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# Symmetries of the Chern-Simons Theory in the Axial Gauge

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*Abstract.* The Green functions of the Chern-Simons theory quantized in the axial gauge are shown to be calculable as the unique, exact solution of the Ward identities which express the invariance of the theory under the topological supersymmetry of Delduc, Gieres and Sorella. This solution coincides with the one which would have been obtained from a tree graph expansion with the principal value prescription for the free propagators.

# 1 Introduction

It was shown in a previous paper [1] that the Chern-Simons model [2, 3, 4], quantized in the axial gauge[1], obeys the topological supersymmetry which was known to hold in the Landau (covariant) gauge [6, 7]. This supersymmetry, whose generators form a three-vector and

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[1]See also [5] for more general noncovariant gauges.

whose anticommutator with the BRS generator yields the translations, has been shown to be at the origin of the utraviolet finiteness of the theory [8, 9]. Another important feature of topological supersymmetry, which makes it physically relevant[2], is its role in the construction of observables [11].

The aim of the present work is to examine further the relevance and the consequences of supersymmetry. This will be done in the axial gauge, since in this particular gauge all calculations can be performed rather explicitly [4, 12, 13, 14, 15]. The extension of our discussion to other gauge choices remains to be done.

We will show that the Ward identities defining the theory, namely the Ward identities for gauge invariance and for supersymmetry, allow to compute exactly all the Green functions, without the need of an action principle and of the usual Feynman graph expansion derived from it. Our main result is that the solution of the Ward identities is explicit and unique. This solution turns out to coincide with the expression which one would have obtained from the mentionned Feynman graph expansion, only tree graphs contributing to it, an the principal value prescription [16] being chosen for the free propagators.

The importance of the topological supersymmetry is stressed by the outcome that the latter, together with the axial gauge condition, essentially suffices to determine the theory. All Green functions of the gauge and ghost fields indeed are fixed by the supersymmetry Ward identities, without demanding gauge or BRS invariance. They moreover coincide with the ones which would follow from the *BRS invariant* action. The *resulting* BRS invariance in turn fixes the Green functions involving the Lagrange multiplier field[3] solely, which are the only ones not determined by supersymmetry.

Another intriguing point of the Chern-Simons model in the axial gauge is the existence of a very large algebra of symmetries. It was already shown in Ref. [1] that an anti-BRS invariance [17] and its associated supersymmetry hold. But these new invariances do not make a closed algebra with the ones already known. They belong to an algebra which is shown in App. A.

# 2 Chern-Simons theory in the axial gauge

The action of the Chern-Simons model in the axial gauge reads[4]

$$\begin{aligned}\Sigma_{\rm CS} = & -\tfrac{1}{2}\int d^3x\,\epsilon^{\mu\nu\rho}{\rm Tr}\,(A_\mu\partial_\nu A_\rho + \frac{2}{3}gA_\mu A_\nu A_\rho) \\ & +\int d^3x\,{\rm Tr}\,(dn^\mu A_\mu + bn^\mu D_\mu c),\end{aligned} \tag{2.1}$$

[2]Concerning the physical irrelevance of topological supersymmetry we disagree with the authors of Ref. [10]

[3]The Lagrange multiplier fields are used for the implementation of the gauge condition.

[4]Conventions: $\mu, \nu, \dots = 1, 2, 3$ , $g_{\mu\nu} = {\rm diag}(1, -1, -1)$ , $\epsilon^{\mu\nu\rho} = \epsilon_{\mu\nu\rho} = \epsilon^{[\mu\nu\rho]}$, $\epsilon_{123} = 1$.

with $D_\mu\cdot = \partial_\mu \cdot + g[A_\mu, \cdot]$ for the covariant derivative. The gauge group is chosen to be simple, all fields belong to the adjoint representation and are written as Lie algebra matrices $\varphi(x) = \varphi^a(x)\tau_a$, with

$$[\tau_a, \tau_b] = f^c_{ab}\tau_c, \qquad \mathrm{Tr}\,(\tau_a\tau_b) = \delta_{ab}.$$

The canonical dimensions and ghost numbers of the fields are given in Table 1.

| | $A$ | $d$ | $b$ | $c$ |
|---|---|---|---|---|
| Dimension | 1 | 2 | 2 | 0 |
| Ghost number | 0 | 0 | −1 | 1 |

Table 1: Dimensions and ghost numbers.

# 3 Symmetries and Ward identities

The action (2.1) is invariant [1] under the BRS and anti-BRS transformation $s$ and $\bar{s}$:

$$\begin{aligned} sA_\mu &= -D_\mu c, & \bar{s}A_\mu &= -D_\mu b, \\ sb &= d, & \bar{s}b &= gb^2, \\ sc &= gc^2, & \bar{s}c &= d, \\ sd &= 0, & \bar{s}d &= 0, \end{aligned} \tag{3.1}$$

as well as under the vector supersymmetries $\nu_\mu$ and $\bar{\nu}_\mu$:

$$\begin{aligned} \nu_\rho A_\mu &= \epsilon_{\rho\mu\nu}n^\nu b, & \bar{\nu}_\rho A_\mu &= \epsilon_{\rho\mu\nu}n^\nu c, \\ \nu_\rho b &= 0, & \bar{\nu}_\rho b &= -A_\rho, \\ \nu_\rho c &= -A_\rho, & \bar{\nu}_\rho c &= 0, \\ \nu_\rho d &= \partial_\rho b, & \bar{\nu}_\rho d &= \partial_\rho c. \end{aligned} \tag{3.2}$$

The BRS transformations $s$ and the supersymmetry transformations $\nu_\mu$ form an algebra which closes on-shell [7]:

$$s^2 = \{\nu_\mu, \nu_\nu\} = 0, \quad \{s, \nu_\mu\} = \partial_\mu + \text{Eq. of motion.} \tag{3.3}$$

There is a similar algebra for $\bar{s}$ and $\bar{\nu}_\mu$. However, if one wants to consider the whole set of transformations (3.1) and (3.2), it must be completed in order to form a closed algebra. This is done in App. A. We keep now only the BRS and supersymmetry transformations $s$ and $\nu_\mu$. We shall also assume that the theory is scale invariant, as it is the case for the classical theory.

The BRS invariance of the theory can be expressed, formally, by the functional identity

$$\mathrm{Tr}\int d^3x\left(-J^\mu[D_\mu c]\cdot Z_c - gJ_c[c^2]\cdot Z_c - J_b\frac{\delta Z_c}{\delta J_d}\right) = 0. \tag{3.4}$$

Here $Z_c(J^\mu, J_b, J_c, J_d)$ is the generating functional of the connected Green functions, $J^\mu$, $J_d$, $J_b$ and $J_c$ denoting the sources of the fields $A_\mu$, $d$, $b$ and $c$, respectively. We have used the notation

$$[O]\cdot Z_c(J^\mu, J_b, J_c, J_d)$$

for the generating functional of the connected Green functions with the insertion of the local field polynomial operator $O$. Usually, such insertions must be renormalized, their renormalization is controlled by coupling them to external fields and the identity (3.4) becomes the Slavnov identity [18]. We shall however see below that, in the axial gauge which we will choose to work in, these insertions are trivial and thus the Slavnov identity is replaced by a local gauge Ward identity.

The axial gauge is defined by the *gauge condition*

$$n^\mu\frac{\delta Z_c}{\delta J^\mu} + J_d = 0. \tag{3.5}$$

This gauge choice breaks Poincaré invariance, but the theory remains invariant under the *transverse Poincaré group*, i.e. under the Poincaré transformations which leave the gauge vector $n$ unchanged.

The invariance under the supersymmetry transformations $\nu_\rho$ given in (3.2) leads to the *supersymmetry Ward identity*

$$\mathrm{Tr}\int d^3x\left(J^\mu\varepsilon_{\rho\mu\nu}n^\nu\frac{\delta}{\delta J_b} + J_c\frac{\delta}{\delta J^\rho} + J_d\partial_\rho\frac{\delta}{\delta J_b}\right)Z_c = 0. \tag{3.6}$$

The projection of the supersymmetry Ward identity along the gauge vector $n$,

$$\mathrm{Tr}\int d^3x J_d\left(-J_c + n^\mu\partial_\mu\frac{\delta Z_c}{\delta J_b}\right) = 0, \tag{3.7}$$

can be put in a more convenient form. Calling $X$ the term between the parenthesis we see that locality, scale invariance and ghost number conservation imply that $X$ is a local polynomial in the sources $J$ and the functional derivatives $\delta/\delta J$, of dimension 3 and ghost number $-1$. Its most general form, compatible with transverse Poincaré invariance and with the gauge condition (3.5) taken into account, reads – in component form:

$$X_a = x\,J_{c^a} + y\,n^\mu\partial_\mu\frac{\delta Z_c}{\delta J_{b^a}} + z_{abc}\,J_d^b\frac{\delta Z_c}{\delta J_{b^c}},$$

where $x$, $y$ and $z_{abc}$ are constants, the latter being a tensor invariant under the gauge group. Equation (3.7), which reads

$$\int d^3x J_{d^a}X_a = 0,$$

holds if and only if $x = y = 0$ and $z_{abc}$ is antisymmetric in $a$ and $b$, *i.e.*, is proportional to the structure constants $f_{[abc]}$. Thus, coming back to the matrix notation, we see that $X$ is proportional to the commutator of $J_d$ with the functional derivative of $Z_c$ with respect to $J_b$. Calling $g$ the proportionality factor, we get in this way the local *antighost equation*

$$-J_c + \left( n^\mu \partial_\mu \frac{\delta}{\delta J_b} - g \left[ J_d, \frac{\delta}{\delta J_b} \right] \right) Z_c = 0. \tag{3.8}$$

We have thus seen that the gauge condition together with the $n$-component (3.7) of the supersymmetry Ward identity imply the local antighost equation. The converse statement being obvious, it follows that, under the axial gauge condition, the local antighost equation is indeed equivalent to the $n$-component of the supersymmetry.

As in any gauge theory with a linear gauge condition there is a *ghost equation.* This follows from the Slavnov identity (3.4), differentiated with respect to the source $J^3$, and from the gauge condition (3.5). The ghost equation reads

$$-J_b + \left( n^\mu \partial_\mu \frac{\delta}{\delta J_c} - g \left[ J_d, \frac{\delta}{\delta J_c} \right] \right) Z_c = 0. \tag{3.9}$$

The ghost equation (3.9) and the antighost equation (3.8) express the "freedom" of the ghosts in the axial gauge [16]: they couple only to the external source $J_d$, *i.e.*, to the $n$-component of the gauge field. Their effect is to factorize out the contributions of the ghost field $c$ to the composite fields appearing in the BRS Ward identity (3.4). We shall thus replace the latter by the local *gauge Ward identity*:

$$\begin{aligned} -\partial_\mu J^\mu + \Bigg( g \left[ J^\mu, \frac{\delta}{\delta J^\mu} \right] + g \left[ J_d, \frac{\delta}{\delta J_d} \right] + g \left\{ J_b, \frac{\delta}{\delta J_b} \right\} \\ + g \left\{ J_c, \frac{\delta}{\delta J_c} \right\} - n^\mu \partial_\mu \frac{\delta}{\delta J_d} \Bigg) Z_c = 0. \end{aligned} \tag{3.10}$$

# 4 Consequences of supersymmetry

Let us show that, taken together with the axial gauge condition (3.5) and the requirement of scale invariance, the supersymmetry Ward identities (3.6) determine all the Green functions of the theory except those containing only the Lagrange multiplier field $d$.

Without loss of generality we can choose the vector $n$ defining the axial gauge as

$$(n^\mu) = (0, 0, 1). \tag{4.1}$$

The coordinates transverse to $n$ will be denoted by

$$x^{\text{tr}} = (x^i,\ i = 1, 2). \tag{4.2}$$

Since the supersymmetry Ward identity (3.6) for $\rho = 3$ is equivalent to the antighost equation (3.8), we shall solve the latter:

$$- J_c + \left(\partial_3 \frac{\delta}{\delta J_b} - g\left[J_d, \frac{\delta}{\delta J_b}\right]\right) Z_c = 0, \tag{4.3}$$

and then the transverse components of the supersymmetry Ward identity[5]:

$$\mathrm{Tr} \int d^3x \left(J^j \varepsilon_{ij} \frac{\delta}{\delta J_b} + J_c \frac{\delta}{\delta J^i} + J_d \partial_i \frac{\delta}{\delta J_b}\right) Z_c = 0. \tag{4.4}$$

## Gauge condition:

We begin by looking at the gauge condition (3.5). It implies the vanishing of all connected Green functions involving the component $A_3$ of the gauge field:

$$\langle A_3^a(x)\varphi_1(x_1)\cdots\varphi_N(x_N)\rangle = 0, \quad \forall \text{ fields } \varphi_k(x_k), \tag{4.5}$$

with one exception:

$$\left\langle A_3^a(x) d^b(y)\right\rangle = -\delta^{ab}\delta^3(x-y). \tag{4.6}$$

## Antighost equation:

The antighost equation (3.8) gives the following equations for the connected Green functions involving one pair of ghost fields:

$$\partial_{x^3}\left\langle b^a(x)c^b(y)\right\rangle = -\delta^{ab}\delta^3(x-y), \tag{4.7}$$

and

$$\begin{aligned} &\partial_{x^3}\left\langle b^a(x)c^b(y)d^{c_1}(z_1)\cdots d^{c_n}(z_n)A_{i_1}^{d_1}(t_1)\cdots A_{i_m}^{d_m}(t_m)\right\rangle \\ &= g\sum_{k=1}^{n} f_{ac_ke}\delta^3(x-z_k) \\ &\quad \left\langle b^e(z_k)c^b(y)d^{c_1}(z_1)\cdots \widehat{d^{c_k}}(z_k)\cdots d^{c_n}(z_n)A_{i_1}^{d_1}(t_1)\cdots A_{i_m}^{d_m}(t_m)\right\rangle \\ &\text{for} \quad (n,m)\neq(0,0), \end{aligned} \tag{4.8}$$

where $\widehat{X}$ means the omission of the argument $X$. The right-hand-side of (4.8) of course vanishes for $(n,m) = (0,m)$, $m \neq 0$.

The general solution of the differential equation (4.7) reads

$$\left\langle b^a(x)c^b(y)\right\rangle = -\delta^{ab}[\theta(x^3-y^3)+\alpha]\delta^2(x^{\mathrm{tr}}-y^{\mathrm{tr}}), \tag{4.9}$$

[5]with $\varepsilon_{ij} = \varepsilon_{[ij]}$, $\varepsilon_{12} = 1$

where $\theta$ is the step function

$$\theta(u) = \begin{cases} 1 & \text{if } u > 0, \\ 0 & \text{if } u < 0, \end{cases} \tag{4.10}$$

and $\alpha$ is an integration constant. The term proportional to $\alpha$ represents the general solution of the homogeneous equation, compatible with transverse Poincaré invariance and scale invariance[6].

Using the same arguments we can write the general solution of the system of equations (4.8) as

$$\begin{aligned} &\left\langle b^a(x)c^b(y)d^{c_1}(z_1)\cdots d^{c_n}(z_n)A^{d_1}_{i_1}(t_1)\cdots A^{d_m}_{i_m}(t_m)\right\rangle \\ &= g\sum_{k=1}^{n} f_{ac_ke}[\theta(x^3-z^3_k)+\alpha^{(n,m)}]\delta^2(x^{\rm tr}-z^{\rm tr}_k) \\ &\quad \left\langle b^e(z_k)c^b(y)d^{c_1}(z_1)\cdots\widehat{d^{c_k}}(z_k)\cdots d^{c_n}(z_n)A^{d_1}_{i_1}(t_1)\cdots A^{d_m}_{i_m}(t_m)\right\rangle \\ &\text{for} \quad (n,m)\neq(0,0). \end{aligned} \tag{4.11}$$

As in (4.8) the right-hand-side vanishes for $n = 0$. The integration constants $\alpha^{(n,m)}$ depend only on the numbers $n$ and $m$, and not on $k$, due to the Bose symmetry of the field $d$.

Equations (4.11) build a recurrence on the number of fields $d$. They imply that from the connected Green functions with two ghost fields, those containing the field $A$ vanish,

$$\langle bc(d)^n(A)^m\rangle = 0 \qquad \text{for} \quad m\neq 0, \tag{4.12}$$

where an obvious shortened notation has been used. On the other hand the Green functions not containing the field $A$ are completely determined by the ghost propagator (4.9) – up to the integration constants $\alpha^{(n)} \equiv \alpha^{(n,0)}$ – through the recurrence relations (4.11) taken for $m = 0$:

$$\begin{aligned} &\left\langle b^a(x)c^b(y)d^{c_1}(z_1)\cdots d^{c_n}(z_n)\right\rangle \\ &= g\sum_{k=1}^{n} f_{ac_ke}[\theta(x^3-z^3_k)+\alpha^{(n)}]\delta^2(x^{\rm tr}-z^{\rm tr}_k) \\ &\quad \left\langle b^e(z_k)c^b(y)d^{c_1}(z_1)\cdots\widehat{d^{c_k}}(z_k)\cdots d^{c_n}(z_n)\right\rangle \\ &\text{for} \quad n\geq 1. \end{aligned} \tag{4.13}$$

A very similar argument shows that the connected Green functions involving more than one pair of ghosts all vanish. One indeed sees from the antighost equation (3.8) that the Green functions of the type $\langle(b)^p(c)^p\rangle$ vanish for $p > 1$. The recurrence relations generalizing (4.11) then imply the result

$$\langle(b)^p(c)^p(d)^n(A)^m\rangle = 0, \quad \text{for } p > 1, \tag{4.14}$$

whith the same shortened notation as in (4.12).

[6]Scale invariance excludes a solution of the type $1/(x^{\rm tr}-y^{\rm tr})^2$. Indeed, due to its short distance singularity, the latter expression is not a well defined distribution. To give it a meaning would need the introduction of a UV subtraction point, *i.e.*, of a dimensionful parameter which would break scale invariance.

## Transverse supersymmetry:

The transverse supersymmetry Ward identity (4.4) yields, for the two-point functions:

$$\begin{aligned}&\left\langle A_i^a(x)A_j^b(y)\right\rangle + \varepsilon_{ij3}\left\langle b^a(x)c^b(y)\right\rangle = 0,\\&\left\langle d^a(x)A_i^b(y)\right\rangle - \partial_{x^i}\left\langle b^a(x)c^b(y)\right\rangle = 0.\end{aligned} \tag{4.15}$$

With the result (4.9), this gives

$$\left\langle A_i^a(x)A_j^b(y)\right\rangle = \varepsilon_{ij}\delta^{ab}\left[\theta(x^3-y^3)+\alpha\right]\delta^2(x^{\rm tr}-y^{\rm tr}), \tag{4.16}$$

$$\left\langle d^a(x)A_i^b(y)\right\rangle = -\delta^{ab}\left[\theta(x^3-y^3)+\alpha\right]\partial_{x^i}\delta^2(x^{\rm tr}-y^{\rm tr}). \tag{4.17}$$

The integration constant $\alpha$ is now fixed to the value

$$\alpha = -\frac{1}{2} \tag{4.18}$$

by the Bose symmetry condition on the propagator $\langle A_iA_j\rangle$. This result corresponds to the Cauchy principal value prescription [16] for the propagator in momentum space. Indeed the Fourier transform of $\left[\theta(u)-\frac{1}{2}\right]\delta^2(x^{\rm tr})$ is equal to

$$\mathrm{vp}\,\frac{i}{p_3} = \mathrm{vp}\,\frac{i}{n^\mu p_\mu}$$

For the higher point connected Green functions the tranverse supersymmetry Ward identity (4.4) gives the relations

$$\begin{aligned}&\left\langle A_i^b(y)d^{c_1}(z_1)\cdots d^{c_n}(z_n)A_{i_1}^{d_1}(t_1)\cdots A_{i_m}^{d_m}(t_m)\right\rangle\\&=\sum_{k=1}^{n}\partial_{z_k^i}\left\langle b^{c_k}(z_k)c^b(y)d^{c_1}(z_1)\cdots\widehat{d^{c_k}}(z_k)\cdots d^{c_n}(z_n)A_{i_1}^{d_1}(t_1)\cdots A_{i_m}^{d_m}(t_m)\right\rangle\\&+\sum_{k=1}^{m}\varepsilon_{ii_k}\left\langle b^{d_k}(t_k)c^b(y)d^{c_1}(z_1)\cdots d^{c_n}(z_n)A_{i_1}^{d_1}(t_1)\cdots\widehat{A_{i_k}^{d_k}}(t_k)\cdots A_{i_m}^{d_m}(t_m)\right\rangle.\end{aligned} \tag{4.19}$$

They allow to compute all the Green functions of $A$ and $d$ from the Green functions (4.9), (4.12) and (4.13) of the ghost fields determined by the antighost equation. We thus obtain

$$\langle (A)^m(d)^n\rangle = 0 \qquad \text{for} \quad m\geq 3, \tag{4.20}$$

and, for the nonvanishing, ghost independent, connected Green functions, the relations

$$\begin{aligned}&\left\langle A_i^b(y)d^{c_1}(z_1)\cdots d^{c_n}(z_n)\right\rangle\\&= g\sum_{k,l=1(l\neq k)}^{n} f_{c_kc_le}[\theta(z_k^3-z_l^3)+\alpha^{(n)}]\partial_i\delta^2(z_k^{\rm tr}-z_l^{\rm tr})\\&\quad\left\langle b^e(z_l)c^b(y)d^{c_1}(z_1)\cdots\widehat{d^{c_k}}(z_k)\cdots\widehat{d^{c_l}}(z_l)\cdots d^{c_n}(z_n)\right\rangle,\end{aligned} \tag{4.21}$$

$(n\geq 2)$,

$$\begin{aligned}
&\left\langle A_i^a(x)A_j^b(y)d^{c_1}(z_1)\cdots d^{c_n}(z_n)\right\rangle \\
&= g\varepsilon_{ij}\sum_{k=1}^{n} f_{bc_ke}[\theta(y^3 - z_k^3) + \alpha^{(n)}]\delta^2(y^{\mathrm{tr}} - z_k^{\mathrm{tr}}) \\
&\quad \left\langle b^e(z_k)c^a(x)d^{c_1}(z_1)\cdots \widehat{d^{c_k}}(z_k)\cdots d^{c_n}(z_n)\right\rangle , \\
&(n \geq 1).
\end{aligned} \tag{4.22}$$

Here again, like for the two-point functions, Bose symmetry for the field $A$ fixes the value of the integration constants:

$$\alpha^{(n)} = -\frac{1}{2} \tag{4.23}$$

It is then easy to see from the above that the following recurrence relations hold:

$$\begin{aligned}
&\langle A_i^a(x)d^{c_1}(z_1)\cdots d^{c_n}(z_n)\rangle \\
&= g\sum_{k=1}^{n} f_{ac_ke}[\theta(x^3 - z_k^3) - \frac{1}{2}]\delta^2(x^{\mathrm{tr}} - z_k^{\mathrm{tr}}) \\
&\quad \left\langle A_i^e(z_k)d^{c_1}(z_1)\cdots \widehat{d^{c_k}}(z_k)\cdots d^{c_n}(z_n)\right\rangle , \\
&(n \geq 2),
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
&\left\langle A_i^a(x)A_j^b(y)d^{c_1}(z_1)\cdots d^{c_n}(z_n)\right\rangle \\
&= g\sum_{k=1}^{n} f_{ac_ke}[\theta(x^3 - z_k^3) - \frac{1}{2}]\delta^2(x^{\mathrm{tr}} - z_k^{\mathrm{tr}}) \\
&\quad \left\langle A_i^e(z_k)A_j^b(y)d^{c_1}(z_1)\cdots \widehat{d^{c_k}}(z_k)\cdots d^{c_n}(z_n)\right\rangle , \\
&(n \geq 1).
\end{aligned} \tag{4.25}$$

They are illustrated together with the ghost recurrence relation (4.13) (with $\alpha^{(n)} = -1/2$) in Figs. 2 to 4. The thin lines correspond to the two-point functions as depicted in Fig. 1. The three-point vertices ($cbA_3$) and ($AAA_3$) correspond respectively to the expressions $gf_{abc}\int d^3z\cdots$ and $g\varepsilon_{ij}f_{abc}\int d^3z\cdots$. A 'hat' on an argument means its omission.

We have thus obtained the general result that, in the axial gauge, the supersymmetry completely fixes the connected Green functions, with the notable exception of those involving the Lagrange multiplier field $d$ only.

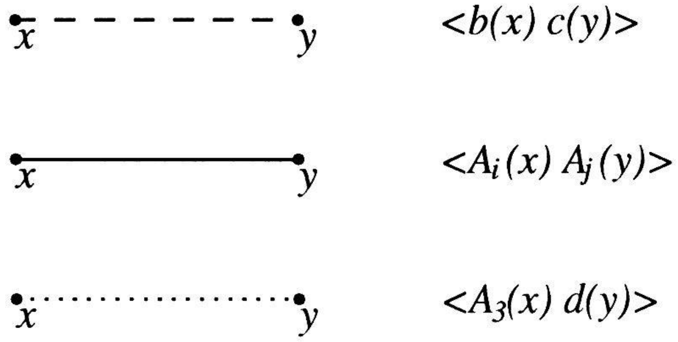


Figure 1: Graphical representation of the two-point functions (4.9), (4.16), (4.6) (with $\alpha = -1/2$).

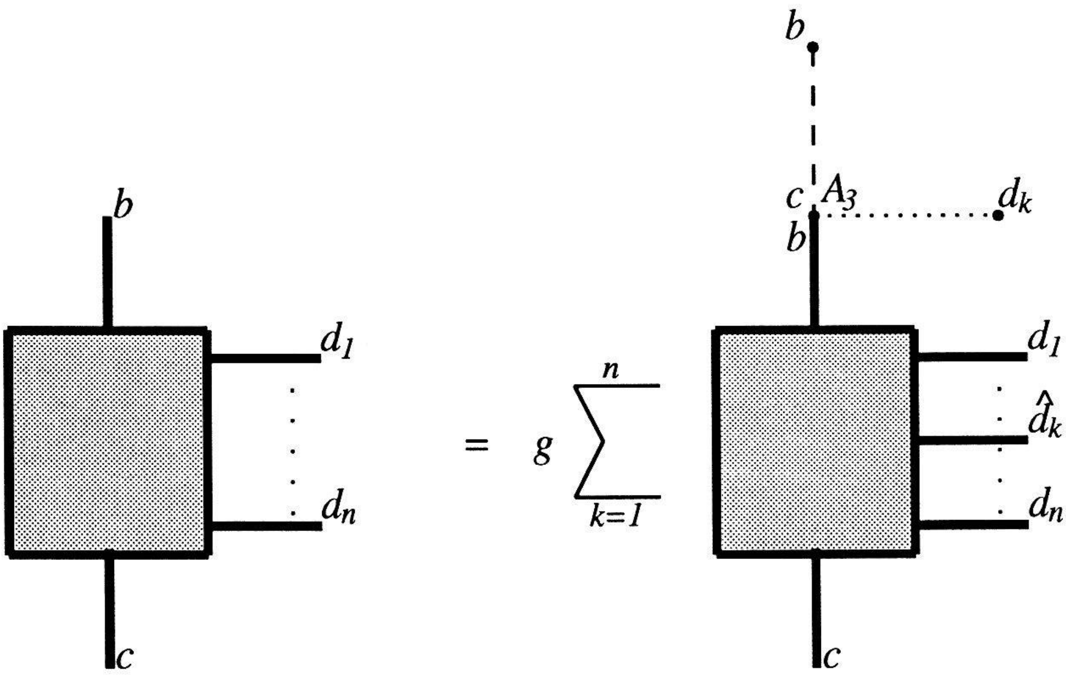


Figure 2: Graphical representation of Eq.(4.13).

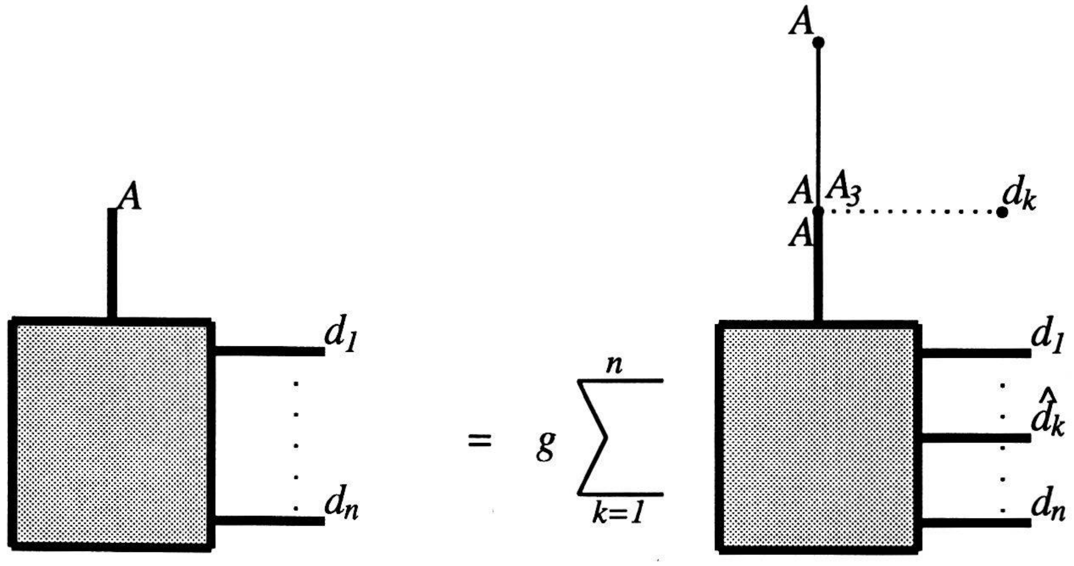


Figure 3: Graphical representation of Eq.(4.24).

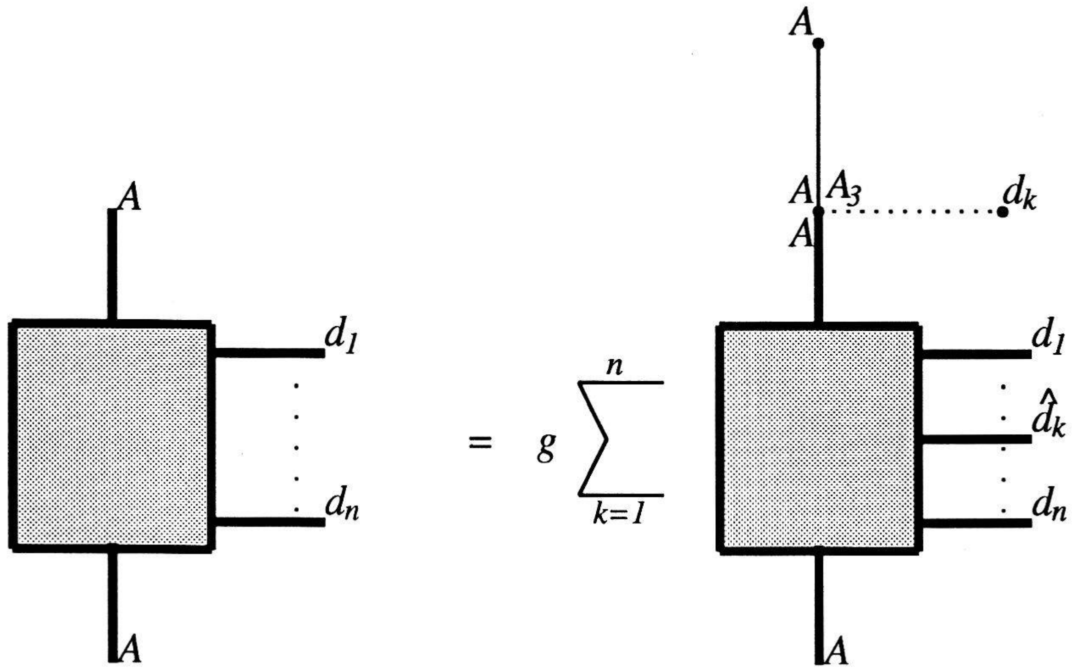


Figure 4: Graphical representation of Eq.(4.25).

# 5 Consequences of gauge invariance

The gauge Ward identity (3.10) yields, for the connected Green functions of the Lagrange multiplier field the equations

$$\begin{aligned}&\partial_{x^3}\left\langle d^a(x)d^{c_1}(z_1)\cdots d^{c_n}(z_n)\right\rangle\\&=g\sum_{k=1}^{n}f_{ac_ke}\delta^3(x-z_k)\left\langle d^e(z_k)d^{c_1}(z_1)\cdots\widehat{d^{c_k}}(z_k)\cdots d^{c_n}(z_n)\right\rangle.\end{aligned}\tag{5.1}$$

The solution which respects scale invariance vanishes identically:

$$\left\langle d^{c_1}(z_1)\cdots d^{c_n}(z_n)\right\rangle=0\quad\forall n.\tag{5.2}$$

This ends the demonstration that the Ward identities of the Chern-Simons theory in the axial gauge determine uniquely all its Green functions. However, beyond the gauge fixing condition and the supersymmetry Ward identities – together with the antighost equation – we have used only a small part of the gauge Ward identities, namely the equations (5.1) for the field $d$. The remaining ones, which involve also the other fields, as well as the ghost equation (3.9) have to be – and indeed have been – checked explicitly.

# 6 Conclusion

The main result of this study is that the Green functions of the three-dimensional Chern-Simons theory in the axial gauge follow as the – unique – solution of the Ward identities defining the model, without reference to any action principle.

More remarkable, they are all determined by the gauge condition and the Ward identities of topological supersymmetry only, with the exception of the Green functions (5.2) of the Lagrange multiplier field. It is merely in order to fix the latters that the gauge Ward identity is effectively needed.

However, notwithstanding the latter point, and looking at the solution of the gauge condition and of the supersymmetry Ward identities, we remark that it consists of exactly the Green functions that one would calculate – with the principal value prescription for the free propopagators [16] – from the gauge fixed action (2.1), which is itself a solution of the gauge Ward identity. This is at best seen from the graphical representations (see Figs. 2 to 4) of the equations (4.13), (4.24) and (4.25). Turning the argument round, we may conclude that enforcing the supersymmetric Ward identities and the gauge condition on the Green functions fixes uniquely the action – solution of the gauge Ward identity – and then all the Green functions of the theory.

Finally, it is worth noticing that the Green functions we have obtained correspond to tree graphs only. The loop graphs which, in a conventional Feynman graph expansion scheme, are expected a priori to contribute – even in an axial gauge, where they in fact cancel if one appropriately takes into account the ghost loops [13] – simply do not show up in the present approach.

# Appendices:

Appendix A is devoted to demonstrate the existence of a larger algebra of symmetries of the Chern Simons model quantized in the axial gauge.
Parallel to the analysis done in the covariant Landau gauge we show in App. B that also in the non-covariant axial gauge an off-shell formulation for the BRS and supersymmetry is possible.

# A The algebra generated by BRS, anti-BRS and supersymmetry

Beside the symmetry transformations (3.1) and (3.2) one can construct a further symmetry defined by

$$\begin{aligned} \hat{\delta} A_\mu &= -D_\mu(2d - g\{b,c\}), \\ \hat{\delta} b &= g\,[d,b]\,, \\ \hat{\delta} c &= g\,[d,c]\,, \\ \hat{\delta} d &= 0. \end{aligned} \tag{A.1}$$

It is straightforward to show that the action (2.1) is invariant under (A.1) and in addition one has the following closure with $s$ and $\bar{s}$:

$$\begin{aligned} &s^2 = \bar{s}^2 = 0, \quad \{s,\bar{s}\} = \hat{\delta}, \\ &\left[s,\hat{\delta}\right] = 0, \qquad \left[\bar{s},\hat{\delta}\right] = 0. \end{aligned} \tag{A.2}$$

In contradiction to the Landau gauge $s$ and $\bar{s}$ do not anticommute.
Nevertheless, $\nu_\mu$ and $\bar{\nu}_\rho$ form a closed algebra with the ghost-number transformation $r$ and a new tensor symmetry $h_{\alpha\beta}$:

$$\begin{aligned} h_{\alpha\beta} A_\mu &= -\varepsilon_{\alpha\mu\nu} n^\nu A_\beta - \varepsilon_{\beta\mu\nu} n^\nu A_\alpha, & rA_\mu &= 0, \\ h_{\alpha\beta} b &= 0, & rb &= -b, \\ h_{\alpha\beta} c &= 0, & rc &= c, \\ h_{\alpha\beta} d &= -\partial_\alpha A_\beta - \partial_\beta A_\alpha, & rd &= 0, \end{aligned} \tag{A.3}$$

leading to the following larger algebra:

$$\begin{aligned}
&\{\nu_\alpha,\nu_\beta\}=0, && \{\bar\nu_\alpha,\bar\nu_\beta\}=0,\\
&\{\nu_\alpha,\bar\nu_\beta\}=h_{\alpha\beta}+\varepsilon_{\alpha\beta\nu}n^\nu r,\\
&[\nu_\alpha,r]=\nu_\alpha, && [\bar\nu_\alpha,r]=-\bar\nu_\alpha,\\
&[\nu_\alpha,h_{\beta\gamma}]=\varepsilon_{\beta\alpha\nu}n^\nu\nu_\gamma+\varepsilon_{\gamma\alpha\nu}n^\nu\nu_\beta, && [\bar\nu_\alpha,h_{\beta\gamma}]=-\varepsilon_{\beta\alpha\nu}n^\nu\bar\nu_\gamma-\varepsilon_{\gamma\alpha\nu}n^\nu\bar\nu_\beta,\\
&[r,h_{\alpha\beta}]=0.
\end{aligned} \tag{A.4}$$

Additionally there exists a further symmetry algebra. The two new symmetries $\tau_\rho$ and $\tau$

$$\begin{aligned}
&\tau_\alpha A_\mu=-\varepsilon_{\alpha\mu\nu}n^\nu gc^2, && \tau A_\mu=0,\\
&\tau_\alpha b=\partial_\alpha c+D_\alpha c, && \tau b=gc^2,\\
&\tau_\alpha c=0, && \tau c=0,\\
&\tau_\alpha d=\partial_\alpha(gc^2), && \tau d=0
\end{aligned} \tag{A.5}$$

lead to the following closed algebra:

$$\begin{aligned}
&\{s,\bar\nu_\alpha\}=\tau_\alpha, && [\tau_\alpha,\tau_\beta]=0,\\
&[\tau_\alpha,\bar\nu_\beta]=3\varepsilon_{\alpha\beta\nu}n^\nu\tau, && \tau^2=0,\\
&[s,\tau_\alpha]=0, && \{s,\tau\}=0,\\
&\{\bar\nu_\alpha,\tau\}=0, && [\tau_\alpha,\tau]=0.
\end{aligned} \tag{A.6}$$

# B Off-shell BRS and supersymmetry algebra

As usually and parallely to the analysis for the Landau gauge the composite fields appearing in the BRS transformations (3.1) are coupled to external fields $\gamma^\mu$, $\sigma$, and thus induce the source term in the action:

$$\Sigma_{\rm ext}=\mathrm{Tr}\int d^3x\,(\gamma^\mu sA_\mu+\sigma sc)\,. \tag{B.1}$$

BRS invariance is then expressed by the Slavnov identity

$$SZ_c=\mathrm{Tr}\int d^3x\left(J^\mu\frac{\delta}{\delta\gamma^\mu}-J_c\frac{\delta}{\delta\sigma}-J_b\frac{\delta}{\delta J_d}\right)Z_c=0, \tag{B.2}$$

where $Z_c(J^\mu,J_b,J_c,J_d,\gamma^\mu,\sigma)$ is the generating functional of the connected Green functions, $J^\mu$, $J_d$, $J_b$ and $J_c$ denoting the sources of the fields $A_\mu$, $d$, $b$ and $c$, respectively.

The presence of these source terms breaks the supersymmetry. Thus one has the following Ward identity:

$$\begin{aligned}\mathcal{W}_\rho Z_c \equiv \mathrm{Tr}\int \ & d^3x \left( J^\mu \varepsilon_{\rho\mu\nu} n^\nu \frac{\delta}{\delta J_b} - J_c \frac{\delta}{\delta J^\rho} + J_d \partial_\rho \frac{\delta}{\delta J_b} \right. \\ & \left. -\sigma \left( \frac{\delta}{\delta \gamma^\rho} - \partial_\rho \frac{\delta}{\delta J_c} \right) - \gamma^\mu \left( \partial_\rho \frac{\delta}{\delta J^\mu} - \varepsilon_{\rho\mu\nu} n^\nu \frac{\delta}{\delta J_d} \right) \right) Z_c \\ & = \Delta_\rho = \mathrm{Tr} \int d^3x \varepsilon_{\rho\mu\nu} J^\mu \gamma^\nu .\end{aligned} \tag{B.3}$$

The action of the Slavnov operator on the "breaking term" $\Delta_\rho$ vanishes identically:

$$\mathcal{S}\Delta_\rho \ = 0. \tag{B.4}$$

Remember the nilpotency of the Slavnov operator

$$\mathcal{S}\mathcal{S}F[J_\phi] = 0. \tag{B.5}$$

for any functional $F[J_\phi]$ of the sources, then the following identities complete the algebra between supersymmetry and BRS.

$$\left(\mathcal{W}_\rho \mathcal{W}_\sigma + \mathcal{W}_\sigma \mathcal{W}_\rho\right) F[J_\phi] = 0. \tag{B.6}$$

$$\begin{aligned}\mathcal{W}_\rho \mathcal{S} F[J_\phi] + \mathcal{S}\left(\mathcal{W}_\rho F[J_\phi] - \Delta_\rho\right) \ &= \\ \left(\mathcal{W}_\rho \mathcal{S} + \mathcal{S}\mathcal{W}_\rho\right) F[J_\phi] \ &= \mathcal{P}_\rho F[J_\phi].\end{aligned} \tag{B.7}$$

Here $\mathcal{P}_\rho$ is the operator of translations defined by

$$\mathcal{P}_\rho Z_c \equiv \mathrm{Tr} \int d^3x \ \left( \textstyle\sum_\phi \partial_\rho J_\phi \frac{\delta}{\delta J_\phi} \right) Z_c. \tag{B.8}$$

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