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The Plasmon in the One Component Plasma

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Abstract

It is proved that in the one component plasma, the macroscopic dipole and current fluctuations behave dynamically as the canonical observables of a harmonic oscillator with frequency $\omega_p = \sqrt{\frac{4\pi e^2 \rho_B}{m}}$, the plasmon-frequency.

1 Introduction

In [FMV] correlation inequalities were used to determine the value of the bulk momentum fluctuations in a quantum mechanical system of density ρ in thermal equilibrium at temperature T. It was shown that the mean square momentum fluctuations per unit volume equal $m\rho kT$ whenever the potential is short-ranged and the correlations have a sufficiently fast decay. This result coincides with the classical equipartition law for quantum systems and is clearly only valid in the absence of phase transitions.

In [MaO] this analysis is extended to the quantum mechanical one component plasma of particles with charge e and mass m in a neutralizing background of density ρ; i.e. the jellium model in thermal equilibrium. Again using correlation inequalities, characterising thermal states of infinitely extended systems [FaV], they considered the current and the dipole fluctuations and obtained : if $J = \sum_i \frac{e}{m} p_i$ and $D = \sum_i e x_i$, then

$$\langle J^2 \rangle \;=\; \omega_p^2 \langle D^2 \rangle$$

where $\omega_p^2 = \frac{4\pi e^2 \rho}{m}$ is the square of the plasmon frequency.

In the one component plasma, D and J are respectively proportional to the coordinate $X = \sum_i x_i$ and momentum $P = \sum_i p_i$ of the center of mass of the whole system. Since the center of mass decouples from the relative coordinates, it will only be subjected to the harmonic force $-m\omega_p^2 X$ due to the charged background. Therefore X and P should behave as the canonical observables of a quantum harmonic oscillator at a macroscopic level with energy :

$$\frac{1}{2m}\left(P^2 + m^2 \omega_p^2 X^2\right)$$

All this follows from the results of [MaO] for equilibrium states.

At this stage it is natural to ask whether one can induce a macrodynamics on the subsystem of J and D and close this dynamics on this subsystem.

In the present paper we prove rigorously that these variables behave *dynamically* as the position and momentum operators of a quantum harmonic oscillator with a frequency equal to ω_p.

The theoretical setting that will be used is the same as in [BrV], where a soluble model is considered in which the same phenomenon appears. Here we treat an unsoluble model. This is a serious complication and forces us to make several assumptions on the states; in particular we will assume that a central limit theorem, analogous to that of [BrV], holds. In view of [GVV1,2] this means assuming enough clustering of the states.

Another important fact of the model is that it is a model with long range forces. So we are in a situation that goes beyond the Goldstone theorem [Gol] where it is possible to have an energy-gap in the density excitation spectrum. We will show that indeed we get a finite frequency $\omega \neq 0$ at $k = 0$. In fact $\omega(k = 0)$ will be the famous plasmon-frequency $\omega_p = \sqrt{\frac{4\pi e^2 \rho}{m}}$.

Our work relies very much on results in [MaO] and our techniques are influenced by [MoS] where the plasmon-frequency is discussed in the frame of the breaking of the Galilei invariance for the jellium model.

For a discussion of our main result see section 5.

2 The Model

We consider an infinite system of particles (bosons or fermions) moving in the three-dimensional space $\mathbb{R}^3$, with Coulomb interaction and in a background of uniform particle density ρ_B. The microscopic algebra of observables $\mathcal{A}$ is assumed to be build on the formal (fermion or boson) creation and annihilation operators ψ^+, ψ : for $x, y \in \mathbb{R}^3$, one has

$$[\psi(x), \psi(y)]_\pm = 0$$

$$[\psi(x), \psi^+(y)]_\pm = \delta(x-y)$$

$$\psi^+(x) = \psi(x)^*$$

The dynamics of the model is governed by the following formal Hamiltonian :

$$\begin{aligned} H &= \frac{1}{2m} \int dx\, \nabla\psi^+(x)\nabla\psi(x) \qquad (2.1) \\ &+\frac{1}{2} \iint dxdy\, \psi^+(x)\psi^+(y)V(x-y)\psi(y)\psi(x) \\ &- \iint dxdy\, \psi^+(x)\psi(x)V(x-y)\rho_B \end{aligned}$$

where $V(x) = \frac{e^2}{|x|}$ and ρ_B is the background density.

In order to define rigorously the dynamics, one has to replace the potential $V(x)$ by a regularised potential $V_L(x) = V(x)f(\frac{x}{L})$, where f is any C^∞−function which is one inside a sphere of radius one and vanishes outside a sphere of radius 1+a [MoS].

The full dynamics is then described by the following regularised equations of motion:

$$\frac{\partial}{i\partial t}\psi(x,t) = [H_L, \psi(x,t)] = \frac{\Delta}{2m}\psi(x,t) - V_L * (\rho - \rho_B)(x,t)\psi(x,t) \qquad (2.2)$$

where

$$\rho(x,t) = \psi^+(x,t)\psi(x,t)$$

$$V_L * (\rho - \rho_B)(x,t) = \int dy V_L(y-x)(\rho(y,t) - \rho_B)$$

These equations yield the continuity equation :

$$\frac{\partial}{\partial t} e\rho(x,t) + \nabla j(x,t) = 0 \tag{2.3}$$

where $j(x,t) = \frac{ie}{2m}\{\nabla\psi^+(x,t)\psi(x,t) - \psi^+(x,t)\nabla\psi(x,t)\}$

A state ω of this system will be described in terms of its correlation functions : $\omega(\psi^+(x_1)\dots\psi^+(x_n)\psi(y_m)\dots\psi(y_1))$.

We will restrict ourselves to a family of states which are :

1. gauge invariant :

$$m \neq n \Rightarrow \omega(\psi^+(x_1)\dots\psi^+(x_n)\psi(y_m)\dots\psi(y_1)) = 0$$

2. The states are $L \to \infty$ limit states of time invariant states ω_L for the Hamiltonians H_L :

$$\forall X \in \mathcal{A} : \lim_{L\to\infty} \omega_L([H_L, X]) = 0$$

3. translation invariant :

$$\forall a \in \mathbb{R}^3 : \omega(\psi^+(x_1+a)\dots\psi^+(x_n+a)\psi(y_n+a)\dots\psi(y_1+a)) =$$

$$\omega(\psi^+(x_1)\dots\psi^+(x_n)\psi(y_n)\dots\psi(y_1)).$$

4. rotation invariant :

$$\forall R \in O(3) : \omega(\psi^+(Rx_1)\dots\psi^+(Rx_n)\psi(Ry_n)\dots\psi(Ry_1)) =$$

$$\omega(\psi^+(x_1)\dots\psi^+(x_n)\psi(y_n)\dots\psi(y_1))$$

5. time reversal invariant :

$$\forall X \in \mathcal{A} : \omega(\sigma(X)) = \omega(X^*)$$

where σ is the time reversal operation.

These five properties of the system can hardly be considered as conditions. The main assumptions appear in the next section.

3 Cluster-properties and sumrules

As in [MaO],[MaG], we will assume that the correlation functions of the state can be expressed by the reduced density matrices $\rho^{(n)}$, with kernels formally written as :

$$\langle y_1, \ldots, y_n | \rho^{(n)} | x_n, \ldots, x_1 \rangle = \omega(\psi^+(x_1) \ldots \psi^+(x_n) \psi(y_n) \ldots \psi(y_1))$$

We make a first assumption on the clustering of the state :

$$\langle y_1, \ldots, y_n, x | \rho^{(n+1)} | x', y_n', \ldots, y_1' \rangle - \rho_B \langle y_1, \ldots, y_n | \rho^{(n)} | y_n', \ldots, y_1' \rangle = O(|x|^{-(3+\epsilon)})$$

for $x' = x + a$, a fixed; $\epsilon > 0$.

For notational convenience, we introduce the density and momentum correlations :

$$\langle \rho^{(n)} p_{i_1}^{r_1} \ldots p_{i_s}^{r_s} \rangle (x_1, \ldots, x_{n-1}) = \langle x_1, \ldots, x_{n-1}, 0 | \rho^{(n)} p_{i_1}^{r_1} \ldots p_{i_s}^{r_s} | 0, x_{n-1}, \ldots, x_1 \rangle$$

$$= \int dy_1 \ldots \int dy_n \langle x_1, \ldots, x_{n-1}, 0 | \rho^{(n)} | y_n, \ldots, y_1 \rangle \langle y_1, \ldots, y_n | p_{i_1}^{r_1} \ldots p_{i_s}^{r_s} | 0, x_{n-1}, \ldots, x_1 \rangle$$

where $r_j = 1, 2, 3$; $i_j = 1, \ldots, n$; $s = 0, 1, 2, \ldots$

Only homogeneous phases of the one component plasma having good screening properties which satisfy the assumptions on the clustering as in [MaO], will be considered. This means :

- the density-density correlations have fast cluster properties as in the classical case :

$$\int dx|x|^2|\langle\rho_T^{(2)}\rangle(x)| \;<\; \infty \tag{3.4}$$

$$\iint dxdy|x||\langle\rho_T^{(3)}\rangle(x,y)| \;<\; \infty \tag{3.5}$$

where the $\rho_T^{(n)}$ are the fully truncated n-point functions, defined in the usual way.

- the momentum-density correlations are integrable :

$$\int dx|\langle\rho^{(2)}p_1^r\rangle(x)| \;<\; \infty \tag{3.6}$$

$$\int dx|x||\langle\rho^{(2)}p_1^r p_1^s\rangle(x) - \rho_B\langle\rho^{(1)}p_1^r p_1^s\rangle| \;<\; \infty \tag{3.7}$$

- The momentum-momentum correlations have, however, a slow clustering of the type

$$\langle\rho^{(2)}p_1^r p_2^s\rangle(x) \;=\; c_1\frac{\partial^2}{\partial x^r\partial x^s}(|x|^{-1}) + O(|x|^{-(3+\epsilon)}) \tag{3.8}$$

$$\langle\rho^{(2)}(p_1^r)^2(p_2^s)^2\rangle(x) \;=\; c_2 + c_3\frac{\partial^2}{\partial x^r\partial x^s}(|x|^{-1}) + O(|x|^{-(3+\epsilon)}) \tag{3.9}$$

with c_1, c_2, c_3 some constants.

We also need some additional cluster properties :

-

$$\iint dy_1dy_2\frac{1}{|y_2|^2}\,|\langle\rho^{(3)}\rangle(y_1,y_2) - \rho_B\langle\rho^{(2)}\rangle(y_2) + 2\langle\rho^{(2)}\rangle(y_2)\delta(y_1) -$$
$$\rho_B\langle\rho^{(2)}\rangle(y_1) - \rho_B^3 + \delta(y_1)\rho_B^2\big| \;<\; \infty \tag{3.10}$$

$$\iint dy_1dy_2\frac{1}{|y_2|^2}\left|\langle\rho(3)p_2^k p_2^{k'}\rangle(y_1,y_2) - \rho_B^2\langle\rho^{(2)}p^k p^{k'}\rangle(y_2)+\right.$$
$$\langle\rho^{(2)}p_2^k p_2^{k'}\rangle(y_2)\delta(y_1) + \langle\rho^{(2)}p_1^k p_1^{k'}\rangle(y_2)\delta(y_1)-$$

$$\rho_B\langle\rho^{(2)}p_2^k p_2^{k'}\rangle(y_1) - \rho_B^2\langle\rho^{(1)}p^k p^{k'}\rangle + \rho_B\langle\rho^{(1)}p^k p^{k'}\rangle\delta(y_1)\Big| \quad < \quad \infty \qquad (3.11)$$

$$\int dy_3 \Big|\omega\Big(\psi^+(o)\int dy_1 \partial_k V(y_1)(\rho(y_1)-\rho_B)\psi(o)$$
$$\psi^+(y_3)\int dy_1\partial_k V(y_1)(\rho(y_3+y_1)-\rho_B)\psi(y_3)\Big)\Big| \quad < \quad \infty \qquad (3.12)$$

These cluster properties are not rigorously proven in the quantum case, but it is generally believed that they hold in the high temperature regime.

We shall also assume that a set of sumrules hold and refer to [MaO],[MaG] for their derivation, where one proves that these sumrules are a necessary consequence of the time-invariance equations obeyed by the reduced density matrices of a state having suitable cluster properties, such as a high temperature equilibrium state. One has :

- charge sumrules :

$$\int dx \,\left(\langle\rho^{(2)}\rangle(x) - \rho_B^2\right) + \rho_B \;=\; 0 \qquad (3.13)$$

$$\int dx \,\left(\langle\rho^{(3)}\rangle(x,y) - \rho_B\langle\rho^{(2)}\rangle(y)\right) + 2\langle\rho^{(2)}\rangle(y) \;=\; 0 \qquad (3.14)$$

- the dipole sumrules :

$$\int dx\, x\left(\langle\rho^{(2)}\rangle(x-y) - \rho_B^2\right) + y\rho_B \;=\; 0 \qquad (3.15)$$

$$\int dx\, x\left(\langle\rho^{(3)}\rangle(x,y) - \rho_B\langle\rho^{(2)}\rangle(y)\right) + y\langle\rho^{(2)}\rangle(y) \;=\; 0 \qquad (3.16)$$

- and :

$$\int dx\,\left(\langle\rho^{(2)}(p_1^r)^2\rangle(x) - \rho_B\langle\rho^{(1)}(p_1^r)^2\rangle\right) + \langle\rho^{(1)}(p_1^r)^2\rangle \;=\; 0 \qquad (3.17)$$

4 Dipole and Current Fluctuations

With the help of rather specific cutoff-functions, we define in this section the local fluctuations of the vector-fields which describe the momentum and position of the charge center of the system.

These vector-fields are respectively given by :

$$J^r : \mathbb{R}^3 \to \mathcal{A} : x \mapsto j^r(x)$$

$$D^r : \mathbb{R}^3 \to \mathcal{A} : x \mapsto x^r \rho(x)$$

where $r = 1, 2, 3$.

As in [MaO], we choose smoothened characteristic functions of a cylinder Λ of radius R and length $2L$ along the r-axis (r=1,2,3) as cutoff-functions to describe the local volumes or the boundary conditions and we define :

$$\tilde{J}^r_\Lambda := \frac{1}{\sqrt{2L}\sqrt{\pi R^2}} \int dx\, g(\frac{x^r}{L})\chi_R(x^{\perp r}) j^r(x) \tag{4.18}$$

$$\tilde{D}^r_\Lambda := \frac{e}{\sqrt{2L}\sqrt{\pi R^2}} \int dx\, h(\frac{x^r}{L})\chi_R(x^{\perp r}) x^r \rho(x) \tag{4.19}$$

where the vector x is split up in a part x^r along the r-axis and a part $x^{\perp r}$ orthogonal to the r-axis. The functions appearing in the local fluctuations need to satisfy the following restrictions :

$$\chi_R(x^{\perp r}) = \begin{cases} 1 & \text{iff } |x^{\perp r}| \leq R \\ 0 & \text{iff } |x^{\perp r}| \geq R + 1 \end{cases}$$

$h(s) \in C^\infty(\mathbb{R})$ and has compact support;

$$h(s) = h(-s);$$

$$h(1) = 1;$$

$$\frac{1}{2}\int ds\{\frac{d}{ds}(sh(s))\}^2 = 1$$

$$g(s) = \frac{d}{ds}(sh(s))$$

Note that by time-reversal invariance and space-reflection invariance we have $\omega(\tilde{J}^r_\Lambda) = \omega(\tilde{D}^r_\Lambda) = 0$ for all L, R. For short we denote Λ for the cylinder (L, R).

Consider now the two-dimensional real linear spaces H^r spanned by the fields J^r and D^r $(r = 1, 2, 3)$. It is an easy calculation to see that under a state ω as specified in section 3 :

$$\lim_{L\to\infty}\lim_{R\to\infty}\frac{1}{i}[\tilde{D}^r_{L,R}, \tilde{J}^r_{L,R}] \quad = \quad \rho_B \mathbb{1}_\omega; \tag{4.20}$$

where the limit is a weak-operator limit in the state ω. The order of taking the limits is important. The local volumes are cylinders in the direction in which the fluctuations are considered. Technically, taking first the R-limit and then the L-limit means that one considers the fluctuations between two infinite planes — approximated by disks with increasing radius R — orthogonal to the r-direction.

This limit provides us in a natural way with a two-dimensional symplectic space (H^r, σ), where σ is a symplectic form determined by the commutator (4.3) : $\sigma(D^r, J^r) = \rho_B$.

From [GVV1] and [GVV2] it follows that we can give a mathematical meaning to $\lim_{\Lambda\to\infty}\tilde{D}^r{}_\Lambda$ and $\lim_{\Lambda\to\infty}\tilde{J}^r{}_\Lambda$ as operators by a central limit theorem. We denote these limits respectively by $\tilde{D}^r$ and $\tilde{J}^r$. Note that (4.3) then reads as

$$\frac{1}{i}[\tilde{D}^r, \tilde{J}^r] \quad = \quad \rho_B \mathbb{1} \tag{4.21}$$

In this paper we do not aim at proving this central limit theorem; we will on the basis of the results in [GVV1] and [GVV2] – i.e. assuming enough clustering – assume the existence of the following limits : for all $\lambda, \mu \in \mathbb{R}$,

$$\lim_{L\to\infty}\lim_{R\to\infty} \omega\left(e^{i(\lambda \tilde{D}^r_\Lambda + \mu \tilde{J}^r_\Lambda)}\right) = \exp\{-\frac{1}{2} s_\omega(\lambda D^r + \mu J^r, \lambda D^r + \mu J^r)\} \tag{4.22}$$

where

$$s_\omega(\lambda D^r + \mu J^r, \lambda D^r + \mu J^r) = \lim_\Lambda \omega\left([\lambda \tilde{D}^r_\Lambda + \mu \tilde{J}^r_\Lambda]^2\right) \tag{4.23}$$

Of course, ω is a state belonging to the family specified in section 3. Formula (4.5) is a non-commutative central limit theorem and although we do not prove this theorem, the resulting formula is proved to be meaningful in [MaO]. There it is proved that the limit in (4.6) exists and hence the right hand side of (4.5) is well defined.

Therefore, one can consider the central limits as a representation of the Weyl algebra build on the symplectic space (H^r, σ) [BrR] and determined by the quasi-free state $\tilde{\omega}$ which is defined by

$$\tilde{\omega}\left(e^{i(\lambda \tilde{D}^r + \mu \tilde{J}^r)}\right) = \exp\{-\frac{1}{2} s_\omega(\lambda D^r + \mu J^r, \lambda D^r + \mu J^r)\} \tag{4.24}$$

The macroscopic fluctuations $\tilde{D}^r$ and $\tilde{J}^r$ are then the two generating boson fields on the two-dimensional symplectic space (H^r, σ) satisfying (4.4).

Note that s_ω is a bilinear form on the real space H^r.

From (4.6) one gets :

$$\tilde{\omega}((\tilde{D}^r)^2) = s_\omega(D^r, D^r) \tag{4.25}$$

$$\tilde{\omega}((\tilde{J}^r)^2) = s_\omega(J^r, J^r) \tag{4.26}$$

In [MaO], it is proved that the time-invariance of the state ω yields the quantum virial theorem :

$$\tilde{\omega}((\tilde{J}^r)^2) = \omega_p^2 \tilde{\omega}((\tilde{D}^r)^2)$$

where $\omega_p^2 = \frac{4\pi e^2 \rho}{m}$, with ω_p the well-known plasmon frequency of charged systems. It is also proved in [MaO] that the macrosopic fluctuation operators $\tilde{D}^r$ and $\tilde{J}^r$ are distributed as the canonical observables of a quantum harmonic oscillator in equilibrium at inverse temperature β if the microscopic state ω is an equilibrium state at the same temperature. They proved :

$$\tilde{\omega}((\tilde{D}^r)^2) = \frac{1}{4\pi}\frac{1}{2}\hbar\omega_p \coth(\frac{\beta}{2}\hbar\omega_p).$$

This means that on the fluctuation level, the macroscopic variables $\tilde{D}^r$ and $\tilde{J}^r$ split off from the other variables of the system.

So, the macroscopic state $\tilde{\omega}$ (4.7) defines a representation of the C.C.R. algebra generated by the fluctuation operators $\tilde{D}^r$ and $\tilde{J}^r$. Let $\tilde{\mathcal{H}}$ be the G.N.S.-representation space of $\tilde{\omega}$ [BrR] and $\tilde{\Omega}$ the cyclic vector in $\tilde{\mathcal{H}}$, then

$$\tilde{\omega}(P(\tilde{D}^r)P'(\tilde{J}^r)) = (\tilde{\Omega}, P(\tilde{D}^r)P'(\tilde{J}^r)\tilde{\Omega}) \tag{4.27}$$

for any polynomial P and P'.

$\tilde{\mathcal{H}}$ is the closure of the set of vectors $\left\{P(\tilde{D}^r)P'(\tilde{J}^r)\tilde{\Omega} \mid P, P' \text{ polynomials}\right\}$.

The next question that immediately arises is whether the *dynamics* also closes on this subsystem of $\tilde{D}^r$ and $\tilde{J}^r$. This is the subject of this paper. We will prove rigorously that these variables behave dynamically as the position and momentum operators of a quantum harmonic oscillator with a frequency which is exactly the plasmon frequency ω_p.

Theorem 4.1 *The microdynamics generated by the derivation $\delta = [H, .]$ where H is given by (2.1), induces on the macroscopic level a dynamics, generated by $\tilde{\delta}$:*

$$\tilde{\delta}\tilde{D}^r = \lim_{\Lambda\to\infty} \delta\tilde{D}^r_\Lambda \overset{\text{not}}{=} \lim_{\Lambda\to\infty} (\widetilde{\delta D^r})_\Lambda \tag{4.28}$$

$$\tilde{\delta}\tilde{J}^r = \lim_{\Lambda\to\infty} \delta\tilde{J}^r_\Lambda \overset{\text{not}}{=} \lim_{\Lambda\to\infty} (\widetilde{\delta J^r})_\Lambda \tag{4.29}$$

Moreover we have the following operator equalities :

$$\tilde{\delta}\tilde{D}^r = \tilde{J}^r \tag{4.30}$$

$$\tilde{\delta}\tilde{J}^r = -\omega_p^2\tilde{D}^r \tag{4.31}$$

i.e. $\tilde{D}^r$ and $\tilde{J}^r$ behave dynamically as the canonical observables of a quantum harmonic oscillator.

■

Note that the plasmon frequency appears as a *discrete point* in the spectrum of the fluctuationdynamics.

Two important remarks need to be made at this point. First of all, the definition of $\tilde{\delta}$ is only meaningful if the right hand sides of (4.11) and (4.12) are well-defined operators. This is not trivial because we treat a model with Coulomb-interaction,which is a long range interaction. For instance we have that δJ^r is highly nonlocal and taking fluctuations of nonlocal observables may cause problems. For short range interactions it is proved in [GVV1] that the way (4.11)(4.12) of inducing a macrodynamics on the algebra of macroscopic fluctuations is successful for all strictly local microscopic observables.

The second remark is that it is also not trivial that the macrodynamics , generated by $\tilde{\delta}$, closes on the subsystem spanned by the fluctuations $\tilde{D}^r$ and $\tilde{J}^r$. Technically, it is a priori not clear that $\tilde{\delta}\tilde{D}^r$ should be in the G.N.S.-representation, described above.

Proof of theorem 4.1 :

If we can prove that :

$$\lim_{\Lambda} \omega\left(\left[(\widetilde{\delta D^r})_\Lambda - (\widetilde{J^r})_\Lambda\right]^2\right) = 0 \tag{4.32}$$

$$\lim_{\Lambda} \omega\left(\left[(\widetilde{\delta^2 D^r})_\Lambda + \omega_p^2(\widetilde{D^r})_\Lambda\right]^2\right) = 0 \tag{4.33}$$

then it is clear that first the definitions (4.11) and (4.12) of the dynamics on $\tilde{D}^r$ and $\tilde{J}^r$ are consistent and secondly, that also formulae (4.13) and (4.14) hold. This is true, because (4.15) and (4.16) mean that the macroscopic fluctuations of $\delta(D^r) - J^r$ and of $\delta^2(D^r)+\omega_p^2 D^r$ have exactly the same distribution as the zero-operator, because the central limit distribution is completely determined by the second moments. This means that the fluctuations of δD^r and $\delta^2 D^r$ are in the operator sense the same as the fluctuations of J^r and $-\omega_p^2 D^r$ respectively.

- The first equality (4.15): $\lim_\Lambda \omega\left(\left[(\widetilde{\delta D^r})_\Lambda - (\widetilde{J^r})_\Lambda\right]^2\right) = 0$ is a simple consequence of the continuity equation (2.3).

$$\lim_{\Lambda} \omega\left(\left[(\widetilde{\delta D^r})_\Lambda - (\widetilde{J^r})_\Lambda\right]^2\right)$$

$$= \lim_{L,R\to\infty} \omega\left(\left[\frac{e}{\sqrt{2L\pi R^2}}\int dx h(\frac{x^r}{L})\chi_R(x^{\perp r})x^r\delta\rho(x) - \frac{1}{\sqrt{2L\pi R^2}}\int dx g(\frac{x^r}{L})\chi_R(x^{\perp r})j^r(x)\right]\right.$$

$$= \lim_{L,R\to\infty} \omega\left(\left[\frac{-1}{\sqrt{2L\pi R^2}}\int dx h(\frac{x^r}{L})\chi_R(x^{\perp r})x^r\nabla j(x) - \frac{1}{\sqrt{2L\pi R^2}}\int dx g(\frac{x^r}{L})\chi_R(x^{\perp r})j^r(x)\right]^2\right)$$

Using the Gauss-theorem ($\int dx \nabla(f(x)j(x)) = 0$ for functions f with compact support) and the translation and rotation invariance of the state, one gets :

$$\lim_{L,R\to\infty}\frac{1}{2L\pi R^2}\iint dxdy x^r h(\frac{x^r}{L})y^r h(\frac{y^r}{L})\sum_{i\neq r}\partial_i\chi_R(x^{\perp r})\partial_i\chi_R(y^{\perp r})\omega(j^i(x-y)j^i(o))$$

And this is also equal to :

$$\sum_{i\neq r}\langle\rho^{(1)}(p^i)^2\rangle \lim_{L,R\to\infty}\frac{1}{2L\pi R^2}\int dy \left(y^r h(\frac{y^r}{L})\partial_i\chi_R(y^{\perp r})\right)^2 \tag{4.34}$$

$$+\sum_{i\neq r}\lim_{L,R\to\infty}\frac{\rho_B}{8L\pi R^2}\int dy \left(y^r h(\frac{y^r}{L})\partial_i\chi_R(y^{\perp r})\right)^2 \tag{4.35}$$

$$+\sum_{i\neq r}\lim_{L,R\to\infty}\frac{1}{8L\pi R^2}\int dy_1\langle\rho_T^{(2)}\rangle(y_1)\int dy_2$$

$$y_2^r h(\frac{y_2^r}{L})\partial_i\chi_R(y_2^{\perp r})(y_2^r+y_1^r)h(\frac{y_2^r+y_1^r}{L})\partial_i\chi_R(y_2^{\perp r}+y_1^{\perp r}) \tag{4.36}$$

$$+\sum_{i\neq r}\lim_{L\to\infty}\int dy_1^r\frac{1}{2L}\int dy_2^r y_2^r h(\frac{y_2^r}{L})(y_2^r+y_1^r)h(\frac{y_2^r+y_1^r}{L})$$

$$\lim_{R\to\infty}\int dy_1^{\perp r}\langle\rho^{(2)}p_1^ip_2^i\rangle(y_1^r,y_1^{\perp r})\frac{1}{\pi R^2}\int dy_2^{\perp r}\partial_i\chi_R(y_2^{\perp r})\partial_i\chi_R(y_2^{\perp r}+y_1^{\perp r}) \tag{4.37}$$

Remark that :

$$\lim_{R\to\infty}\frac{1}{\pi R^2}\int dy^{\perp r}\chi_R(y^{\perp r})\chi(y^{\perp r}+a^{\perp r}) = 1 \quad \forall a^{\perp r} \tag{4.38}$$

and if there is one or more partial derivatives of χ_R in the integral, then ($n+m\geq 1$)

$$\begin{aligned}\left|\frac{1}{\pi R^2}\int dy^{\perp r}\,\chi_R^{(n)}(y^{\perp r})\chi_R^{(m)}(y^{\perp r}+a^{\perp r})\right| \\ < \frac{M}{\pi R^2}(\pi(R+1)^2-R^2) \\ \xrightarrow{R\to\infty} 0\end{aligned} \tag{4.39}$$

By this remark and cluster property (3.1) and by dominated convergence, it is trivial that (4.17),(4.18) and (4.19) are zero in the limit, since one first has to take the limit $R \to \infty$.

Under the assumption (3.5) on momentum-momentum correlations, one sees that $\langle \rho^{(2)}(p_1^r)^2(p_2^r)^2\rangle(y_1^r, y_1^{\perp r})$ is integrable in $y_1^{\perp r}$ for fixed y_1^r. Moreover since

$$\begin{aligned}\int dy^{\perp r} \frac{\partial^2}{\partial {y^r}^2}(|y|^{-1}) &= -4\pi\delta(y^r) \\ &= 0 \text{ for } y^r \neq 0\end{aligned}$$

the integral

$$\int dy_1^{\perp r} \langle \rho^{(2)} p_1^r p_2^r\rangle(y_1^r, y_1^{\perp r})$$

is integrable in y_1^r. Thus, one obtains that also (4.20) tends to zero by dominated convergence.

- The second equality (4.16): $\lim_\Lambda \omega\left(\left[(\widetilde{\delta^2 D^r})_\Lambda + \omega_p^2(\widetilde{D^r})_\Lambda\right]^2\right) = 0$ or equivalently :

$$\begin{aligned}\lim_\Lambda \Big\{\omega_p^4\omega\left([(\widetilde{D^r})_\Lambda]^2\right) + \omega_p^2\omega\left((\widetilde{\delta^2 D^r})_\Lambda(\widetilde{D^r})_\Lambda\right) + \omega_p^2\omega\left((\widetilde{D^r})_\Lambda(\widetilde{\delta^2 D^r})_\Lambda\right) + \omega\left([(\widetilde{\delta^2 D^r})_\Lambda]^2\right)\Big\} \\ = 0\end{aligned}$$

is much harder to prove.

1. The first term yields :

$$\omega_p^4 \lim_\Lambda \omega((\widetilde{D^r})_\Lambda(\widetilde{D^r})_\Lambda) = \omega_p^4 s_\omega(D^r, D^r) = \omega_p^4\tilde{\omega}((\tilde{D}^r)^2)$$

2. The second and third term give the same contribution. One gets :

$$\begin{aligned}&2\omega_p^2 \lim_\Lambda \omega((\widetilde{\delta^2 D^r})_\Lambda(\widetilde{D^r})_\Lambda) = \\ &\lim_{L,R\to\infty} \frac{2\omega_p^2 e^2}{2L\pi R^2} \iint dx dy\, x^r h(\frac{x^r}{L}) y^r h(\frac{y^r}{L})\chi_R(x^{\perp r})\chi_R(y^{\perp r})\omega(\delta^2[\rho(x)]\rho(y))\end{aligned}$$

$$= \lim_{L,R\to\infty} \frac{-2\omega_p^2 e^2}{2L\pi R^2} \iint dxdy\, x^r h(\frac{x^r}{L}) y^r h(\frac{y^r}{L}) \chi_R(x^{\perp r}) \chi_R(y^{\perp r}) \omega(\delta[\rho(x)]\delta[\rho(y)])$$

by the time-invariance of the state;

$$= \lim_{L,R\to\infty} \frac{-2\omega_p^2}{2L\pi R^2} \iint dxdy\, x^r h(\frac{x^r}{L}) \chi_R(x^{\perp r}) y^r h(\frac{y^r}{L}) \chi_R(y^{\perp r}) \omega(\nabla j(x) \nabla j(y))$$

where we plugged in the continuity equation (2.5). Using again the Gauss-theorem and the rotation invariance of the state and the same arguments as for (4.20), one gets :

$$= \lim_{L,R\to\infty} \frac{-2\omega_p^2}{2L\pi R^2} \iint dxdy \sum_{k=1}^{3} \partial_k (x^r h(\frac{x^r}{L}) \chi_R(x^{\perp r})) \partial_k (y^r h(\frac{y^r}{L}) \chi_R(y^{\perp r})) \omega(j^k(x) j^k(y))$$

$$= -2\omega_p^2 s_\omega(J^r, J^r)$$

$$= -2\omega_p^4 s_\omega(D^r, D^r)$$

3. What remains to prove is :

Lemma 4.1

$$\lim_\Lambda \omega((\widetilde{\delta^2 D^r})_\Lambda (\widetilde{\delta^2 D^r})_\Lambda) = \omega_p^4 s_\omega(D^r, D^r)$$

We calculate :

$$\langle \delta^2 D^r, \delta^2 D^r \rangle_\omega = \tag{4.40}$$

$$\lim_{L,R\to\infty} \frac{e^2}{2L\pi R^2} \iint dy_1 dy_2\, y_1^r h(\frac{y_1^r}{L}) \chi_R(y_1^{\perp r}) y_2^r h(\frac{y_2^r}{L}) \chi_R(y_2^{\perp r})$$

$$\omega\left(\left\{ -\frac{1}{2m^2} [\Delta^2 \psi^+(y_1)\psi(y_1) - 2\Delta\psi^+(y_1)\Delta\psi(y_1) + \psi^+(y_1)\Delta^2\psi(y_1)] + \right.\right.$$

$$\frac{\nabla}{2m}[\psi^+(y_1)\nabla(V*[\rho-\rho_B])(y_1)\psi(y_1)]\Big\}$$
$$\Big\{-\frac{1}{2m^2}[\Delta^2\psi^+(y_2)\psi(y_2)-2\Delta\psi^+(y_2)\Delta\psi(y_2)+\psi^+(y_2)\Delta^2\psi(y_2)]+$$
$$\frac{\nabla}{2m}[\psi^+(y_2)\nabla(V*[\rho-\rho_B])(y_2)\psi(y_2)]\Big\}\Big)$$

The only term in (4.23) contributing to the limit, will be :

$$\lim_{L,R\to\infty}\frac{e^2}{2L\pi R^2}\iint dy_1dy_2y_1^rh(\frac{y_1^r}{L})\chi_R(y_1^{\perp r})y_2^rh(\frac{y_2^r}{L})\chi_R(y_2^{\perp r})$$
$$\omega\left(\frac{\nabla}{2m}\,[\psi^+(y_1)\nabla(V*[\rho-\rho_B])(y_1)\psi(y_1)]\,\frac{\nabla}{2m}\,[\psi^+(y_2)\nabla(V*[\rho-\rho_B])(y_2)\psi(y_2)]\right)$$

$$=\lim_{L,R\to\infty}\frac{e^2}{2L\pi R^2}\iint dy_3dy_4\omega\left(\psi^+(y_3)\int dy_1\nabla\gamma(y_1)\nabla V(y_3-y_1)[\rho(y_1)-\rho_B+\rho_B]\psi(y_3)\right.$$
$$\left.\psi^+(y_4)\int dy_2\nabla\gamma(y_2)\nabla V(y_4-y_2)[\rho(y_2)-\rho_B+\rho_B]\psi(y_4)\right)$$

where we used $\gamma(y)=y^rh(\frac{y^r}{L})\chi_R(y^{\perp r})$, to shorten notation.

$$=\lim_{L,R\to\infty}\frac{e^2}{2L\pi R^2}\iint dy_3dy_4\omega\left(\psi^+(y_3)\int dy_1\nabla\gamma(y_1)\nabla V(y_3-y_1)\rho_B\psi(y_3)\right.$$
$$\left.\psi^+(y_4)\int dy_2\nabla\gamma(y_2)\nabla V(y_4-y_2)\rho_B\psi(y_4)\right)\qquad(4.41)$$
$$+\lim_{L,R\to\infty}\frac{e^2}{2L\pi R^2}\iint dy_3dy_4\left\{\omega\left(\psi^+(y_3)\int dy_1\nabla\gamma(y_1)\nabla V(y_3-y_1)[\rho(y_1)-\rho_B]\psi(y_3)\right.\right.$$
$$\left.\left.\psi^+(y_4)\int dy_2\nabla\gamma(y_2)\nabla V(y_4-y_2)\rho_B\psi(y_4)\right)+\text{c.c.}\right\}\qquad(4.42)$$
$$+\lim_{L,R\to\infty}\frac{e^2}{2L\pi R^2}\iint dy_3dy_4\,\omega\left(\psi^+(y_3)\int dy_1\nabla\gamma(y_1)\nabla V(y_3-y_1)[\rho(y_1)-\rho_B]\psi(y_3)\right.$$
$$\left.\psi^+(y_4)\int dy_2\nabla\gamma(y_2)\nabla V(y_4-y_2)[\rho(y_2)-\rho_B]\psi(y_4)\right)\qquad(4.43)$$

Consider in the first term (4.24) of this expression, the following integral :

$$\int dy\nabla\gamma(y)\nabla V(x-y)\rho_B\ =$$

$$
\begin{aligned}
-\int dy\gamma(y)\Delta V(x-y)\rho_B &= \\
4\pi e^2\rho_B \int dy\gamma(y)\delta(y-x) &= \\
&\qquad 4\pi e^2\rho_B\gamma(x)
\end{aligned}
$$

So (4.24) becomes :

$$
\lim_{L,R\to\infty} \frac{e^2}{2L\pi R^2}\left(\frac{4\pi e^2\rho_B}{m^2}\right)^2 \iint dy_3 dy_4\, \gamma(y_3)\gamma(y_4)\omega(\rho(y_3)\rho(y_4)) =
$$

$$
\omega_p^4 s_\omega(D^r, D^r) \qquad \blacksquare
$$

We want to stress that it is exactly at this point where the long-range of the potential plays a fundamental role.

The other terms (4.25) and (4.26) are zero, as will be proved below.

4. We now proceed to prove that all the other terms in (4.23) vanish in the limit.

   (a) We deal first with :

$$
\begin{aligned}
&\lim_{L,R\to\infty} \frac{e^2}{2L\pi R^2} \iint dy_1 dy_2\, \gamma(y_1)\gamma(y_2)(2m)^{-4} \\
&\quad \omega\left(\left[\Delta^2\psi^+(y_1)\psi(y_1) - 2\Delta\psi^+(y_1)\Delta\psi(y_1) + \psi^+(y_1)\Delta^2\psi(y_1)\right]\right. \\
&\qquad \left.\left[\Delta^2\psi^+(y_2)\psi(y_2) - 2\Delta\psi^+(y_2)\Delta\psi(y_2) + \psi^+(y_2)\Delta^2\psi(y_2)\right]\right)
\end{aligned}
$$

By applying several times the Gauss-theorem, we see that this last expression simplifies to :

$$
\lim_{L,R\to\infty} \frac{e^2}{2L\pi R^2}(2m)^{-4} \iint dy_1 dy_2\, \Delta^2\gamma(y_1)\Delta^2\gamma(y_2)\omega(\rho(y_1)\rho(y_2))
$$

$$-4 \lim_{L,R\to\infty} \frac{e^2}{2L\pi R^2}(2m)^{-4} \iint dy_1 dy_2 \, \mathrm{Re}[\Delta^2\gamma(y_1) \sum_{k,k'} \partial^2_{k,k'}\gamma(y_2)$$
$$\omega\left(\rho(y_1)(\psi^+(y_2)\partial^2_{k,k'}\psi(y_2) + \partial^2_{k,k'}\psi^+(y_2)\psi(y_2))\right)]$$
$$+4 \lim_{L,R\to\infty} \frac{e^2}{2L\pi R^2}(2m)^{-4} \iint dy_1 dy_2 \sum_{k,k',j,j'} \partial^2_{k,k'}\gamma(y_1)\partial^2_{j,j'}\gamma(y_2)$$
$$\omega\left([\psi^+(y_1)\partial^2_{k,k'}\psi(y_1) + \partial^2_{k,k'}\psi^+(y_1)\psi(y_1)]\right.$$
$$\left.[\psi^+(y_2)\partial^2_{j,j'}\psi(y_2) + \partial^2_{j,j'}\psi^+(y_2)\psi(y_2)]\right)$$

This can be written in the following way :

$$\frac{e^2}{(2m)^4}\int dy_1 \left(\langle\rho^{(2)}\rangle(y_1) - \rho_B^2 + \delta(y_1)\rho_B\right)$$
$$\lim_{L,R\to\infty} \frac{1}{2L\pi R^2}\int dy_2 \Delta^2\gamma(y_2)\Delta^2\gamma(y_2+y_1) \tag{4.44}$$
$$-\sum_{k,k'} \frac{e^2}{(2m)^4}\int dy_1 \left(\langle\rho^{(2)} p_1^k p_1^{k'}\rangle(y_1) - \rho_B\langle\rho^{(1)}p^k p^{k'}\rangle - \delta(y_1)\langle\rho^{(1)}p^k p^{k'}\rangle\right)$$
$$\cdot \lim_{L,R\to\infty}\int dy_2 \partial^2_{k,k'}\gamma(y_2)\Delta^2\gamma(y_2+y_1) \tag{4.45}$$
$$+\lim_{L\to\infty}\frac{e^2}{(2m)^4}\frac{4}{L}\sum_{k,k',j,j'}\left\{\iint dy_1^r dy_2^r \lim_{R\to\infty}\frac{1}{\pi R^2}\iint dy_1^{\perp r} dy_2^{\perp r}\right.$$
$$\partial^2_{j,j'}\gamma(y_2)\partial^2_{k,k'}\gamma(y_2+y_1)\langle\rho^{(2)}p_1^k p_1^{k'} p_2^j p_2^{j'}\rangle(y_1) \tag{4.46}$$
$$+\int dy^r \lim_{R\to\infty}\frac{1}{\pi R^2}\int dy^{\perp r}\partial^2_{j,j'}\gamma(y)\partial^2_{k,k'}\gamma(y)\langle\rho^{(1)}p^k p^{k'} p^j p^{j'}\rangle \tag{4.47}$$
$$\left. + \text{ integrals with higher order derivatives on the testfunctions}\right\}$$

We already remarked ((4.21) and (4.22)) that in order to contribute to the limit there must not be any derivatives of χ_R. This means that for (4.27) we only need to analyse the limit :

$$\lim \frac{1}{2L}\int dy_2^r \frac{d^4}{dy_2^4}\left(y_2^r h(\frac{y_2^r}{L})\right)\frac{d^4}{dy_2^4}\left((y_2^r+y_1^r)h(\frac{y_2^r+y_1^r}{L})\right)$$

We make a limited Taylor expansion ($0 \le \theta \le 1$) of $(y_2^r + y_1^r)h(\frac{y_2^r+y_1^r}{L}) =$

$y_2^r h(\frac{y_2^r}{L}) + y_1^r(y_2^r + \theta y_1^r)h(\frac{y_2^r+\theta y_1^r}{L})$ and obtain :

$$\lim \frac{1}{2} L^{-6} \int ds \frac{d^4}{ds^4}(sh(s)) \frac{d^4}{ds^4}(sh(s) + y_1^r(s + \frac{\theta y_1^r}{L})h(s + \frac{\theta y_1^r}{L})) \quad \overset{L\to\infty}{\longrightarrow} 0$$

So by dominated convergence, (4.27) does not contribute since the remaining integral is nothing but sumrule (3.10).

By analogous arguments—a finite Taylor expansion, rotation invariance and sumrule (3.14)—one verifies that also the second term (4.28) is zero in the limit.

The third term (4.29) is slightly more complicated. Only the case $k = k' = j = j' = r$ needs to be studied;the other cases are trivially zero as we first take $\lim R \to \infty$. Under the assumption (3.6) on momentum-momentum correlations, one sees that $\langle \rho^{(2)} (p_1^r)^2 (p_2^r)^2 \rangle (y_1^r, y_1^{\perp r})$ is integrable in $y_1^{\perp r}$ for fixed y_1^r. Moreover since

$$\begin{aligned} \int dy^{\perp r} \frac{\partial^2}{\partial {y^r}^2}(|y|^{-1}) &= -4\pi\delta(y^r) \\ &= 0 \text{ for } y^r \neq 0 \end{aligned}$$

the integral

$$\int dy_1^{\perp r} \, \langle \rho^{(2)} (p_1^r)^2 (p_2^r)^2 \rangle (y_1^r, y_1^{\perp r})$$

is integrable in y_1^r Thus, one obtains that

$$\frac{1}{2L} \int dy_1^r \int dy_2^r \frac{d^2}{{dy_2^r}^2} \left(y_2^r h(\frac{y_2^r}{L}) \right) \frac{d^2}{{dy_2^r}^2} \left((y_2^r + y_1^r) h(\frac{y_2^r + y_1^r}{L}) \right)$$
$$\int dy_1^{\perp r} \langle \rho^{(2)} (p_1^r)^2 (p_2^r)^2 \rangle (y_1^r, y_1^{\perp r}) \frac{1}{\pi R^2} \int dy_2^{\perp r} \, \chi_R(y_2^r) \chi_R(y_2^r + y_1^r)$$

tends to zero by dominated convergence.

The other terms (4.30) in that third term are trivially zero.

(b) The second term is :

$$\lim_{L,R\to\infty} \frac{e^2}{2L\pi R^2} \iint dy_1 dy_2\, \gamma(y_1)\gamma(y_2) \frac{-1}{4m^2}$$

$$\{\omega\left([\Delta^2\psi^+(y_1)\psi(y_1) - 2\Delta\psi^+(y_1)\Delta\psi(y_1) + \psi^+(y_1)\Delta\psi(y_1)]\right.$$

$$\left.\nabla\left(\psi^+(y_2)\nabla\left(V * [\rho - \rho_B]\right)(y_2)\psi(y_2)\right)\right) + \text{c.c.}\}$$

It is straightforward to see that this is equal to :

$$\frac{e^2\omega_p^2}{4m^2}\int dy_1 \left(\langle\rho^{(2)}\rangle - \rho_B^2 - \delta(y_1)\rho_B\right) \lim_{L,R\to\infty}\int dy_2 \gamma(y_2)\Delta^2\gamma(y_2+y_1) \quad +$$

$$\frac{2e^2\omega_p^2}{4m^2}\sum_{k,k'}\int dy_1 \left(\langle\rho^{(2)}p_1^k p_1^{k'}\rangle(y_1) - \rho_B\langle\rho^{(1)}p^k p^{k'}\rangle + \delta(y_1)\langle\rho^{(1)}p^k p^{k'}\rangle\right)$$

$$\lim_{L,R\to\infty}\frac{1}{2\pi R^2}\int dy_2\gamma(y_2)\partial^2_{k,k'}\gamma(y_2+y_1) \quad +$$

$$\frac{2e^2}{4m^2}\int dy_2 \frac{y_2}{|y_2|^3}\int dy_1 \left(\langle\rho^{(3)}\rangle(y_1,y_2) - \rho_B\langle\rho^{(2)}\rangle(y_2) + 2\langle\rho^{(2)}\rangle(y_2)\delta(y_1)\right.$$

$$\left. - \rho_B\left[\langle\rho^{(2)}\rangle(y_1) - \rho_B^2 + \delta(y_1)\rho_B\right]\right)$$

$$\lim_{L,R\to\infty}\frac{1}{2L\pi R^2}\int dy_3\Delta\gamma(y_3)\nabla\gamma(y_3+y_2-y_1) \quad +$$

$$\sum_{k,k'}\frac{8e^2}{4m^3}\int dy_2\frac{y_2}{|y_2|^3}\int dy_1\left(\langle\rho^{(3)}p_2^k p_2^{k'}\rangle(y_1,y_2) - \rho_B^2\langle\rho^{(1)}p^k p^{k'}\rangle\right.$$

$$+ \langle\rho^{(2)}p_2^k p_2^{k'}\rangle(y_2)\delta(y_1)\langle\rho^{(2)}p_1^k p_1^{k'}\rangle(y_2)\delta(y_1) - \rho_B\langle\rho^{(2)}p_2^k p_2^{k'}\rangle(y_1)$$

$$\left.-\rho_B^2\langle\rho^{(1)}p^k p^{k'}\rangle + \rho_B\langle\rho^{(1)}p^k p^{k'}\rangle\delta(y_1)\right)$$

$$\lim_{L,R\to\infty}\frac{1}{2L\pi R^2}\int dy_3\partial^2_{k,k'}\gamma(y_3)\nabla\gamma(y_3+y_2-y_1)$$

By the cluster properties (3.1) to (3.4) and by dominated convergence one can take the limits $R\to\infty$ and $L\to\infty$. All the terms yield zero : the first, the third and the fourth by taking the limit and for the second term we get zero by sumrule (3.14).

(c) We now turn back to the terms (4.25)(4.26) to prove that they are zero :

$$\lim_{L,R\to\infty} \frac{e^2}{2L\pi R^2} \iint dy_3 dy_4 \left\{ \omega \left(\psi^+(y_3) \int dy_1 \nabla\gamma(y_1)\nabla V(y_3 - y_1)[\rho(y_1) - \rho_B]\psi(y_3) \right.\right.$$

$$\left.\left. \psi^+(y_4) \int dy_2 \nabla\gamma(y_2)\nabla V(y_4 - y_2)\rho_B\psi(y_4) \right) + \text{c.c.} \right\}$$

$$= \frac{-4\pi e^4 \rho_B}{m^2 2L\pi R^2} \int dy_1 \frac{y_1}{|y_1|^3} \int dy_4 \left(\langle \rho^{(3)} \rangle (y_1, y_4) - \rho_B \langle \rho^{(2)} \rangle (y_4) + \right.$$

$$\left. 2\langle \rho^{(2)} \rangle (y_1)\delta(y_4) - \rho_B \left[\langle \rho^{(2)} \rangle (y_1) - \rho_B^2 + \rho_B \delta(y_1) \right] \right)$$

$$\lim_{L,R\to\infty} \int dy_3 \nabla\gamma(y_3)\gamma(y_3 + y_4 - y_1)$$

By the cluster properties (3.1)(3.2) and by dominated convergence, this is

$$\frac{\omega_p^2 e^2}{m} \int dy_1 \frac{y_1^r}{|y_1|^3} \int dy_4 \left(\langle \rho^{(3)} \rangle (y_1, y_4) - \rho_B \langle \rho^{(2)} \rangle (y_4) + \right.$$

$$\left. 2\langle \rho^{(2)} \rangle (y_1)\delta(y_4) - \rho_B \left[\langle \rho^{(2)} \rangle (y_1) - \rho_B^2 + \rho_B \delta(y_1) \right] \right)$$

which is zero by sumrules (3.11) and (3.10).

The second term (4.26) was :

$$\lim_{L,R\to\infty} \frac{e^2}{m^2 2L\pi R^2} \iint dy_3 dy_4 \omega \left(\psi^+(y_3) \int dy_1 \nabla\gamma(y_1)\nabla V(y_1 - y_3)[\rho(y_1) - \rho_B]\psi(y_3) \right.$$

$$\left. \psi^+(y_4) \int dy_2 \nabla\gamma(y_2)\nabla V(y_2 - y_4)[\rho(y_2) - \rho_B]\psi(y_4) \right)$$

$$= \left(\frac{e}{m}\right)^2 \sum_{k,k'} \int dy_4 \omega \left(\psi^+(o) \int dy_1\, \partial_k V(y_1)[\rho(y_1) - \rho_B]\psi(o) \right.$$

$$\left. \psi^+(y_4) \int dy_2 \partial_{k'} V(y_2)[\rho(y_2 + y_4) - \rho_B]\psi(y_4) \right)$$

$$\lim_{L,R\to\infty} \frac{1}{2L\pi R^2} \int dy_3\, \partial_k \gamma(y_1 + y_3)\partial_{k'}\gamma(y_3 + y_2 - y_4)$$

$$= (\frac{e}{m})^2 \int dy_4\, \omega \left(\left[\psi^+(o) \int dy_1 \partial_r V(y_1)[\rho(y_1) - \rho_B]\psi(o) \right] \right.$$
$$\left. \tau_{y_4} \left[\psi^+(o) \int dy_2 \partial_r V(y_2)[\rho(y_2) - \rho_B]\psi(o) \right] \right)$$

One can show that this expression is zero, as a consequence of (3.9) and of the space-reflection-symmetry of the state. We make a simple substitution $x_1 := -y_1$ and we get :

$$(\frac{e}{m})^2 \int dy_4 \int dx_1 \frac{(-x_1^r)}{|x_1|^3} \omega \left(\psi^+(o)\rho(-x_1)\psi(o) \int dy_2 \frac{y_2^r}{|y_2|^3} \rho(y_2 + y_4)\psi(y_4) \right)$$

Substitute $x_4 := -(y_4 + y_2)$:

$$-(\frac{e}{m})^2 \int dx_1 \frac{x_1^r}{|x_1|^3} \int dy_2 \frac{y_2^r}{|y_2|} \int dx_4$$
$$\omega \left(\psi^+(o)\rho(-x_1)\psi(o)\psi^+(-x_4 - y_2)\rho(-x_4)\psi(-x_4 - y_2) \right)$$
$$= -(\frac{e}{m})^2 \int dx_1 \frac{x_1^r}{|x_1|^3} \int dy_2 \frac{y_2^r}{|y_2|} \int dx_4$$
$$\omega \left(\psi^+(o)\rho(x_1)\psi(o)\psi^+(x_4 + y_2)\rho(x_4)\psi(x_4 + y_2) \right)$$

(by the space-reflection invariance)

$$= 0$$

■

5 Discussion

In this paper we explained for a large class of boundary conditions the mathematical role of the plasmon frequency : it is an eigenvalue of the dynamical system generated by the fluctuation operators. In particular we proved that the average position and momentum fluctuations — at least at high temperatures and low densities — generate a dynamical system, dynamically independent from the other variables of the system. These variables behave dynamically as the momentum and position of a harmonic oscillator with frequency equal to the plasmon frequency.

Technically a first indication towards our results can be found in [ALM; eq.2.34], where dynamical independence of some correlation functions is derived. Earlier work [JLM] revealed heuristic arguments for this.

Furthermore, this paper gives a rigorous and explicit formulation of the plasmons as boson fields and particles. It is explicitly derived that this plasmon field is the average position fluctuation of the center of mass of the system, whereas the conjugate field is the average momentum fluctuation. We showed that there is no need for an artificial supplementary construction or quantization-procedure in order to obtain the quantum plasmon. Our work shows that this macroscopic quantum character is a consequence of the microscopic system.

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