U(t,\omega)$. One checks that $U(0,\omega) =\mathrm{Id}$, $\| U(t,\omega)\| \le e^{|t|(1+Ch)}$, that $U^{-1}(t,\omega) = \exp\left(-\int_0^t (1+A(u,\omega))du\right)$, and that

$$\frac{d}{dt}U^{-1}(t,\omega) = -(1+A(t,\omega))U^{-1}(t,\omega) = -U^{-1}(t,\omega)(1+A(t,\omega)).$$

So, by Lemma 4.1, we get, for admissible h, t and ω,

$$\frac{d}{dt}\left(U^{-1}(t,\omega)\tilde{H}_\Lambda(t,\omega)U^{-1}(t,\omega)\right) = \mathcal{O}\left(e^{-\frac{h_0}{h}}\right). \tag{4.17}$$

Integrating (4.17), we get, for $0 < h < h'$ small enough, $-2 \leq t \leq 2$ and $\omega \in G^{\mathbb{Z}^d}$,

$$\tilde{H}_\Lambda(t,\omega) = U(t,\omega)\tilde{H}_\Lambda(0,\omega)U(t,\omega) + t\mathcal{O}\left(e^{-\frac{h_0}{h}}\right). \tag{4.18}$$

Denote by $\mu_k(t,\omega)$ the k$^{\text{th}}$ eigenvalue of $\tilde{H}_\Lambda(t,\omega)$ (the eigenvalues being ordered increasingly), and by $\tilde{\mu}_k(t,\omega)$ the k$^{\text{th}}$ eigenvalue of $U(t,\omega)\tilde{H}_\Lambda(0,\omega)U(t,\omega)$. By Ostrowsky's Theorem (see, for example [Ho-Jo] p.224) and the estimate (1) of Lemma 4.1,

$$\begin{aligned} e^{C|t|h}e^{2t}\mu_k(0,\omega) \leq \tilde{\mu}_k(t,\omega) \leq e^{-C|t|h}e^{2t}\mu_k(0,\omega) \quad &\text{if } \mu_k(0,\omega) \leq 0 \\ e^{-C|t|h}e^{2t}\mu_k(0,\omega) \leq \tilde{\mu}_k(t,\omega) \leq e^{C|t|h}e^{2t}\mu_k(0,\omega) \quad &\text{if } \mu_k(0,\omega) \geq 0 \end{aligned} \tag{4.19}$$

where $C > 0$ is some constant independent of t, h, ω and Λ.

Combining (4.18) and (4.19), we get that, for $0 < h_0' < h_0$ small enough, for $0 < h < h' < h_0'$,

1) if $\mu_k(0,\omega) = \mu_k(\omega) \geq 1$

$$\begin{aligned} \mu_k(t,\omega) - \mu_k(0,\omega) \geq t \quad &\text{if } t \geq 0 \\ \mu_k(t,\omega) - \mu_k(0,\omega) \leq t \quad &\text{if } t \leq 0 \end{aligned} \tag{4.20}$$

2) if $\mu_k(0,\omega) = \mu_k(\omega) \leq -1$

$$\begin{aligned} \mu_k(t,\omega) - \mu_k(0,\omega) \leq -t \quad &\text{if } t \geq 0 \\ \mu_k(t,\omega) - \mu_k(0,\omega) \geq -t \quad &\text{if } t \leq 0. \end{aligned} \tag{4.21}$$

Define $I^- = (-\infty, -1]$, $I^+ = [1, +\infty)$ and $I = I^- \cup I^+$. For $E \in \mathbb{R}$, define the counting function

$$N(E,\omega) = \sharp\{1 \leq k \leq |\Lambda|;\ \mu_k(\omega) \leq E\}.$$

Pick $E \in I^+$. Then, by (4.20), for $\eta \in]0,1[$,

$$\{1 \leq k \leq |\Lambda|;\ E-\eta < \mu_k(\omega) \leq E+\eta\} \subset \\ \subset \{1 \leq k \leq |\Lambda|;\ \mu_k(-\eta,\omega) \leq E \text{ and } \mu_k(\eta,\omega) > E\}$$

so

$$N(E+\eta,\omega) - N(E-\eta,\omega) \leq N(E, \tilde{SG}_\Lambda(-\eta)\omega) - N(E, \tilde{SG}_\Lambda(\eta)\omega).$$

Then, following Wegner [We], if P denote the probability defined by the random variable $(\omega_\alpha)_{\alpha\in\mathbb{Z}^d}$, we compute

$$\begin{aligned} P\left(\{\omega;\ \mathrm{dist}\left(E,\sigma(\tilde{H}_\Lambda(\omega))\right) < \eta\}\right) &\leq \\ &\leq \int (N(E+\eta,\omega) - N(E-\eta,\omega))\, dP(\omega) \qquad (4.22) \\ &\leq \int \left(N(E,\tilde{SG}_\Lambda(-\eta)\omega) - N(E,\tilde{SG}_\Lambda(\eta)\omega)\right) dP(\omega). \end{aligned}$$

We know that the distribution of each of the variables $(\omega_\alpha)_{\alpha\in\mathbb{Z}^d}$ is given by the same density g, and that $\tilde{SG}_\Lambda$ acts only on the components $(\omega_\alpha)_{\alpha\in\Lambda}$, so

$$\begin{aligned} \int N(E,\tilde{SG}_\Lambda(-\eta)\omega)dP(\omega) &= \int_{(\mathbb{R}^d)^{\mathbb{Z}^d}} N(E,\tilde{SG}_\Lambda(-\eta)\omega) \prod_{\alpha\in\mathbb{Z}^d} g(\omega_\alpha)d\omega_\alpha \\ &= \int_{(\mathbb{R}^d)^{\mathbb{Z}^d}} N(E,\omega)\mathrm{Det}\left(\nabla_\omega \tilde{SG}_\Lambda(\eta)\right)(\omega) \prod_{\alpha\in\mathbb{Z}^d} g(\tilde{SG}_\Lambda(\eta)\omega_\alpha)d\omega_\alpha \\ &= \int_{(\mathbb{R}^d)^\Lambda} \left(\int_{(\mathbb{R}^d)^{\mathbb{Z}^d\setminus\Lambda}} N(E,\omega) \prod_{\alpha\in\mathbb{Z}^d\setminus\Lambda} g(\omega_\alpha)d\omega_\alpha\right) \qquad (4.23) \\ &\qquad \prod_{\alpha\in\Lambda} g(SG_\Lambda(\eta)\omega_\alpha)\mathrm{Det}\,(\nabla_\omega SG_\Lambda(\eta))\,(\omega) \prod_{\alpha\in\Lambda} d\omega_\alpha \\ &= \int_{(\mathbb{R}^d)^\Lambda} N_\Lambda(E,\omega)g_\Lambda(\omega)\mathrm{Det}\,(\nabla_\omega SG_\Lambda(\eta))\,(\omega)d\omega_\Lambda, \end{aligned}$$

where

$$N_\Lambda(E,\omega) = \int_{(\mathbb{R}^d)^{\mathbb{Z}^d\setminus\Lambda}} N(E,\omega) \prod_{\alpha\in\mathbb{Z}^d\setminus\Lambda} g(\omega_\alpha)d\omega_\alpha$$

and

$$g_\Lambda(\omega) = \prod_{\alpha\in\Lambda} g(\omega_\alpha) \text{ and } d\omega_\Lambda = \prod_{\alpha\in\Lambda} d\omega_\alpha.$$

As $\mid N(E,\omega) \mid\leq\mid \Lambda \mid$, we know that $\mid N_\Lambda(E,\omega) \mid\leq\mid \Lambda \mid$, so

$$\begin{aligned} &\int N(E,\tilde{SG}_\Lambda(-\eta)\omega) - N(E,\tilde{SG}_\Lambda(\eta)\omega)dP(\omega) = \\ &= \int_{(\mathbb{R}^d)^\Lambda} N_\Lambda(E,\omega)\,(g_\Lambda(SG_\Lambda(\eta)\omega)\mathrm{Det}\,(\nabla_\omega SG_\Lambda(\eta))\,(\omega) - \\ &\qquad\qquad - g_\Lambda(SG_\Lambda(-\eta)\omega)\mathrm{Det}\,(\nabla_\omega SG_\Lambda(-\eta))\,(\omega))\, d\omega_\Lambda \\ &\leq\mid \Lambda \mid \int_{(\mathbb{R}^d)^\Lambda} \sum_{k=0}^{|\Lambda|_{\rho_0}-1} \mid g_\Lambda(SG_\Lambda(\eta_k)\omega)\mathrm{Det}\,(\nabla_\omega SG_\Lambda(\eta_k))\,(\omega) - \\ &\qquad\qquad - g_\Lambda(SG_\Lambda(\eta_{k+1})\omega)\mathrm{Det}\,(\nabla_\omega SG_\Lambda(\eta_{k+1})) \mid d\omega_\Lambda, \end{aligned}$$

where $|\Lambda|_{\rho_0} = |\Lambda|^{\sup(1,\frac{1}{\rho_0})}$ and $\eta_k = \eta\left(1 - \frac{2k}{|\Lambda|_{\rho_0}}\right)$.

Using the semigroup property of SG_Λ and the changes of variables $\omega \mapsto SG_\Lambda(\eta_{k+1})\omega$ for each k, we get

(4.24)

$$\int N(E, \tilde{SG}_\Lambda(-\eta)\omega) - N(E, \tilde{SG}_\Lambda(\eta)\omega)dP(\omega) \leq$$

$$\begin{aligned}
&\leq |\Lambda| \sum_{k=0}^{|\Lambda|_{\rho_0}-1} \int_{(\mathbb{R}^d)^\Lambda} |\, g_\Lambda(SG_\Lambda(\frac{2\eta}{|\Lambda|_{\rho_0}})\omega)\mathrm{Det}\left(\nabla_\omega SG_\Lambda(\frac{2\eta}{|\Lambda|_{\rho_0}})\right)(\omega) - g_\Lambda(\omega)\,|\, d\omega_\Lambda \\
&\leq |\Lambda|^{1+\sup(1,\frac{1}{\rho_0})} \,. \\
&\cdot \left(\int_{(\mathbb{R}^d)^\Lambda} |\, g_\Lambda(SG_\Lambda(\frac{2\eta}{|\Lambda|_{\rho_0}})\omega) - g_\Lambda(\omega)\,|\, \mathrm{Det}\left(\nabla_\omega SG_\Lambda(\frac{2\eta}{|\Lambda|_{\rho_0}})\right)(\omega)d\omega_\Lambda \right. \\
&\qquad\qquad \left. + \int_{(\mathbb{R}^d)^\Lambda} g_\Lambda(\omega)\,|\,\mathrm{Det}\left(\nabla_\omega SG_\Lambda(\frac{2\eta}{|\Lambda|_{\rho_0}})\right)(\omega) - 1\,|\, d\omega_\Lambda \right).
\end{aligned}$$

using the positivity of $g_\Lambda(\omega)$ and $\mathrm{Det}\,(\nabla_\omega SG_\Lambda(\eta))\,(\omega)$.

By (4.7), we know that, for $\omega \in G^\Lambda$,

$$\left(1 - Ch\frac{2\eta}{|\Lambda|_{\rho_0}}\right)^{|\Lambda|} \leq \mathrm{Det}\left(\nabla_\omega SG_\Lambda(\frac{2\eta}{|\Lambda|_{\rho_0}})\right)(\omega) \leq \left(1 + Ch\frac{2\eta}{|\Lambda|_{\rho_0}}\right)^{|\Lambda|}$$

so

$$|\,\mathrm{Det}\left(\nabla_\omega SG_\Lambda(\frac{2\eta}{|\Lambda|_{\rho_0}})\right)(\omega) - 1\,| \leq Ch\eta\,|\Lambda|^{1-\sup(1,\frac{1}{\rho_0})} . \tag{4.25}$$

Denote $g_\infty(\omega_\alpha, \eta) = \sup_{v\in B(0,1)} |\, g(\omega_\alpha + \eta v) - g(\omega_\alpha)\,|$; by (4.6), we know that, for some $C > 0$,

$$|\, g(SG_\Lambda(\frac{2\eta}{|\Lambda|_{\rho_0}})\omega_\alpha) - g(\omega_\alpha)\,| \leq g_\infty(\omega_\alpha, \frac{2Ch\eta}{|\Lambda|_{\rho_0}}). \tag{4.26}$$

Notice that, assumption (H.3) d) implies

$$\int_{\mathbb{R}^d} g_\infty(\omega_\alpha, \eta)d\omega_\alpha \leq \left(\frac{\eta}{\eta_0}\right)^{\rho_0} . \tag{4.27}$$

So

$$\begin{aligned}
g_\Lambda(SG_\Lambda(\frac{2\eta}{|\Lambda|_{\rho_0}})\omega) - g_\Lambda(\omega) &= \prod_{\alpha\in\Lambda} g(SG_\Lambda(\frac{2\eta}{|\Lambda|_{\rho_0}})\omega_\alpha) - \prod_{\alpha\in\Lambda} g(\omega_\alpha) \\
&= \sum_{\substack{\Lambda'\cup\Lambda''=\Lambda \\ \Lambda'\cap\Lambda''=\emptyset \\ |\Lambda'|\geq 1}} \prod_{\alpha\in\Lambda'} \left(g(SG_\Lambda(\frac{2\eta}{|\Lambda|_{\rho_0}})\omega_\alpha) - g(\omega_\alpha)\right) \prod_{\beta\in\Lambda''} g(\omega_\beta)
\end{aligned}$$

so, by (4.26),

$$| g_\Lambda(SG_\Lambda(\frac{2\eta}{|\Lambda|_{\rho_0}})\omega) - g_\Lambda(\omega) | \leq \sum_{\substack{\Lambda'\cup\Lambda''=\Lambda \\ \Lambda'\cap\Lambda''=\emptyset \\ |\Lambda'|\geq 1}} \prod_{\alpha\in\Lambda'} g_\infty(\omega_\alpha, \frac{2Ch\eta}{|\Lambda|_{\rho_0}}) \prod_{\beta\in\Lambda''} g(\omega_\beta).$$

Integrating this inequality over all $(\omega_\alpha)_{\alpha\in\Lambda}$ and using (5.25) and (5.27), we get, for some $C' > 1$,

$$\begin{aligned} \int_{(\mathbb{R}^d)^{|\Lambda|}} | g_\Lambda(SG_\Lambda(\frac{2\eta}{|\Lambda|_{\rho_0}})\omega) - g_\Lambda(\omega) | \operatorname{Det}\left(\nabla_\omega SG_\Lambda(\frac{2\eta}{|\Lambda|_{\rho_0}})\right)(\omega)d\omega_\Lambda \leq \\ \leq C' \sum_{k=1}^{|\Lambda|} \binom{|\Lambda|}{k} \left(\frac{2Ch\eta}{\eta_0 |\Lambda|_{\rho_0}}\right)^{\rho_0 k} \\ \leq C' \left(\frac{2Ch\eta}{\eta_0}\right)^{\rho_0} |\Lambda|^{1-\rho_0 \sup(1,\frac{1}{\rho_0})}. \end{aligned} \tag{4.28}$$

Then plugging (4.28) and (4.25) into (4.24), and then (4.24) into (4.22), we get, for $E \in I^+$,

$$\begin{aligned} &P\left(\{\omega;\ \operatorname{dist}\left(E, \sigma(\tilde{H}_\Lambda(\omega))\right) < \eta\right) \leq \\ &\leq |\Lambda|^{1+\sup(1,\frac{1}{\rho_0})} \left(Ch\eta |\Lambda|^{1-\sup(1,\frac{1}{\rho_0})} + C'\left(\frac{2Ch\eta}{\eta_0}\right)^{\rho_0} |\Lambda|^{1-\rho_0\sup(1,\frac{1}{\rho_0})}\right) \\ &\leq |\Lambda|^{1+\sup(1,\frac{1}{\rho_0})} \eta^{\inf(1,\rho_0)}, \end{aligned}$$

for h small enough depending only on η_0, ρ_0 and d.

Of course the same estimate holds for $E \in I^-$. This ends the proof of the Wegner estimate.

B) The proof of Lemma 3.4.

We recall that, for $|\gamma - \beta| = 1$, we defined

$$W_{\gamma,\beta} = \{\omega_\gamma \in G;\ \exists\omega_\beta \in G, \text{ such that } d_\omega(\gamma,\beta) = S_0\}.$$

Then, if λ denotes the Lebesgue measure in $\mathbb{R}^d$,

$$\lambda(W_{\gamma,\beta}) = 0.$$

Indeed, if we suppose that $\lambda(W_{\gamma,\beta}) > 0$ then, for almost every $x \in \lambda(W_{\gamma,\beta})$,

$$\frac{\lambda(W_{\gamma,\beta} \cap B(x,\epsilon))}{\lambda(B(x,\epsilon))} \to 0 \text{ when } \epsilon \to 0. \tag{5.29}$$

(see, for example, [Ru]).

Let $\omega_\gamma \in W_{\gamma,\beta}$ such that (4.29) holds and $\omega_\beta \in G$ such that $d_\omega(\gamma,\beta) = S_0$. By Lemma 2.1, the vector $v = \frac{1}{|\nabla_{\omega_\gamma} d_\omega(\gamma,\beta)|}\nabla_{\omega_\gamma} d_\omega(\gamma,\beta)$ is well defined. By (4.29), in any cone of axis v, we will find a point of $W_{\gamma,\beta} \subset G$ as close as we want to ω_γ. So using the Taylor formula for $d_\omega(\gamma,\beta)$, we will find $\omega'_\gamma \in G$ such that

$$d_\omega(\gamma + \omega'_\gamma, \beta + \omega_\beta) > S_0,$$

which is in contradiction with the definition of S_0.

Then, using the regularity of the measure defined by the random variable ω_γ, when $\delta \to 0$, $P_{\omega_\gamma}(W_{\gamma,\beta}(\delta)) \to 0$ (here P_{ω_γ} denotes the conditionnal probability knowing the variables $(\omega_\alpha)_{\alpha\neq\gamma}$). Define the event

$$\mathcal{E}_{\Lambda_l(\alpha)}(\delta) = \{(\omega_\gamma)_{\gamma\in\Lambda_l(\alpha)};\ \exists\gamma \in \Lambda_l(\alpha) \text{ and } \beta \in \mathbb{Z}^d \\ \text{such that } \mid \beta - \gamma \mid = 1 \text{ and } \omega_\gamma \in W_{\gamma,\beta}(\delta)\}.$$

Clearly

$$P\left(\mathcal{E}_{\Lambda_l(\alpha)}(\delta)\right) \leq \sum_{\gamma\in\Lambda_l(\alpha)} \sum_{|\beta-\gamma|=1} P_{\omega_\gamma}\left(W_{\gamma,\beta}(\delta)\right).$$

So $P\left(\mathcal{E}_{\Lambda_l(\alpha)}(\delta)\right) \to 0$ when $\delta \to 0$. Choose now δ small enough such that

$$P\left(\mathcal{E}_{\Lambda_l(\alpha)}(\delta)\right) \leq \frac{1}{l^p}.$$

We know that, there exists some $C_0 > 0$ and $0 < h'_0 < h_0$ such that, for $0 < h < h' < h'_0$ and $\omega \in G^{\mathbb{Z}^d}$,

$$\| \tilde{M}_{\Lambda_l(\alpha)}(\omega) \| \leq e^{-\frac{h_0}{h}},$$

and

$$\| \tilde{W}_{\Lambda_l(\alpha)}(\omega) \| \leq C_0 \sup_{\substack{|\gamma-\beta|=1 \\ (\gamma,\beta)\in\Lambda_l(\alpha)\times\Lambda_l(\alpha)}} \left(e^{-\frac{1}{h}\left(d_\omega(\gamma,\beta)-(S_0+h')\right)}\right).$$

So for $\omega \in G^{\mathbb{Z}^d}$ such that $(\omega_\gamma)_{\gamma\in\Lambda_l(\alpha)} \notin \mathcal{E}_{\Lambda_l(\alpha)}(\delta)$, we get, for some $C > 0$,

$$\| \tilde{H}_{\Lambda_l(\alpha)}(\omega) \| \leq Ce^{-\frac{1}{h}(\delta-h')}. \tag{4.30}$$

Pick $0 < h_l < \inf(\frac{\delta}{2(l+1)}, h'_0)$. For $0 < h < h' < h_l$ and $E \in (-\infty, 1] \cup [1, +\infty)$, we get

$$\tilde{G}_{\Lambda_l(\alpha)}(E) = \left(E - \tilde{H}_{\Lambda_l(\alpha)}(\omega)\right)^{-1} = \frac{1}{E} + \sum_{n\geq1} \frac{\left(\tilde{H}_{\Lambda_l(\alpha)}(\omega)\right)^n}{E^{n+1}}.$$

Using (4.30), one computes

$$\| \tilde{G}_{\Lambda_l(\alpha)}(E) \| \leq \frac{1}{E - Ce^{-\frac{1}{h}(\delta-h')}} \leq \frac{1}{E - Ce^{-\frac{\delta}{2h_l}}},$$

and

$$\sum_{\epsilon\frac{l}{2}\leq|\beta-\alpha|\leq\frac{l}{2}} \mid \tilde{G}_{\Lambda_l(\alpha)}(E;\alpha,\beta) \mid e^{2\frac{h_l}{h}|\alpha-\beta|} \leq C_d l^d e^{-\frac{1}{h}(\delta-h_l(l+1))} = C_d l^d e^{-\frac{\delta}{2h_l}}.$$

So, for $h_l > 0$ small enough, with probability larger than $1 - \frac{1}{l^p}$ (i.e for $\omega \in G^{\mathbb{Z}^d}$ such that $(\omega_\gamma)_{\gamma\in\Lambda_l(\alpha)} \notin \mathcal{E}_{\Lambda_l(\alpha)}(\delta)$), we obtain

$$\| \tilde{G}_{\Lambda_l(\alpha)}(E) \| \leq e^{l^\beta},$$

and

$$\sum_{\epsilon\frac{l}{2}\leq|\beta-\alpha|\leq\frac{l}{2}} \mid \tilde{G}_{\Lambda_l(\alpha)}(E;\alpha,\beta) \mid e^{2\frac{h_l}{h}|\alpha-\beta|} < 1.$$

This ends the proof of Lemma 3.4.

V) Proof of Theorem 1.5.

To prove this, we will suitably renormalize the reduced operator $H(t,\omega)$ obtained in Theorem 1.4 and show that the conditions required to apply Theorem 3.1 to this renormalization are satisfied.

Case 1: in this case, we renormalize, for $0 < h < h'$,

$$\tilde{H}(t,\omega) = e^{\frac{S_0+h'}{h}}\left(H(t,\omega) - \mu(h)\right).$$

So, if $0 < 2h' < \lim_{h\to 0}(-h\log a(h)) - S_0 = \delta_0$, we get

$$\| \tilde{D}(t) \| \leq e^{-\frac{\delta_0}{2h}}.$$

Then counting $\tilde{D}(t)$ into the negligible terms of $\tilde{H}(t,\omega)$, we use the techniques we used to prove Theorem 1.3 to get the announced result.

Case 2: in this case, we use the same renormalization as in [Kl 2]. Pick $\delta_0 = \lim_{h\to 0}(h\log a(h)) + S_0$ and set

$$\tilde{H}(\tilde{t},\omega) = \frac{1}{h\cdot a(h)}(H(t,\omega) - \mu(h)) = D(\tilde{t}) + \tilde{W}(\omega) + \tilde{M}(\tilde{t},\omega),$$

where $\tilde{t}_\gamma = \frac{1}{h\cdot a(h)}b(t_\gamma)$, $\tilde{W}(\omega) = \frac{1}{h\cdot a(h)}W(\omega)$ and $\tilde{M}(\tilde{t},\omega) = \frac{1}{h\cdot a(h)}M(b^{-1}(h\cdot a(h)\tilde{t}_\gamma),\omega)$.
Using the estimates given in Theorem 1.4, we get, for h small enough and some h_0,

$$\| \tilde{M}(\tilde{t},\omega) \| \leq e^{-\frac{h_0}{h}}. \tag{5.1}$$

That $\tilde{H}(\tilde{t},\omega)$ satisfies assumption (H".1) (see section IV) and a decoupling property, is proved in the same way as in [Kl 2]. To prove Wegner estimate for $\tilde{H}(\tilde{t},\omega)$ on $\mathbb{R}$, we use the same method as in [Kl 2] (Appendix II)B)). Now, if we prove

Lemma 5.1. *For any $l > 1$, there exists $h_l > 0$ such that, for $0 < h < h' < h_l$, for $\alpha,\gamma) \in \mathbb{Z}^d \times \mathbb{Z}^d$ such that $|\alpha - \gamma| > \frac{5}{4}l$,*

$$P(\{\forall E \in \mathbb{R},\ \Lambda_l(\alpha) \text{ or } \Lambda_l(\gamma) \textit{ is } (E, 2\frac{h_l}{h}, \beta, \frac{1}{4})\textit{-regular}\}) > 1 - l^{-p}.$$

then, applying Theorem 3.1, we conclude Theorem 1.5 using the uniform exponential decrease at infinity of the functions $(\varphi_{\gamma,t,\omega})_{\gamma\in\mathbb{Z}^d}$ exhibited in Theorem 1.4.

Proof of Lemma 5.1.

First notice that, if $\lim_{h\to 0} h \log a(h) > -S_0$ then, for some $\delta > 0$ and h small enough,

$$\| \tilde{W}(h) \| \leq e^{-\frac{\delta_0}{h}}.$$

One could then directly apply the proof of Lemma 1.11 ([Kl 2]) to get Lemma 1.6. This partly justifies the fact that one should consider three regimes instead of two (remark that follows Theorem 1.5).

Define $\mathcal{E}^{\omega}_{l;\alpha,\gamma}(\delta) = \mathcal{E}_{\Lambda_l(\alpha)}(\delta) \cup \mathcal{E}_{\Lambda_l(\gamma)}(\delta)$ where the event $\mathcal{E}_{\Lambda_l(\gamma)}(\delta)$ has been defined in the proof of Lemma 4.4. Pick δ small enough such that

$$P_\omega\left(\mathcal{E}^{\omega}_{l;\alpha,\gamma}(\delta)\right) = P_\omega\left(\mathcal{E}_{\Lambda_l(\alpha)}(\delta)\right) + P_\omega\left(\mathcal{E}_{\Lambda_l(\gamma)}(\delta)\right) < \frac{1}{2l^p}, \tag{5.2}$$

where P_ω is the probability computed with respect to the ω variables.

For $\delta' > 0$, define the event

$$\begin{aligned}\mathcal{E}^{t}_{l;\alpha,\gamma}(\delta') = \{t; \exists E \in \mathbb{R} \text{ such that } &\| \left(D_{\Lambda_l(\alpha)}(\tilde{t}) - E\right)^{-1} \| \geq \delta' \\ &\text{and } \| \left(D_{\Lambda_l(\gamma)}(\tilde{t}) - E\right)^{-1} \| \geq \delta'\} \\ \subset \{t;\ \exists E \in \mathbb{R} \text{ and } \exists(\mu,\mu') \in \Lambda_l(\alpha) \times \Lambda_l(\gamma) &\text{ such that } |\tilde{t}_\mu - \tilde{t}_{\mu'}| \leq 2\delta'\}.\end{aligned}$$

Using then the proof of Lemma 1.11 (see [Kl 2]), we estimate

$$P_t\left(\mathcal{E}^{t}_{l;\alpha,\gamma}(\delta')\right) \leq (2Cl2d)h, \tag{5.3}$$

for some $C > 0$ independent of h, l, α and γ (here P_t is the probability computed with respect to the t variables).

Pick $\delta' = \frac{1}{2}e^{l^\beta}$. By (5.3), there exists $h_l > 0$ such that, for $h \in (0, h_l)$,

$$P_t\left(\mathcal{E}^{t}_{l;\alpha,\gamma}(\delta')\right) \leq \frac{1}{2l^p}.$$

Now define the event

$$\mathcal{E}_{l;\alpha,\gamma} = \left((\mathbb{R}^d)^{\mathbb{Z}^d} \setminus \mathcal{E}^{t}_{l;\alpha,\gamma}(\delta')\right) \times \left(G^{\mathbb{Z}^d} \setminus \mathcal{E}^{\omega}_{l;\alpha,\gamma}(\delta)\right).$$

Then, as the t's and ω's are independent random variables,

$$P_{t,\omega}(\mathcal{E}_{l;\alpha,\gamma}) \geq (1 - \frac{1}{2l^p})(1 - \frac{1}{2l^p}) \geq 1 - \frac{1}{l^p}.$$

For $(t,\omega) \in \mathcal{E}_{l;\alpha,\gamma}$, for any $E \in \mathbb{R}$, one has

$$\| \left(D_{\Lambda_l(\alpha)}(\tilde{t}) - E\right)^{-1} \| \leq \frac{1}{2}e^{l^\beta} \text{ or } \| \left(D_{\Lambda_l(\gamma)}(\tilde{t}) - E\right)^{-1} \| \leq \frac{1}{2}e^{l^\beta}.$$

Assume it is the first that holds. Then

$$\begin{aligned}(6.4)\quad \left(\tilde{H}_{\Lambda_l(\alpha)}(\tilde{t},\omega) - E\right)^{-1} &= \left(D_{\Lambda_l(\alpha)}(\tilde{t}) - E\right)^{-1} \cdot \\ &\cdot \left(\mathrm{Id} + \left(\tilde{W}_{\Lambda_l(\alpha)}(\tilde{t},\omega) + \tilde{M}_{\Lambda_l(\alpha)}(\tilde{t},\omega)\right) \cdot \left(D_{\Lambda_l(\alpha)}(\tilde{t}) - E\right)^{-1}\right).\end{aligned}$$

As $\omega \notin \mathcal{E}^{\omega}_{l;\alpha,\gamma}(\delta)$, we know that

$$\| \tilde{W}_{\Lambda_l(\alpha)}(\omega) \| \leq e^{-\frac{\delta}{2h}}.$$

Using (5.1), and expanding (5.4) using a Neumann series, we get

$$(5.5)\qquad \tilde{G}_{\Lambda_l(\alpha)}(E) = \left(\tilde{H}_{\Lambda_l(\alpha)}(\tilde{t},\omega) - E\right)^{-1} = \left(D_{\Lambda_l(\alpha)}(\tilde{t}) - E\right)^{-1} + R,$$

where

$$(5.6)\qquad \| R \| \leq e^{-\frac{1}{Ch}},$$

for some $C > 0$ (independent of l, α and h).

Then for h small enough, we get

$$\| \tilde{G}_{\Lambda_l(\alpha)}(E) \| \leq e^{l^\beta}.$$

Moreover, by (5.5) and (5.6), as $(D_{\Lambda_l(\alpha)}(\tilde{t}) - E)^{-1}$ is diagonal,

$$\sum_{\frac{l}{8} \leq |\alpha-\mu| \leq \frac{l}{2}} | \tilde{G}_{\Lambda_l(\alpha)}(E;\alpha,\mu) | \, e^{\frac{h_l}{h}|\alpha-\mu|} \leq C_d l^d e^{-\frac{1-2h_l Cl}{Ch}} < 1,$$

if $0 < h < h_l$, for some $h_l > 0$ small enough (independent of α).

So, for $0 < h < h_l$, with probability larger that $1 - \dfrac{1}{l^p}$ (i.e. for $(t,\omega) \in \mathcal{E}_{l;\alpha,\gamma}$), either $\Lambda_l(\alpha)$ or $\Lambda_l(\gamma)$ is $(E, 2\frac{h_l}{h}, \beta, \frac{1}{4})$-regular. This completes the proof of Lemma 5.1.

■

REFERENCES

[A] V.I. Arnold, *Equations différentielles ordinaires*, Editions MIR, Moscou, 1981.

[Ai-Mo] M. Aizenman, S. Molchanov, *Localization at Large Disorder and at Extreme Energies: an Elementary Derivation*, Preprint (1992).

[Ca] U. Carlsson, *An infinite number of wells in the semi-classical limit*, Asympt. Anal. **3** (1990), 189–214.

[C-L] R. Carmona, J. Lacroix, *Spectral Theory of Random Schrödinger Operators*, Birkhäuser, Boston Basel Berlin, 1990.

[Co-His] J. M. Combes, P. D. Hislop, *Localization for some continuous random hamiltonians in d-dimensions.*, Preprint, 1993.

[Fa 1] W. Faris, *Localization estimates for a random discrete wave equation at high frequency*, J. Stat. Phys. **46** (1987), 477–491.

[Fa 2] ———, *Localization estimates for off-diagonal disorder*, Mathematics of Random Media (W. Kohler, B. White, eds.), Lectures in Applied Mathematics, Vol. 27, A.M.S, Providence, R.I., 1991, pp. 391–406.

[Ha] B. Halperin, *Properties of a particle in a one-dimensional random potential*, Adv. Chem. Phys. **13** (1967), 123–178.

[He] B. Helffer, *Semi-classical analysis for the Schrödinger Operator and Applications. LNM 1336*, Springer, Berlin Heidelberg New-York, 1987.

[He-Sj 1] B. Helffer, J. Sjöstrand, *Multiple wells in the semi-classical limit I*, Comm. P.D.E **9** (1984), 337–408.

[He-Sj 2] ———, *Puits multiples en limite semi-classique II. Interaction moléculaire. Symétries. Perturbation*, Ann. Inst. Henri Poincaré sér. Phys. Théor. **42** (1985), 127–212.

[Ho-Jo] R. Horn, C. Johnson, *Matrix Analysis*, Cambridge University Press, Cambrige London New-York, 1985.

[Ki 1] W. Kirsch, *On a class of Random Schrödinger Operators*, Adv. Appl. Math. **6** (1985), 177–187.

[Ki 2] ———, *Random Schrödinger Operators*, Schrödinger Operators. LNP 345 (H. Holden, A. Jensen, eds.), Proceedings, Sonderborg, Denmark 1988, Springer, Berlin Heidelberg New-York, 1989, pp. 264–371.

[Ki-Ma 1] W. Kirsch, F. Martinelli, *On the Spectrum of Schrödinger Operators with a Random Potential*, Commun. Math. Phys. **85** (1982), 329–350.

[Ki-Ma 2] ———, *On the density of states of Schrödinger operators with a random potential*, J. Phys. A: Math. Gen. **15** (1982), 2139–2156.

[Kl 1] F. Klopp, *Étude semi-classique d'une perturbation d'un opérateur de Schrödinger périodique*, Ann. Inst. Henri Poincaré sér. Phys. Théor. **55** (1991), 459–509.

[Kl 2] ———, *Localisation pour des opérateurs de Schrödinger aléatoires dans $L^2(\mathbb{R}^d)$: un modèle semi-classique*, Preprint series of the Mittag-Leffler Institute, report No 1 (1992).

[Li-Mat] E. Lieb, D. Mattis, *Mathematical physics in one-dimension*, Academic Press, New-York, 1966.

[P-Fi] L. Pastur, A. Figotin, *Spectra of Random and Almost-Periodic Operators*, Springer, Berlin Heidelberg New-York, 1992.

[Ru] W. Rudin, *Real and Complex Analysis*, Mc Graw-Hill, New-York, 1983.

[We] F. Wegner, *Bounds on the density of states in disordered systems*, Z. Phys. **B44** (1981), 9–15.