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Massless fermion emission on 1+1 dimensional curved space-times

Th. Gallay*

Département de physique théorique, Université de Genève
CH-1211 Geneva 4, Switzerland

G. Wanders

Institut de physique théorique, Université de Lausanne
CH-1015 Lausanne, Switzerland

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Abstract. This paper examines the production of massless fermions by the curvature of asymptotically flat $1+1$ dimensional space-times. Expectation values of the current and the energy-momentum tensor in the incoming vacuum as well as inclusive probabilities of detecting outgoing particles in given states are evaluated. Point-like curvatures give very different results according to the coordinate system in which they are static for some time. The radiation of fermion-antifermion pairs is either a steady process with a sort of thermal spectrum or a transient phenomenon accompanying the switching on and off of the curvature. The existence of outgoing states occupied with high probability is also investigated and properties of the fermionic effective action accessible to our approach are established.

1 Introduction

The purpose of this paper is to present the outcome of an investigation of quantized massless Dirac fermion fields on curved $1+1$ dimensional space-times with particular emphasis on the pair creation mechanism.

*Part of this work originates in Th. Gallay's diploma thesis (Ecole Polytechnique Fédérale de Lausanne, 1990).

Quantized fields in curved space-times have been discussed extensively [1] and the subject is really not new. Production of spin-$\frac{1}{2}$ particles in $3+1$ dimensions has been discussed recently in [2] and fermions in $1+1$ dimensions have been considered in [3]. There is one feature of a model of massless fermions in $1+1$ dimensions which, to our knowledge, has not been exploited entirely: the general solution of the Dirac equation in conformal coordinates can be displayed explicitly and it is particularly simple. Due to this favorable circumstance, the configuration of outgoing fermions and antifermions created by the curvature can be analysed in great detail and in a non-perturbative way.

Our setup is as simple as possible: asymptotically flat space-times, curvature with compact support and globally defined conformal coordinates. A complication arises from the fact that conformal coordinates cannot be asymptotically minkowskian in all directions. This leads us to equip our spaces with two distinct conformal coordinate systems: an "incoming" system that is minkowskian in the past of the curvature, and an "outgoing" one, minkowskian in its future. It is natural to express the incoming fermion field in the incoming system, and the outgoing field in the outgoing system. This removes the ambiguity in the definition of the in- and outgoing particles in a natural way.

A drawback of massless fermions in $1+1$ dimensions is that, due to an infrared singularity, the correspondence between the outgoing fields and the incoming ones is, in general, not unitarily implementable. This means that the curvature induces the creation of an infinite number of fermion-antifermion pairs; there is no outgoing vacuum in the Fock space of the ingoing vacuum. In spite of that, the total mean amounts of energy and momentum which are produced are finite. The mean outgoing energy and momentum densities can be computed. The outgoing particle content of the ingoing vacuum can be analysed by means of the inclusive probabilities for the detection of outgoing fermions or antifermions in given states. Analysing these various quantities, one gets a rather precise picture of the pair creation process.

In the main applications of our general results we will be dealing with point-like curvatures concentrated on the time axis and switched on during a finite time interval. Two situations will be considered: in the first one an effective curvature strength in the incoming coordinate system is kept constant during some time; in the second case, it is a scalar strength that is quasi-static. Surprisingly, the choice between these two possibilities is in no way innocent: they lead to completely different pair creation patterns. In the first case one discovers a steady pair emission whereas in the second case, pair emission is a transient phenomenon accompanying the switching on and off of the curvature. Another surprise is that in the first type of point-like curvature, the inclusive probabilities of a suitably selected sample of outgoing states exhibit the form of a thermal distribution reminiscent of a Hawking radiation [5].

We explain in the following Sections how these findings come about. Our space-times and their coordinates are presented in Section 2. The Dirac fermion field machinery is reviewed and the in-out correspondence is defined in Section 3. In Section 4, mean outgoing quantities like current and energy-momentum are computed in the incoming vacuum. A master formula for inclusive probabilities is established in Section 5 and applied to our point-like curvatures in Section 6. In Section 7 we prove that a strong curvature pulse has outgoing states which are occupied with probabilities close to 1. The results obtainable within our approach are completed in Section 8 with a discussion of the implementability of the in-out correspondence. If implementability holds, an effective action can be defined and we show that the functional derivative with respect to the metric of its imaginary part is correctly related to the energy-momentum tensor. Computational technicalities and proofs are collected in four appendices.

2 Setting the stage

The 1 + 1 dimensional space-times we shall consider are parameterized by conformal coordinates x^μ ($\mu = 0, 1$). They are covered by a single map extending over the whole (x^0, x^1)-plane. The scalar curvature $R(x)$ is assumed to have a compact support D. The metric has the form

$$g_{\mu\nu}(x) = \mathrm{e}^{\Omega(x)}\eta_{\mu\nu}, \tag{2.1}$$

$\eta_{\mu\nu} = \mathrm{diag}\,(1, -1)$. The Liouville field $\Omega(x)$ is related to the curvature by

$$\left(\partial_0^2 - \partial_1^2\right)\Omega(x) = -R(x)\mathrm{e}^{\Omega(x)}. \tag{2.2}$$

At given R this is a nonlinear equation for Ω. We define our coordinates in such a way that Ω vanishes outside the absolute future of D. This implies that Ω is the retarded solution of (2.2). As shown in Fig. 1, it reduces to functions $\Omega_\pm(x^\pm)$ of x^+, resp. x^-, in the causal shadows D_+ and D_- of D ($x^\pm = (1/\sqrt{2})\,(x^0 \pm x^1)$): it is equal to a constant Ω_0 in between. Our space-times are asymptotically flat but the conformal coordinates x^μ become minkowskian only in the past of D: generically they are not minkowskian in its future.

The assumptions we have made are consistent if the retarded solution Ω of the nonlinear equation (2.2) is non-singular over the whole plane. This is not automatically the case and imposes constraints on the curvature R. The way singularities may appear is best seen if eq. (2.2) is rewritten as an integral equation

$$\Omega(x) = -\frac{1}{2}\int_{C_x} \mathrm{d}^2x'\, R(x')\mathrm{e}^{\Omega(x')}, \tag{2.3}$$

where C_x is the past light cone of the point x. A negative R produces a positive Ω which gets amplified exponentially. This can result in a divergence of Ω at finite times as will be illustrated in examples at the end of this Section.

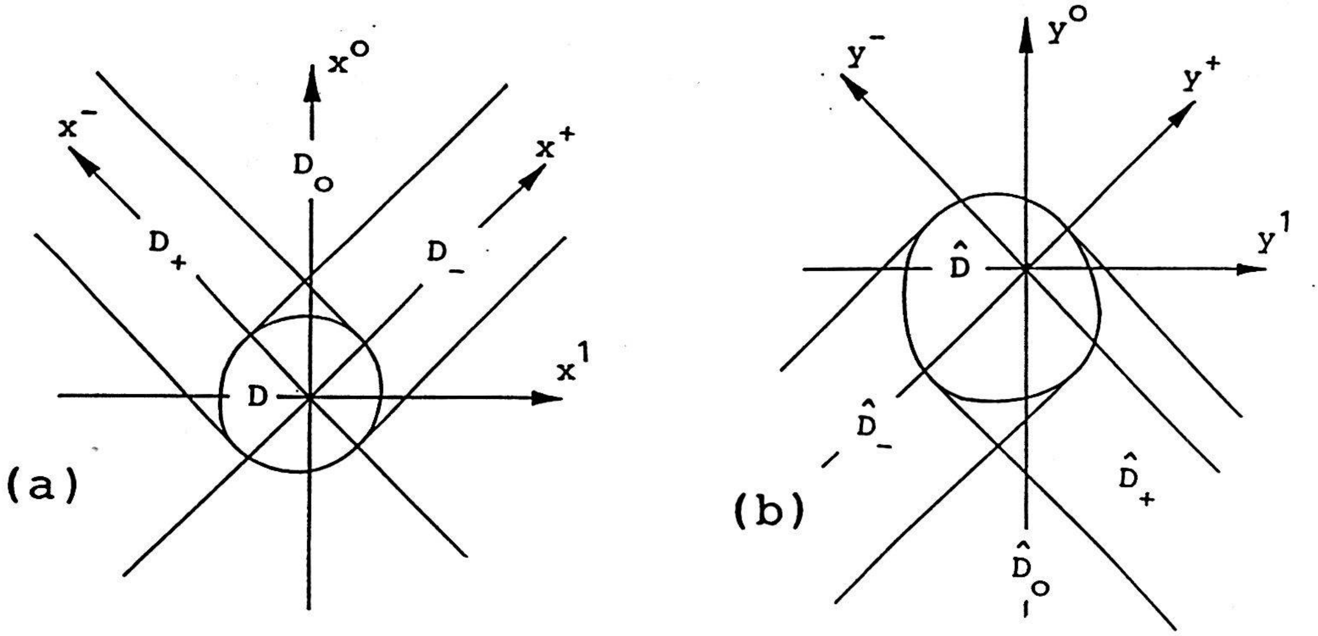

Fig. 1: An asymptotically flat space-time depicted in two conformal coordinate systems (a): in the incoming coordinates x the Liouville field Ω is nonzero in the union of D, D_+, D_- and D_0 ($D = \operatorname{supp} R$); $\Omega(x) = \Omega_+(x^+) + \Omega_-(x^-)$ in $D_+ \cup D_-$ and $\Omega(x) = \Omega_0$ in D_0; (b): a similar situation holds in the incoming coordinate system y with $D, D_\pm, D_0$ replaced by $\widehat{D}, \widehat{D}_\pm, \widehat{D}_0$. $\widehat{D}$ is the image of D under the change of variables (2.4).

We can define, and shall indeed use, a complementary set of conformal coordinates y^μ which are minkowskian in the future of D (Fig. 1). The two sets of coordinates are related by two increasing functions $\eta^\pm$ of a single variable:

$$y^+ = \eta^+(x^+), \qquad y^- = \eta^-(x^-). \tag{2.4}$$

The Liouville field $\widehat{\Omega}(y)$ of the new coordinates is obtained from $\Omega(x)$ by the relation

$$\widehat{\Omega}(y) = \Omega(x) - \alpha_+(x^+) - \alpha_-(x^-), \tag{2.5}$$

where $\alpha_\pm(x^\pm) = \ln(\partial_\pm \eta^\pm(x^\pm))$. In eq. (2.5) and in many forthcoming ones a function of y is equated to a function of x: it is understood that the values of these variables are related by eq. (2.4). The functions $\alpha_\pm$ have to be such that $\widehat{\Omega}$ vanishes in the future of the support D. This condition fixes the new coordinates up to a global Poincaré transformation. Up to a translation we have

$$\eta^\pm(x^\pm) = \int_0^{x^\pm} du\, e^{\alpha_\pm(u)}, \tag{2.6}$$

with

$$
\begin{aligned}
\alpha_-(x^-) &= \begin{cases} \Lambda & \text{for} \quad x^- \leq a^- \\ \Omega_-(x^-) + \Lambda & \text{for} \quad a^- < x^- \leq b^- \\ \Omega_0 + \Lambda & \text{for} \quad x^- > b^- \end{cases} \\
\alpha_+(x^+) &= \begin{cases} -\Omega_0 - \Lambda & \text{for} \quad x^+ \leq a^+ \\ \Omega_+(x^+) - \Omega_0 - \Lambda & \text{for} \quad a^+ < x^+ \leq b^+ \\ -\Lambda & \text{for} \quad x^+ > b^+ \end{cases}
\end{aligned}
\tag{2.7}
$$

The intervals $[a^\mp, b^\mp]$ are the projections of D on the x^-– and x^+-axis. Λ is an arbitrary constant: a change of Λ produces a global Lorentz transformation. For any choice of Λ, $y^\pm$ is linear in $x^\pm$ for $x^\pm < a^\pm$ and $x^\pm > b^\pm$.

It is clear that (2.7) leads to a $\widehat{\Omega}$ vanishing in the future of D; in fact $\widehat{\Omega}$ is the advanced solution of (2.2) written in terms of the y-variables. We call the x- and y-variables respectively *incoming* and *outgoing conformal variables.*

Before we close this Section we want to prepare the ground for the illustrations which will appear in the following Sections. They deal mainly with point-like curvatures concentrated on the time axis and switched on during a finite time. It is convenient to display such a curvature as follows:

$$R(x) = \sqrt{2}\,\mathrm{e}^{-\frac{1}{2}\Omega(x)} \rho(x^0/\sqrt{2})\,\delta(x^1). \tag{2.8}$$

The factors $\sqrt{2}$ facilitate the passage to the light-cone coordinates. The exponential factor has the effect that the strength ρ is a scalar under conformal coordinate transformations $x^\pm \to z^\pm = f(x^\pm)$ which transform the x-time axis into the z-time axis [6]. We may also write (2.8) in terms of an effective strength $\bar{\rho}(x^0/\sqrt{2}) = \exp(-\Omega(x^0, 0)/2)\,\rho(x^0/\sqrt{2})$ which is no longer a scalar:

$$R(x) = \sqrt{2}\,\bar{\rho}(x^0/\sqrt{2})\,\delta(x^1). \tag{2.9}$$

We shall be interested in situations where either the effective strength or the scalar strength is static during some time. This leads us to define two types of point-like curvatures.

Type I. The effective strength $\bar{\rho}$ is quasi-static in terms of the incoming coordinates: $\operatorname{supp}\bar{\rho} = [0, L]$, $\bar{\rho}(u) = \bar{\rho} =$ constant for $u \in [L_1, L_2]$, $0 < L_1 < L_2 < L$.

Type II. The scalar strength is quasi-static: $\operatorname{supp}\rho = [0, L]$, $\rho(u) = \rho =$ constant for $u \in [L_1, L_2]$.

Clearly a type I point-like curvature has a scalar strength which is not quasi-static and, conversely, a type II curvature has an effective strength which is not quasi-static in the incoming coordinates. This last effective strength will be quasi-static in coordinates z^μ such that their Liouville field vanishes on the image of the interval $[L_1, L_2]$ of the time axis. As we shall shortly see, these coordinates exist: they differ from our incoming and outgoing coordinates because their Liouville field is nonzero in the past as well as in the future of the curvature.

For both types of point-like curvatures the retarded Liouville field has the form

$$\Omega(x) = \theta(x^1)F(x^-) + \theta(-x^1)F(x^+), \tag{2.10}$$

where θ denotes the standard step function: $\theta(x) = 0$ for $x \leq 0$ and $\theta(x) = 1$ for $x > 0$.

The asymptotic forms $\Omega_\pm$ are identical and equal to F. Each type has its own F:

$$F_I(u) = -\theta(u) \log\left(1 + \int_0^u dv\, \bar{\rho}(v)\right), \tag{2.11}$$

$$F_{II}(u) = -\theta(u) \log\left(1 + \frac{1}{2}\int_0^u dv\, \rho(v)\right)^2. \tag{2.12}$$

These results illustrate the observation on the existence of Ω made at the beginning of this Section: Ω is nonsingular only if the integral of ρ (or $\bar{\rho}$) from 0 to L is not too negative.

If we want the time axis of the outgoing coordinates to be the image of the time axis of the incoming coordinates, we must set $\Lambda = -\Omega_0/2$ in (2.7).

The z-coordinates mentioned above are given by

$$z^\pm = \zeta(x^\pm), \qquad \zeta(u) = \int_0^u dv\, e^{\frac{1}{2}F(v)}. \tag{2.13}$$

The Liouville field $\tilde{\Omega}$ of this coordinate system is

$$\tilde{\Omega}(z) = \frac{1}{2}\epsilon(x^1)\left(F(x^-) - F(x^+)\right) \tag{2.14}$$

where $\epsilon(x) = \theta(x) - \theta(-x)$ and the effective strength $\tilde{\rho}$ is equal to the scalar strength ρ: it is quasi-static in the type II case. $\tilde{\Omega}$ is a truly static field, equal to $(1/2\sqrt{2})\rho z^1$, in the double cone $\zeta(L_1) < z^\pm < \zeta(L_2)$ (ρ is the static value of $\rho(u)$).

We consider both types of point-like curvatures as natural choices. It is a surprising outcome of our work that they lead to totally different pair creation patterns.

3 The fermion field and its in-out relationship

We want to construct a massless Dirac fermion field on the $1+1$ dimensional curved space time presented in Section 2. We start collecting a few well-known facts [1, 2, 3]. In arbitrary coordinates the Dirac equation is

$$\gamma^{\mu}(x)\left(\partial_{\mu}+\omega_{\mu}(x)\right)\psi(x)=0, \tag{3.1}$$

where $\gamma^{\mu}(x)=e_{a}^{\mu}(x)\bar{\gamma}^{a}$, $e_{a}^{\mu}(x)$ are the components of a zweibein of vector fields $\epsilon_{a}(x)$ ($a=0,1$, $e_{a}^{\mu}e_{b}^{\nu}g_{\mu\nu}=\eta_{ab}$, $e_{a}^{\mu}e_{b}^{\nu}\eta^{ab}=g^{\mu\nu}$) and $\bar{\gamma}^{a}$ are flat space Dirac matrices. The spin connection ω_{μ} is given by

$$\omega_{\mu}=\frac{1}{4}\gamma_{\nu}\left[\partial_{\mu}\gamma^{\nu}+\Gamma_{\rho\mu}^{\nu}\gamma^{\rho}\right] \tag{3.2}$$

in terms of the Christoffel symbols $\Gamma_{\mu\nu}^{\rho}$.

If the variables x^{μ} are conformal coordinates, for instance the incoming variables of Section 2, one can choose a zweibein of vectors parallel to the coordinate lines, $\bar{\epsilon}_{a}=\exp(-\frac{1}{2}\Omega)\delta_{a}^{\mu}\,\partial/\partial x^{\mu}$. An arbitrary zweibein $\{\epsilon_{a}\}$ is obtained from $\{\bar{\epsilon}_{a}\}$ by a local Lorentz transformation Λ:

$$\epsilon_{a}(x)=\Lambda_{a}^{b}(x)\bar{\epsilon}_{b}(x), \qquad \Lambda(x)=\begin{pmatrix}\operatorname{ch}\lambda & \operatorname{sh}\lambda\\ \operatorname{sh}\lambda & \operatorname{ch}\lambda\end{pmatrix}(x). \tag{3.3}$$

The spin connection becomes

$$\omega_{\mu}=\frac{1}{2}\bar{\gamma}^{5}\left(\partial_{\mu}\lambda+\frac{1}{2}\varepsilon_{\mu}{}^{\nu}\partial_{\nu}\Omega\right), \tag{3.4}$$

$\bar{\gamma}^{5}=\bar{\gamma}^{0}\bar{\gamma}^{1}$, $\varepsilon_{\mu}{}^{\nu}=\eta_{\mu\lambda}\varepsilon^{\lambda\nu}$, $\varepsilon^{\mu\nu}=-\varepsilon^{\nu\mu}$, $\varepsilon^{01}=1$.

In the chiral spinor basis where $\bar{\gamma}^{5}=\operatorname{diag}(1,-1)$, the two components ψ_{-} and ψ_{+} of ψ decouple and the general solution of the Dirac equation (3.1) is expressed in terms of a free massless field ϕ:

$$\psi_{\pm}(x)=\exp\left[-\frac{1}{4}\Omega(x)\pm\frac{1}{2}\lambda(x)\right]\phi_{\pm}(x^{\pm}). \tag{3.5}$$

We see that a massless Dirac field is the same as a pair of independent Weyl fermion fields. Consequently we may restrict ourselves to one chiral component, for instance ψ_{-}

which gives rise to the right-moving outgoing particles. We do that and ψ stands for ψ_- from now on.

The curved space canonical equal time anticommutation relations of the ψ-field imply flat space canonical anticommutation relations for the ϕ-field, in particular:

$$\left\{\phi^\dagger(x^-), \phi(x'^-)\right\} = \frac{1}{\sqrt{2}}\delta(x^- - x'^-). \tag{3.6}$$

We notice that the ϕ-field is gauge independent, i.e. it is unaffected by a change of the zweibein $\{\epsilon_a\}$. Such a change modifies only the hyperbolic angle λ in the exponential in (3.5). In the past of D where $\Omega = 0$, the only difference between ψ and ϕ comes from the gauge dependence of ψ. This leads us to identify the ϕ-field as a gauge independent incoming free field, and we call it ϕ_{in} from now on. The creation and annihilation operators appearing in its Fourier representation

$$\phi_{\text{in}}(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{2^{1/4}}\int_0^\infty dk\left(a_{\text{in}}(k)e^{-ikx^-} + b_{\text{in}}^\dagger(k)e^{ikx^-}\right), \tag{3.7}$$

create and annihilate incoming fermions and antifermions:

$$\left\{a_{\text{in}}^\dagger(k), a_{\text{in}}(k')\right\} = \left\{b_{\text{in}}^\dagger(k), b_{\text{in}}(k')\right\} = \delta(k - k'). \tag{3.6'}$$

What we have done in the incoming coordinate system can also be done in the outgoing one. The result is an expression of the Dirac field $\widehat{\psi}$ in this system similar to (3.5):

$$\widehat{\psi}(y) = \exp\left[-\frac{1}{4}\widehat{\Omega}(y) - \frac{1}{2}\hat{\lambda}(y)\right]\widehat{\phi}_{\text{out}}(y^-). \tag{3.8}$$

The Liouville field $\widehat{\Omega}$ is given in (2.5). $\widehat{\psi}$ is evaluated with respect to a zweibein $\{\hat{\epsilon}_a\}$ which is obtained from the canonical $\{\hat{\bar{\epsilon}}_a\}$ by a local Lorentz transformation specified by $\hat{\lambda}$. $\widehat{\phi}_{\text{out}}$ is the gauge invariant outgoing field.

We are primarily interested in the correspondence between ϕ_{in} and $\widehat{\phi}_{\text{out}}$. It is determined by the relationship between ψ and $\widehat{\psi}$ resulting from the transformation law of a fermion field under the $x \to y$ change of coordinates and the $\{\epsilon_a\} \to \{\hat{\epsilon}_a\}$ gauge transformation. One obtains

$$\widehat{\psi}(y) = \exp\left[-\frac{1}{2}\left(\hat{\lambda}(y) + \frac{1}{2}(\alpha_-(x^-) - \alpha_+(x^+)) - \lambda(x)\right)\right]\psi(x). \tag{3.9}$$

$(\alpha_- - \alpha_+)/2$ is the hyperbolic angle of the Lorentz transformation relating the canonical zweibeins $\{\bar{\epsilon}_a\}$ and $\{\hat{\bar{\epsilon}}_a\}$. Combining (3.5), (3.8) and (3.9) we obtain the in-out correspondence:

$$\widehat{\phi}_{\mathrm{out}}(y^-) = \mathrm{e}^{-\frac{1}{2}\alpha_-(x^-)}\phi_{\mathrm{in}}(x^-). \tag{3.10}$$

We observe that this is consistent with canonical anticommutation relations for $\widehat{\phi}_{\mathrm{out}}$ in the y-coordinates if ϕ_{in} satisfies such relations in the x-coordinates. The outgoing field $\widehat{\phi}_{\mathrm{out}}$ has a Fourier representation similar to (3.7) in terms of the y^--variable. The in-out correspondence (3.9) determines the Bogoliubov transformation relating the creators and annihilators of the in- and outgoing fields

$$\begin{pmatrix} a_{\mathrm{out}} \\ b^\dagger_{\mathrm{out}} \end{pmatrix} = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \begin{pmatrix} a_{\mathrm{in}} \\ b^\dagger_{\mathrm{in}} \end{pmatrix}. \tag{3.11}$$

The kernels $K_i(k,p)$ are nonzero for $(k,p) \in \mathbb{R}_+ \times \mathbb{R}_+$. Their values in this quadrant are obtained from a single function $U(k,p)$:

$$\begin{aligned} K_1(k,p) &= U(k,p), & K_2(k,p) &= U(k,-p), \\ K_3(k,p) &= U(-k,p), & K_4(k,p) &= U(-k,-p), \end{aligned} \tag{3.12}$$

with

$$U(k,p) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}[k\eta^-(x)-px]}\,\mathrm{e}^{\frac{1}{2}\alpha_-(x)}. \tag{3.13}$$

According to (2.7) and (2.3), the explicit form of α_- in the shadow of D is

$$\alpha_-(x^-) = -\frac{1}{2}\int_D \mathrm{d}^2u\, \theta(x^- - u^-)\, R(u)\, \mathrm{e}^{\Omega(u)} + \Lambda. \tag{3.14}$$

It is instructive to compare our results with the Bogoliubov transformation describing the in-out relationship of a massless charged fermion field on Minkowski space in a background abelian gauge field [7]. Formulas (3.12) and (3.13) are valid in this case, with U replaced by

$$V(k,p) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}(k-p)x}\,\mathrm{e}^{\mathrm{i}\omega_-(x)}. \tag{3.15}$$

Up to a constant, the phase ω_- is given by

$$\omega_-(x^-) = e\int \mathrm{d}^2u\, \theta(x^- - u^-)\, E(u), \tag{3.16}$$

where E is the electric field and e the fermion charge.

Whereas the structures are very similar, there are striking differences:

(i) There is a real exponential $\exp(\frac{1}{2}\alpha_-(x))$ in (3.13) instead of the phase factor $\exp(\mathrm{i}\omega_-(x))$ in (3.15).

(ii) V is simply the Fourier transform of this phase factor whereas (3.13) contains the hybrid construct $[k\eta^-(x) - px]$ reflecting the fact that on a curved space the in- and out-fields are defined in terms of different variables.

Consequently we may expect the discovery of truly new features distinguishing fermions on a curved space from Minkowski fermions in a gauge field. The following Sections will confirm this conjecture.

4 Current and energy momentum

We start our investigation of the pair creation mechanism encoded in the in-out automorphism (3.10) with those aspects which do not depend on its implementability. We begin with the outgoing current: its right component is given by

$$\hat{\jmath}_-^{\,\mathrm{out}}(y^-) = \sqrt{2} : \hat{\phi}^\dagger_{\mathrm{out}}(y^-)\,\hat{\phi}_{\mathrm{out}}(y^-) :_{\mathrm{out}} \tag{4.1}$$

where $: \quad :_{\mathrm{out}}$ means normal ordering with respect to the outgoing creators and annihilators. This current coincides with the full fermion current in the future of D, the support of the curvature.

If $\langle\, \hat{\jmath}_- \,\rangle$ is the expectation value of $\hat{\jmath}_-^{\,\mathrm{out}}$ in the incoming vacuum:

$$\langle\, \hat{\jmath}_-(y^-)\,\rangle = \left(\Omega_{\mathrm{in}}, \hat{\jmath}_-^{\,\mathrm{out}}(y^-)\Omega_{\mathrm{in}}\right), \tag{4.2}$$

eq. (4.1) can be rewritten as

$$\begin{aligned} \hat{\jmath}_-^{\,\mathrm{out}}(y^-) &= \sqrt{2} : \hat{\phi}^\dagger_{\mathrm{out}}(y^-)\,\hat{\phi}_{\mathrm{out}}(y^-) :_{\mathrm{in}} + \langle\, \hat{\jmath}_-(y^-)\,\rangle \\ &= \mathrm{e}^{-\alpha_-(x^-)} j_-^{\mathrm{in}}(x^-) + \langle\, \hat{\jmath}_-(y^-)\,\rangle. \end{aligned} \tag{4.3}$$

Eq. (3.10) has been used in the last equality and j_-^{in} is the incoming counterpart of $\hat{\jmath}_-^{\,\mathrm{out}}$: $j_-^{\mathrm{in}}(x^-) = : \phi^\dagger_{\mathrm{in}}(x^-)\,\phi_{\mathrm{in}}(x^-) :_{\mathrm{in}}$.

Straightforward computations sketched in Appendix A lead to the surprising conclusion that $\langle\, \hat{\jmath}_-(y^-)\,\rangle$ vanishes identically. Obviously, this does not mean that there are no outgoing particles created by the curvature: fermions and antifermions are produced on top of each other so that there is no net outgoing current. Recovering both light-cone components, eq. (4.3) reduces to

$$\hat{\jmath}_{\pm}^{\text{out}}(y^{\pm}) = \mathrm{e}^{-\alpha_{\pm}(x^{\pm})} \hat{\jmath}_{\pm}^{\text{in}}(x^{\pm}). \tag{4.4}$$

Remembering the transformation law of a covariant vector v under a change of conformal coordinates $[\hat{v}_{\pm}(y) = (\mathrm{d}y^{\pm}/\mathrm{d}x^{\pm})^{-1} v_{\pm}(x^{\pm})]$ we see that $\hat{\jmath}_{\pm}^{\text{out}}$ and $j_{\pm}^{\text{in}}$ are the components of the same vector field in the incoming and outgoing coordinate systems.

Appendix A tells us that the vanishing of $\langle \hat{\jmath}_{\pm} \rangle$ is due to the relation $K_3(k,p) = K_2^*(k,p)$ between the kernels of the Bogoliubov transformation. There is no such relation in the case of an abelian background field in Minkowski space and the mean outgoing current is nonzero. In fact, it cannot vanish because of the chiral anomaly.

We turn now to the energy-momentum tensor and define its outgoing form in the same way as the outgoing current:

$$\hat{\theta}_{--}^{\text{out}}(y^-) = \frac{\mathrm{i}}{\sqrt{2}} \; : \hat{\phi}_{\text{out}}^{\dagger} \partial_{y^-} \hat{\phi}_{\text{out}} - \left(\partial_{y^-} \hat{\phi}_{\text{out}}\right)^{\dagger} \hat{\phi}_{\text{out}} \; :_{\text{out}}(y^-). \tag{4.5}$$

There is a similar expression for $\hat{\theta}_{++}^{\text{out}}$ in terms of the left-moving $\hat{\phi}_{\text{out}}$ whereas $\hat{\theta}_{+-}^{\text{out}} = 0$. The mean value $\langle \hat{\theta}_{--} \rangle$ of $\hat{\theta}_{--}^{\text{out}}$ in the incoming vacuum is nonzero: according to Appendix A one finds

$$\langle \hat{\theta}_{--}(y^-) \rangle = \frac{1}{48\pi} \left[2\partial_{x^-}^2 \Omega_- - (\partial_{x^-} \Omega_-)^2 \right] (x^-)\, \mathrm{e}^{-2\alpha_-(x^-)}. \tag{4.6}$$

Remember that $\langle \hat{\theta}_{--} \rangle$ is a tensor component in the outgoing coordinate system. The corresponding component in the incoming system is obtained by removing the exponential in (4.6). As a result we recover the asymptotic form of the energy-momentum tensor in the incoming vacuum obtained by point-splitting [8]:

$$\langle \theta_{--}(x) \rangle = \frac{1}{48\pi} \left[2\partial_{x^-}^2 \Omega(x) - (\partial_{x^-} \Omega(x))^2 \right]. \tag{4.7}$$

As $\langle \theta_{+-}(x) \rangle = (1/48\pi) \sqrt{-g(x)}\, R(x)$ (trace anomaly), this component vanishes asymptotically in our context.

If we rewrite (4.6) in terms of derivatives with respect to y^- we have

$$\langle \hat{\theta}_{--}(y^-) \rangle = \frac{1}{48\pi} \left[\left(\partial_{y^-} \Omega_- \left(\xi^-(y^-) \right) \right)^2 + 2\partial_{y^-}^2 \Omega_- \left(\xi^-(y^-) \right) \right]. \tag{4.8}$$

The function ξ^- is the inverse of η^- defined in (2.6). Up to a factor 1/2, $\langle\,\widehat{\theta}_{--}\,\rangle$ is the mean energy density or the mean momentum density of the outgoing right-movers. It has a compact support such that outgoing mean energy and momentum are strictly localized in the causal shadow D_- of the curvature. Whereas the expectation value (4.8) is not positive definite, its integral over y^- is positive because $\partial_{y^-}\Omega_-$ has a compact support. The y-variables being minkowskian in the future, this integral makes sense: it gives the total mean energy carried by the outgoing right-moving particles produced in the incoming vacuum.

In the case of a type I point-like curvature, defined in Section 2, eq. (4.6) becomes

$$\langle\,\widehat{\theta}_{--}(y^-)\,\rangle = \frac{1}{48\pi}\left[\bar{\rho}^2(x^-) - 2\left(1 + \int_0^{x^-} \mathrm{d}u\,\bar{\rho}(u)\right)\partial_{x^-}\bar{\rho}(x^-)\right]\mathrm{e}^{\Omega_0}. \tag{4.9}$$

The parameter Λ in (2.7) has been set equal to $-\Omega_0/2$.

If the curvature strength $\bar{\rho}$ is kept constant during some time, the mean energy and momentum density will be constant and positive over the corresponding y^--interval, independently of the sign of $\bar{\rho}$. It will be negative somewhere outside this interval, as an effect of switching the curvature on or off. As long as the strength $\bar{\rho}$ is static it induces a steady production of momentum and energy. The total right-moving outgoing mean energy is

$$\begin{aligned}\langle\,E_-^{\mathrm{out}}\,\rangle &= \frac{1}{2}\int_{-\infty}^{+\infty}\mathrm{d}y^1\langle\,\widehat{\theta}_{--}(y^-)\,\rangle \\ &= \frac{1}{48\pi\sqrt{2}}\int_{-\infty}^{+\infty}\mathrm{d}x^-\,\frac{(\bar{\rho}(x^-))^2\,\mathrm{e}^{\Omega_0/2}}{1+\int_0^{x^-}\mathrm{d}u\,\bar{\rho}(u)}.\end{aligned} \tag{4.10}$$

If we turn to a type II point-like curvature, equation (4.9) is replaced by

$$\langle\,\widehat{\theta}_{--}(y^-)\,\rangle = -\frac{1}{24\pi}\left(1 + \frac{1}{2}\int_0^{x^-}\mathrm{d}u\,\rho(u)\right)^3\partial_{x^-}\rho(x^-)\,\mathrm{e}^{\Omega_0}, \tag{4.11}$$

and we get a completely new picture. The energy density now vanishes where ρ is static: energy emission is no longer a steady process but a transient phenomenon accompanying the switching on and off of the curvature. This turns out to agree with a result obtained by Isler *et al.* [3] if one observes that their “eternal star” is a type II point-like curvature. Eq. (4.11) implies that the total energy is simply

$$\langle\,E_-^{\mathrm{out}}\,\rangle = \frac{1}{48\pi\sqrt{2}}\int_{-\infty}^{+\infty}\mathrm{d}x^-\,\left(\rho(x^-)\right)^2\,\mathrm{e}^{\Omega_0/2}. \tag{4.12}$$

This formula, taken alone, is misleading: it gives the impression of a steady energy production.

In $1+1$ dimensions, the mean value of the full energy momentum tensor $\widehat{\theta}_{\mu\nu}(y)$ can be determined completely everywhere from the covariant equation of continuity $\nabla_\mu \theta^{\mu\nu} = 0$ combined with the trace anomaly and suitable boundary conditions [4]. In our case this computation leads to an asymptotic form which coincides with (4.8). As an illustration we display the complete mean energy momentum produced by point-like curvatures of type I and II turned on and off instantaneously ($\bar{\rho}(u)$ ou $\rho(u)$ with rectangular profile, amplitude $\bar{\rho}$ or ρ). For both types we may write

$$
\begin{aligned}
\langle \widehat{\theta}_{00}(y) \rangle &= 2\langle \widehat{\theta}_{+-}(y) \rangle + \langle \widehat{\theta}_{11}(y) \rangle \\
\langle \widehat{\theta}_{11}(y) \rangle &= \frac{1}{48\pi}\left[\theta(y^+ - y^-)H(y^-) + \theta(y^- - y^+)H(y^+)\right] \\
\langle \widehat{\theta}_{01}(y) \rangle &= \frac{-1}{48\pi}\left[\theta(y^+ - y^-)H(y^-) - \theta(y^- - y^+)H(y^+)\right]
\end{aligned}
\tag{4.13}
$$

If the curvature is of type I :

$$
\begin{aligned}
\langle \widehat{\theta}^I_{+-}(y) \rangle &= \frac{1}{24\pi}\frac{\bar{\rho}}{\sqrt{1+\bar{\rho}L}}\delta(y^- - y^+)\theta(y^-)\theta(\widehat{L} - y^-) \\
H^I(u) &= \frac{\bar{\rho}}{\sqrt{1+\bar{\rho}L}}\left[\delta(u - \widehat{L}) - \delta(u) + \frac{1}{2}\frac{\bar{\rho}}{\sqrt{1+\bar{\rho}L}}\theta(u)\theta(\widehat{L} - u)\right],
\end{aligned}
\tag{4.14}
$$

where $\widehat{L} = \eta(L) = (1+\bar{\rho}L)^{1/2}(1/\bar{\rho})\log(1+\bar{\rho}L)$. For a type II curvature we have

$$
\begin{aligned}
\langle \widehat{\theta}^{II}_{+-}(y) \rangle &= \frac{1}{24\pi}\frac{\rho}{1+\frac{1}{2}\rho(L - y^-)}\delta(y^- - y^+)\theta(y^-)\theta(L - y^-) \\
H^{II}(u) &= \rho\left[\delta(u - L) - \frac{1}{1+\frac{1}{2}\rho L}\delta(u)\right].
\end{aligned}
\tag{4.15}
$$

The particular values of the factor $\exp\Omega_0$ appearing in eq. (4.9) have been incorporated in the formulae (4.13-15). They corroborate our previous results: they lead to the picture shown in Fig. 2. For both types of curvature $\langle \widehat{\theta}_{00} \rangle$ contains a term localized on the time axis, on top of the curvature: it is static for type I whereas its strength increases with time for type II. Two bursts are emitted when the curvature is turned on and off. The second burst is opposite to the first one in the case of type I and substantially larger in strength than the first burst in the type II case, if $|\rho|L \gg 1$. A type I curvature produces a term that is homogeneous as long as the curvature is switched on: it corresponds to a steady emission of positive energy. No such emission is observed if the curvature is of type II.

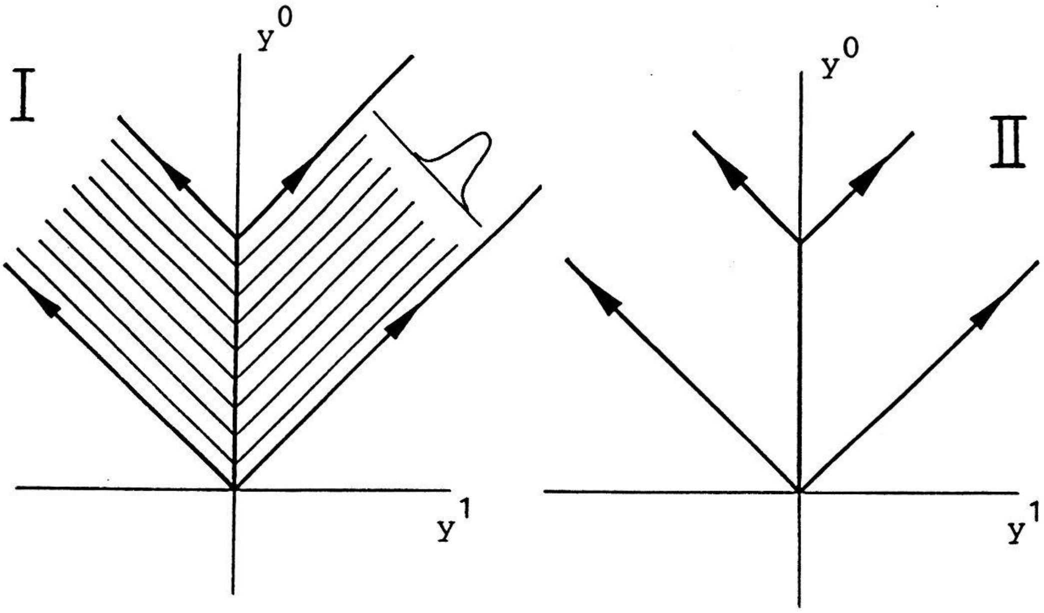

Fig. 2: The mean energy momentum produced in the incoming vacuum by static point-like curvatures of type I and II. The thick lines represent the delta function terms in (4.13-15). For type I there is an additional steady production of energy-momentum. This agrees with the fact, established in Section 6, that a fermion whose wave function has a squared modulus as indicated is observed with a finite probability. The corresponding probability is negligibly small for type II.

One of the lessons of this Section is that one has to be cautious in dealing with static sources. It is only in a particular coordinate system that a given source is effectively static and sources which are static in different coordinate systems can lead to completely different physical outputs.

5 Inclusive probabilities

As already said in the Introduction, the main advantage of massless fermions over bosons in $1+1$ dimensions is that the outgoing particle content of the incoming vacuum state Ω_{in} can be analysed in detail, beyond the expectation values evaluated in the preceding Section. As we shall see in Section 8, a generic curvature creates an infinite number of particles; the distribution of these particles can be caracterized by means of inclusive probabilities, the simplest one being the probability $W[f]$ of detecting an outgoing fermion in a given state, for instance the right-moving state f ($f \in L^2(\mathbb{R}_+, dk)$, $\|f\| = 1$) [7]:

$$W[f] = (\Omega_{\text{in}}, P_{\text{out}}[f]\Omega_{\text{in}}) . \tag{5.1}$$

$P_{\text{out}}[f]$ is the projector on the subspace of the Fock space of Ω_{in} in which the outgoing state f is occupied. As we are dealing with fermions, $P_{\text{out}}[f]$ is simply equal to $a_{\text{out}}^{\dagger}[f]\, a_{\text{out}}[f]$ where $a_{\text{out}}[f] = \int_0^\infty \mathrm{d}k\, f(k)\, a_{\text{out}}(k)$. Using (3.11) we obtain

$$W[f] = \|K_2^{\dagger} f^*\|^2 = 1 - \|K_1^{\dagger} f^*\|^2. \tag{5.2}$$

The second expression is a consequence of the unitarity of the matrix formed by the K s.

Detailed properties of the functional $W[f]$ have been derived in [7] for the case of Minkowski fermions in an abelian gauge field. The kernel of $K_1^{\dagger}$ is in general non-empty. This implies the existence of outgoing states occupied with probability 1. Their number is determined by the winding number associated with the phase ω_- appearing in (3.15).

In the present case, the situation is less transparent. For instance we have no exact result on $\ker K_1^{\dagger}$ although we believe it to be empty. What we have obtained is a master formula for $W[f]$ which allows the detailed discussion of special examples, as will be shown in Sections 6 and 7. This formula involves the y-Fourier transform of f:

$$\tilde{f}(y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{d}k\, f(k)\, \mathrm{e}^{iky}. \tag{5.3}$$

As explained in Appendix B, the fact that $\tilde{f}$ is regular in the upper y half-plane allows us to transform the definition (5.2) into the formula

$$W[f] = \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}y \int \mathrm{d}y'\, G(y, y')\, \tilde{f}^*(y)\, \tilde{f}(y'), \tag{5.4}$$

with

$$G(y, y') = \frac{\exp\left[-\frac{1}{2}(\gamma(y) + \gamma(y'))\right]}{\xi(y) - \xi(y')} - \frac{1}{y - y'}. \tag{5.5}$$

The function ξ inverts the relation $y = \eta^-(x)$, $x = \xi(y)$, and γ is equal to α_- written in terms of y: $\gamma(y) = \alpha_-(\xi(y))$. Eq. (2.6) implies that $(\mathrm{d}\xi(y)/\mathrm{d}y) = \exp(-\gamma(y))$; as a consequence, the denominator of the first fraction in (5.5) is equal to $\int_{y'}^{y} \mathrm{d}z\, \exp(-\gamma(z))$. This shows that $G(y, y')$ is regular in the whole (y, y')-plane.

Finally we observe that, due to the equalities (A.2), an outgoing fermion in a given state has the same inclusive probability as an outgoing antifermion in the same state. This agrees with the vanishing of the expectation value of the outgoing current found in Section 4.

6 Point-like curvatures and occurrence of a thermal spectrum

In our first application of formula (5.4) we consider a type I point-like curvature in the static case ($L_1 = 0$, $L_2 = L$ in the notation of Section 2). We show that a suitable family of states f leads to an inclusive probability $W[f]$ resembling a thermal distribution. We take wave functions $\tilde{f}$ which are well localized inside the causal shadow $0 < y < \widehat{L}$ of the curvature ($\widehat{L} = \eta(L)$). With such a state, we obtain a good approximation of $W[f]$ if we replace $G(y, y')$ everywhere in (5.4) by its exact expression valid on $[0, \widehat{L}] \times [0, \widehat{L}]$. According to (2.11):

$$\begin{aligned} \xi(y) &= \frac{1}{\bar{\rho}}\left(e^{\bar{\rho}_0 y} - 1\right) \\ \gamma(y) &= -\bar{\rho}_0 y - \frac{1}{2}\Omega_0 \end{aligned} \qquad \text{for } 0 \le y \le \widehat{L} = \frac{1}{\bar{\rho}_0}\log(1 + \bar{\rho} L), \tag{6.1}$$

eq. (5.5) gives

$$G(y, y') = \frac{\bar{\rho}_0}{2\,\mathrm{sh}\left(\frac{1}{2}\bar{\rho}_0(y - y')\right)} - \frac{1}{y - y'}, \tag{6.2}$$

for $(y, y') \in [0, \widehat{L}] \times [0, \widehat{L}]$. We have set $\bar{\rho}_0 = |\bar{\rho}| \exp(\Omega_0/2)$ and $\bar{\rho}$ is the constant value of the effective curvature strength. With this substitution, the integrand of (5.4) becomes a meromorphic function of y' in the upper half-plane. The integral over y' is a sum of residues:

$$W[f] = \sum_{n=1}^{\infty} (-1)^{n+1} \int_{-\infty}^{+\infty} \mathrm{d}y\, \tilde{f}^*(y)\, \tilde{f}\left(y + \mathrm{i}\frac{2\pi}{\bar{\rho}_0} n\right). \tag{6.3}$$

If we observe that the remaining integrals are equal to $\int_0^{\infty} \mathrm{d}k\, |f(k)|^2$ $\exp\left(-(2\pi/\bar{\rho}_0) n k\right)$ we may rewrite (6.3) as

$$W[f] = \int_0^{\infty} \mathrm{d}k \frac{|f(k)|^2}{1 + \exp((2\pi/\bar{\rho}_0)k)}. \tag{6.3$'$}$$

We see that $W[f] \le \frac{1}{2}$ and that $W[f] \sim \exp\left(-(2\pi/\bar{\rho}_0)\langle\, k\,\rangle\right)$ if the mean value $\langle\, k\,\rangle$ of k is large.

To go beyond the asymptotic form of $W[f]$ for large mean energies, it is convenient to choose a specific state:

$$\tilde{f}_\omega(y) = \sqrt{\delta/\pi}\, \frac{1}{y - y_0 + \mathrm{i}\delta}\, e^{\mathrm{i}\omega\sqrt{2}y}, \tag{6.4}$$

$0 < y_0 < \widehat{L}$, $0 < \delta \ll \mathrm{Min}\,(y_0, \widehat{L} - y_0)$, $\omega > 0$. The ugly factor $\sqrt{2}$ in the exponent ensures that ω is an energy (remember that $\sqrt{2}\,y^- = y^0 - y^1$). The mean energy of this state is $\omega + (2\sqrt{2}\delta)^{-1}$: it is close to ω if δ is large. Under these conditions, the sum in (6.3) is close to a geometrical series and one has

$$W[f_\omega] = \frac{1}{1 + (\pi/\bar{\rho}_0\delta)}\,\frac{1}{\exp(\beta\omega) + 1} + O\left(\frac{1}{\bar{\rho}_0\delta}\,\mathrm{e}^{-\beta\omega}\right), \tag{6.5}$$

with $\beta = 2\sqrt{2}\pi/\bar{\rho}_0$. If the strength $\bar{\rho}_0$ and the time $\sqrt{2}\widehat{L}$ during which the curvature is switched on are large, we may have $\bar{\rho}_0\widehat{L} \gg 1$ and the half-width δ of the wave function $\tilde{f}$ can be chosen such that $\bar{\rho}_0\delta \gg 1$. Under these circumstances, the first term in the right-hand side of (6.5) dominates and we are left with

$$W[f_\omega] \simeq \frac{1}{\mathrm{e}^{\beta\omega} + 1} \tag{6.6}$$

for all values of ω, ω being a good approximation of the mean energy of the state f_ω at the scale $1/\beta$. Equation (6.6) gives the announced thermal-like distribution, with temperature $1/\beta$ and zero chemical potential. Notice that any dependence on the parameters y_0 and δ which fix the shape of $\tilde{f}$ has disappeared.

The validity of (6.6) is not restricted to the special form (6.4) of $\tilde{f}_\omega$. If this function is of the form $g(y)\,\exp\left(\mathrm{i}\sqrt{2}\omega y\right)$ where g is suitably localized, regular in the upper half-plane and slowly variable at the scale ω^{-1}, ω is close to the mean energy and it follows from eq. (6.3) that W is approximated by (6.6).

As already experienced with the energy-momentum density, a type II point-like curvature leads to inclusive probabilities which differ dramatically from the probabilities of a type I curvature. Instead of (6.1) we now have from (2.12)

$$\left.\begin{aligned} \xi(y) &= y\,\mathrm{e}^{\frac{1}{2}\Omega_0}\left[1 - \frac{1}{2}\rho_0 y\right]^{-1} \\ \gamma(y) &= \log\left[1 - \frac{1}{2}\rho_0 y\right]^2 - \frac{1}{2}\Omega_0 \end{aligned}\right\} \quad \text{for } 0 \le y \le \widehat{L} = \frac{L\,\mathrm{e}^{\frac{1}{2}\Omega_0}}{1 + \frac{1}{2}\rho L}, \tag{6.7}$$

with $\rho_0 = \rho\,\exp(\Omega_0/2)$. Insertion into (5.5) shows that $G(y, y')$ is now identically zero on $[0, \widehat{L}] \times [0, \widehat{L}]$. This implies that a state $\tilde{f}$ which is well localized inside $[0, \widehat{L}]$ has a negligibly small probability W, vanishing in the approximation used before.

Our present findings corroborate entirely the results on the mean energy obtained in the previous Section. The steady energy production by the type I curvature is in agreement with the fact that $W[f]$ does not depend on the location and shape of $\tilde{f}$ as long as it is well localized inside the region where this steady energy emission is observed. Similarly, the circumstance that no particles are detected in the shadow of the static strength ρ of a type II curvature, fits perfectly with the fact that the energy release by this curvature is a transient side effect.

Although it is tempting to identify the pairs created by a type I curvature as a sort of Hawking radiation [5], it is not clear whether this is really justified. A thermal distribution is normally defined in terms of exclusive probabilities whereas equation (6.6) gives inclusive occupation probabilities for a special sample of states. The correct interpretation of equation (6.6) deserves further investigation.

For the time being, we may observe that (6.6) is a direct consequence of the fact that the outgoing variable y^- depends logarithmically on the incoming variable x^-. A thermal-like distribution will appear as soon as y^- exhibits such behaviour over a large part of the causal shadow of the curvature; this does not require a singular curvature. In this respect, our results illustrate the ubiquitous character of thermal spectra which has already been noted [9].

7 In search of high probability states

The inclusive probabilities we have evaluated up to now are all smaller than 1/2. This is a general trend for states of the form (6.4) and it is not restricted to point-like curvatures. One may wonder if there were states with probability larger than 1/2, possibly close to 1. We notice that the states we have considered so far are localized inside the shadow D_- of the curvature. Although the mean outgoing energy-momentum has D_- as support, this does not exclude the detection of particles outside this shadow. It turns out that large values of $W[f]$ are precisely obtained for those states whose extension is large compared to D_-.

We illustrate this in the limiting case where the Liouville field of the incoming variables is constant, equal to Ω_0 inside the future light-cone of the origin and zero on and outside this light-cone:

$$\Omega(x) = \begin{cases} \Omega_0 & \text{for} \quad x^2 > 0, \; x^0 > 0 \\ 0 & \text{for} \quad x^2 \leq 0, \; \text{and} \; x^2 > 0, \; x^0 < 0. \end{cases} \tag{7.1}$$

Referring to eq. (2.2) we see that such a field is produced by a curvature pulse in the limit where the pulse is concentrated at $x = 0$: formally $R(x) = -2\Omega_0\delta^{(2)}(x)$. If Λ is set equal to $-\Omega_0/2$ in (2.7) the ingredients of the kernel $G(y, y')$, eq. (5.5), become

$$\xi(y) = \mathrm{e}^{-\gamma(y)} y, \qquad \gamma(y) = \frac{1}{2}\,\epsilon(y)\,\Omega_0. \tag{7.2}$$

Consequently, the definition (5.5) shows that $G(y, y')$ vanishes in the first and third quadrants of the (y, y')-plane. In the fourth quadrant it is equal to

$$G(y, y') = \frac{\sqrt{\sigma}}{\sigma y - y'} - \frac{1}{y - y'}, \qquad y > 0,\ y' < 0. \tag{7.3}$$

with $\sigma = \exp(-\Omega_0)$. $G(y, y')$ being antisymmetric, (7.3) fixes its value in the second quadrant.

The wave function $\tilde{f}$ being regular in the upper half-plane, its real part is the Hilbert transform of its imaginary part. Consequently $W[f]$ is determined by Im $\tilde{f}$ alone. If f_+ and f_- are the restrictions of Im $\tilde{f}$ to $\mathbb{R}_+$ and $\mathbb{R}_-$ respectively, one finds, as shown at the end of Appendix B,

$$W[f] = W_+[f_+] + W_-[f_-], \tag{7.4}$$

with

$$W_+[f_+] = \frac{1}{\pi^2} \int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}y' \frac{1}{y - y'} \ln \frac{y}{y'} \left[f_+(y) - \sqrt{\sigma}\, f_+(\sigma y) \right] f_+(y') \tag{7.5}$$

and a similar expression for $W_-[f_-]$. The functions f_+ and f_- being independent, we may take $f_- = 0$ and ask for an f_+ producing the largest possible W_+. The form of (7.5) suggests that one should choose f_+ such that the two terms in the square bracket interfere constructively:

$$\sqrt{\sigma}\, f_+(\sigma y) = -f_+(y). \tag{7.6}$$

The function

$$g(y) = \frac{1}{\sqrt{y}} \cos\left(\frac{\pi}{\Omega_0} \log\left(\frac{y}{a}\right)\right) \theta(y), \tag{7.7}$$

where a is a constant length, fulfills equation (7.6) but fails to be square integrable. In fact, this equation has no solution belonging to $L^2(\mathbb{R}_+)$. The best we can do is to choose a square integrable function that satisfies (7.6) over a large interval $[l, L]$, $L \gg l > 0$, for instance:

$$f_+(y) = \begin{cases} C g(y) & \text{for } \ l \le y \le L, \\ 0 & \text{for } \ y < l \text{ and } y > L. \end{cases} \tag{7.8}$$

The normalisation of f_+ is $\|f_+\| = 1/\sqrt{2}$ ($\|\tilde{f}\| = 1$ implies $\|\mathrm{Im}\,\tilde{f}\| = \|\mathrm{Re}\,\tilde{f}\| = 1/\sqrt{2}$ because $\tilde{f}$ is regular in $\mathrm{Im}\,y > 0$). This condition fixes the constant C; if $L/l > \exp \Omega_0$ one finds

$$C = (\log(L/l))^{-\frac{1}{2}}[1 + O(\Omega_0/\log(L/l))] \tag{7.9}$$

Inserting (7.8) into (7.5) we arrive at

$$W_+[f_+] = \frac{1}{\mathrm{ch}^2\,(\pi^2/\Omega_0)} + O(1/\log(L/l)) + O\,(\Omega_0/\log(L/l))\,. \tag{7.10}$$

As $L \gg l$, we see that the outgoing state (7.8) has an occupation probability which is close to 1 if the strength Ω_0 of the curvature pulse is large. Notice the highly non-perturbative behaviour of g and W_+ as functions of Ω_0. We have achieved the goal set at the beginning of the Section: if Ω_0 is large we can exhibit high probability states. Admittedly, these are rather exotic states, poorly localized and exhibiting peculiar oscillations.

8 Existence of the outgoing vacuum and the vacuum persistence amplitude

In this last Section we discuss first the implementability of the in-out automorphism (3.11), i.e. the existence of a unitary operator U such that

$$a_{\mathrm{out}}(k) = U^\dagger a_{\mathrm{in}}(k) U, \qquad b_{\mathrm{out}}(k) = U^\dagger b_{\mathrm{in}}(k) U. \tag{8.1}$$

In this case there is an outgoing vacuum state Ω_{out} in the Fock space of Ω_{in}: $\Omega_{\mathrm{out}} = U^\dagger \Omega_{\mathrm{in}}$.

Implementability holds iff the non-diagonal kernels K_2 and K_3 of the Bogoliubov transformation (3.11) are Hilbert-Schmidt [10, 11]. In our case, K_2 and K_3 have the same Hilbert-Schmidt norm and it is sufficient to check if

$$\mathrm{Tr}(K_2 K_2^\dagger) = \int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}p\,|K_2(k,p)|^2 < \infty. \tag{8.2}$$

We prove in appendix C that this is fulfilled iff $\Omega_0 = 0$, that is if Ω vanishes between the two future shadows of the curvature. Referring to (2.3) this means that the space-time integral of the curvature has to vanish:

$$\int \mathrm{d}^2x\,\sqrt{-g(x)}\,R(x) = 0. \tag{8.3}$$

This condition can also be phrased in terms of the spin connection. As $R(x) = 0$ if x belongs to D^c, the complement of D, ω_μ is locally a pure gauge there:

$$\omega_\mu(x) = S^{-1}(x)\,\partial_\mu\, S(x). \tag{8.4}$$

We may take

$$S(x) = \exp\left[\int_{P_0}^{P(x)} \mathrm{d}z^\mu \omega_\mu(z)\right], \tag{8.5}$$

the integral being taken along a curve in D^c connecting a fixed point P_0 to $P(x)$. This matrix is diagonal and it is single-valued in D^c if

$$\oint_C \mathrm{d}z^\mu \omega_\mu(z) = 0 \tag{8.6}$$

for a closed contour C surrounding D. According to the expression (3.4) of ω_μ this condition is equivalent to (8.3). Consequently, the automorphism (3.11) is unitarily implementable iff the spin connection is globally a pure gauge outside the support D of the curvature.

Finally, at fixed x, $S(x)$ is the spinor representation of an element $\Lambda(x)$ of the Lorentz group $SO(1,1)$ and (8.5) defines a mapping $D^c \to SO(1,1)$. The group $SO(1,1)$ being simply connected, this mapping has a differentiable continuation in the interior of D. The resulting $S(x)$ defines a gauge transformation on the whole space-time which brings ω_μ to zero outside D. The existence of such a gauge transformation is another necessary and sufficient condition for the implementability of the in-out correspondence.

Quite a similar situation prevails for massless fermions in a non-abelian gauge field on flat space [12]. The condition (8.3) is very restrictive. Generically it is not satisfied, an infinite number of fermion-antifermion pairs is created and there is no outgoing vacuum state Ω_{out} in the Fock space of the incoming vacuum Ω_{in}. The origin of this phenomenon is clearly the infrared behaviour of our massless fermions.

We assume from now on that condition (8.3) is satisfied. If $\ker K_1^\dagger = \emptyset$, the Bogoliubov transformation (3.11) is weak, Ω_{out} exists and $(\Omega_{\text{out}}, \Omega_{\text{in}}) \neq 0$ [11]. This scalar product defines an effective action A:

$$\exp(\mathrm{i}A) = (\Omega_{\text{out}}, \Omega_{\text{in}}). \tag{8.7}$$

The automorphism (3.11) defines the operator U up to a phase ϕ, and only the modulus of the right-hand side of (8.7), i.e. the vacuum persistence amplitude, is explicitly determined by means of the Bogoliubov transformation. Therefore our approach gives only the value of the imaginary part of A. The outgoing vacuum is a coherent pair state given by [13]:

$$\Omega_{\text{out}} = \mathrm{e}^{-\mathrm{i}\phi} \left(\det(K_1 K_1^\dagger)\right)^{\frac{1}{2}} \exp\left(\int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}p\, L(k,p)\, a_{\text{in}}^\dagger(k)\, b_{\text{in}}^\dagger(p)\right) \Omega_{\text{in}}, \tag{8.8}$$

where $L = -K_1^{-1} K_2 = K_3^\dagger K_4^{\dagger -1}$. This implies

$$\text{Im } A = -\frac{1}{2} \log\left(\det(K_1 K_1^\dagger)\right). \tag{8.9}$$

Finally we want to check the consistency of this result with the relation connecting the functional derivative of A with respect to the metric to the energy-momentum tensor:

$$\frac{\delta A}{\delta g^{\mu\nu}(x)} = \frac{1}{2}\sqrt{-g(x)}\, \frac{(\Omega_{\text{out}}, \theta_{\mu\nu}\Omega_{\text{in}})}{(\Omega_{\text{out}}, \Omega_{\text{in}})}. \tag{8.10}$$

Here $\theta_{\mu\nu}$ is the full energy-momentum tensor. We restrict ourselves to the component θ_{--}:

$$\theta_{--} = \frac{\mathrm{i}}{\sqrt{2}} : \psi_-^\dagger \partial_- \psi_- - \left(\partial_- \psi_-^\dagger\right) \psi_- :_{\text{in}} + \langle \theta_{--} \rangle. \tag{8.11}$$

The fermion field ψ_- is given by (3.5) and $\langle \theta_{--} \rangle = (\Omega_{\text{in}}, \theta_{--}\Omega_{\text{in}})$. This expectation value does not enter into our test because it is real. Using (8.9), (3.5) and (8.8), one finds that the imaginary parts of both sides of equation (8.10) are equal if

$$\text{Tr}\left(K_1^{-1} \frac{\delta K_1}{\delta g^{--}(x)} + K_1^{\dagger -1} \frac{\delta K_1^\dagger}{\delta g^{--}(x)}\right) = \frac{1}{2}\sqrt{-g(x)}\, \text{Re}\, (\partial_u - \partial_v)\, \tilde{L}(u,v)\Big|_{u=v=x^-} \tag{8.12}$$

where $\tilde{L}(u,v)$ is the Fourier transform of the kernel $L(k,p)$ appearing in the expression (8.8) of Ω_{out}. The validity of this equality is by no means obvious. We sketch in Appendix D how one can show that (8.12) is an identity although the inverse K_1^{-1} is not known explicitly. This substantiates the consistency of our treatment of fermions on a curved space as an external field problem with the functional integral approach.

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Appendix A. Computing expectation values

We start with the right-moving current $\langle \hat{\jmath}_-(y) \rangle$ defined in eq. (4.3). The definition (4.1) of the outgoing current and the Bogoliubov transformation (3.11) give

$$\langle \hat{\jmath}_-(y) \rangle = \frac{1}{2\pi}\int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}p \left\{ \left[\left(K_2 K_2^\dagger \right)^T - \left(K_3 K_3^\dagger \right) \right] (k,p)\, \mathrm{e}^{\mathrm{i}(k-p)y} + \left[\left(K_4 K_2^\dagger \right)(k,p)\, \mathrm{e}^{\mathrm{i}(k+p)y} - \left(K_1 K_3^\dagger \right)(k,p)\, \mathrm{e}^{-\mathrm{i}(k+p)y} \right] \right\} \tag{A.1}$$

The expressions (3.12–13) of the kernels K_i imply

$$K_1(k,p) = K_4^*(k,p), \qquad K_2(k,p) = K_3^*(k,p), \tag{A.2}$$

and, as a consequence, the first square bracket in the integrand of (A.1) is identically zero. Eq. (A.2) implies that the two terms in the second square bracket of the integrand are complex conjugates. The expectation value $\langle \hat{\jmath}_- \rangle$ being real, the difference of their integrals has to vanish; this can also be checked explicitly. We conclude that $\langle \hat{\jmath}_- \rangle = 0$.

The expectation value $\langle \widehat{\theta}_{--}(y) \rangle$ of the outgoing energy-momentum component $\widehat{\theta}_{--}^{\mathrm{out}}$ defined in eq. (4.5) is obtained from (A.1) by changing the sign of the second term in both square brackets and by multiplying the first bracket by $(k+p)/2$ and the second one by $(k-p)/2$. Inserting the explicit forms (3.12–13) of the kernels and performing the k and p integrations one ends up with a sum of four integrals over variables y' and y'', each of them containing a factor $(y'-y\pm \mathrm{i}\epsilon)^{-2}(y''-y\pm \mathrm{i}\epsilon)^{-2}$. These factors combine together in such a way that, finally,

$$\langle \theta_{--}(y) \rangle = \frac{1}{4\pi}\int_{-\infty}^{+\infty} \mathrm{d}y' \int_{-\infty}^{+\infty} \mathrm{d}y'' \frac{\mathrm{e}^{-\frac{1}{2}(\gamma(y')+\gamma(y''))}}{\xi(y')-\xi(y'')}(y'-y'')\delta'(y'-y)\delta'(y''-y), \tag{A.3}$$

where γ and ξ are the functions defined after equation (5.5). The evaluation of the remaining integrals gives eq. (4.6) where we have used the fact that the derivatives of Ω_- and α_- coincide.

If one goes through the successive steps of the lengthy calculation we have just outlined, it is impressive to observe how a sum of nonlocal functionals of α_- turns into a local function of α_- and its derivatives.

Appendix B. Formulas for inclusive probabilities

We rewrite the definition (5.2) of $W[f]$:

$$W[f] = \int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}p\, f(k) \left(K_2 K_2^\dagger\right)(k,p) f^*(p). \tag{B.1}$$

The form (3.12–13) of K_2 gives

$$\begin{aligned}\left(K_2 K_2^\dagger\right)(k,p) &= -\frac{1}{2}\delta(k-p) + \frac{\mathrm{i}}{4\pi^2} P \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}x' \frac{1}{x-x'} \\ &\quad \times \exp\left[\mathrm{i}k\eta^-(x) - \mathrm{i}p\eta^-(x') + \frac{1}{2}\alpha_-(x) + \frac{1}{2}\alpha_-(x')\right].\end{aligned} \tag{B.2}$$

Switching to variables $y = \eta^-(x)$ and $y' = \eta^-(x')$ and using the identity

$$\delta(k-p) = -\frac{\mathrm{i}}{2\pi^2} P \int_{-\infty}^{+\infty} \mathrm{d}y \int_{-\infty}^{+\infty} \mathrm{d}y' \frac{1}{y-y'} \mathrm{e}^{\mathrm{i}ky - \mathrm{i}py'}, \tag{B.3}$$

we find:

$$\left(K_2 K_2^\dagger\right)(k,p) = \frac{\mathrm{i}}{4\pi^2} \int_{-\infty}^{+\infty} \mathrm{d}y \int_{-\infty}^{+\infty} \mathrm{d}y'\, \mathrm{e}^{\mathrm{i}ky}\, G(y,y')\, \mathrm{e}^{-\mathrm{i}py'}, \tag{B.4}$$

the kernel $G(y,y')$ being defined in (5.5). Eq. (5.4) is a direct consequence of (B.4) and (B.1).

We show now how formulas (7.4) and (7.5) follow from (5.4). Using (7.3), eq. (5.4) becomes

$$W[f] = \frac{1}{\pi} \int_0^{+\infty} \mathrm{d}y \int_{-\infty}^0 \mathrm{d}y' \left[\frac{\sqrt{\sigma}}{\sigma y - y'} - \frac{1}{y-y'}\right] \left[\tilde{f}_-(y')\mathrm{Re}\, \tilde{f}(y) - \tilde{f}_+(y)\mathrm{Re}\, \tilde{f}(y')\right] \tag{B.5}$$

because, by definition, $\mathrm{Im}\, \tilde{f}(y) = \theta(y) f_+(y) + \theta(-y) f_-(y)$. On the other hand, the regularity of $\tilde{f}$ in $\mathrm{Im}\, y > 0$ implies that $\mathrm{Re}\, \tilde{f}$ is the Hilbert transform of $\mathrm{Im}\, \tilde{f}$. Inserting the resulting expression of $\mathrm{Re}\, \tilde{f}$ into (B.4) one discovers that the terms depending on the product $f_+ f_-$ cancel and one is left with the form (7.4–5) of $W[f]$.

Appendix C. Condition for the implementability of the Bogoliubov transformation (3.11)

We show that $\|K_2\|_{\text{H.S.}} < \infty$ iff $\Omega_0 = 0$ and if Ω is sufficiently regular. First we prove that $|K_2(k,p)|^2$ is locally integrable. K_2 is defined in (3.12) and (3.13). As $\delta(k+p) = 0$ for $(k,p) \in \mathbb{R}_+ \times \mathbb{R}_+$, we may write

$$K_2(k,p) = \frac{1}{2\pi}\,\theta(k)\,\theta(p) \int_{-\infty}^{+\infty} \mathrm{d}x \left(\mathrm{e}^{ik\eta^-(x)+\frac{1}{2}\alpha_-(x)} - \mathrm{e}^{ikx}\right) \mathrm{e}^{ipx}. \tag{C.1}$$

We set $\Lambda = 0$ and $a^- = 0$ in (2.7), then $\alpha_-(x) = \Omega_-(x)$. If $x < 0$, $\Omega_-(x) = 0$ and $\eta^-(x) = x$ whereas $\Omega_-(x) = \Omega_0$ and $\eta^-(x) = \mu x + c$, $\mu = \exp(\Omega_0)$, for $x > b^-$. The integral in (C.1) is the sum $(A+B)$ of two integrals: $A = (1/2\pi)\int_0^{b^-} \mathrm{d}x \ldots$, $B = (1/2\pi)\int_{b^-}^{\infty} \mathrm{d}x \ldots$. $A(k,p)$ is bounded and

$$B(k,p) = \frac{\mathrm{e}^{ib^-p}}{2\mathrm{i}\pi}\left(\frac{\mathrm{e}^{ib^-k}}{k+p+\mathrm{i}\epsilon} - \frac{\sqrt{\mu}\,\mathrm{e}^{\mathrm{i}(\mu b^-+c)k}}{\mu k+p+\mathrm{i}\epsilon}\right). \tag{C.2}$$

If $\Omega_0 = 0$ $(\mu = 1)$, B is bounded and K_2 is locally square integrable. If $\Omega_0 \neq 0$,

$$|B(k,p)| \geq \frac{1}{2\pi}\left|\frac{1}{k+p} - \frac{\sqrt{\mu}}{\mu k+p}\right|, \tag{C.3}$$

and $B(k,p)$ is not square integrable at the origin. Consequently K_2 is not Hilbert-Schmidt.

We assume now $\Omega_0 = 0$ and show that $\|K_2\|_{\text{H.S.}}$ is finite. The same manipulations as those described in Appendix B lead to the following expression for the Hilbert-Schmidt norm of K_2:

$$\|K_2\|^2_{\text{H.S.}} = -\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\mathrm{d}y \int_{-\infty}^{+\infty}\mathrm{d}y'\, \frac{1}{y-y'}\,G(y,y'), \tag{C.4}$$

the kernel G being defined in eq. (5.5).

If we assume that $\gamma(y)$ belongs to $C^2(\mathbb{R})$, $G(y,y') \sim (y-y')$ for $y \simeq y'$ and the integrand of (C.4) is bounded. It is easy to construct upper bounds for G which imply the convergence of the integral in (C.4).

Appendix D. Equation (8.12) is an identity

We have to find an expression for $\delta K_1/\delta g^{--}(x)$ and show that this expression turns eq. (8.12) into an identity. The variation of K_1 requires some care because this kernel depends on the metric in its conformal form. Our coordinates x^μ are no longer conformal if the metric $g_{\mu\nu} = (\exp\Omega)\eta_{\mu\nu}$ is changed into $g_{\mu\nu} + \delta g_{\mu\nu}$. The variation of the metric must be combined with an infinitesimal change of coordinates such that the new coordinates are again conformal and the variation of Ω can be identified. On finds

$$\frac{\delta\Omega_-(x^-)}{\delta g^{--}(z)} = \frac{1}{2}\sqrt{-g(z)}\,\mathrm{e}^{-\Omega_-(x^-)}\partial_{x^-}\left(\delta(x^- - z^-)\,\mathrm{e}^{\Omega_-(x^-)}\right). \tag{D.1}$$

Using (3.12–13) and (2.7), with $\Lambda = \Omega_0 = 0$, this leads to

$$\begin{aligned}\frac{\delta K_1(k,p)}{\delta g^{--}(x)} &= \frac{\sqrt{-g(x)}}{8\pi}\exp\left[\mathrm{i}k\eta^-(x^-) - \mathrm{i}px^- + \frac{1}{2}\Omega_-(x^-)\right]\\ &\quad\left(\mathrm{i}k\mathrm{e}^{\Omega_-(x^-)} + \mathrm{i}p + \frac{1}{2}\,\partial_{x^-}\Omega_-(x^-)\right).\end{aligned} \tag{D.2}$$

Now we transform the right-hand side of (8.12) so that it becomes identical to its left-hand side when (D.2) is taken into account. Inserting the explicit form of K_2 into the definition $L = -K_1^{-1}K_2$ one finds

$$\begin{aligned}\tilde{L}(u,v) &= \frac{1}{2\pi}\int_0^\infty \mathrm{d}k\int_0^\infty \mathrm{d}p\,L(k,p)\,\mathrm{e}^{-\mathrm{i}(ku+pv)}\\ &= -\frac{1}{4\pi^2}\int_0^\infty \mathrm{d}k\,\mathrm{e}^{-\mathrm{i}ku}\int_{-\infty}^{+\infty}\mathrm{d}z\int_0^\infty \mathrm{d}q\,K_1^{-1}(k,q)\mathrm{e}^{\mathrm{i}q\eta^-(z)}\\ &\qquad\times\int_0^\infty \mathrm{d}p\,\mathrm{e}^{\mathrm{i}p(z-v)}\,\mathrm{e}^{\frac{1}{2}\Omega_-(z)}.\end{aligned} \tag{D.3}$$

We write $\int_0^\infty \mathrm{d}p\,\exp(\mathrm{i}p(z-v)) = 2\pi\delta(z-v) - \int_0^\infty \mathrm{d}p\,\exp(-\mathrm{i}p(z-v))$. The second term produces a $K_1(q,p)$ in (D.3) which, in conjunction with $K_1^{-1}(k,q)$, leads to a $\delta(k-p)$ after integration over q. This transforms eq. (D.3) into

$$\begin{aligned}\tilde{L}(u,v) &= -\frac{1}{2\pi}\int_0^\infty \mathrm{d}k\int_0^\infty \mathrm{d}q\left(K_1^{-1}\right)(k,q)\,\mathrm{e}^{-\mathrm{i}(ku-q\eta(v))}\,\mathrm{e}^{\frac{1}{2}\Omega_-(v)}\\ &\qquad - \frac{\mathrm{i}}{2\pi}\frac{1}{u-v-\mathrm{i}\epsilon}.\end{aligned} \tag{D.4}$$

If we operate with $\frac{1}{4}\sqrt{-g(x)}\,(\partial_u - \partial_v)$ on the first term in the right-hand side of this equation and set $u = v = x^-$, we get an expression which, according to (D.2), is identical to the first term of the left-hand side of eq. (8.12). If we derive an expression similar to (D.4) for $\tilde{L}^*$ from $L = K_3^\dagger K_4^{\dagger-1}$ the second term in the right-hand side of (D.4) is canceled in the sum $L + L^*$ and right- and left-hand sides of (8.12) coincide.

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