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Distortion analyticity for N−particle Hamiltonians

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Abstract
We define resonances for N−particle Hamiltonians with pair potentials which are not dilation analytic.

1 Introduction

We consider in this paper N−particle Hamiltonians

$$H = \sum_{i=1}^{N} \frac{1}{2m_i}\Delta_{x_i} + \sum_{i<j} V_{ij}(x_i - x_j)$$

on $L^2(\mathbf{R}^{N\nu})$. A rigorous mathematical theory of resonances for such Hamiltonians has been developped in the pioneering papers by Aguilar-Combes [Ag-Co] for 2-particle Hamiltonians and Balslev-Combes [Ba-Co] for arbitrary N. In this approach resonances are defined as complex eigenvalues of some non self-adjoint deformation of H. However in these two papers it is essential that the pair potentials are *dilation analytic*, which roughly means that the $V_{ij}(y)$ have to extend holomorphically in a cone $\{|Imz| \leq C|Rez|\}$. For some applications (like for example N−particle Hamiltonians where some particles have infinite mass) it proved necessary to extend the class of potentials in order to accept potentials which are analytic only near infinity. This was done for 2−particle Hamiltonians by a number of authors (see [Si], [S], [Cy], [Hu]), by introducing variants of the dilation method.

However it seems that no such results are known for N−particle Hamiltonians when $N \geq 3$. For example there is no definition of resonances in the literature even when the pair potentials have compact support.

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The goal of this note is to fill this gap by defining resonances for N−particle Hamiltonians when the pair potentials are analytic only near infinity. This is done using the analytic distortion method of Hunziker [Hu] with a vector field which respects the N−particle structure of the potential. This vector has been originally introduced by Graf [Gr] to prove propagation estimates for N−particle Hamitonians and is now a fundamental tool in the scattering theory for such systems.

Let us now describe more in details the class of Hamiltonians which we will consider. We study a straightforward extension of N−body Hamiltonians called *Agmon Hamiltonians* (*cf* [Ag]). One considers a finite dimensional real vector space X with a positive definite quadratic form $g(x,x)$, and a finite familly $\{X_a\}$, $a \in A$ of linear vector subspaces of X which is closed under intersection and obeying $\cap_{a\in\mathcal{A}} X_a = \{0\}$ and $X \in \{X_a\}$. One denotes by X^a the space $X_a^{\perp}$, by π^a, π_a the orthogonal projections on X^a and X_a.

On $\mathcal{A}$ one puts a partial ordering by saying that $b \leq a$ if $X^b \subset X^a$. With this ordering $\mathcal{A}$ is a lattice and one gets that $X_{a_{\max}} = \{0\}$ and $X_{a_{\min}} = X$. Let $D_x = \frac{1}{i}\partial_x$ and let $\langle x\rangle = (1+g(x,x))^{1/2}$. For $a \in \mathcal{A}$, one denotes by $\sharp a$ the maximal number k such that $a_1 = a < a_2 \cdots < a_k = a_{\max}$.

If $N = \sharp a_{min}$, one defines a (generalized) N−body Hamiltonian by :

$$H = \frac{1}{2}\tilde{g}(D_x, D_x) + V(x),$$

where : $V(x) = \sum_{a\in\mathcal{A}} V_a(\pi^a x)$ and $\tilde{g}$ is the dual quadratic form on X' associated with g. For simplicity of notations, we will simply denote $\tilde{g}(D_x, D_x)$ by D_x^2.

For $a \in \mathcal{A}$, we denote by H_a the Hamiltonian $H - I_a(x)$, where $I_a(x) = \sum_{b\not\leq a} V_b(x^b)$. One has also $H_a = \frac{1}{2}D_a^2 + H^a$, where H^a is the Hamiltonian acting on $L^2(X^a)$ defined by $H^a = \frac{1}{2}D^{a2} + V^a(x^a)$ for $V^a(x^a) = \sum_{b<a} V_b(x^b)$.

We will assume that the potentials V_a satisfy the following hypotheses :

$$\textit{H1)}\ V_a \in L^\infty(X^a).$$

H2) V_a extends holomorphically in

$$\{z \in \mathbf{C}^{n^a} \mid |Re z| \leq R,\ |Im z| < \epsilon_0 |Re z|\},$$

for some R, ϵ_0, and satisfies in this region :

$$\textit{H3)}\ \lim_{z\to\infty} V_a(z) = 0.$$

The condition that $V_a \in L^\infty(X^a)$ is purely for illustrative purposes. The extension to singular potentials is easy.

Let us now give the plan of this paper. In Section 2, we recall the definition of the Graf's vector field and prove two important properties. In Section 3 we define resonances as eigenvalues of the distorted Hamiltonian and prove that they coincide with poles of the meromorphic continuation of the resolvent.

2 The distortion vector field

To define the complex distortion, we will use a vector field originally introduced by Graf [Gr] to prove propagation estimates for N−body Hamiltonians. For the reader's convenience, we will briefly recall its construction.

Let us first introduce some notations. For $a \leq b$ one defines :

$$x_a^b := \pi^b x_a = x^b - x^a = \pi_a x^b.$$

Note that :

$$(x_a^b)^2 = x_a^2 - x_b^2 = x^{b2} - x^{a2}.$$

One puts then :

$$J_a(x) := \prod_{a<f} F((x_a^f)^2 > q_a^f) \prod_{g<a} F((x_g^a)^2 \leq q_g^a), \tag{2.1}$$

where $F(x \in A)$ denotes the characteristic function of A. The constants q_b^a are chosen equal to :

$$q_b^a = q^a - q^b,$$

where

$$q^a := \begin{cases} q^{\sharp a-1} \text{ if } a \neq a_{\min} \\ 0 \text{ if } a = a_{\min}. \end{cases}$$

For a mollifier $\phi \in C_0^\infty(X)$ with :

$$\phi \geq 0, \int \phi(x)dx = 1, \int x\phi(x)dx = 0, \text{supp}\phi \subset \{|x| \leq \sigma\},$$

one then defines :

$$j_a(x) := J_a \star \phi(x).$$

We will use the following properties of j_a (see [Gr]) :

Lemma 2.1

i) $\sum_{a\in\mathcal{A}} j_a(x) = 1$.

ii) $\exists C_0$ *such that* $|x^a| \leq C_0$ *on* $\text{supp}j_a$.

iii) $\exists C_1$ *such that* $\forall b \not\subset a$, $|x^b| \geq C_1$, *on* $\text{supp}j_a$.

We then define the distortion vector field :

$$v_C(x) := \sum_{a\in\mathcal{A}} j_a(\frac{x}{C})x_a.$$

the constant C will have to be chosen large enough later. For ease of notations we will usually forget the subscript C and write simply $v(x)$ for $v_C(x)$. We remark that if $N = 2$, then $V_C(x)$ is identical to the distortion vector field of [Hu]. Note the following estimate, which follows directly from Lemma 2.1 :

$$|\partial_x^\alpha v_C(x)| \leq C_\alpha,\ \forall \alpha \in \mathbf{N}. \tag{2.2}$$

Let us now recall the definition of the *distortion* associated with v_C given in [Hu]. Since $\nabla_x v_C(x) = O(1)$, the mapping

$$X \ni x \mapsto x + \theta v_C(x)$$

is invertible for $\theta \in \mathbf{R}$, $|\theta| \leq c_0$. So we can define the unitary transformation U_θ on $L^2(X)$ by

$$U_\theta u(x) := J_\theta^{\frac{1}{2}} u(x + \theta v_C(x)),$$

for $J_\theta = \det(\delta_{ik} + \partial_k v^i)$. One then puts

$$\begin{aligned} H_\theta &:= U_\theta H U_\theta^{-1} = \\ &\tfrac{1}{2} U_\theta D_x^2 U_\theta^{-1} + V_\theta(x). \end{aligned}$$

Here

$$\begin{aligned} U_\theta D_x^2 U_\theta^{-1} &= J_\theta^{-\frac{1}{2}} D_i J_\theta^{ik} J_\theta J_\theta^{mk} D_k J_\theta^{-\frac{1}{2}}, \\ V_\theta(x) &= V(x + \theta v_C(x)), \end{aligned}$$

where (J_θ^{ik}) is the inverse of the matrix $(J_\theta)_{ik}$ and summation over repeated indices is understood.

The goal is now to extend H_θ to complex values of θ. The first important property of v is the following :

Proposition 2.2 *Assume that :*

$$V(x) = \sum_{a \in \mathcal{A}} V_a(x^a),$$

where V_a satisfy the conditions H). Then for C large enough, there exist c_0, c_1 such that :

$$\theta \mapsto V_\theta(x) := V(x + \theta v_C(x))$$

extends from $\theta \in \mathbf{R}$ as a function holomorphic in

$$\{\theta \in \mathbf{C} \mid |Re\theta| \leq c_0,\ |Im\theta| \leq c_1\}$$

with values in $L^\infty(X)$.

proof. it suffices to consider a potential $V_b(x^b)$. For $\theta \in \mathbf{R}$, we write :

$$V_b(x^b + \theta\pi^b v(x)) = V_b(x^b + \theta \textstyle\sum_{a\in\mathcal{A}} j_a(\frac{x}{C})\pi^b x_a) =$$

$$V_b(x^b + \theta \textstyle\sum_{a\in\mathcal{A}, b\not\leq a} j_a(\frac{x}{C})\pi^b x_a),$$

since $\pi^b x_a = 0$ if $b \leq a$. Next we observe that on $supp j_a(\frac{x}{C})$, one has :

$$|x^a| \leq C_0 C,\ |x^b| \geq C_1 C,\ \forall b \not\leq a. \tag{2.3}$$

So :

$$|\pi^b x_a| = |\pi^b x - \pi^b x_a| \leq |x^b| + C_0 C. \tag{2.4}$$

By (2.3), we see that

$$\sum_{a\in\mathcal{A}, b\not\leq a} j_a(\frac{x}{C})\pi^b x_a$$

is supported in $\{|x^b| \geq C_1 C\}$, so that

$$V_b(x^b + \theta \sum_{a\in\mathcal{A}, b\not\leq a} j_a(\frac{x}{C})\pi^b x_a)$$

is clearly analytic in θ for $|x^b| \leq C_1 C$. By *H)*, we can now pick C large enough such that if $|x^b| \geq C_1 C$ then

$$\theta \mapsto V_b(x^b + \theta y^b)$$

is holomorphic in $|Im\theta| \leq c_0$ for some $c_0 > 0$, uniformly for $|y^b| \leq c_2|x^b|$. But using (2.4), we have :

$$|\sum_{a\in\mathcal{A}, b\not\leq a} j_a(\frac{x}{C})\pi^b x_a| \leq c_2|x^b|,$$

in $\{|x^b| \geq C_1 C\}$, so that

$$\theta \mapsto V_b(x^b + \theta\pi^b v(x))$$

is holomorphic in θ as claimed.□

To state the second property, we introduce another partition of unity. It is well known (see for example [C.F.K.S]), that there exist a partition of unity :

$$1 = \sum_{\sharp a\leq 2} q_a(x), \tag{2.5}$$

with the following properties :

$q_{a_{\max}} \in C_0^\infty(|x| \leq 2)$,

for $\sharp a = 2$, $supp q_a \subset \{x \in X \mid |x| \geq 1, |x^b| \geq \epsilon_0|x|, \forall b \not\leq a\}$,

for $\sharp a = 2$, $q_a \in C^\infty(X)$, $|\nabla_x q_a| \leq C\langle x\rangle^{-1}$.

We will denote by $q_{a,R}(x)$ the scaled functions $q_a(\frac{x}{R})$. In the next proposition, we will denote by $v_C^a(x^a)$ a vector field on X^a defined exactly as v_C, replacing X by X^a and the set of indices $\mathcal{A}$ by the subset $\{b \in \mathcal{A} \mid b \leq a\}$. Accordingly in the definition of the constants q_b^d for $b, d \leq a$, one has to replace $\sharp b$ by $\sharp^a b$ defined as the maximal number k such that :

$$b_1 = b < a_2 \cdots < b_k = a.$$

Note that by the Jordan-Dedekind chain condition, one has :

$$\sharp b + 1 = \sharp^a b + \sharp a. \tag{2.6}$$

Proposition 2.3 *For any $C > 0$, there exist $R > 0$ such that $\forall a$ with $\sharp a = 2$ one has :*

$$v_C(x) = x_a + v_C^a(x^a)$$

on $supp q_{a,R}$.

proof. let us consider

$$v_C(x) = \sum_b j_b(\frac{x}{C})x_b.$$

On $supp q_{a,R}$ one has

$$|x| \geq R,\ |x^b| \geq \epsilon_0|x|, \text{ if } b \not\leq a.$$

So for R large enough, one has :

$$\begin{aligned} v_C(x) &= \textstyle\sum_{b\leq a} j_b(\frac{x}{C})x_b = \\ &\textstyle\sum_{b\leq a} j_b(\frac{x}{C})x_b^a + \\ &\textstyle(\sum_{b\leq a} j_b(\frac{x}{C}))x_a = \\ &\textstyle\sum_{b\leq a} j_b(\frac{x}{C})x_b^a + x_a \end{aligned} \tag{2.7}$$

on $supp q_{a,R}$. For $b \leq a$, we write :

$$j_b(\frac{x}{C}) = \int J_b(\frac{x}{C} + y)\phi(y)dy$$

and replace J_b by its expression given in (2.1). If $x \in supp q_{a,R}$, $y \in supp\phi$, and $f \not\leq a$, one has :

$$(\frac{x_b^f}{C} + y_b^f)^2 \geq \frac{1}{2}\frac{(\epsilon_0 R)^2}{C^2} - \frac{1}{2}\sigma^2 - (\frac{x^b}{C} + y^b)^2 \geq$$
$$\frac{1}{2}\frac{(\epsilon_0 R)^2}{C^2} - \frac{1}{2}\sigma^2 - C_0^2,$$

since $|x^f| \geq \epsilon_0 R$ and $|\frac{x^b}{C} + y^b| \leq C_0$ if $J_b(\frac{x}{C} + y) \neq 0$. If we put

$$J_b^a(x^a) := \Pi_{b<f\leq a} F((x_b^f)^2 > q_b^f) \Pi_{g<b} F((x_g^b)^2 \leq q_g^b),$$

we obtain that on $supp q_{a,R}$, $j_b(x)$ is equal to

$$\int J_b^a(\frac{x^a}{C} + y^a)\phi(y)dy,$$

which is a function similar to j_b if we replace X by X^a, the mollifier ϕ by

$$\phi^a(x^a) = \int_{X_a} \phi(x)dx_a,$$

and (see (2.6)) the constant C by $Cq^{1-\sharp a}$. Using (2.7), this completes the proof of the Proposition. □

3 The resonances

In this section we describe the spectrum of the distorted Hamiltonian H_θ and define the resonances as the discrete eigenvalues of H_θ. We show that the resonances are the poles of the meromorphic continuation of matrix elements of the resolvent $\langle \phi, (z - H)^{-1}\psi \rangle$ for suitable analytic vectors ϕ, ψ.

As in [Hu], we denote by F the space of entire functions in $\mathbf{C}^n$ which decay faster than any power of $\langle z \rangle$ in some cone

$$\{z \in \mathbf{C} \mid |Imz| \leq \epsilon \langle Rez \rangle\}$$

for some $\epsilon > 0$. We define the set A of analytic vectors by

$$A := \{f \in L^2(X) \mid f(x) = \psi(x) \text{ for some } \psi \in F\}.$$

As in [Hu], one has :

Lemma 3.1 *i) for any $f \in A$, the map*

$$\theta \mapsto U_\theta f \in L^2(X)$$

is analytic in $K := \{\theta \in \mathbf{C} \mid |Re\theta| \leq \epsilon_0, |Im\theta| \leq \epsilon_1\}$.
ii) for any $\theta \in K$, the image of A under U_θ is dense in $L^2(X)$.

We first analyse the spectral properties of H_θ.

Theorem 3.2 *i) H_θ with domain $H^2(X)$ is closed and one has :*

$$\|u\|_{H^2(X)} \leq C(\|H_\theta u\| + \|u\|), \forall u \in H^2(X).$$

ii) H_θ is m-sectorial with a sector :

$$S = \{z \in \mathbf{C} \mid |\arg(z)| \leq b < \pi/2\}.$$

iii) the essential spectrum of H_θ is equal to

$$\sigma_{\mathrm{ess}}(H_\theta) = \bigcup_{\sharp a=2} \sigma(H_\theta^a) + \frac{1}{(1+\theta)^2}\mathbf{R}^+.$$

Definition 3.3 *The points in $\sigma(H_\theta)\backslash\sigma_{\mathrm{ess}}(H_\theta)$ are called the* resonances *of H.*

proof. let us first prove *i)* and *ii)*. Since V_θ is a bounded operator, it suffices to prove the corresponding statements with H_θ replaced by $H_{0,\theta} = \frac{1}{2}U_\theta D_x^2 U_\theta^{-1}$. The Hamiltonian $H_{0,\theta}$ is a second order differential operator with principal symbol equal to :

$$T(x,\xi) = (A(x)\xi, \xi),$$

where $A(x) = B^{-1}(x)$, $B(x) = {}^tJJ$, and $J = \mathbf{1} + \theta\nabla v$. So A is diagonal in a basis of eigenvectors of the selfadjoint matrix ∇v with eigenvalues :

$$1 + 2\theta\lambda_i + 2\theta^2\lambda_i^2, \tag{3.1}$$

where $\lambda_1 \leq \cdots \leq \lambda_n$ are the eigenvalues of ∇v. So we see that for $|\theta| \leq \epsilon_0$:

$$|T(x,\xi)| \geq |\xi|^2,$$

which proves *i)* by standard elliptic theory. To prove *ii)*,we remark that by (3.1) the principal symbol $T(x,\xi)$ takes its values in a convex cone strictly included in S. Then *ii)* follows from this observation using for example Gärding's inequality (see [Hö]).

Let us now prove *iii)* by induction on the number of particles. For $N = 2$, *iii)* is proven in [Hu]. Let us assume that Theorem 3.2 holds for all $M-$particle Hamiltonians with $M \leq N-1$. For $a \neq a_{\max}$, let us denote by H_θ^a the distorted Hamiltonian on $L^2(X^a)$ obtained with the vector field v^a, and by $\tilde{H}_{a,\theta}$ the Hamiltonian :

$$\tilde{H}_{a,\theta} := H_\theta^a + \frac{1}{2(1+\theta)^2}D_a^2.$$

Using the partition of unity defined in (2.5) and Proposition 2.3, we obtain :

$$H_\theta = \sum\nolimits_{\sharp a=2} q_{a,R}\tilde{H}_{a,\theta} +$$

$$q_{a_{\max},R}H_\theta + \sum\nolimits_{\sharp a=2} I_{a,\theta}q_{a,R}.$$

Using the fact that H_θ^a is m-sectorial by the induction hypothesis and Ichinose's lemma (see [R-S]), we get that :

$$\sigma(\tilde{H}_{a,\theta}) = \sigma(H_\theta^a) + \frac{1}{(1+\theta)^2}\mathbf{R}^+.$$

By the induction hypothesis we also have :

(3.2) $$\|u\|_{H^2(X)} \leq C(\|\tilde{H}_{a,\theta}u\| + \|u\|),\ \forall u \in H^2(X),$$

which shows as in [Hu] that :

$$\sigma_{ess}(H_\theta) = \bigcup_{\sharp a=2} \sigma(H_\theta^a) + \frac{1}{(1+\theta)^2}\mathbf{R}^+.$$

This completes the proof of the Theorem. □

Finally we can identify the resonances of H with poles of the meromorphic continuation of the resolvent.

Theorem 3.4 *For any $\phi, \psi \in A$ the quantity $\langle\psi, (z-H)^{-1}\psi\rangle$ extends meromorphically in z from $\{z \in \mathbf{C} \mid Imz > 0\}$ to $\mathbf{C}\backslash\sigma_{ess}(H_\theta)$ with poles at the resonances of H. One has :*

$$\sigma_{disc}(H_\theta) = \bigcup_{\phi,\psi \in A} \{\text{ poles of } \langle\psi, (z-H)^{-1}\psi\rangle\}.$$

If λ is a discrete eigenvalue of (H_{θ_0}), and θ varies continuously, λ remains a discrete eigenvalue of H_θ as long as λ stays in $\mathbf{C}\backslash\sigma_{ess}(H_\theta)$.

proof. the proof is exactly the same as in [Hu, Thm 4]. □

References

[Ag] S. Agmon: *Lectures on exponential decay of solutions of second order elliptic equations*, Princeton University Press, Princeton 1982.

[Ag-Co] J. Aguilar-J.M. Combes: A class of analytic perturbations for one-body Schrödinger operators, Comm. in Math. Phys. 22 (1971) 269-279.

[Ba-Co] E. Balslev-J.M. Combes: Spectral properties of many body Schrödinger operators with dilation analytic potentials, Comm. in Math. Phys. 22 (1971) 280-294.

[Cy] H.L. Cycon: Resonances defined by modified dilations, Helv. Phys. Acta 58 (1986)969-981.

[C.F.K.S] H.L. Cycon, R. Froese, W. Kirsch, B. Simon : Schrödinger operators with applications to Quantum Mechanics and Global Geometry, Texts and Monographs in Physics, Springer 1987.

[Gr] G.M. Graf: Asymptotic completeness for $N-$body short range quantum systems : A new proof, Comm. in Math. Phys. Vol 132 (1990), 73-101.

[Hö] L. Hörmander: *Linear partial differential operators*, Berlin Springer Verlag 1963.

[Hu] W. Hunziker: Distortion analyticity and molecular resonances curves, Ann. Inst. Henri Poincaré 45 (1986), 339-358

[R-S] M. Reed-B. Simon: *Methods of modern mathematical physics* Vol IV, New York Academic Press 1978.

[S] I.M. Sigal: Complex transformations method and resonances in one- body quantum systems, Ann. Inst. Henri Poincaré 41 (1984), 103- 114.

[Si] B. Simon: The definition of molecular resonance curves by the method of exterior complex scaling, Phys.Lett. 71A (1979) 211-214.