Zeitschrift: Helvetica Physica Acta
Band: 66 (1993)
Heft: 2

Artikel: Renormalization theory for hierarchical models
Autor: Pordt, A.
DOI: https://doi.org/10.5169/seals-116565

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 12.12.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Helv Phys Acta
Vol. 66 (1993)

0018-0238/93/020105-50$1.50+0.20/0
(c) 1993 Birkhäuser Verlag, Basel

Renormalization Theory for Hierarchical Models

A. Pordt[1]

Université de Lausanne
BSP, Institut de Physique Théorique
CH-1015 Lausanne, Switzerland

(9. VII. 1992)

ABSTRACT

A nonperturbative renormalization theory of Euclidean field theories in hierarchical approximation is presented in this paper. The ultraviolet limit is related to the thermodynamic limit of a polymer system on a multigrid. General methods to present a hierarchical model as a multigrid polymer system are discussed. We provide a sufficient condition for the existence of the continuum limit by a bound for Moebius transforms. The use of renormalization conditions for polymer activities is described. We study a recursive calculation scheme for polymer activities. We state a general cluster expansion formula, which is useful for estimating Moebius transforms.

[1] Supported by the Swiss National Science Foundation

1. Introduction

Perturbative renormalization theory deals with divergences in Feynman perturbation expansions of field theoretic models. Renormalization shows the way how to define a "bare" interaction such that the effective theory becomes finite. The approach starts with a regularized model which is finite by definition. The parameter which regularizes the model is called a cutoff. Regularization is achieved by suppressing low momentum modes (infrared cutoff) or high momentum modes (ultraviolet cutoff). For example, we may define an infrared cutoff by restricting the model to a finite volume and an ultraviolet cutoff by putting the model on a lattice. There are various other possibilities to provide a regularized model. The ultimate aim is to remove the cutoff in such a way that the physical quantities of the model stay finite. A model is called ultraviolet or infrared (perturbatively) renormalizable, if it is possible to find a "bare" interaction which is parametrized by a finite number of coupling constants such that the effective interaction stays finite to all orders of perturbation theory when the cutoff is removed. A perturbative renormalization theory was presented by Zimmermann [1] who introduced the notion of "forest" for Feynman graphs. Gallavotti and Nicolò [2–5] invented a perturbative renormalization theory in terms of a tree graph expansion. Each tree graph may be represented by a sum of Feynman graphs. The GN tree graph expansion comes from a successive iteration of truncated expectation values of the interactions.

The GN tree expansion is related to Wilson's renormalization group approach [6–7]. In Wilson's renormalization group approach high momentum modes are integrated out step by step. Each such step is called a renormalization group step. The renormalization group procedure starts with a local "bare" interaction. We may consider the model with this "bare" interaction as a model where ultraviolet and infrared cutoff are equal (and finite). Each renormalization group step lowers the infrared cutoff. The interactions after applying renormalization group steps are called effective interactions. They contain nonlocal terms which are induced by the nonlocality of the free propagator. The idea of Wilson's renormalization group approach is that the range of this nonlocality of the effective interactions is of the order of the momentum range of the fields, which will be integrated out in the next renormalization group step. In other words, the renormalization group method analyzes a system, where an infinite (or a large) number of degrees of

freedom are strongly coupled, by reducing it to a system, where only a finite (or small) number of degrees of freedom are strongly coupled after an infinite (or large) number of renormalization group steps.

Power series expansions for field theoretic models are in general not convergent. For a rigorous construction of field theoretic models one needs convergent expansion methods. Such a method, called phase space cell expansion, was introduced by Glimm and Jaffe [8–9] and further developped and applied to various models [10–13]. A rigorous block spin approach was introduced by Gawędzki and Kupiainen [14–16]. The main technical problem of these methods is to deal with large field configurations (large field problem).

A strategy to avoid divergences, coming from perturbation expansions in powers of coupling constants, is to control the flow of effective Boltzmannians instead of the flow of actions. The action is the logarithm of the Boltzmannian. Introduction of this logarithm in the renormalization group equation is the source of divergences in perturbation expansion. In cluster expansion methods the usual logarithm *log* is replaced by a new logarithm *LOG* (cp. [17,18]). *LOG* is a function which maps partition functions to polymer activities. Polymer systems come from expansion methods of (classical) statistical mechanics and were introduced by Gruber and Kunz [19–20]. Polymer systems can be applied to quantum field theories [21–24]. The idea to introduce a polymer system, is to describe a system with infinite (or a large) number of degrees of freedom by subsystems which contain only a small number of degrees of freedom. The original system can be recovered by performing a thermodynamic limit for these finite subsystems. We see that Wilson's renormalization group method and the method of polymer systems have a similar underlying idea.

Effective interactions are nonlocal, but well localized. This nonlocality of the effective interaction causes the large field problem. Because of the nonlocality of effective interactions suitable definitions of polymer systems are complicated [25]. To avoid such problems, coming from nonlocalities, we consider here only the renormalization theory of hierarchical models. A hierarchical model [26–28] is simply defined by omitting all nonlocal terms in the effective interactions. Effective interactions for models in hierarchical approximation are by definition local and each renormalization group step concerns only a small number of degrees of freedom. A further reason to study a renormalization theory of hierarchical models is that renormalization can be studied in its pure form.

Polymers for a field theory are subsets of a special space. This space will be the multigrid. The use of multigrids for field theory were introduced by Mack and Pordt [21–24] (cp. also [17] for hierarchical Φ^4-models and complete massless lattice Φ^4_4-model). In the following, we shall explain what a multigrid is.

Consider a d-dimensional hypercube x^* with unit side length in $\mathbf{R}^d$. Divide this hypercube x^* into L^d, $L \in \{2, 3, \ldots\}$, hypercubes $y_1, \ldots, y_{L^d}$ which have equal side length L^{-1}. Furthermore, divide all obtained hypercubes $y_i, i \in \{1, \ldots, L^d\}$ into hypercubes which have equal side length L^{-2} and so on. After the nth step we obtain a split-up of the hypercube x^* into hypercubes with side length L^{-n}. Representing the hypercubes by its center points, we obtain after n steps a lattice Λ_n with lattice spacing L^{-n}. The disjoint union of all these lattices Λ_n, where the lattice points represent hypercubes, is called a *multigrid* $\Lambda = \Lambda_0 + \Lambda_1 + \Lambda_2 + \ldots$ with *base space* $x^* \subset \mathbf{R}^d$. The lattice Λ_n is called the *nth layer* of the multigrid. For $y \in \Lambda_{n+1}$ and $y' \in \Lambda_n$, a relation $y \underline{\in} y'$ is given, if and only if the hypercube y is contained in y'. The multigrid $\Lambda = \Lambda_0 + \Lambda_1 + \ldots$ can be used for a definition of a polymer system for a continuum field theory on a finite volume. The corresponding multigrid for a theory on a finite volume and with ultraviolet cutoff would be the multigrid $\Lambda_{\leq N} = \Lambda_0 + \Lambda_1 + \ldots + \Lambda_N$.

For infrared problems we consider the following multigrid. Split $\mathbf{R}^d$ into hypercubes x with unit side length. Denote the corresponding lattice by Λ_0. Define Λ_{-1} by disjoint hypercubes with side length L which contain L^d hypercubes of Λ_0. In this way, define Λ_{-n} for all $n \in \mathbf{N}$. Then, we obtain the multigrid $\Lambda = \Lambda_0 + \Lambda_{-1} + \Lambda_{-2} + \ldots$ with base space $\mathbf{Z}^d$. Define Λ_n, for all $n > 0$, by the splitting procedure, for all hypercubes of Λ_0, which was described above. The multigrid $\Lambda = \sum_{j \in \mathbf{Z}} \Lambda_j$ with base space $\mathbf{R}^d$ is considered for models where ultraviolet and infrared cutoffs are both removed.

The "bare" action of a field theoretic model consists of the inverse free propagator and a local "bare" interaction. The free propagator is splitted up into high and low frequency (momentum) parts. Correspondingly, fields can be splitted into high and low frequency parts. After this split-up of the free propagator, all couplings in a field theoretic model come from the nonlocality of the free propagator parts and the coupling of different high and low frequency parts of the fields. In a hierarchical model approximation the free propagator parts are replaced by local operators. Then, we have to deal only with couplings which come from the different frequency parts of the propagators and fields. These couplings of different

frequency parts are responsible for ultraviolet resp. infrared divergences in quantum field theory. In multigrid language, the couplings of blocks $x, y \in \Lambda$, where y is contained in x and y is very small compared to x, and the large number of such blocks y, are responsible for ultraviolet and/or infrared divergences in field theory.

To formulate a field theoretic model as a multigrid polymer system, one has to define polymers and partition functions of polymers. The ultraviolet and/or infrared limit can be considered as a thermodynamic limit of a multigrid polymer system. The definition of a multigrid polymer system is recursive, starting with the smallest polymers. Partition functions of polymers are defined in such a way that they obey renormalization group equations. Polymer activities for hierarchical models are Moebius transforms of the partitions function. When the polymer system is well-defined, the corresponding polymer activities are small for large polymers. By *large* polymers we mean polymers which contain a large number of elements or polymers of large extension. The polymer activities are proportional to a power of a small constant, where the power is proportional to the number of elements in the polymer. This constant depends on the coupling constants and is small for weakly coupled models.

The property that polymer activities are small for polymers with large extension can be achieved by imposing renormalization conditions on partition functions respectively polymer activities.

This paper is organized as follows. Section 2 discusses the geometry of a multigrid and provides definitions of multigrid polymer sets, which will be useful in later sections. Furthermore, a sufficient condition for the existence of the thermodynamic limit of multigrid polymer systems is presented in proposition 2.9.

Two versions of defining multigrid polymer systems for hierarchical models are studied in section 3. In the first version, the partition fuctions for polymers are explicitly cutoff-dependent. To show the existence of the thermodynamic limit, in this version, one has firstly to remove the cutoff and then to show that the sum over all polymers stays finite. In the second version, no explicit cutoff is required. The cutoff comes only from the finiteness of the polymers. In this case, there are no "bare" coupling constants introduced a priori. This is achieved by introducing a polymer-dependent function (renormalization group flow function), which relates the corresponding "bare" and effective coupling constants of one renormalization group step. Like in the renormalization of perturbation expansions, where it is nec-

essary only to renormalize divergent parts of Feynman graphs, it is not necessary to renormalize all polymer activities in a multigrid polymer system. Considering this item, we introduce the notion of (n)- renormalization part polymer systems.

Section 4 studies the formulation of renormalization conditions (by introducing operators $\mathcal{L}$) for multigrid polymer systems. These renormalization conditions determine the renormalization group flow functions. Therefore, they are part of a definition of a multigrid polymer system. As examples for renormalization conditions, we consider hierarchical approximations of the Φ^4-theory (cp. [17]) and the twodimensional $O(N)$ σ-model (cp. [29–31]).

Section 5 is the technical core of the paper. It describes in four steps, how to compute recursively Moebius transforms of multigrid partition functions. We will do this for two kinds of polymer systems.

Finally, section 6 presents an explicit representation of Moebius transforms by using cluster expansion formulas. Proposition 6.1 states a general cluster expansion formula for Moebius transforms. Corollary 6.4 gives a cluster expansion formula for Moebius transforms, which are introduced in section 5. This completes the recursive procedure for calculating Moebius transforms of multigrid partition functions.

We shall use the following set-theoretic conventions in this paper: $\mathbf{N} := \{0, 1, 2, \ldots\}$, $\mathbf{R} :=$ set of all real numbers, $\mathbf{C} :=$ set of all complex numbers, $\mathbf{R}_+ :=$ set of all non-negative real numbers, $+ :=$ union of disjoint sets, $\exists :=$ exists, $|X| :=$ number of elements in the set X, $\emptyset :=$ empty set, $\subseteq :=$ subset or equal, $\subset :=$ subset, but not equal, and the Kronecker delta defined by

$$\delta_{X,Y} := \begin{cases} 1, & \text{if } X = Y; \\ 0, & \text{otherwise,} \end{cases}$$

for all sets X and Y. We write "iff" for "if and only if".

2. Hierarchically ordered sets, entropy factors and tree graphs

Before we represent a field theoretic model by a polymer system on a multigrid, we have to discuss the geometric structure of a multigrid and have to define sets of polymers. This is done, in this section, by introducing some definitions and notations. We present here three different kinds of polymers. The special feature of describing hierarchical models is revealed by the fact that all polymers defined

in this section contain only one maximal hypercube. We can define an equivalence relation of the polymer set, such that each equivalence class is related to a tree graph. These tree graphs for polymers correspond to tree graphs introduced by Gallavotti and Nicolò [2–5]. The difference of these two kinds of tree graphs is that the Gallavotti-Nicolò tree graphs are part of a perturbation expansion, while the tree graphs introduced here are part of a nonperturbative expansion.

We start this section by defining hierarchically ordered sets. We shall see that the multigrid $\Lambda = \Lambda_0 + \Lambda_1 + \ldots$ with base space $x^* \subset \mathbf{R}^d$ (x^* = unit hypercube) is an example of a hierarchically ordered set.

Definition 2.1. *Let Λ be a denumerable set with a relation $\underline{\in}$ such that there exists $x^* \in \Lambda$ with $x^* \not\underline{\in} y$, for all $y \in \Lambda$, and for all $y \in \Lambda - \{x^*\}$, there exists one and only one element $y' \in \Lambda - \{y\}$ such that $y \underline{\in} y'$. Furthermore, for all $y \in \Lambda$, there exists $n \in \mathbf{N}$ such that*

$$y = y_0 \underline{\in} y_1 \underline{\in} y_2 \underline{\in} \cdots \underline{\in} y_{n-1} \underline{\in} y_n = x^*,$$

where $y_a \neq y_b$ if $a \neq b$, $a, b \in \{0, \ldots, n\}$. Then $(\Lambda, \underline{\in})$ is called a hierarchically ordered set (h.o.s.).

The unique element y' with $y \underline{\in} y'$ is denoted by $[y]$. We will suppose that, for all $y \in \Lambda$, the number of elements in

$$\overline{y} := \{y' \mid y' \underline{\in} y\} \tag{2.1}$$

is finite and equal to L^d, $L \in \{2, 3, \ldots\}$, $d \in \{1, 2, \ldots\}$. This is the case for the multigrid Λ with base space $x^* \subset \mathbf{R}^d$ (x^*= unit hypercube), previously defined in the introduction.

The relation $\underline{\in}$ induces an (partial) order relation $\prec$ in Λ, in the following way. Let us recall that $\prec$ is called a partial ordering of Λ, iff (a) $x \preceq x$, (b) $x \preceq y$ and $y \preceq z$ imply $x \preceq z$, and (c) $x \preceq y$ and $y \preceq x$ imply $x = y$ for all $x, y, z \in \Lambda$. For $y, y' \in \Lambda$, define $y \prec y'$, iff $y \underline{\in} y'$ or there exists $y_1, \ldots, y_n \in \Lambda$, such that

$$y \underline{\in} y_1 \underline{\in} \cdots \underline{\in} y_n \underline{\in} y'.$$

The order relation $\prec$ is called the *vertical order relation* in Λ. Obviously, $\max(\Lambda)$:= set of all maximal elements in $\Lambda = \{x^*\}$.

For all $n \in \mathbf{N}$ and a h.o.s. $(\Lambda, \underline{\in})$, we define $\Lambda_n \subset \Lambda$ recursively. Define $\Lambda_0 := \{x^*\}$ and for $n \geq 1$,

$$\Lambda_n := \{y \in \Lambda |\ \exists x \in \Lambda_{n-1}\colon\ y \underline{\in} x\}. \tag{2.2}$$

Λ_n is called the *nth layer* of Λ. Λ is equal to the disjoint union of all layers $\Lambda_n, n \in \mathbf{N}$. We obtain the following split-up

$$\Lambda = \sum_{n \geq 0} \Lambda_n. \tag{2.3}$$

Obviously, $|\Lambda_n| = L^{nd}$. $|X|$ denotes the number of elements in a set X.

For $x \in \Lambda$, define its *frequency index* $j(x) \in \mathbf{N}$ by $x \in \Lambda_{j(x)}$, i.e. x is contained in the $j(x)$th layer of Λ.

Definition 2.2. *Let $(\Lambda, \underline{\in})$ be a h.o.s.. A subset X of Λ is called convex, iff $(X, \underline{\in})$ is a h.o.s..*

The name convex is motivated by the following lemma.

Lemma 2.3 . *A subset X of Λ is convex, iff*
(a) $|\max(X)| = 1$
(b) For all $y, y' \in X$ with $y \prec y'$, the following condition holds: if $y'' \in \Lambda$ such that $y \prec y'' \prec y'$, then $y'' \in X$.

Proof: (i) Suppose that X is convex. By definition 2.2 there exists a unique element $x \in X$, such that $\max(X) = \{x\}$. This proves (a). Consider $y, y' \in X$ with $y \prec y'$ and $y'' \in \Lambda$ such that $y \prec y'' \prec y'$. By definition 2.1, there exist $y_1, \ldots, y_n \in X$, such that $y \underline{\in} y_1 \underline{\in} \ldots \underline{\in} y_n \underline{\in} y'$ and the elements $y_1, \ldots y_n$ are uniquely determined. Then, the definition of $\prec$ shows that there exists an $i \in \{1, \ldots, n\}$, such that $y'' = y_i \in X$. This proves (b).
(ii) Suppose that (a) and (b) hold. By (a) there exists $x \in X$ with $\max(X) = \{x\}$. Consider $y \in X - \{x\}$. We have to prove that $[y] \in X$. This is obvious, if $[y] = x$. Suppose that $[y] \neq x$. Then, we have $y \prec [y] \prec x$. Since $y, x \in X$, we see that (b) implies $[y] \in X$. □

In the following, we introduce polymers on Λ in a general way. For a set Λ, define the set $\mathcal{P}_{fin}(\Lambda)$ of all finite subsets of Λ

$$\mathcal{P}_{fin}(\Lambda) := \{X |\ X \subseteq \Lambda, |X| < \infty\}. \tag{2.4}$$

A *set of polymers* $\mathcal{P}$ (in Λ) is a subset of $\mathcal{P}_{fin}(\Lambda) - \{\emptyset\}$. We use the following definitions for $X, X' \subseteq \Lambda$, $X' \subseteq X$

$$\begin{aligned} \mathcal{P}(X) :=& \{Y \in \mathcal{P} |\ Y \subseteq X\}, \\ \mathcal{P}(X, X') :=& \{Y \in \mathcal{P} |\ X' \subseteq Y \subseteq X\}. \end{aligned} \tag{2.5}$$

For example, for a multigrid Λ, we may define the set of all polymers $\mathcal{P}$, consisting of all finite convex subsets X of Λ. We call these polymers *convex-polymers*. All convex sets are nonempty. Therefore, all convex-polymers are nonempty. Furthermore, for all $x \in \Lambda$, define

$$\mathcal{P}_x := \{X \in \mathcal{P} |\ \max(X) = \{x\}\}. \tag{2.6}$$

Then $\mathcal{P}_x$ may also be considered as a set of polymers. For a subset $X \subseteq \Lambda$ and $y \in \Lambda$, let us introduce the notation

$$X_y := \{y' \in X |\ y' \preceq y\}. \tag{2.7}$$

We will define recursively an equivalence relation $\sim$ on the set $\mathcal{P}_1 := \{X \subseteq \Lambda |\ |X| < \infty,\ |\max(X)| = 1\}$. Suppose that $<$ is a total order relation of Λ. Let us recall that $<$ is a total ordering iff $<$ is a partial order relation and $x \leq y$ or $y \leq x$ for all $x, y \in \Lambda$. Since Λ is denumerable, there exists always a total ordering $<$ of Λ. Consider $X, X' \in \mathcal{P}_1$ with

$$\begin{aligned} \max(X) &= \{x\}, \\ \max(X') &= \{x'\}, \\ \max(X - \{x\}) &= \{y_1, \ldots, y_m\}, \\ \max(X' - \{x'\}) &= \{y'_1, \ldots, y'_{m'}\}, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} y_1 < \ldots < y_m, \\ y'_1 < \ldots < y'_{m'}. \end{aligned} \tag{2.9}$$

Then the sets $X, X' \in \mathcal{P}_1$ obey the following equations

$$\begin{aligned} X &= \{x\} + \sum_{a=1}^{m} X_{y_a}, \\ X' &= \{x'\} + \sum_{a=1}^{m'} X'_{y'_a}. \end{aligned} \tag{2.10}$$

We call X and X' equivalent, $X \sim X'$, iff

$$m = m',\ j(x) = j(x'),\ j(y_a) = j(y'_a) \tag{2.11}$$

and

$$X_{y_a} \sim X'_{y'_a}, \tag{2.12}$$

for all $a \in \{1, \ldots, m\}$. Denote the equivalence class of $X \in \mathcal{P}_1$ by $[X]$.

An *extended equivalence relation* $\sim_e$ is recursively defined as follows. Consider X and X' as above. $X \sim_e X'$, iff there exists $h \in \mathbf{Z}$ such that

$$m = m',\ j(x) = j(x') + h,\ j(y_a) = j(y'_a) + h \tag{2.13}$$

and

$$X_{y_a} \sim_e X'_{y'_a}, \tag{2.14}$$

for all $a \in \{1, \ldots, m\}$.

For a finite subset X of Λ with $|\max(X)| = 1$, define the *entropy factor* $E(X)$ by the number of elements in the equivalence class $[X]$

$$E(X) := |\{X' \subseteq \Lambda |\ X' \sim X\}|. \tag{2.15}$$

For instance, $E(\{x^*, y\}) = L^{j(y)d}$, for all $y \in \Lambda$ and $x^* = \max(\Lambda)$, $j(x^*) = 0$. We see that $E(\{x^*, y\})$ can be very large for very large $j(y)$, i.e. very small hypercubes y. The largeness of the entropy factor $E(X)$, for all polymers X, which contain elements x, y with $|j(x) - j(y)|$ large, is the reason for the divergence of multigrid polymer systems. This divergence is related to the infrared or ultraviolet divergence in field theory.

Let $\mathcal{P}$ be the set of all convex-polymers. For $x \in \Lambda$, we have an equivalence relation $\sim$ on $\mathcal{P}_x$ as previously defined. The equivalence class for $X \in \mathcal{P}_x$ may be represented by a tree graph $\tau(X)$, i.e.

$$X \sim X' \iff \tau(X) = \tau(X'), \tag{2.16}$$

for all $X, X' \in \mathcal{P}_x$. We consider here only "plane" trees. Trees which differ by a reordering of distinct subtrees are distinct. Then tree graphs may be considered as embedded in a plane. To define the tree graph $\tau(X)$, we have to specify the vertices and lines of $\tau(X)$. Define the *core* of $X \in \mathcal{P}_x$ by

$$core(X) := \{x\} + \{y \in X - \{x\} |\ |\overline{y} \cap X| \neq 1\}, \tag{2.17}$$

i.e. $y \in core(X)$, iff $y = x$ or $y \in X$ and there exists *not* one and only one $y' \in X$ with $y' \underline{\in} y$. The vertices of $\tau(X)$ corresponds to the elements of $core(X)$. For $y, y' \in core(X)$ with $y \prec y'$, there is a line (yy'), iff there exists no $y'' \in core(X)$, such that $y \prec y'' \prec y'$.

For each vertex y of $\tau(x)$, assign a number $n \in \mathbf{N}$ defined by $n = j(y)$. This mapping is called the *momentum assignment* of τ. We consider $x = \max(X)$ as the root of $\tau(X)$ and if y' is a successor of y, then $j(y) < j(y')$. The fact that equivalence classes $[X]$ can be represented by tree graphs $\tau(X)$ is stated in

Lemma 2.4 . *For all $X, X' \in \mathcal{P}_x$, we have*

$$X \sim X' \iff \tau(X) = \tau(X'). \tag{2.18}$$

Denote by $\mathcal{L}(\tau), \mathcal{V}(\tau)$ the set of all lines resp. vertices of τ. For $l = (vv') \in \mathcal{L}(\tau)$, define the *length* of line l by $|l| := |j(v) - j(v')|$.

For a subset $X \subseteq \Lambda$, define the *boundary* of X by

$$\partial X := \{y \in X | \ \nexists y_1 \in X : y \underline{\in} y_1 \ \textit{or} \ \nexists y_2 \in X : y_2 \underline{\in} y\}, \tag{2.19}$$

i.e. ∂X consists of all minimal and maximal elements of X. The *interior* of X is defined by

$$int(X) := X - \partial X. \tag{2.20}$$

The following lemma is a direct consequence of the foregoing definitions.

Lemma 2.5 . *Let $\mathcal{P}$ be the set of all convex-polymers. For all $X, X_1, X_2 \in \mathcal{P}$, we have*

$$\begin{aligned} &\partial X_1 = \partial X_2 \Rightarrow X_1 = X_2 \\ &\partial X \subseteq core(X) \\ &core(X_1) = core(X_2) \Rightarrow X_1 = X_2. \end{aligned} \tag{2.21}$$

We see that a convex-polymer P is uniquely determined by smaller sets like its core or even its boundary. Therefore, some elements of a convex polymer P are redundant for the definition of P. Furthermore, we see that all interior elements of P are irrelevant for the determination of P. But we shall see later on that not all interior elements of P are irrelevant in a renormalization point of view. The minimal subset of P, which is necessary for renormalization, is the core set of P, $core(P)$. Therefore, introduce the set of all *core-polymers* by

$$\mathcal{P}_c := \{P \subseteq \Lambda | \ \exists X \in \mathcal{P} : P = core(X)\}. \tag{2.22}$$

In analogy to the definition (2.6), define for all $x \in \Lambda$

$$\mathcal{P}_{c,x} := \{X \in \mathcal{P}_c |\ \max(X) = \{x\}\}. \tag{2.23}$$

We see that $X \in \mathcal{P}_{c,x}$ and $y \in int(X)$ imply $X - \{y\} \notin \mathcal{P}_c(X)$. Therefore, in order to construct all elements of $\mathcal{P}_c(X) := \{Y \in \mathcal{P}_c |\ Y \subseteq X\}$ from X, we have to eliminate step by step all minimal elements of X.

By lemma 2.5, there exists for $P \in \mathcal{P}_c$ one and only one $X \in \mathcal{P}$ with $core(X) = P$. We call this polymer X the *hull* of the core-polymer P, $X = hull(P)$. Obviously, we have for the mappings $core: \mathcal{P} \to \mathcal{P}_c$ and $hull: \mathcal{P}_c \to \mathcal{P}$

$$\begin{aligned} core \circ hull &= \mathbf{1}|_{\mathcal{P}_c} \\ hull \circ core &= \mathbf{1}|_{\mathcal{P}}. \end{aligned} \tag{2.24}$$

Thus *core* and *hull* are bijective mappings and $core = (hull)^{-1}$.

For all $y, x \in \Lambda$ with $y \prec x$, define the *tower*

$$[y, x] := \{y' \in \Lambda |\ y \preceq y' \preceq x\}. \tag{2.25}$$

The tower $[y, x]$ is linearly ordered by $\prec$. We say that the tower $[y, x]$ has *length*

$$l([y, x]) := j(y) - j(x). \tag{2.26}$$

For core-polymers P, the number $k = l([y, x])$, where $y, x \in P$ and $int([y, x]) \cap P = \emptyset$, can be arbitrarily large. Imposing for this number $k \leq n$, leeds to the following definition of a third kind of polymers. These polymers will be called (n)-renormalization part-polymers. To define this kind of polymers, we start by defining a special subset of the set of all convex-polymers $\mathcal{P}$ in

Definition 2.6. *A polymer $X \in \mathcal{P}_x, x \in \Lambda$, is called a renormalization part of length n (≥ 1), iff there exists $y \in core(X)$ such that $l([y, x]) = n$ and*

$$X = [[y], x] + X_y. \tag{2.27}$$

The set of all renormalization parts of length n is denoted by $\mathcal{P}_n^{(ren)}$. We will use the notation

$$\mathcal{P}_{\geq n}^{(ren)} := \bigcup_{m:m \geq n} \mathcal{P}_m^{(ren)}. \tag{2.28}$$

For all $n \in \{1, 2, \ldots\}$, define a mapping $h_n \colon \mathcal{P} \to \mathcal{P}_{fin}(\Lambda)$ by recursion. Consider $X \in \mathcal{P}_x, x \in \Lambda$. If $X \in \mathcal{P}^{ren}_{\geq n}$ and $X = [[y], x] + X_y$ with $l([y, x]) \geq n$, then define

$$h_n(X) := \{x\} + h_n(X_y) \tag{2.29}$$

and if $X \in \mathcal{P}_x - \mathcal{P}^{ren}_{\geq n}$, define

$$h_n(X) := \{x\} + \sum_{\substack{y \in \overline{x}: \\ X_y \neq \emptyset}} h_n(X_y). \tag{2.30}$$

Start the recursive definition by

$$h_n(\{x\}) := \{x\}, \tag{2.31}$$

for all $x \in \Lambda$. We say that a polymer $X \in \mathcal{P}$ contains no renormalization parts of length $\geq n$, iff $h_n(X) = X$. We see that the mapping h_n is injective.

The mapping h_n does the following. Consider a tower $[y, y'] \subseteq X \in \mathcal{P}$, $y, y' \in core(X)$ and suppose that there is no $y'' \in core(X)$ with $j(y) = j(y'')$, $y \neq y''$ and $[y'', y'] \subseteq X$. If $l([y, y']) \geq n$, then eliminate all elements of $int([y, y'])$ from X. Doing this for all such towers $[y, y'] \subseteq X$, we obtain $h_n(X)$.

For $x \in \Lambda$, we may also define $h_n(\mathcal{P}_x)$ as the image set of $\mathcal{P}_x$. Obviously, we have $h_n(\mathcal{P}_x) = h_n(\mathcal{P})_x$.

Consider the elements of the image set $h_n(\mathcal{P})$ as a new kind of polymers. These polymers are called *(n)-renormalization part-polymers.*

Since *core* and $h_n \colon \mathcal{P} \to h_n(\mathcal{P})$ are bijective mappings, we can also define the entropy factor $E(X)$, the equivalence relation $\sim$ and the tree graph $\tau(X)$, for all core-polymers resp. (n)-renormalization part polymers X.

An upper bound for the entropy factor is given by the following

Lemma 2.7 . *For all $X \in \mathcal{P}_x$, we have*

$$E(X) \leq \prod_{l \in \mathcal{L}(\tau(X))} L^{|l|d}. \tag{2.32}$$

In this section we have defined three examples of polymers. The notion of a Moebius transform for these kinds of polymers is important for the following sections. For general polymer sets $\mathcal{P}$, define a $\mathcal{P}$-Moebius transform by

Definition 2.8. *Let $\mathcal{P}$ be a general set of polymers. For a complex-valued function $B: \mathcal{P} \to \mathbf{C}$, we call $\widetilde{B}: \mathcal{P} \to \mathbf{C}$ the $\mathcal{P}$-Moebius transform of B, if for all $X \in \mathcal{P}$*

$$B(X) = \sum_{P \in \mathcal{P}(X)} \widetilde{B}(P). \tag{2.33}$$

When it is evident from the context, we write Moebius transform instead of $\mathcal{P}$-Moebius transform. Rewriting eq. (2.33) as

$$\widetilde{B}(X) = B(X) - \sum_{\substack{P \in \mathcal{P}(X): \\ |P| < |X|}} \widetilde{B}(P), \tag{2.34}$$

we see that the Moebius transform $\widetilde{B}$ is uniquely defind by eq. (2.33).

When speaking of a Moebius transform in cases of the set of all convex-polymers $\mathcal{P}$, the set of all core-polymers $\mathcal{P}_c$ and the set of all (n)-renormalization part polymers, we mean in the following the $\mathcal{P}_x$-, $\mathcal{P}_{c,x}$- and $h_n(\mathcal{P})_x$-Moebius transform, where $x \in \Lambda$. For instance, the Moebius transform $\widetilde{B}$ of $B: \mathcal{P} \to \mathbf{C}$ is defined by

$$B(X) = \sum_{P \in \mathcal{P}_x(X)} \widetilde{B}(P), \tag{2.35}$$

for all $X \in \mathcal{P}_x$.

Let us remark that in the following definitions and proposition the term polymer may be replaced by the terms convex-polymer, core-polymer and (n)-renormalization part-polymer.

The formal thermodynamic limit of $B: \mathcal{P} \to \mathbf{C}$ is given, for all $x \in \Lambda$, by

$$B(\Lambda_x) := \lim_{X \nearrow \Lambda_x} B(X) := \sum_{P \in \mathcal{P}_x} \widetilde{B}(P). \tag{2.36}$$

To show the existence of this limit, we will show that the series on the rhs of eq. (2.36) is absolutely convergent. In the next proposition, we shall present a sufficient condition for the existence of the thermodynamic limit. For $P \in \mathcal{P}_x$, define the *degree* of P by

$$deg(P) := |\min(P)|. \tag{2.37}$$

Consider the function $B: \mathcal{P}_x \to \mathbf{C}$. We call B *$\sim$-invariant*, iff $P \sim P'$ implies $B(P) = B(P')$. We have the following relations

$$|\mathcal{V}(\tau(P))| = |\mathcal{L}(\tau(P))| + 1, \qquad deg(P) \geq \frac{1}{2}|\mathcal{V}(\tau(P))|. \tag{2.38}$$

Proposition 2.9. *Let $q : \mathbf{N} \to \mathbf{R}_+$ be positive function and λ be a positive constant. Suppose that $B: \mathcal{P}_x \to \mathbf{C}$ is $\sim$-invariant and the following bound for the Moebius transform $\widetilde{B}$*

$$|\widetilde{B}(P)| \leq [\prod_{l \in \mathcal{L}(\tau(P))} q(|l|)]\ \lambda^{deg(P)} \tag{2.39}$$

holds, for all $P \in \mathcal{P}_x$. Define $C(L,q) := \sum_{i=1}^{\infty} L^{di} q(i)$ and suppose

$$C(L,q)\lambda^{1/2} \leq 1/4. \tag{2.40}$$

Then the thermodynamic limit of B exists and we have the following bound

$$|B(\Lambda_x)| \leq 2\lambda^{1/2}. \tag{2.41}$$

Proof: Using $\sim$-invariance, we obtain

$$\sum_{P \in \mathcal{P}_x} |\widetilde{B}(P)| = \sum_{\tau} \sum_{\substack{P \in \mathcal{P}_x: \\ \tau(P)=\tau}} |\widetilde{B}(P)| = \sum_{\tau} E(\tau)\ |\widetilde{B}(\tau)|. \tag{2.42}$$

With help of lemma 2.4 and lemma 2.7, bounds (2.39), (2.42) and (2.38), we get

$$\sum_{P \in \mathcal{P}_x} |\widetilde{B}(P)| \leq \sum_{\tau} [\prod_{l \in \mathcal{L}(\tau)} L^{d|l|} q(|l|)]\ \lambda^{\frac{1}{2}|\mathcal{V}(\tau)|}. \tag{2.43}$$

Define, for all $k \in \{1, 2, \ldots\}$,

$$\widetilde{I}_k := \sum_{\tau: |\mathcal{V}(\tau)|=k} [\prod_{l \in \mathcal{L}(\tau)} L^{d|l|} q(|l|)]\ \lambda^{\frac{1}{2}k} \tag{2.44}$$

and for all $n \in \{1, 2, \ldots\}$,

$$I_n := \sum_{k: 1 \leq k \leq n} \widetilde{I}_k. \tag{2.45}$$

By eq. (2.43), we have to prove that

$$I_n \leq 2\lambda^{\frac{1}{2}} \tag{2.46}$$

holds, for all $n \in \{1, 2, \ldots\}$. This is done by induction. Obviously, $\widetilde{I}_1 = I_1 = K\lambda^{\frac{1}{2}}$. Suppose that eq. (2.46) holds for all $n \in \{1, \ldots, N-1\}$. Split up the tree τ into its subtrees $\tau_1, \ldots, \tau_m$, we see that for all $n \geq 2$,

$$\widetilde{I}_n \leq \sum_{m \geq 1} \sum_{\substack{k_1, \ldots, k_m: \\ \sum k_i = n-1}} \sum_{l_1, \ldots, l_m \geq 1} \lambda^{\frac{1}{2}} [\prod_{i=1}^{m} L^{d|l_i|} q(|l_i|) \widetilde{I}_{k_i}]. \tag{2.47}$$

Thus, using eq. (2.40) , we obtain

$$
\begin{aligned}
I_N &\leq \lambda^{\frac{1}{2}} + \sum_{m=1}^{\infty} \sum_{\substack{k_1,\ldots,k_m: \\ \sum k_i = n-1}} \sum_{l_1,\ldots,l_m \geq 1} \lambda^{\frac{1}{2}} [\prod_{i=1}^{m} L^{d|l_i|} q(|l_i|)] I_{N-1}^m \\
&= \lambda^{\frac{1}{2}} + \sum_{m=1}^{\infty} \lambda^{\frac{1}{2}} \left(C(L,q) I_{N-1} \right)^m \leq 2\lambda^{\frac{1}{2}}.
\end{aligned}
\tag{2.48}
$$

This proves the assertion. □

As we have seen in the proof of proposition 2.9, all polymers P in formulas containing $\sim$-invariant functions $B : \mathcal{P} \to \mathbf{C}$ can be replaced by trees $\tau(P)$. In the next sections, we shall formulate all relations in terms of polymers. We could always replace these relations in terms of tree graphs for $\sim$-invariant functions B.

In this section, we have defined polymer sets on a multigrid Λ. The aim of the next section is to represent hierarchical models by multigrid polymer systems.

3. Renormalization group equations and polymer systems

Hierarchical renormalization group equations are related to polymer systems on the multigrid. The definition of a polymer system is in no way unique. We present in this section two definitions of a polymer system. The first one needs an explicit definition of the bare coupling constants. In the second definition of a polymer system bare coupling constants will be polymer-dependent and recursively defined. This is in analogy with Gallavotti and Nicolò [2–5] where the bare coupling constants are expressed as a formal power series of effective (renormalized) coupling constants.

Let $\mathcal{O} \subseteq \mathbf{R}^m$ be a space of parameters (coupling constants), which will not be further specified here. Consider parameters $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{O} \subseteq \mathbf{R}^m$ and assign to each λ a function Z_λ of M-component fields

$$
\lambda \mapsto Z_\lambda : \ \mathbf{R}^M \to \mathbf{R}_+ := \{a \in \mathbf{R} |\ a \geq 0\}. \tag{3.1}
$$

The choice $\dot{\mathbf{R}}^M$ for the field space is not essential for the following discussion. Let us remark that the choice of $\mathcal{O}$ will depend on the model defined by the function Z_λ. How the region $\mathcal{O}$ depends on Z_λ is an important question, which will not be answered in this paper.

In the following, we shall define the hierarchical renormalization group equations in d dimensions. To be specific, choose a special hierarchical renormalization group equation introduced by Gallavotti et al [28]. But all results stated here do not depend on the special manner of defining a hierarchical approximation. For $N \in \mathbf{N}$ and $\gamma \geq 0$, define recursively $Z_k^{(N)}$ by

$$Z_k^{(N)}(\Psi) := c_{0,k}^{(N)} \left[\int d\mu_\gamma(\Phi) Z_{k+1}^{(N)}(\Phi + L^{1-d/2}\Psi) \right]^{L^d}, \tag{3.2}$$

for $k \in \{0, \ldots, N-1\}$. The recursion starts with

$$Z_N^{(N)}(\Psi) := Z_{\lambda_N}(\Psi), \tag{3.3}$$

for a parameter $\lambda_N \in \mathcal{O}$. We call $Z_{\lambda_N}(\Psi)$ the *bare partition function.* $d\mu_\gamma(\Phi)$ is the Gaussian measure with mean zero. It is defined by

$$d\mu_\gamma(\Phi) := (2\pi\gamma)^{-M/2} \; d^M\Phi \; \exp\{-\frac{1}{2\gamma}\Phi^2\}. \tag{3.4}$$

We have supposed here that the Gaussian measures of $Z_k^{(N)}$ exist. $c_{0,k}^{(N)}$ are normalization constants which will not be further specified.

The ultraviolet limit is defined by choosing parameters λ_N, such that the following limit

$$\lim_{N\to\infty} Z_k^{(N)}(\Psi) =: Z_k(\Psi) \tag{3.5}$$

exists, for all $k \in \mathbf{N}, \Psi \in \mathbf{R}^M$. For a given N, the parameters λ_N are called *bare coupling constants* and $Z_k^{(N)}$ is called *effective partition function* with *ultraviolet cutoff* N. The *effective interaction* is defined by $V(\Psi) = -\ln Z(\Psi)$.

We study two versions to define a polymer system, which represent the renormalization equations eqs. (3.2) and (3.3) of a model defined by the function Z_λ. In the first version, we need an explicit ultraviolet cutoff, which is defined by imposing for polymers that they are contained in the first N layers of the multigrid. In the second version, we do not need an explicit ultraviolet cutoff. Throughout this section, when we speak of polymers, we mean convex-polymers.

3.1. Polymer system defined by an explicit ultraviolet cutoff

Define, for all $y \in \Lambda^{(N)} := \Lambda_0 + \Lambda_1 + \ldots + \Lambda_N$ and coupling constants $\lambda_y \in \mathcal{O}$ *relevant partition functions* by

$$Z^{rel}(y|\Psi) := Z_{\lambda_y}(\Psi). \tag{3.6}$$

For all $x \in \Lambda^{(N)}$ and $X \in \mathcal{P}_x^{(N)} = \mathcal{P}_x(\Lambda^{(N)})$, define recursively partition functions $Z^{(N)}(X|\Psi)$ by

$$Z^{(N)}(X|\Psi) := Z^{rel}(x|\Psi), \tag{3.7}$$

if $X = \{x\}, x \in \Lambda_N$ and

$$\begin{aligned} Z^{(N)}(X|\Psi) :=& c_0(X) \Big[\prod_{\substack{y \subseteq x: \\ X_y = \emptyset}} \int d\mu_\gamma(\Phi) Z^{rel}(y|\Phi + L^{1-d/2}\Psi) \Big] \\ & \Big[\prod_{\substack{y \subseteq x: \\ X_y \neq \emptyset}} \int d\mu_\gamma(\Phi) Z^{(N)}(X_y|\Phi + L^{1-d/2}\Psi) \Big], \end{aligned} \tag{3.8}$$

if $|X| \geq 2$. $c_0(X)$ is a normalization factor which will not be further specified here. Comparing this definition with the definition of $Z_k^{(N)}$ (eqs. (3.2) and (3.3)), we see that the relation

$$Z_k^{(N)}(\Psi) = Z^{(N)}(\Lambda_x^{(N)}|\Psi) \tag{3.9}$$

holds, for all $x \in \Lambda_k, \Lambda_x^{(N)} := \{y \in \Lambda^{(N)} |\ y \preceq x\}$, if the normalization constants are related by

$$c_{0,k}^{(N)} = c_0(\Lambda_x^{(N)}) \tag{3.10}$$

and $\lambda_y = \lambda_N$, for all $y \in \Lambda_N$. We see that $Z^{(N)}(\Lambda_x^{(N)}|\Psi)$ depends only on the bare coupling constants λ_y, $y \in \Lambda_N$. We set $\lambda_y = \lambda_N$, for all $y \in \Lambda_N$. The other coupling constants $\{\lambda_y,\ y \in \Lambda^{(N-1)}\}$ determine the polymer system on $\Lambda^{(N)}$. Obviously, we have $Z(X|\Psi) = Z(X'|\Psi)$, for all $X, X' \in P$ with $X \sim X'$.

For $y \in \Lambda^{(N-1)}$, the parameters λ_y are chosen in such a way that $Z^{rel}(x|\Psi)$ will be a "good" approximation to $Z^{(N)}(X|\Psi)$, for all polymers $X \in \mathcal{P}_x^{(N)}$. Then, the parameters λ_y are no longer independent from each other. For example, let $\lambda_j^{(N)}$ be the running (effective) coupling constant of layer Λ_j with ultraviolet cutoff N. The running coupling constants are related by

$$\lambda_{j-1}^{(N)} = G_j^{(N)}(\lambda_j^{(N)}). \tag{3.11}$$

The functions $G_j^{(N)}$ determine the renormalization group flow of the running coupling constants $\lambda_j^{(N)}$. Then a suitable choice for the coupling constants λ_y, for all $y \in \Lambda^{(N)}$, would be $\lambda_y := \lambda_{j(y)}^{(N)}$.

Define the *irrelevant partition function* $R^{(N)}(X|\Psi)$ by

$$R^{(N)}(X|\Psi) := -Z^{rel}(x|\Psi) + Z^{(N)}(X|\Psi), \tag{3.12}$$

for all $X \in \mathcal{P}_x^{(N)}, x \in \Lambda^{(N)}$.

The definition of the relevant partition function $Z^{(N)}$ implies the following renormalization group equations for the irrelevant partition functions $R^{(N)}$

$$R^{(N)}(X|\Psi) = 0, \tag{3.13}$$

if $X = \{x\}, x \in \Lambda_N$ and

$$R^{(N)}(X|\Psi) = -Z^{rel}(x|\Psi) + \sum_{P:\ P \subseteq \overline{x}} c_0(X) \left[\prod_{y \in \overline{x} - P} \int d\mu_\gamma(\Phi) Z^{rel}(y|\Phi + L^{1-d/2}\Psi) \right] \left[\prod_{y \in P} \int d\mu_\gamma(\Phi) R^{(N)}(X_y|\Phi + L^{1-d/2}\Psi) \right], \tag{3.14}$$

if $|X| \geq 2$. We could equivalently define $R^{(N)}$ by eqs. (3.13) and (3.14) first and define $Z^{(N)}$ by eq. (3.12) . Obviously, $Z^{(N)}$ would obey the renormalization group equations eqs. (3.7) and (3.8) .

The proof for the existence of the ultraviolet limit consists of two parts. Firstly, choose the bare coupling constants $\lambda_y = \lambda_N$, for all $y \in \Lambda_N, N \in \mathbf{N}$, such that the limit

$$\lim_{\substack{N \to \infty \\ N \geq h(P)}} Z^{(N)}(P|\Psi) =: Z(P|\Psi) \tag{3.15}$$

exists, for all $P \in \mathcal{P}_x, x \in \Lambda^{(N)}, h(P) := \max\{j|\ \Lambda_j \cap P \neq \emptyset\}$. Let us recall that the coupling constant λ_y, $y \in \Lambda^{(N)}$, is λ_N-dependent. Otherwise, the limit in eq. (3.15) would be trivial. In the next step, we have to show that the thermodynamic limit

$$\lim_{P \nearrow \Lambda_x} Z(P|\Psi) =: Z(\Lambda_x|\Psi) \tag{3.16}$$

exists. This can be done by proving

$$\sum_{P \in \mathcal{P}_x} |\widetilde{R}(P|\Psi)| < \infty, \tag{3.17}$$

where $\widetilde{R}$ is the Moebius transform of R.

We have seen in this section that we need two limit processes to prove the existence of the ultraviolet limit. In the next section, we shall define a polymer system for hierarchical models, such that we need only one limit process to prove the existence of the ultraviolet limit. For the polymer system defined in this subsection we have to know the bare coupling constants a priori. The definition of a polymer system presented in the next subsection shows that also the bare coupling constants may be constructed in a canonical way.

3.2. Definition of a polymer system with no explicit ultraviolet cutoff

The reason that two limit processes were required for performing the ultraviolet limit in the last section, was the introduction of bare coupling constants λ_N, or equivalently the cutoff-dependence of the partition functions. Instead of introducing coupling constants λ_y for all $y \in \Lambda$, we introduce in this section functions $F_X: \mathcal{O} \to \mathcal{O}$ for all polymers $X \in \mathcal{P}$. The functions F_X relate the coupling constants of before and after one renormalization group step. We call F_X *renormalization group flow function of* X. Suppose in this section, that the functions F_X are arbitrarily defined. In the next section, we shall see that the renormalization group functions F_X can be fixed by imposing renormalization conditions on the partition functions.

Suppose that for all polymers $X \in \mathcal{P}$, there are differentiable functions $F_X : \mathcal{O} \to \mathcal{O}$. For all $\lambda \in \mathcal{O}, X \in \mathcal{P}_x, x \in \Lambda$, define recursively *partition functions*

$$Z_\lambda(X|\Psi) := c_0(X) \, [\prod_{y \underline{\in} x} \int d\mu_\gamma(\Phi) Z_{F_X(\lambda)}(X_y|\Phi + L^{1-d/2}\Psi)]. \tag{3.18}$$

Start the recursive definition by

$$Z_\lambda(\emptyset|\Psi) := Z_\lambda(\Psi). \tag{3.19}$$

Define *irrelevant partition functions* R_λ by

$$R_\lambda(X|\Psi) := -Z_\lambda(\Psi) + Z_\lambda(X|\Psi). \tag{3.20}$$

By eqs. (3.19) and (3.20) follows $R_\lambda(\emptyset|\Psi) = 0$. Then the renormalization group

equations for R_λ and $X \in \mathcal{P}$ with $\max(X) = \{x\}$ read

$$R_\lambda(X|\Psi) = -Z_\lambda(\Psi) + \sum_{P:\ P\subseteq \overline{x}} c_0(X)\ [\prod_{y\in\overline{x}-P} \int d\mu_\gamma(\Phi) Z_{F_X(\lambda)}(\Phi + L^{1-d/2}\Psi)]$$
$$[\prod_{y\in P} \int d\mu_\gamma(\Phi) R_{F_X(\lambda)}(X_y|\Phi + L^{1-d/2}\Psi)]. \tag{3.21}$$

We could equivalently define R_λ by eq. (3.21) first and then define Z_λ by eq. (3.20). Obviously, Z_λ would obey the renormalization group equations (3.18).

The functions F_X are chosen in such a way that $Z_\lambda(\Psi)$ is a "good" approximation to $Z_\lambda(X|\Psi)$, for all polymers X. We see in the next section, how this can be achieved by introducing renormalization conditions.

Suppose that $F_.$ is $\sim_e$-invariant, i.e. $F_X = F_{X'}$, if $X, X' \in \mathcal{P}$ with $X \sim_e X'$. This implies $\sim_e$-invariance of $Z_\lambda(.|\Psi)$.

We discuss now how the bare coupling constant λ_N and the effective partition function $Z_0^{(N)}$ can be recovered. The bare coupling constant $\lambda_N(\lambda)$ is a function of λ. For all $N \in \mathbf{N}$ and $\lambda \in \mathcal{O}$, define $F_N := F_{\Lambda^{(N)}}$ and

$$\lambda_N(\lambda) := (F_{N-1} \circ F_{N-2} \circ \ldots \circ F_1 \circ F_0)(\lambda). \tag{3.22}$$

Suppose that F_k is invertible, for all $k \in \{0, \ldots, N-1\}$. Then, we obtain the effective coupling constants by

$$\lambda_j^{(N)} = F_j^{-1} \circ F_{j+1}^{-1} \circ \ldots \circ F_{N-1}^{-1}(\lambda_N), \tag{3.23}$$

for all $j \in \{0, \ldots, N-1\}$. We see that the renormalization group flow equation eq. (3.11) holds with

$$G_j^{(N)} = F_j^{-1}. \tag{3.24}$$

Moreover, we obtain

$$Z_0^{(N)}(\Psi) = Z_\lambda(\Lambda^{(N)}|\Psi). \tag{3.25}$$

The proof of the ultraviolet limit is equivalent to find functions F_X, for all polymers X, such that the thermodynamic limit

$$Z_\lambda(\Lambda|\Psi) := \lim_{X\nearrow\Lambda} Z_\lambda(X|\Psi) \tag{3.26}$$

exists, for all $\lambda \in \mathcal{O}$. The thermodynamic limit may be shown by proving

$$\sum_{P\in\mathcal{P}_x} |\widetilde{R}_\lambda(P|\Psi)| < \infty, \tag{3.27}$$

for an arbitrarily chosen $x \in \Lambda$, where $\widetilde{R}_\lambda$ is the Moebius transform of R_λ.

The foregoing definition of a polymer system may also used to find fixed points Z^* of the hierarchical renormalization group equation $Z' = H(Z)$, where H is defined by

$$H(Z)(.) := \left[\int d\mu_\gamma(\Phi) Z(\Phi + L^{1-d/2}(.))\right]^{L^d}. \tag{3.28}$$

Suppose that the thermodynamic limit of Z_λ exists and that the following limit

$$F(\lambda) := \lim_{X \nearrow \Lambda} F_X(\lambda) \tag{3.29}$$

exists, for all $\lambda \in \mathcal{O}$. F is called the *renormalization group flow function.* By eq. (3.18) we get in the limit $X \nearrow \Lambda$

$$Z_\lambda(\Lambda|\Psi) = c_0(\Lambda) \left[\int d\mu_\gamma(\Phi) Z_{F(\lambda)}(\Lambda|\Phi + L^{1-d/2}\Psi)\right]^{L^d}. \tag{3.30}$$

Let $\lambda^* \in \mathcal{O}$ be a fixed point of F, i.e. $F(\lambda^*) = \lambda^*$. Then we see, using eq. (3.30), that

$$Z^*(\Psi) := c_0(\Lambda)^{\frac{1}{L^d-1}} \; Z_{\lambda^*}(\Lambda|\Psi) \tag{3.31}$$

is a fixed point of the hierarchical renormalization group equation (3.28).

There is one Gaussian integral for each renormalization group step. Up to now, the partition functions $Z_\lambda(X|\Psi)$ are defined by performing stepwise renormalization group equations. We could also rewrite the hierarchical renormalization group equations by iterating n renormalization group steps and define partition functions correspondingly. We follow this idea to define a polymer system for (n)-renormalization part-polymers. For that, we want to distinguish two kinds of polymers. The first kind of polymers needs renormalization and the second kind needs no renormalization. This is further specified by the following discussion. For weakly coupled models, we can suppose that R_λ is small. Then we see that the main contributions on the rhs of the renormalization group eq. (3.21) for R_λ comes from terms with $|P| = 1$. Iterating renormalization group eq. (3.21) further, we see that the most "dangerous" polymers $X \in \mathcal{P}_x$ have the property that there emerges only one line from the root x of the tree graph $\tau(X)$.

For the following lemma, we introduce some notations. Define, for all $n \in \mathbf{N}$, $X \subseteq \Lambda$, where $\max(X) \subseteq \Lambda_j$,

$$\begin{aligned} \overline{X} &:= \sum_{y \in X} \overline{y}, \\ X^n &:= X \cap \Lambda_{j+n}, \\ X^{\geq n} &:= X \cap (\Lambda_{j+n} + \Lambda_{j+n+1} + \ldots), \\ X^{\leq n} &:= X \cap (\Lambda_j + \ldots + \Lambda_{j+n}) \end{aligned} \tag{3.32}$$

and for all $\Phi_y \in \mathbf{R}^M, y \in \Lambda$, X renormalization part of length n_0, $n \leq n_0$

$$\begin{aligned} F_X^{(n)} &:= F_{X^{\geq n-1}} \circ \ldots \circ F_{X^{\geq 1}} \circ F_X, \\ c_0^{(n)}(X) &:= c_0(X^{\geq n-1}) \cdot \ldots \cdot c_0(X^{\geq 1}) \cdot c_0(X), \\ \Phi_y^{(n)} &:= L^{(1-d/2)(n-j(y))} \Phi_y. \end{aligned} \tag{3.33}$$

Iterating the renormalization group eq. (3.18) for renormalization parts X, we obtain the following

Lemma 3.1 *Let $X \in \mathcal{P}_x, x \in \Lambda$, be a renormalization part of length n_0 and $X = [[y], x] + X_y, l([y, x]) = n_0$. Then, we have, for all $n \in \{1, 2, \ldots, n_0\}$, $\Psi = \Phi_x$ and $y = y_n \underline{\in} y_{n-1} \underline{\in} \ldots \underline{\in} y_1 \underline{\in} y_0 = x$,*

$$\begin{aligned} Z_\lambda(X|\Psi) = c_0^{(n)}(X) &\Big[\prod_{y \in \overline{X}^{\leq n-1}} \int d\mu_\gamma(\Phi_y) \Big] \\ &\Big\{ \Big[\prod_{k=0}^{n-1} \prod_{y \in \overline{y}_k - \{y_{k+1}\}} Z_{F_X^{(k+1)}(\lambda)}(\Phi_y + \sum_{i=0}^{k} \Phi_{y_i}^{(k+1)}) \Big] \\ &Z_{F_X^{(n)}(\lambda)}(X_{y_n} | \sum_{i=0}^{n} \Phi_{y_i}^{(n)}) \Big\}. \end{aligned} \tag{3.34}$$

Proof (by induction): For $n = 1$ eq. (3.34) is equivalent to eq. (3.18). Suppose that eq. (3.34) holds for $n < n_0$. By definition eq. (3.18), we have

$$\begin{aligned} &Z_{F_X^{(n)}(\lambda)}(X_{y_n} | \sum_{i=0}^{n} \Phi_{y_i}^{(n)}) = \\ &\qquad c_0^{(n)}(X_{y_n}) \Big[\int d\mu_\gamma(\Phi) Z_{F_{X_{y_n}} \circ F_X^{(n)}(\lambda)}(X_y | \Phi + L^{1-d/2} \sum_{i=0}^{n} \Phi_{y_i}^{(n)}) \Big]. \end{aligned} \tag{3.35}$$

Using $X_{y_n} = X^{\geq n}$, $c_0(X^{\geq n}) \cdot c_0^{(n)}(X) = c_0^{(n+1)}(X)$, $F_{X^{\geq n}} \circ F_X^{(n)} = F_X^{(n+1)}$ and $L^{1-d/2}\Phi_{y_i}^{(n)} = \Phi_{y_i}^{(n+1)}$, we obtain by insertion of eq. (3.35) into eq. (3.34)

$$\begin{aligned} Z_\lambda(X|\Psi) =& c_0^{(n+1)}(X)\Big[\prod_{y\in\overline{X}^{\leq n-1}}\int d\mu_\gamma(\Phi_y)\Big]\Big[\prod_{y\in\overline{X}^n}\int d\mu_\gamma(\Phi_y)\Big] \\ & \Big\{\Big[\prod_{k=0}^{n-1}\prod_{y\in\overline{y}_k-\{y_{k+1}\}} Z_{F_X^{(k+1)}(\lambda)}(\Phi_y + \sum_{i=0}^{k}\Phi_{y_i}^{(k+1)})\Big] \\ & \Big[\prod_{y\in\overline{y}_n-\{y_{n+1}\}} Z_{F_X^{(n+1)}(\lambda)}(\Phi_y + \sum_{i=0}^{n}\Phi_{y_i}^{(n+1)})\Big] \\ & Z_{F_X^{(n+1)}(\lambda)}(X_{y_{n+1}}|\sum_{i=0}^{n+1}\Phi_{y_i}^{(n+1)})\Big\}. \end{aligned} \tag{3.36}$$

Since $\overline{X}^{\leq n} = \overline{X}^{\leq n-1} + \overline{X}^n$, eq. (3.36) implies the assertion for $n+1$. □

Lemma 3.1 and eq. (3.20) imply the following recursion relation for $R_\lambda(X|\Psi)$, if $X \in \mathcal{P}_x, x \in \Lambda_0$, is a renormalization part of length n_0 and $n \leq n_0$

$$R_\lambda(X|\Psi) = \delta Z_\lambda^{(n)}(X|\Psi) + \delta R_\lambda^{(n)}(X|\Psi), \tag{3.37}$$

where

$$\begin{aligned} \delta Z_\lambda^{(n)}(X|\Psi) :=& - Z_\lambda(\Psi) + c_0^{(n)}(X)\Big[\prod_{y\in\overline{X}^{\leq n-1}}\int d\mu_\gamma(\Phi_y)\Big] \\ & \Big\{\Big[\prod_{k=0}^{n-1}\prod_{y\in\overline{y}_k-\{y_{k+1}\}} Z_{F_X^{(k+1)}(\lambda)}(\Phi_y + \sum_{i=0}^{k}\Phi_{y_i}^{(k+1)})\Big] \\ & Z_{F_X^{(n)}(\lambda)}(\sum_{i=0}^{n}\Phi_{y_i}^{(n)})\Big\} \end{aligned} \tag{3.38}$$

and

$$\begin{aligned} \delta R_\lambda^{(n)}(X|\Psi) :=& c_0^{(n)}(X)\Big[\prod_{y\in\overline{X}^{\leq n-1}}\int d\mu_\gamma(\Phi_y)\Big] \\ & \Big\{\Big[\prod_{k=0}^{n-1}\prod_{y\in\overline{y}_k-\{y_{k+1}\}} Z_{F_X^{(k+1)}(\lambda)}(\Phi_y + \sum_{i=0}^{k}\Phi_{y_i}^{(k+1)})\Big] \\ & R_{F_X^{(n)}(\lambda)}(X_{y_n}|\sum_{i=0}^{n}\Phi_{y_i}^{(n)})\Big\}. \end{aligned} \tag{3.39}$$

We want to define recursively partition functions ${}_nZ_\lambda(X|\Psi)$ for (n)-renormalization part-polymers X. In analogy with eq. (3.18), define in the following a partition function

$$ {}_nZ_\lambda(.|\Psi)\colon h_n(\mathcal{P}) \cup \{\emptyset\} \to \mathbf{R}_+. $$

Consider an (n)-renormalization part-polymer $X \in h_n(\mathcal{P})$. If $|\{y \in \overline{x}|\ X_y \neq \emptyset\}| \geq 2$ or $X = \{x, y_1, \ldots, y_{k-1}\} + X_{y_k}$, $y_k \underline{\in} y_{k-1} \underline{\in} \ldots \underline{\in} y_1 \underline{\in} x$, $k < n$, then define

$$ {}_nZ_\lambda(X|\Psi) := c_0(X)\ [\prod_{y \underline{\in} x} \int d\mu_\gamma(\Phi)_n Z_{F_X(\lambda)}(X_y|\Phi + L^{1-d/2}\Psi)]. \tag{3.40} $$

If $X = \{x\} + X_{y_n}, |j(y_n) - j(x)| = n$, then define

$$ \begin{aligned} {}_nZ_\lambda(X|\Psi) := c_0^{(n)}(X) & \Big[\prod_{y \in \overline{X}^{\leq n}} \int d\mu_\gamma(\Phi_y)\Big] \\ & \Big\{\Big[\prod_{k=0}^{n-1} \prod_{y \in \overline{y}_k - \{y_{k+1}\}} Z_{F_X^{(k+1)}(\lambda)}(\Phi_y + \sum_{i=0}^{k} \Phi_{y_i}^{(k+1)})\Big] \\ & {}_nZ_{F_X^{(n)}(\lambda)}(X_{y_n}|\sum_{i=0}^{n} \Phi_{y_i}^{(n)})\Big\}. \end{aligned} \tag{3.41} $$

Start the recursive definition by

$$ {}_nZ_\lambda(\emptyset|\Psi) := Z_\lambda(\Psi). \tag{3.42} $$

We have used the abbreviations (3.33) of $c_0^{(n)}(X)$ and $F_X^{(n)}$, for all $X \in h_n(\mathcal{P})$. Then, we see that

$$ {}_nZ_\lambda(X|\Psi) = Z_\lambda(h_n^{-1}(X)|\Psi), \tag{3.43} $$

for all $X \in h_n(\mathcal{P})$. This can be shown using lemma 3.1.

We want to define an irrelevant partition function ${}_nR_\lambda\colon h_n(\mathcal{P}) \to \mathbf{R}$. Define

$$ {}_nR_\lambda(X|\Psi) := -Z_\lambda(\Psi) +_n Z_\lambda(X|\Psi), \tag{3.44} $$

for all $X \in h_n(\mathcal{P})$. Then the renormalization group equation for ${}_nR_\lambda, X \in h_n(\mathcal{P})$, reads

$$ \begin{aligned} {}_nR_\lambda(X|\Psi) = -Z_\lambda(\Psi) + \sum_{P:P \subseteq \overline{x}} c_0(X)\ & [\prod_{y \in \overline{x} - P} \int d\mu_\gamma(\Phi)_n Z_{F_X(\lambda)}(\Phi + L^{1-d/2}\Psi)] \\ & [\prod_{y \in P} \int d\mu_\gamma(\Phi)_n R_{F_X(\lambda)}(X_y|\Phi + L^{1-d/2}\Psi)], \end{aligned} \tag{3.45} $$

if $|\{y \in \overline{x}|\ X_y \neq \emptyset\}| \geq 2$ or $X = \{x, y_1, \ldots, y_{k-1}\} + X_{y_k}$, $y_k \underline{\in} y_{k-1} \underline{\in} \cdots \underline{\in} y_1 \underline{\in} x$, $k < n$ and

$$
\begin{aligned}
{}_nR_\lambda(X|\Psi) = -Z_\lambda(\Psi) + c_0^{(n)}(X) \Big[\prod_{y \in \overline{X}^{\leq n}} \int d\mu_\gamma(\Phi_y) \Big] \\
\Big\{ \Big[\prod_{k=0}^{n-1} \prod_{y \in \overline{y}_k - \{y_{k+1}\}} Z_{F_X^{(k+1)}(\lambda)}(\Phi_y + \sum_{i=0}^{k} \Phi_{y_i}^{(k+1)}) \Big] \\
\Big[{}_nZ_{F_X^{(n)}(\lambda)}(\sum_{i=0}^{n} \Phi_{y_i}^{(n)}) +_n R_{F_X^{(n)}(\lambda)}(X_{y_n} | \sum_{i=0}^{n} \Phi_{y_i}^{(n)}) \Big] \Big\},
\end{aligned}
\tag{3.46}
$$

if $X = \{x\} + X_{y_n}, |j(y_n) - j(x)| = n$.

Let us remark that for $n = 1$ we have $h_1(\mathcal{P}) = \mathcal{P}$ and ${}_1Z_\lambda(X|\Psi) = Z_\lambda(X|\Psi)$, i.e. both polymer systems defined above are equal in this case.

4. Renormalization conditions, operator $\mathcal{L}$ and the function F_X

The definition of the polymer system given in the last section depends on the functions $F_X, X \in \mathcal{P}$. F_X will be chosen, such that the thermodynamic limit of Z_λ exists. In the case $F_X(\lambda) \equiv \lambda \in \mathcal{O}$, for all $X \in \mathcal{P}$, we call the corresponding polymer system *not renormalized.* In general, the thermodynamic limit of a not renormalized polymer system does not exist.

For example, consider the Φ^4-model in 2 and 3 dimensions. Scalar Φ^4-models are defined by

$$Z_{(m^2,\lambda)}(\Phi) = \exp\{-\frac{\lambda}{4!}\Phi^4 - \frac{m^2}{2}\Phi^2\}, \tag{4.1}$$

where $\Phi \in \mathbf{R}$. In two dimensions, we choose $F_X \equiv F$, where

$$F(m^2, \lambda) = (L^{-2}m^2 - \frac{\lambda}{2}L^{-2}\gamma, L^{-2}\lambda), \tag{4.2}$$

and in three dimensions

$$F(m^2, \lambda) = (L^{-2}m^2 - \frac{\lambda}{2}L^{-2}\gamma + \frac{\lambda^2}{3!}L^{-2}\gamma^3, L^{-1}\lambda), \tag{4.3}$$

for all polymers X. Then it can be shown that the corresponding thermodynamic limit exists [17]. The reason for such a simple choice of F_X is that the Φ^4-model is perturbatively superrenormalizable in two and three dimensions. For superrenormalizable models, the function F_X may be chosen as a polynom. Using eqs. (4.2)

and (4.3), we obtain the following renormalization group flow equations of bare coupling constants in two dimensions

$$\begin{aligned} m_{j+1}^2 &= L^{-2} m_j^2 - \frac{\lambda_j}{2} L^{-2} \gamma, \\ \lambda_{j+1} &= L^{-2} \lambda_j, \end{aligned} \tag{4.4}$$

and in three dimensions

$$\begin{aligned} m_{j+1}^2 &= L^{-2} m_j^2 - \frac{\lambda_j}{2} L^{-2} \gamma + \frac{\lambda_j^2}{3!} L^{-2} \gamma^3, \\ \lambda_{j+1} &= L^{-1} \lambda_j, \end{aligned} \tag{4.5}$$

for all $j \in \mathbf{N}$. $(m_0^2, \lambda_0) = (m^2, \lambda)$ are the renormalized coupling constants. Define

$$\begin{aligned} \widetilde{m}_j^2 &:= L^{-2j} m_j^2, \\ \widetilde{\lambda}_j &:= L^{(4-d)j} \lambda_j. \end{aligned} \tag{4.6}$$

By eqs. (4.4) and (4.5), we obtain in 2 dimensions

$$\begin{aligned} \widetilde{m}_{j+1}^2 &= \widetilde{m}_j^2 - \frac{\widetilde{\lambda}_j \gamma}{2}, \\ \widetilde{\lambda}_{j+1} &= \widetilde{\lambda}_j \end{aligned} \tag{4.7}$$

and in 3 dimensions

$$\begin{aligned} \widetilde{m}_{j+1}^2 &= \widetilde{m}_j^2 - L^j \frac{\widetilde{\lambda}_j \gamma}{2} + \frac{\widetilde{\lambda}_j^2 \gamma^3}{3!}, \\ \widetilde{\lambda}_{j+1} &= \widetilde{\lambda}_j. \end{aligned} \tag{4.8}$$

Using eqs. (4.7) and (4.8), the bare coupling constants expressed by the renormalized coupling constants in two dimensions are

$$\begin{aligned} m_N^2 &= m^2 - N \frac{\lambda \gamma}{2}, \\ \lambda_N &= L^{-2N} \lambda, \end{aligned} \tag{4.9}$$

and in three dimensions

$$\begin{aligned} m_N^2 &= m^2 - \frac{L^N - 1}{L - 1} \frac{\lambda \gamma}{2} + N \frac{\lambda^2 \gamma^3}{3!}, \\ \lambda_N &= L^{-N} \lambda. \end{aligned} \tag{4.10}$$

By these equations, we see the ultraviolet asymptotic freedom and the logarithmic (linear) divergence of the bare mass squared for the Φ^4-model in two (three) dimensions.

The functions F_X are in general not uniquely defined. Imposing some *renormalization conditions* on partition functions, we may define the functions $F_X = (F_X^1, \ldots, F_X^m)$ (or some components F_X^a) uniquely. For that, define linear operators $\mathcal{L}_{\lambda,X}$. The renormalization conditions are given by

$$\mathcal{L}_{\lambda,X}\{R_\lambda(X|.)\}(\Psi) = 0, \tag{4.11}$$

for all polymers $X \in \mathcal{P}, \lambda \in \mathcal{O}, \Psi \in \mathbf{R}^M$.

The polymer-dependence of the operator $\mathcal{L}_{\lambda,X}$ may, for instance, come in by

$$\mathcal{L}_{\lambda,X} := \begin{cases} \mathcal{L}_\lambda, & \text{if } X \in \mathcal{P}^{(div)}; \\ \mathbf{0}, & \text{if } X \in \mathcal{P} - \mathcal{P}^{(div)}, \end{cases} \tag{4.12}$$

where $\mathcal{P}^{(div)} \subseteq \mathcal{P}$, and

$$\mathbf{0}\{Z(.)\}(\Psi) := 0, \tag{4.13}$$

for all functions $Z\colon \mathbf{R}^M \to \mathbf{R}$. In other words, we "renormalize" only polymers $P \in \mathcal{P}^{(div)}$. Then the renormalization condition (4.11) is trivially fulfilled, for $X \in \mathcal{P} - \mathcal{P}^{(div)}$. For example, choose $\mathcal{P}^{(div)} := \mathcal{P}^{(ren)}_{\geq n} := \{P \in \mathcal{P} |\ P$ is renormalization part of length $\geq n\}$.

Suppose that the operators $\mathcal{L}_{\lambda,X}$ fix the functions F_X, for all polymers X, by renormalization conditions eq. (4.11) . Using eq. (3.20), we see that

$$\mathcal{L}_{\lambda,X}\{Z_\lambda(X|.)\}(\Psi) = \mathcal{L}_{\lambda,X}\{Z_\lambda(.)\}(\Psi). \tag{4.14}$$

We shall give some examples for the operators $\mathcal{L}_{\lambda,X}$. These examples are the Φ^4-model and the 2-dimensional nonlinear $O(N)$ σ-model. In the first example, the operators $\mathcal{L}_{\lambda,X}$ do not depend on λ and polymers X. In the second example, the operators $\mathcal{L}_{\lambda,X}$ depend on the coupling constant λ but are polymer-independent. We could also define polymer-dependent operators $\mathcal{L}_{\lambda,X}$ in our examples by imposing renormalization conditions on partition functions only for special polymers.

For the Φ^4-model, we may choose $\mathcal{L}_{\lambda,X} \equiv \mathcal{L}$, where

$$\mathcal{L}\{Z(.)\}(\Psi) = \exp\{-c\Psi^2\}\left[\mathbf{1} + \frac{\Psi^2}{2}\frac{\partial^2}{\partial\widetilde{\Psi}^2}\right]\left[\exp\{c\widetilde{\Psi}^2\}Z(\widetilde{\Psi})\right]\Big|_{\widetilde{\Psi}=0}, \tag{4.15}$$

and c is a positive constant. Let us notice that in this case eq. (4.14) is equivalent to the more conventional renormalization conditions

$$Z_{(m^2,\lambda)}(X|0) = 1, \quad -\frac{\partial^2}{\partial\Psi^2}Z_{(m^2,\lambda)}(X|\Psi)|_{\Psi=0} = m^2, \tag{4.16}$$

for all polymers X. Therefore, the renormalization condition eq. (4.11) defines the component F_X^1 (mass renormalization) and the normalization constant $c_0(X)$. Choosing

$$\mathcal{L}\{Z(.)\}(\Psi) = \exp\{-c\Psi^2\}\left[\mathbf{1} + \frac{\Psi^2}{2}\frac{\partial^2}{\partial\widetilde{\Psi}^2} + \frac{\Psi^4}{4!}\frac{\partial^4}{\partial\widetilde{\Psi}^4}\right]\left[\exp\{c\widetilde{\Psi}^2\}Z(\widetilde{\Psi})\right]\Big|_{\widetilde{\Psi}=0}, \quad (4.17)$$

we would fix $F_X = (F_X^1, F_X^2)$ (mass and coupling constant renormalization) and the normalization constant $c_0(X)$. The renormalization condition eq. (4.14) is in this case equivalent to eq. (4.16) and

$$-\frac{\partial^4}{\partial\Psi^4}\ln Z_{(m^2,\lambda)}(X|\Psi)|_{\Psi=0} = \lambda, \quad (4.18)$$

for all polymers X. The reason for the prefactor $\exp\{-c\Psi^2\}$ in the definitions eqs. (4.15) and (4.17) of $\mathcal{L}$ is to control large fields by a stability bound. In perturbation expansion, this term is not necessary. In the example considered here, the renormalization conditions eq. (4.11) do not depend on c.

We see that the symmetry relation $Z_\lambda(\Psi) = Z_\lambda(-\Psi)$ for the "bare" partition function holds also for all partition functions $Z_\lambda(X|\Psi)$.

As a second example, we want to consider the case where the operator $\mathcal{L}_{\lambda,X} \equiv \mathcal{L}_\lambda$ depends on the coupling constants and does not depend on polymers. For instance, consider the 2-dimensional nonlinear $O(N)$ σ-model. This model is defined by the function $Z_\lambda\colon \mathbf{R}^N \to \mathbf{R}_+$, where

$$Z_\lambda(\Psi) := \exp\{-\frac{c^*}{4f^2}(\Psi^2 - f^2)^2\}, \quad (4.19)$$

for $c^* := (L^2-1)/(2\gamma)$, $\lambda := 1/f \in [0,\lambda_0]$. λ_0 is a small positive real number. Then, define $\mathcal{L}_\lambda$ by

$$\mathcal{L}_\lambda\{Z(.)\}(\Psi) := \exp\{-\frac{c}{4f^2}(\Psi^2 - f^2)^2\}\left[\mathbf{1} + \sum_{a=1}^{N}(\Psi^a - fu^a)\frac{\partial}{\partial\widetilde{\Psi}^a}\right]$$
$$\left[\exp\{\frac{c}{4f^2}(\widetilde{\Psi}^2 - f^2)^2\}Z(\widetilde{\Psi})\right]\Big|_{\widetilde{\Psi}^a = fu^a}, \quad (4.20)$$

for all $O(N)$-invariant functions Z, $\Psi = (\Psi^1,\ldots,\Psi^N) \in \mathbf{R}^N, \Psi \neq 0$, and unit vector $u := \Psi/\|\Psi\| = (u^1,\ldots,u^N) \in \mathbf{R}^N$, $\|u\| := \sum_{a=1}^N (u^a)^2 = 1$. If $\Psi = 0$, take an arbitrary unit vector u on the rhs of eq. (4.20). c is a positive constant.

We rewrite eq. (4.20) by

$$\mathcal{L}_\lambda\{Z(.)\}(\Psi) := \exp\{-\frac{c}{4f^2}(\Psi^2-f^2)^2\}\left[Z(fu)+\sum_{a=1}^{N}(\Psi^a-fu^a)\frac{\partial}{\partial\widetilde{\Psi}^a}Z(\widetilde{\Psi})|_{\widetilde{\Psi}^a=fu^a}\right]. \tag{4.21}$$

Then, we see that the renormalization condition eq. (4.14) is equivalent to

$$Z_\lambda(X|(f,\mathbf{0})) = 1, \quad \frac{\partial}{\partial\rho}Z_\lambda(X|(f+\rho,\mathbf{0})) = 0, \tag{4.22}$$

for all polymers X and $\mathbf{0} \in \mathbf{R}^{N-1}$ is the null vector.

The renormalization condition eq. (4.14) fixes the function $F_X\colon [0,\lambda_0] \to [0,\lambda_0]$ and normalization factor $c_0(X)$.

By definition, the bare partition function $Z_\lambda(\Psi)$ and the partition function $Z_\lambda(X|\Psi)$ are $O(N)$-invariant, for all polymers X.

We see in our examples, that the renormalization conditions eqs. (4.11) and (4.14) do not depend on the positive constant c, which appears on the rhs of the definitions (4.15) , (4.17) and (4.20).

In the following, we want to discuss symmetry properties. Let $\mathcal{S}\colon \mathbf{R}^M \to \mathbf{R}^M$ be a field transformation. For a function $F\colon \mathbf{R}^M \to \mathbf{R}$, define a function $\mathcal{S}(F)\colon \mathbf{R}^M \to \mathbf{R}$ by

$$\mathcal{S}(F)(\Psi) := F(\mathcal{S}\Psi), \tag{4.23}$$

for all $\Psi \in \mathbf{R}^M$. Suppose that a model is defined by a function $Z_\lambda\colon \mathbf{R}^M \to \mathbf{R}_+$, which is $\mathcal{S}$-symmetric, i.e.

$$\mathcal{S}(Z_\lambda) = Z_\lambda \tag{4.24}$$

and suppose that the Gaussian measure $d\mu_\gamma(\Phi)$ is $\mathcal{S}$-symmetric, i.e.

$$d\mu_\gamma(\mathcal{S}\Phi) = d\mu_\gamma(\Phi). \tag{4.25}$$

We see that, if

$$[\mathcal{L}_{\lambda,X}, \mathcal{S}] = 0 \tag{4.26}$$

holds, then $Z_\lambda(X|.)$ is $\mathcal{S}$-symmetric, for all polymers X. Models, for which the thermodynamic limit $X \nearrow \Lambda$ exists and the eqs. (4.24) and (4.26) are valid, are called *anomaly-free*. It is possible that the existence of the thermodynamic limit requires the definition of operators $\mathcal{L}_{\lambda,X}$ which violates eq. (4.26). For such models,

the partition functions $Z_\lambda(X|\Psi)$ are not $\mathcal{S}$-symmetric. Nevertheless, the "bare" partition function Z_λ may be $\mathcal{S}$-symmetric. In this case, we say that such models have *anomalies.*

In the following, we discuss how to determine explicitly the renormalization group flow functions F_X by the use of renormalization conditions (4.11) . Suppose that the renormalization conditions eq. (4.11) are equivalent to

$$g_X(\lambda, \lambda')|_{\lambda'=F_X(\lambda)} = 0. \tag{4.27}$$

Consider $(\lambda_0, \lambda'_0) \in \mathcal{O}^2$, such that $g_X(\lambda_0, \lambda'_0) \approx 0$, $\partial_{\lambda'} g_X(\lambda_0, \lambda'_0) \neq 0$ and define

$$f_X(\lambda, \lambda') := \lambda' - (\partial_{\lambda'} g_X(\lambda_0, \lambda'_0))^{-1} g_X(\lambda, \lambda'). \tag{4.28}$$

For fixed λ, we consider the fixed point equation $f_X(\lambda, \lambda') = \lambda'$. The solution λ' of this fixed point equation fulfils $g_X(\lambda, \lambda') = 0$. In the following, suppose that the iteration procedure for this fixed point equation is well defined and converges to its solution $\lambda' = F_X(\lambda)$.

Let λ be in a small neighborhood of λ_0. Define $\lambda_{i,X}(\lambda)$ recursively by

$$\begin{aligned} \lambda_{0,X}(\lambda) &:= \lambda_0, \\ \lambda_{i+1,X}(\lambda) &:= f_X(\lambda, \lambda_{i,X}(\lambda)). \end{aligned} \tag{4.29}$$

If $\lim_{i\to\infty} \lambda_{i,X}(\lambda)$ exists, we have

$$F_X(\lambda) = \lim_{i\to\infty} \lambda_{i,X}(\lambda), \tag{4.30}$$

for all λ in a small neighborhood of λ_0.

In the remainder of this section we shall discuss the question how to construct the Moebius transform $\widetilde{F}_X$ of the functions F_X. Let $\widetilde{f}_X$ be the Moebius transform of f_X. For fixed $X' \in \mathcal{P}$ and $X \in \mathcal{P}_x$ with $X \supseteq X'$ and $Y \in \mathcal{P}(X, X')$ define $\overset{\sim(i)}{f}_{Y,X'}(\lambda)$ by

$$\widetilde{f}_{X'}(\lambda, \lambda_{i,X}(\lambda)) = \sum_{Y\in\mathcal{P}(X,X')} \overset{\sim(i)}{f}_{Y,X'}(\lambda). \tag{4.31}$$

Then the Moebius transform $\widetilde{\lambda}_{i,X}(\lambda)$ of $\lambda_{i,X}(\lambda)$ is recursively defined by

$$\begin{aligned} \widetilde{\lambda}_{0,X}(\lambda) &:= \lambda_0 \delta_{x,X}, \\ \widetilde{\lambda}_{i+1,X}(\lambda) &:= \sum_{X'\in\mathcal{P}(X)} \overset{\sim(i)}{f}_{X,X'}(\lambda). \end{aligned} \tag{4.32}$$

Then we obtain

$$\widetilde{F}_X(\lambda) = \lim_{i\to\infty} \widetilde{\lambda}_{i,X}(\lambda), \tag{4.33}$$

if the corresponding limit exists. This is shown by

$$\begin{aligned}
\sum_{X\in\mathcal{P}(Q)} \widetilde{\lambda}_{i+1,X}(\lambda) &= \sum_{X\in\mathcal{P}(Q)} \sum_{X'\in\mathcal{P}(X)} \overset{\sim}{\widetilde{f}}{}^{(i)}_{X,X'}(\lambda) \\
&= \sum_{X'\in\mathcal{P}(Q)} \sum_{X\in\mathcal{P}(Q,X')} \overset{\sim}{\widetilde{f}}{}^{(i)}_{X,X'}(\lambda) \\
&= \sum_{X'\in\mathcal{P}(Q)} \widetilde{f}_{X'}(\lambda, \lambda_{i,Q}(\lambda)) \\
&= f_Q(\lambda, \lambda_{i,Q}(\lambda)).
\end{aligned} \tag{4.34}$$

An explicit cluster expansion formula for the Moebius transform $\overset{\sim}{\widetilde{f}}{}^{(i)}_{X,X'}(\lambda)$ is given in section 6, corollary 6.4.

5. Renormalization group equations for the Moebius transform $\widetilde{R}_\lambda$

We have seen in the first section, that to show the thermodynamic limit of a polymer system, we have to estimate Moebius transforms. In this section, we present an explicit method to construct recursively the Moebius transform of $\widetilde{R}_\lambda$ for convex-polymer and (n)-renormalization part polymer systems. We perform these constructions in four steps. These steps are presented in detail for convex-polymer systems. For the (n)-renormalization part polymer system, we explain these steps only briefly, since they are analogous with the steps for the convex-polmer system.

The general approach is described as follows. Suppose that we know already $\widetilde{R}_\lambda(P|\Psi)$, for all $P \in \mathcal{P}$ with $|P| < N$. We want to construct $\widetilde{R}_\lambda(X|\Psi)$, for all $X \in \mathcal{P}$ with $|X| = N$. Furthermore, suppose that $c_0(X)$ and F_X are known. By definition of the Moebius transform, we know $R_\lambda(P|\Psi)$, for all $P \in \mathcal{P}$ with $|P| < N$. Using the renormalization group equations of R_λ, we may compute $R_\lambda(X|\Psi)$, for all $X \in \mathcal{P}$ with $|X| = N$. By definition of the Moebius transform, we obtain $\widetilde{R}_\lambda(X|\Psi)$. In the following, use the abbreviation

$$\mathcal{R}_{\lambda,X} := \mathbf{1} - \mathcal{L}_{\lambda,X} \tag{5.1}$$

and for a function $F: \mathbf{R}^M \to \mathbf{R}$, define

$$\mu_\gamma(F) := \int d\mu_\gamma(\Phi) F(\Phi + L^{1-d/2}(.)). \tag{5.2}$$

If $\mathcal{L}_X$ is polymer-independent, i.e. $\mathcal{L}_{\lambda,X} \equiv \mathcal{L}_\lambda$, then we see that the Moebius transform $\widetilde{\mathcal{R}}_\lambda$ of $\mathcal{R}_\lambda := \mathbf{1} - \mathcal{L}_\lambda$ obeys

$$\widetilde{\mathcal{R}}_{\lambda,X} = \delta_{1,|X|} \mathcal{R}_\lambda. \tag{5.3}$$

5.1. Convex-polymer system

We shall discuss the recursive construction of the Moebius transform $\widetilde{R}_\lambda$ in four steps. In the first step, perform the Gaussian integration and set $c_0(X) = 1$ and $F_X(\lambda) = \lambda' \in \mathcal{O}$. We take into account, in the first step, the polymer-dependence of the operator $\mathcal{L}_{\lambda,X}$. In the second step, we have regard to the polymer-dependence of the normalization constant $c_0(X)$ and in the third step, the polymer-dependence of the function F_X. In the fourth step, finally, we recover the Moebius transform $\widetilde{R}_\lambda$ for larger polymers.

1. Step (Integration):

This step is the most important one. Here, we get suppression factors, which control the entropy factors. This works only if the operators $\mathcal{L}_{\lambda,X}$ are defined correctly and if the model is renormalizable. Consider the not normalized and not renormalized renormalization group equations (3.21), i.e. setting

$$c_0(X) = 1, \quad F_X(\lambda) = \lambda' \tag{5.4}$$

in eq. (3.21) for all $X \in \mathcal{P}$. Denote $\mathbf{1} - \mathcal{L}_{\lambda,X}$ of the rhs of eq. (3.21) by $B^u_{\lambda,\lambda'}(X|\Psi)$. Then determine the corresponding Moebius transform $\widetilde{B}^u_{\lambda,\lambda'}(X|\Psi)$.

Define $B^u_{\lambda,\lambda'}(X|\Psi)$, for $\lambda, \lambda' \in \mathcal{O}$, $X \in \mathcal{P}$ by

$$B^u_{\lambda,\lambda'}(X|\Psi) := \mathcal{R}_{\lambda,X}\left\{-Z_\lambda(.) + \sum_{P:P\subseteq\overline{x}} \mu_\gamma(Z_{\lambda'})(.)^{|\overline{x}-P|} \prod_{y\in P} \mu_\gamma(R_{\lambda'}(X_y|.))\right\}(\Psi). \tag{5.5}$$

The following lemma gives the relation $\widetilde{R}_{\lambda'} \mapsto \widetilde{B}^u_{\lambda,\lambda'}$ where $\widetilde{B}^u_{\lambda,\lambda'}$ is the Moebius transform of $B^u_{\lambda,\lambda'}$. Let $\widetilde{\mathcal{R}}_\lambda$ be the Moebius transform of $\mathcal{R}_\lambda$.

Lemma 5.1 . *The Moebius transform $\widetilde{B}^u_{\lambda,\lambda'}$ of $B^u_{\lambda,\lambda'}$ is given by*

$$\widetilde{B}^u_{\lambda,\lambda'}(X|\Psi) = -\widetilde{\mathcal{R}}_{\lambda,X}\{Z_\lambda(.)\}(\Psi) + \sum_{\substack{Y,P\in\mathcal{P}(X):\\ Y\cup P=X}} \widetilde{\mathcal{R}}_{\lambda,Y}\Big\{\mu_\gamma(Z_{\lambda'})(.)^{|\{y\in\overline{x}|\ P_y=\emptyset\}|}\prod_{\substack{y\in\overline{x}:\\ P_y\neq\emptyset}}\mu_\gamma(\widetilde{R}_{\lambda'}(P_y|.))\Big\}(\Psi), \tag{5.6}$$

and if $\mathcal{L}_{\lambda,X}$ is polymer-independent ($\mathcal{L}_{\lambda,X}\equiv\mathcal{L}_\lambda$),

$$\widetilde{B}^u_{\lambda,\lambda'}(X|\Psi) = \mathcal{R}_\lambda\Big\{-\delta_{1,|X|}\{Z_\lambda(.)\} + \mu_\gamma(Z_{\lambda'})(.)^{|\{y\in\overline{x}|\ X_y=\emptyset\}|}\prod_{\substack{y\in\overline{x}:\\ X_y\neq\emptyset}}\mu_\gamma(\widetilde{R}_{\lambda'}(X_y|.))\Big\}(\Psi), \tag{5.7}$$

for all polymers X.

2. Step (Normalization):

For this step, we need the normalization constant $c_0(X)$. Define $B_{\lambda,\lambda'}(X|\Psi)$ by $\mathbf{1}-\mathcal{L}_{\lambda,X}$ of the rhs of the not renormalized renormalization equations eq. (3.21), where $F_X(\lambda)=\lambda'$. Then determine the Moebius transform $\widetilde{B}_{\lambda,\lambda'}$ in terms of the Moebius transform $\widetilde{B}^u_{\lambda,\lambda'}$. Denote the Moebius transform of c_0 by $\widetilde{c}_0$. For $\lambda,\lambda'\in\mathcal{O}, X\in\mathcal{P}$, define $B_{\lambda,\lambda'}(X|\Psi)$ by

$$B_{\lambda,\lambda'}(X|\Psi) := \mathcal{R}_{\lambda,X}\Big\{-Z_\lambda(.) + \sum_{P:P\subseteq\overline{x}} c_0(X)\ \mu_\gamma(Z_{\lambda'})(.)^{|\overline{x}-P|}\prod_{y\in P}\mu_\gamma(R_{\lambda'}(X_y|.))\Big\}(\Psi). \tag{5.8}$$

The following lemma gives the relation $\widetilde{B}^u_{\lambda,\lambda'}\mapsto\widetilde{B}_{\lambda,\lambda'}$.

Lemma 5.2 . *The Moebius transform $\widetilde{B}_{\lambda,\lambda'}$ of $B_{\lambda,\lambda'}$ for $\lambda,\lambda'\in\mathcal{O}$ and $X\in\mathcal{P}$ is*

$$\widetilde{B}_{\lambda,\lambda'}(X|\Psi) = \widetilde{\mathcal{R}}_{\lambda,X}\{Z_\lambda(.)\}(\Psi) + \sum_{\substack{P_1,P_2\in\mathcal{P}(X):\\ P_1\cup P_2=X}}\Big[\widetilde{c}_0(P_1)\Big[\widetilde{\mathcal{R}}_{P_2}\{Z_\lambda(.)\}(\Psi) + \widetilde{B}^u_{\lambda,\lambda'}(P_2|\Psi)\Big]\Big], \tag{5.9}$$

and if $\mathcal{L}_{\lambda,X}$ is polymer-independent ($\mathcal{L}_{\lambda,X}\equiv\mathcal{L}_\lambda$),

$$\widetilde{B}_{\lambda,\lambda'}(X|\Psi) = \mathcal{R}_\lambda\Big\{\Big(-\delta_{1,|X|}+\widetilde{c}_0(X)\Big)Z_\lambda(.) + \sum_{\substack{P_1,P_2\in\mathcal{P}(X):\\ P_1\cup P_2=X}}\widetilde{c}_0(P_1)\widetilde{B}^u_{\lambda,\lambda'}(P_2|.)\Big\}(\Psi). \tag{5.10}$$

3. Step (Renormalization):

For this step, we need the renormalization group flow function F_X. For a fixed polymer $Q \in \mathcal{P}$, define the Moebius transform $\widetilde{\widetilde{B}}_\lambda(P,Q|\Psi)$ of $\widetilde{B}_{\lambda,F_P(\lambda)}(Q|\Psi)$, for all polymers $P \supseteq Q$. A cluster expansion formula of $\widetilde{\widetilde{B}}_\lambda$ in terms of $\widetilde{B}_{\lambda,\lambda'}$ is given in section 6, corollary 6.4. By renormalization condition (4.11), we see

$$\mathcal{L}_{\lambda,X}\{B_{\lambda,F_X(\lambda)}(X|.)\}(\Psi) = 0 \tag{5.11}$$

and therefore, by definition,

$$B_{\lambda,F_X(\lambda)}(X|\Psi) = R_\lambda(X|\Psi), \tag{5.12}$$

for all polymers X.

Let us remark that we could have replaced the operator $\mathcal{R}_{\lambda,X}$ by $\mathbf{1}$ in the definitions eqs. (5.5) and (5.8) of $B^u_{\lambda,\lambda'}$ and $B_{\lambda,\lambda'}$. Then we would also get the relations eqs. (5.11) and (5.12).

In this third step, take the polymer-dependence of F into account. For this, define the following Moebius transform. For a fixed polymer X', define the Moebius transform $\widetilde{\widetilde{B}}_\lambda(.,X'|\Psi)$ of $\widetilde{B}_{\lambda,F_.(\lambda)}(X'|\Psi)$, for all $X \in \mathcal{P}$ with $X \supseteq X'$, by

$$\widetilde{B}_{\lambda,F_X(\lambda)}(X'|\Psi) = \sum_{Q\in\mathcal{P}(X,X')} \widetilde{\widetilde{B}}_\lambda(Q,X'|\Psi). \tag{5.13}$$

An explicit cluster expansion formula of $\widetilde{\widetilde{B}}_\lambda$ is given in section 6, corollary 6.4.

4. Step (Induction):

In the last step determine $\widetilde{R}_\lambda$ by $\widetilde{\widetilde{B}}_\lambda$. We present the relation $\widetilde{\widetilde{B}}_\lambda \mapsto \widetilde{R}_\lambda$ by

Lemma 5.3 *For all polymers X, the following relation holds*

$$\widetilde{R}_\lambda(X|\Psi) = \sum_{Q\in\mathcal{P}(X)} \widetilde{\widetilde{B}}_\lambda(X,Q|\Psi). \tag{5.14}$$

In the following, we present the proofs of lemma 5.1, lemma 5.2 and lemma 5.3.

Proof of lemma 5.1: We have by eq. (5.5) and the definitions of the Moebius transforms $\widetilde{\mathcal{R}}_\lambda$ and $\widetilde{R}_\lambda$

$$\begin{aligned} B^u_{\lambda,\lambda'}(X|\Psi) := \sum_{Y\in\mathcal{P}(X)} \widetilde{\mathcal{R}}_{\lambda,Y}\Big\{ -Z_\lambda(.) + \sum_{P:P\subseteq\overline{x}} \sum_{y\in\overline{x}-P\mapsto Q_y\in\mathcal{P}(X_y)} \\ \mu_\gamma(Z_{\lambda'})(.)^{|\overline{x}-P|} \prod_{y\in P} \mu_\gamma(\widetilde{R}_{\lambda'}(Q_y|.))\Big\}(\Psi). \end{aligned} \tag{5.15}$$

We may replace the second and third sum on the rhs of eq. (5.15) by a sum over all polymers, which are contained in X,

$$B^u_{\lambda,\lambda'}(X|\Psi) = \sum_{Y\in\mathcal{P}(X)} \widetilde{\mathcal{R}}_{\lambda,Y}\Big\{-Z_\lambda(.) + \sum_{P:P\in\mathcal{P}(X)} \mu_\gamma(Z_{\lambda'})(.)^{|\{y\in\overline{x}|\ P_y=\emptyset\}|} \prod_{\substack{y\in\overline{x}:\\ P_y\neq\emptyset}} \mu_\gamma(\widetilde{R}_{\lambda'}(P_y|.))\Big\}(\Psi). \tag{5.16}$$

Furthermore,

$$B^u_{\lambda,\lambda'}(X|\Psi) = \sum_{Q\in\mathcal{P}(X)} \Big[-\widetilde{\mathcal{R}}_{\lambda,Q}\{Z_\lambda(.)\}(\Psi) + \sum_{\substack{Y,P\in\mathcal{P}(Q):\\ Y\cup P=Q}} \widetilde{\mathcal{R}}_{\lambda,Y}\Big\{\mu_\gamma(Z_{\lambda'})(.)^{|\{y\in\overline{x}|\ P_y=\emptyset\}|} \prod_{\substack{y\in\overline{x}:\\ P_y\neq\emptyset}} \mu_\gamma(\widetilde{R}_{\lambda'}(P_y|.))\Big\}(\Psi)\Big]. \tag{5.17}$$

This proves the eq. (5.6). If $\mathcal{L}_{\lambda,X}$ is polymer-independent, then by eqs. (5.3) and (5.6) follows eq. (5.7). □

Proof of lemma 5.2 : Using definitions (5.5) and (5.8), we obtain

$$\begin{aligned} B_{\lambda,\lambda'}(X|\Psi) &= -\mathcal{R}_{\lambda,X}\{Z_\lambda(.)\}(\Psi) \\ &\quad + c_0(X)[\mathcal{R}_{\lambda,X}\{Z_\lambda(.)\}(\Psi) + B^u_{\lambda,\lambda'}(X|\Psi)] \\ &= \sum_{Y\in\mathcal{P}(X)} \Big[-\widetilde{\mathcal{R}}_{\lambda,Y}\{Z_\lambda(.)\}(\Psi) \\ &\quad + \sum_{\substack{P_1,P_2\in\mathcal{P}(X):\\ P_1\cup P_2=Y}} \Big[\widetilde{c}_0(P_1)\Big(\mathcal{R}_{\lambda,P_2}\{Z_\lambda(.)\}(\Psi) + B^u_{\lambda,\lambda'}(P_2|\Psi)\Big)\Big]\Big]. \end{aligned} \tag{5.18}$$

This proves eq. (5.9). If $\mathcal{L}_{\lambda,X}$ is polymer-independent, then by eqs. (5.3) and (5.9) follow eq. (5.10). □

Proof of lemma 5.3 : From the definitions of $\widetilde{B}_\lambda, \widetilde{\widetilde{B}}_\lambda$ and B_λ follow

$$\begin{aligned} \sum_{X'\in\mathcal{P}(X)} \sum_{Q\in\mathcal{P}(X')} \widetilde{\widetilde{B}}_\lambda(X',Q|\Psi) &= \sum_{Q\in\mathcal{P}(X)} \sum_{X'\in\mathcal{P}(X,Q)} \widetilde{\widetilde{B}}_\lambda(X',Q|\Psi) \\ &= \sum_{Q\in\mathcal{P}(X)} \widetilde{B}_{\lambda,F_X(\lambda)}(Q|\Psi) = B_{\lambda,F_X(\lambda)}(X|\Psi) \\ &= R_\lambda(X|\Psi) = \sum_{X'\in\mathcal{P}(X)} \widetilde{R}_\lambda(X'|\Psi). \end{aligned} \tag{5.19}$$

This proves the assertion. □

The scheme for a recursive construction of the Moebius transform $\widetilde{R}_\lambda$ is

$$\widetilde{R}_\lambda \mapsto \widetilde{B}^u_{\lambda,\lambda'} \mapsto \widetilde{B}_{\lambda,\lambda'} \mapsto \widetilde{\widetilde{B}}_\lambda \mapsto \widetilde{R}_\lambda.$$

5.2. (n)-Renormalization part polymer system

In this section, we shall show how to construct the Moebius transform ${}_n\widetilde{R}_\lambda$ recursively. Since the steps for the calculation of ${}_n\widetilde{R}_\lambda$ are in analogy with the steps in the foregoing section, we state the formulas for the Moebius transforms without proofs.

We use the following abbreviation. For a function $F\colon \mathbf{R}^M \to \mathbf{R}$ and

$$y_n \underline{\in} \dots \underline{\in} y_1 \underline{\in} y_0 = x, \quad X := \{x, y_1, \dots, y_n\}, \quad \underline{\lambda} := (\lambda_1, \dots, \lambda_n),$$

define

$$\begin{aligned}\mu^{(n)}_{x,\underline{\lambda}}(F) := &\Big[\prod_{y\in\overline{X}^{\leq n-1}} \int d\mu_\gamma(\Phi_y)\Big] \\ &\Big[\prod_{k=0}^{n-1} \prod_{y\in\overline{y}_k-\{y_{k+1}\}} Z_{\lambda_{k+1}}(\Phi_y + \sum_{i=1}^{k} \Phi^{(k+1)}_{y_i} + L^{(k+1)(1-d/2)}(.))\Big] \\ &F(\sum_{i=1}^{k} \Phi^{(n)}_{y_i} + L^{n(1-d/2)}(.)).\end{aligned} \tag{5.20}$$

Let ${}_n\widetilde{\mathcal{R}}$ be the $h_n(\mathcal{P})$-Moebius transform of the operator $\mathcal{R} = \mathbf{1} - \mathcal{L}$. We shall define ${}_nB^u_{\lambda,\lambda'}(X|\Psi)$ and ${}_nB_{\lambda,\lambda'}(X|\Psi)$, for all $X \in h_n(\mathcal{P})$ and $\lambda \in \mathcal{O}$. If $|\{y \in \overline{x}|\ X_y \neq \emptyset\}| \geq 2$ or $X = [[y], x] + X_y, y \in X, |j(y) - j(x)| < n$, then ${}_nB_{\lambda,\lambda'}(X|\Psi)$ (${}_nB^u_{\lambda,\lambda'}(X|\Psi)$) is equal to $\mathcal{R}_{\lambda,X}$ of the rhs of eq. (3.45), where $F_X(\lambda) = \lambda'$ ($c_0(X) = 1, F_X(\lambda) = \lambda'$). If $X = \{x\} + X_{y_n}$, $|j(y_n) - j(x)| = n$, then $\lambda' = (\lambda_1, \dots, \lambda_n) \in \mathcal{O}^n$,

$${}_nB_{\lambda,\lambda'}(X|\Psi) := {}_n\mathcal{R}_{\lambda,X}\Big\{-Z_\lambda(.) + c^{(n)}_0(X)\mu^{(n)}_{x,\underline{\lambda}}\Big(Z_{\lambda_n}(.) + {}_nR_{\lambda_n}(X_{y_n}|.)\Big)\Big\}. \tag{5.21}$$

${}_nB^u_{\lambda,\lambda'}(X|\Psi)$ is equal to the rhs of eq. (5.21), where $c^{(n)}_0(X) = 1$.

1. Step (Integration):

Let ${}_n\widetilde{B}^u_{\lambda,\lambda'}$ be the Moebius transform of ${}_nB^u_{\lambda,\lambda'}$. The 1. step is given by the relation ${}_n\widetilde{R}_\lambda \mapsto {}_n\widetilde{B}^u_{\lambda,\lambda'}$. Suppose that $X \in h_n(\mathcal{P})$. If $|\{y \in \overline{x}|\ X_y \neq \emptyset\}| \geq 2$ or $X = [[y], x] + X_y, y \in X, |j(y) - j(x)| < n$, then

$$
\begin{aligned}
{}_n\widetilde{B}^u_{\lambda,\lambda'}(X|\Psi) = &- {}_n\widetilde{\mathcal{R}}_{\lambda,X}\{Z_\lambda(.)\}(\Psi) \\
&+ \sum_{\substack{Y,P\in\mathcal{P}(X):\\ Y\cup P=X}} {}_n\widetilde{\mathcal{R}}_{\lambda,Y}\left\{\mu_\gamma(Z_{\lambda'})(.)^{|\{y\in\overline{x}|\ P_y=\emptyset\}|} \prod_{\substack{y\in\overline{x}:\\ P_y\neq\emptyset}} \mu_\gamma(\widetilde{R}_{\lambda'}(P_y|.))\right\}(\Psi).
\end{aligned}
\tag{5.22}
$$

If $X = \{x\} + X_{y_n}$, $|j(y_n) - j(x)| = n$, then

$$
\begin{aligned}
{}_n\widetilde{B}_{\lambda,\lambda'}(X|\Psi) :=& {}_n\widetilde{\mathcal{R}}_{\lambda,X}\{-Z_\lambda(.) + \mu^{(n)}_{x,\underline{\lambda}}(Z_{\lambda_n}(.))\}(\Psi) \\
&+ \sum_{\substack{Y,P\in h_n(\mathcal{P})(X):\\ Y\cup P=X}} {}_n\widetilde{\mathcal{R}}_{\lambda,Y}\{\mu^{(n)}_{x,\underline{\lambda}}(\widetilde{R}_{\lambda_n}(P_{y_n}|.))\}(\Psi).
\end{aligned}
\tag{5.23}
$$

In the next step, we have to take the normalization constants ${}_nc_0(X)$ into account.

2. Step (Normalization):

Let ${}_n\widetilde{c}_0$ be the $h_n(\mathcal{P})$-Moebius transform of the normalization constant. Then in analogy with lemma 5.2, we obtain the relation ${}_n\widetilde{B}^u_{\lambda,\lambda'} \mapsto {}_n\widetilde{B}_{\lambda,\lambda'}$.

3. Step (Renormalization):

By renormalization condition (4.11), we see

$$
\mathcal{L}_{\lambda,X}\{{}_nB_{\lambda,\lambda'}(X|.)\}|_{\lambda'=(F^{(1)}_X(\lambda),\ldots,F^{(n)}_X(\lambda))} \equiv 0 \tag{5.24}
$$

and therefore, by definition,

$$
{}_nB_{\lambda,\lambda'}(X|\Psi)|_{\lambda'=(F^{(1)}_X(\lambda),\ldots,F^{(n)}_X(\lambda))} = {}_nR_\lambda(X|\Psi), \tag{5.25}
$$

where $F^{(k)}_X$, $k \in \{1,\ldots,n\}$ is defined by eq. (3.33). For a fixed polymer X', define the Moebius transform ${}_n\widetilde{\widetilde{B}}_\lambda(., X'|\Psi)$ of $\widetilde{B}_{\lambda,\lambda'}(X'|\Psi)$, where

$$
\lambda' = (F^{(1)}_X(\lambda),\ldots,F^{(n)}_X(\lambda)),
$$

for all $X \in h_n(\mathcal{P})$ with $X \supseteq X'$, by

$$
{}_n\widetilde{B}_{\lambda,\lambda'}(X'|\Psi)|_{\lambda'=(F^{(1)}_X(\lambda),\ldots,F^{(n)}_X(\lambda))} = \sum_{Q\in\mathcal{P}(X,X')} \widetilde{\widetilde{B}}_\lambda(Q, X'|\Psi). \tag{5.26}
$$

Then, the 3. step is given by ${}_n\widetilde{B}_{\lambda,\lambda'} \mapsto {}_n\widetilde{\widetilde{B}}_\lambda$.

4. Step (Induction):

The last step is given by the relation ${}_n\widetilde{\widetilde{B}}_\lambda \mapsto {}_n\widetilde{R}_\lambda$. In analogy with lemma 5.3, we have, for all $X \in h_n(\mathcal{P})$,

$$ {}_n\widetilde{R}_\lambda(X|\Psi) = \sum_{Q \in h_n(\mathcal{P}(X))} {}_n\widetilde{\widetilde{B}}_\lambda(X, Q|\Psi). \tag{5.27} $$

The scheme for a recursive construction of the Moebius transform ${}_n\widetilde{R}_\lambda$ is

$$ {}_n\widetilde{R}_\lambda \mapsto {}_n\widetilde{B}^u_{\lambda,\lambda'} \mapsto {}_n\widetilde{B}_{\lambda,\lambda'} \mapsto {}_n\widetilde{\widetilde{B}}_\lambda \mapsto {}_n\widetilde{R}_\lambda. $$

6. Explicit representations of Moebius transforms

The Moebius transform $\widetilde{B}: \mathcal{P} \to \mathbf{C}$ of a function $B: \mathcal{P} \to \mathbf{C}$ can be expressed by an alternating sum in terms of $B(P)$, $P \in \mathcal{P}$. Since alternating sums are hard to estimate, we shall present in this section a more convenient way to express a Moebius transform in terms of B. We shall answer the question how to relate the Moebius transform $\widetilde{E}$ and $\widetilde{e}$, if E and e are related by $E = H \circ e$ where $H: \mathbf{C} \to \mathbf{C}$ is a differentiable function . This problem is solved by proposition 6.1 which gives a cluster expansion formula for $\widetilde{E}$ in terms of $\widetilde{e}$. We are going to apply this formula to the Moebius transforms $\widetilde{\widetilde{f}}^{(i)}_X$ and $\widetilde{\widetilde{B}}_X$ (see corollary 6.4).

In the following, we introduce some notations and summarize properties of Moebius transforms. We show that Moebius transforms and Fourier transforms share analogous properties.

For $B \in M(\mathcal{P}, \mathbf{C})$, where

$$ M(\mathcal{P}, \mathbf{C}) := \{B : \mathcal{P} \to \mathbf{C}\}, \tag{6.1} $$

let $\mathcal{M}(B)$ be the Moebius transform of B. The inverse $\mathcal{M}^{-1}$ of $\mathcal{M}$ is given by

$$ \mathcal{M}^{-1}(B)(X) = \sum_{Y \in \mathcal{P}(X)} B(Y). \tag{6.2} $$

This is an immediate consequence of the definition 2.8 of Moebius transforms in section 2. Let $\mathcal{P} \subseteq \mathcal{P}_{fin} := \mathcal{P}_{fin}(\Lambda) - \{\emptyset\}$ be a general polymer set. Suppose that

$(\mathcal{P}, \subseteq)$ contains a least element $X^* \in \mathcal{P}$, i.e. $X^* \subseteq P$, for all $P \in \mathcal{P}$. Then for the function $I \in M(\mathcal{P}, \mathbf{C})$ defined by $I(X) := 1$, for all $X \in \mathcal{P}$, the Moebius transform $\widetilde{I} = \mathcal{M}(I)$ is $\widetilde{I} = \delta_{X^*}$, where

$$\delta_X(Y) = \delta_{X,Y}, \tag{6.3}$$

for all $X, Y \in \mathcal{P}$. This corresponds to the property of the Fourier transform, that the Fourier transform of 1 is the Dirac distribution.

We shall define an algebra $\mathcal{A}(\mathcal{P})$ for polymer sets $\mathcal{P}$. $M(\mathcal{P}, \mathbf{C})$ is a vector space with a sum and scalar product defined by

$$(B_1 + B_2)(X) := B_1(X) + B_2(X), \qquad (\lambda B)(X) := \lambda B(X),$$

for all $B, B_1, B_2 \in M(\mathcal{P}, \mathbf{C}), X \in \mathcal{P}, \lambda \in \mathbf{C}$. Then $\{\delta_P, P \in \mathcal{P}\}$ is a basis of $M(\mathcal{P}, \mathbf{C})$. Define a product $\cdot$ by

$$(B_1 \cdot B_2)(X) := B_1(X) B_2(X). \tag{6.4}$$

Then $\mathcal{A}(\mathcal{P}) := (M(\mathcal{P}, \mathbf{C}), \cdot, +)$ is an algebra . The Moebius transform $\mathcal{M}\colon \mathcal{A}(\mathcal{P}) \to \mathcal{A}(\mathcal{P})$ is a linear mapping. Let us remark that $\mathcal{A}(\mathcal{P})$ is a commutative algebra with identity I, in the case that $(\mathcal{P}, \subseteq)$ contains a least element.

Define a convolution-type multiplication $*$ in $M(\mathcal{P}, \mathbf{C})$ by

$$(B_1 * B_2)(X) := \sum_{\substack{P_1, P_2 \in \mathcal{P}(X): \\ P_1 + P_2 = X}} B_1(P_1) B_2(P_2), \tag{6.5}$$

for all $X \in \mathcal{P}$. Then $\mathcal{A}_*(\mathcal{P}) := (M(\mathcal{P}, \mathbf{C}), *, +)$ is a commutative algebra. Let us notice that the convolution-type multiplication in $M(\mathcal{P}, \mathbf{C})$ is trivial, if there exists no disjoint $P_1, P_2 \in \mathcal{P}$. $\mathcal{A}_*(\mathcal{P}_{fin})$ contains no identity element if $|\Lambda| \geq 2$. But $\delta_\emptyset$ is the identity of the algebra $\mathcal{A}_*(\mathcal{P}_{fin} \cup \{\emptyset\})$.

Define a homomorphism $\widehat{\ }\colon (\mathbf{C}, +) \to (M(\mathcal{P}_{fin}, \mathbf{C}), *)$ by

$$\widehat{q}\,(X) := q^{|X|}, \tag{6.6}$$

for all $q \in \mathbf{C}, X \in \mathcal{P}_{fin}$. Define the function $\mathcal{M}_{fin}\colon M(\mathcal{P}_{fin}, \mathbf{C}) \to M(\mathcal{P}_{fin}, \mathbf{C})$ by

$$\mathcal{M}_{fin}(B) := \widehat{(-1)} * B, \tag{6.7}$$

for all $B \in M(\mathcal{P}_{fin}, \mathbf{C})$. Since $\widehat{\ }$ is a homomorphism, we obtain

$$\mathcal{M}_{fin}^{-1}(B) = \widehat{(+1)} * B, \tag{6.8}$$

for all $B \in M(\mathcal{P}_{fin}, \mathbf{C})$. Define the injective mapping $\iota: M(\mathcal{P}, \mathbf{C}) \to M(\mathcal{P}_{fin}, \mathbf{C})$ by

$$\iota(B)(P) := \begin{cases} B(P), & \text{if } P \in \mathcal{P}; \\ 0, & \text{otherwise} \end{cases} \tag{6.9}$$

and the characteristic function $\chi_{\mathcal{P}} \in M(\mathcal{P}_{fin}, \mathbf{C})$ by

$$\chi_{\mathcal{P}}(P) := \begin{cases} 1, & \text{if } P \in \mathcal{P}; \\ 0, & \text{otherwise.} \end{cases} \tag{6.10}$$

We get the following relation for the Moebius transform $\mathcal{M}$

$$\mathcal{M}(B) = \chi_{\mathcal{P}} \cdot \mathcal{M}_{fin}(\iota(B)), \tag{6.11}$$

for all $B \in M(\mathcal{P}, \mathbf{C})$. Thus we get by eqs. (6.7) and (6.8)

$$\mathcal{M}(B)(X) = \sum_{Y: Y \in \mathcal{P}(X)} (-1)^{|X|-|Y|} B(Y). \tag{6.12}$$

We have used the explicit expression

$$(\widehat{q} * B)(X) = \sum_{Y: Y \subseteq X} q^{|X|-|Y|} B(Y), \tag{6.13}$$

for all $q \in \mathbf{C}, B \in M(\mathcal{P}_{fin}, \mathbf{C}), X \in \mathcal{P}_{fin}$.

We may define another convolution-type multiplication $*_\cup$ in $M(\mathcal{P}, \mathbf{C})$, for all $\mathcal{P} \subseteq \mathcal{P}_{fin}$ by

$$(B_1 *_\cup B_2)(X) := \sum_{\substack{P_1, P_2 \subseteq \mathcal{P}(X): \\ P_1 \cup P_2 = X}} B_1(P_1) B_2(P_2), \tag{6.14}$$

for all $B_1, B_2 \in M(\mathcal{P}, \mathbf{C})$ and $X \in \mathcal{P}$. Then $\mathcal{A}_{*_\cup}(\mathcal{P}) := (M(\mathcal{P}, \mathbf{C}), *_\cup, +)$ is a commutative algebra. $\delta_\emptyset$ is the identity of $\mathcal{A}_{*_\cup}(\mathcal{P}_{fin} \cup \{\emptyset\})$ and δ_{X^*} defined by eq. (6.3) is the identity of $\mathcal{A}_{*_\cup}(\mathcal{P})$, if $X^* \in \mathcal{P}$ is the least element of $\mathcal{P}$. We obtain the following relations

$$\mathcal{M}^{-1}(B_1 *_\cup B_2) = \mathcal{M}^{-1}(B_1) \cdot \mathcal{M}^{-1}(B_2) \tag{6.15}$$

and

$$\mathcal{M}(B_1 \cdot B_2) = \mathcal{M}(B_1) *_\cup \mathcal{M}(B_2), \tag{6.16}$$

for all $B_1, B_2 \in M(\mathcal{P}, \mathbf{C})$. These equations correspond to the property of Fourier transforms that the Fourier transform of a product is equal to the convolution product of the Fourier transforms.

In the following, suppose that a polymer system on Λ is given by a subset $\mathcal{P}$ of $\mathcal{P}_{fin}$, such that

$$\bigcap_{P \in \mathcal{P}} P =: X^* \neq \emptyset \tag{6.17}$$

and $X^* \in \mathcal{P}$, i.e. X^* is the least element of $(\mathcal{P}, \subseteq)$. Consider the following problem. Suppose that the functions $H: \mathbf{C} \to \mathbf{C}$, $e: \mathcal{P} \to \mathbf{C}$ and $E: \mathcal{P} \to \mathbf{C}$ are related by

$$E(X) = H(e(X)), \tag{6.18}$$

for all $X \in \mathcal{P}$. How is the Moebius transform $\widetilde{E}$ of E related to the Moebius transform $\widetilde{e}$ of e ? Define the induced mapping $H_*: M(\mathcal{P}, \mathbf{C}) \to M(\mathcal{P}, \mathbf{C})$ by $H_*(e) := H \circ e$ for all $e \in M(\mathcal{P}, \mathbf{C})$. We are looking for a functional $\widetilde{H}_*: M(\mathcal{P}, \mathbf{C}) \to M(\mathcal{P}, \mathbf{C})$, such that

$$\widetilde{H_*(e)} = \widetilde{H}_*(\widetilde{e}) \tag{6.19}$$

or equivalently

$$\widetilde{H}_* = \mathcal{M} \circ H_* \circ \mathcal{M}^{-1}, \tag{6.20}$$

where $\mathcal{M}$ is the Moebius transform. To formulate an answer of this question, we introduce some definitions and notations. Define the set of all *n-clusters* by

$$\mathcal{C}_n(X^*) := \{(P_1, \ldots, P_n) |\ P_i \in \mathcal{P}, P_1 = X^*, P_{i+1} \cap \Lambda - \bigcup_{a=1}^{i} P_a \neq \emptyset\} \tag{6.21}$$

and for $X \subseteq \Lambda$, the set of all *n-clusters in X*

$$\overline{\mathcal{C}}_n(X, X^*) := \{\mathbf{P} \in \mathcal{C}_n(X^*) |\ supp\ \mathbf{P} \subseteq X\}, \tag{6.22}$$

where $supp\ (P_1, \ldots, P_n) := \bigcup_{a=1}^{n} P_a$. The set of all *complete n-clusters* in X is defined by

$$\mathcal{C}_n(X, X^*) := \{\mathbf{P} \in \mathcal{C}_n |\ supp\ \mathbf{P} = X\}. \tag{6.23}$$

We omit X^* in the notations of n-clusters, i.e. write in the following, for notational simplicity, $\mathcal{C}_n(X^*) \equiv \mathcal{C}_n$, $\overline{\mathcal{C}}_n(X, X^*) \equiv \overline{\mathcal{C}}_n(X)$ and $\mathcal{C}_n(X, X^*) \equiv \mathcal{C}_n(X)$.

For an n-cluster $\mathbf{P} = (P_1, \ldots, P_n)$ and parameters $s_1, \ldots, s_{n-1} \in [0,1]$ and a function $\widetilde{e}: \mathcal{P} \to \mathbf{C}$, define recursively

$$\widetilde{e}_{s_1,\ldots,s_k,\mathbf{P}}(Y) := \begin{cases} \widetilde{e}_{s_1,\ldots,s_{k-1},\mathbf{P}}(Y), & \text{if } Y \subseteq \bigcup_{a=1}^{k} P_a; \\ s_k \widetilde{e}_{s_1,\ldots,s_{k-1},\mathbf{P}}(Y), & \text{otherwise,} \end{cases} \tag{6.24}$$

for all $k \in \{1, \ldots, n-1\}, Y \in \mathcal{P}$. We use the notations

$$\begin{aligned} \underline{s}_{n-1} &:= (s_1, \ldots, s_{n-1}) \\ \int d\underline{s}_{n-1} &:= \int_0^1 ds_1 \ldots ds_{n-1} \\ f(\underline{s}_{n-1}) &:= \prod_{a=3}^{n} (s_1 \ldots s_{a-2}). \end{aligned} \tag{6.25}$$

When it is clear from the context or when it is irrelevant what n is, we omit the subscript $n-1$, i.e. we write $\underline{s}_{n-1} = \underline{s}$. Define $e_{\underline{s},\mathbf{P}}: \mathcal{P} \to \mathbf{C}$ by

$$e_{\underline{s},\mathbf{P}}(X) := \sum_{Y \in \mathcal{P}(X)} \widetilde{e}_{\underline{s},\mathbf{P}}(Y), \tag{6.26}$$

i.e. $\widetilde{e}_{\underline{s},\mathbf{P}}$ is the Moebius transform of $e_{\underline{s},\mathbf{P}}$. $e_{\underline{s},\mathbf{P}}$ is called the *interpolated function* of e.

The answer of our foregoing question, i.e. the relation of $\widetilde{E}$ and $\widetilde{e}$ is given in

Proposition 6.1. *Let $H: \mathbf{C} \to \mathbf{C}$ be a smooth function and $E, e \in M(\mathcal{P}, \mathbf{C})$ be complex valued functions, such that*

$$E = H \circ e. \tag{6.27}$$

Then, we obtain

$$\widetilde{E} = \widetilde{H}_*(\widetilde{e}), \tag{6.28}$$

where $\widetilde{H}_*: M(\mathcal{P}, \mathbf{C}) \to M(\mathcal{P}, \mathbf{C})$ is defined by

$$\widetilde{H}_*(B)(X) := \sum_{n \geq 1} \sum_{\substack{\mathbf{P} \in \mathcal{C}_n(X): \\ \mathbf{P} = (P_1, \ldots, P_n)}} \int d\underline{s} f(\underline{s}) [\prod_{a=2}^{n} B(P_a)] \partial_\lambda^{n-1} H(\lambda)|_{\lambda = \mathcal{M}^{-1}(B)_{\underline{s},\mathbf{P}}(X)}, \tag{6.29}$$

for all $X \in \mathcal{P}$ and $B \in M(\mathcal{P}, \mathbf{C})$.

In order to prove proposition 6.1, we will show $\widetilde{E} = \widetilde{H}(\widetilde{e})$ for the special case $H = \exp$.

Lemma 6.2 . *Let $E, e \in M(\mathcal{P}, \mathbf{C})$ be complex valued functions, such that*

$$E(X) = \exp\{e(X)\}, \tag{6.30}$$

for all $X \in \mathcal{P}$. Then, we have for all $k \in \{1, 2, \ldots\}$,

$$\begin{aligned} E(X) = &\sum_{n=1}^{k-1} \sum_{\substack{\mathbf{P} \in \overline{\mathcal{C}}_n(X): \\ \mathbf{P}=(P_1,\ldots,P_n)}} \int d\underline{s}_{n-1} f(\underline{s}_{n-1}) [\prod_{a=2}^{n} \widetilde{e}(P_a)] \exp\{e_{\underline{s}_{n-1},\mathbf{P}}(supp\ \mathbf{P})\} \\ &+ \sum_{\substack{\mathbf{P} \in \overline{\mathcal{C}}_k(X): \\ \mathbf{P}=(P_1,\ldots,P_k)}} \int d\underline{s}_{k-1} f(\underline{s}_{k-1}) [\prod_{a=2}^{k} \widetilde{e}(P_a)] \exp\{e_{\underline{s}_{k-1},\mathbf{P}}(X)\}. \end{aligned} \tag{6.31}$$

Remark: For $k > |X|$, we have $\overline{\mathcal{C}}_k(X) = \emptyset$. Thus, the second term in eq. (6.31) vanishes. Using the definition 2.8 of the Moebius transform $\widetilde{E}$, we see then by eq. (6.31) that proposition 6.1 holds, for $H = \exp$.

Proof of lemma 6.2: The proof is done by induction. The assertion holds trivially for $k = 1$. Suppose that eq. (6.31) is valid for k. For $\mathbf{P} = (P_1, \ldots, P_k) \in \mathcal{C}_k(X)$ and $\underline{s}_{k-1} = (s_1, \ldots, s_{k-1}), s_k \in [0, 1]$, define

$$\widetilde{e}_{\underline{s}_{k-1},s_k,\mathbf{P}}(Y) := \begin{cases} \widetilde{e}_{\underline{s}_{k-1},\mathbf{P}}(Y), & \text{if } Y \subseteq P_1 \cup \ldots \cup P_k; \\ s_k \widetilde{e}_{\underline{s}_{k-1},\mathbf{P}}(Y), & \text{otherwise} \end{cases} \tag{6.32}$$

and

$$e_{\underline{s}_{k-1},s_k,\mathbf{P}}(X) := \sum_{Y \in \mathcal{P}(X)} \widetilde{e}_{\underline{s}_{k-1},s_k,\mathbf{P}}(Y). \tag{6.33}$$

Then,

$$\begin{aligned} \exp\{e_{\underline{s}_{k-1},\mathbf{P}}(X)\} &= \exp\{e_{\underline{s}_{k-1},s_k,\mathbf{P}}(X)\}|_{s_k=1} \\ &= \exp\{e_{\underline{s}_{k-1},\mathbf{P}}(supp\ \mathbf{P})\} + \int_0^1 ds_k \partial_{s_k} \exp\{e_{\underline{s}_{k-1},s_k,\mathbf{P}}(X)\} \\ &= \exp\{e_{\underline{s}_{k-1},\mathbf{P}}(supp\ \mathbf{P})\} \\ &+ \sum_{P_{k+1}:P_{k+1} \in \mathcal{P}(X)} \int_0^1 ds_k \partial_{s_k} \widetilde{e}_{\underline{s}_k,\mathbf{P}}(P_{k+1}) \exp\{e_{\underline{s}_k,\mathbf{P}'}(X)\}, \end{aligned} \tag{6.34}$$

where $\underline{s}_k = (s_1, \ldots, s_k), \mathbf{P}' := (P_1, \ldots, P_k, P_{k+1})$. Since

$$\partial_{s_k} \widetilde{e}_{\underline{s}_k, \mathbf{P}}(P_{k+1}) = \begin{cases} 0, & \text{if } \mathbf{P}' \notin \mathcal{C}_{k+1}(X); \\ s_1 \ldots s_{k-1} \widetilde{e}(P_{k+1}), & \text{if } \mathbf{P}' \in \mathcal{C}_{k+1}(X), \end{cases} \tag{6.35}$$

we get by inserting eq. (6.34) into the second term on the rhs of eq. (6.31), the assertion for $k+1$. □

Proof of proposition 6.1: Eq. (6.27) is equivalent to

$$E(X) = \exp\{e(X)\frac{\partial}{\partial\lambda}\}\ H(\lambda)|_{\lambda=0}. \tag{6.36}$$

From the remark after lemma 6.2 follows

$$\begin{aligned} \exp\{e(X)\frac{\partial}{\partial\lambda}\}\ H(\lambda)|_{\lambda=0} = & \sum_{X' \in \mathcal{P}(X)} \sum_{n \geq 1} \sum_{\substack{\mathbf{P} \in \mathcal{C}_n(X'): \\ \mathbf{P} = (P_1, \ldots, P_n)}} \int d\underline{s} f(\underline{s}) \\ & [\prod_{a=2}^{n} \widetilde{e}(P_a)] \partial_\lambda^{n-1} H(\lambda)|_{\lambda = e_{\underline{s}, \mathbf{P}}(X)}. \end{aligned} \tag{6.37}$$

This proves the assertion. □

For bounds on Moebius transforms, the following lemma is useful.

Lemma 6.3 *For all $n \geq 2$, we have*

$$\int d\underline{s}_{n-1} f(\underline{s}_{n-1}) = \frac{1}{(n-1)!}. \tag{6.38}$$

Proof: Use

$$f(\underline{s}_{n-1}) = \prod_{a=3}^{n} (s_1 \cdots s_{n-2}) = s_1^{n-2} s_2^{n-3} \cdots s_{n-2}^1. \tag{6.39}$$

This proves the assertion. □

We state as an application of the abstract cluster expansion formula eq. (6.29) the following

Corollary 6.4. *Suppose that $\widetilde{\widetilde{f}}^{(i)}$ and $\widetilde{\widetilde{B}}_\lambda$ are defined by eqs. (4.31) and (5.13). Then the following cluster expansion formulas hold, for all polymers $X, X' \in \mathcal{P}$ with $X \supseteq X'$,*

$$\widetilde{\widetilde{f}}^{(i)}_{X,X'}(\lambda) = \sum_{n \geq 1} \sum_{\substack{\mathbf{P} \in \mathcal{C}_n(X): \\ \mathbf{P} = (P_1, \ldots, P_n)}} \int d\underline{s} f(\underline{s}) [\prod_{a=2}^{n} \widetilde{\lambda}_{i,P_a}(\lambda)] \partial_{\widetilde{\lambda}}^{n-1} \widetilde{f}^{(i)}_{X'}(\lambda, \widetilde{\lambda})|_{\widetilde{\lambda} = (\lambda_{i,X})_{\underline{s},\mathbf{P}}(\lambda)}, \tag{6.40}$$

$$\widetilde{\widetilde{B}}_\lambda(X,X'|\Psi)=\sum_{n\geq 1}\sum_{\substack{\mathbf{P}\in\mathcal{C}_n(X):\\ \mathbf{P}=(P_1,\ldots,P_n)}}\int d\underline{s}f(\underline{s})[\prod_{a=2}^{n}\widetilde{F}_{P_a}(\lambda)]\partial_{\widetilde{\lambda}}^{n-1}\widetilde{B}_{\lambda,\widetilde{\lambda}}(X'|\Psi)|_{\widetilde{\lambda}=(F_X)_{\underline{s},\mathbf{P}}(\lambda)},\tag{6.41}$$

where $\mathcal{C}_n(X)\equiv\mathcal{C}_n(X,X')$.

In some cases, the conservation of positivity in cluster expansion formulas is important. Suppose that $e\in M(\mathcal{P},\mathbf{C})$ is positive, i.e. $e(P)\in\mathbf{R}_+$, for all $P\in\mathcal{P}$. Then the interpolating function $e_{\underline{s},\mathbf{P}}$ is positive, for all n-clusters $\mathbf{P}=(P_1,\ldots,P_n)$ and $\underline{s}=(s_1,\ldots,s_{n-1})$. This is shown by

$$e_{\underline{s}_k,\mathbf{P}}(X)=s_k e_{\underline{s}_{k-1},\mathbf{P}}(X)+(1-s_k)\big[e_{\underline{s}_{k-1},\mathbf{P}}(\bigcup_{a=1}^{k}P_a)+e_{\underline{s}_{k-1},\mathbf{P}}(X-\bigcup_{a=1}^{k}P_a)\big],\tag{6.42}$$

for all $X\supset\bigcup_{a=1}^{k}P_a$ and

$$e_{\underline{s}_k,\mathbf{P}}(X)=e_{\underline{s}_{k-1},\mathbf{P}}(X),\tag{6.43}$$

for all $X\subseteq\bigcup_{a=1}^{k}P_a$. We have used here the abbreviation $\underline{s}_k:=(s_1,\ldots,s_k)$. In other words, $e_{\underline{s},\mathbf{P}}$ is constructed by a successive application of linear convex combinations. Since linear convex combinations preserve positivity, we see by induction that the interpolating function $e_{\underline{s},\mathbf{P}}$ is positive, for positive e.

7. Summary and Outlook

We have studied in this paper the renormalization theory of models in hierarchical approximation. Iteration of the hierarchical renormalization group equation leeds to definitions of multigrid polymer systems. We have presented two versions of introducing an ultraviolet cutoff for multigrid polymer systems. The first (more conventional) version of a polymer system, was defined by starting the renormalization group equations with a "bare" Boltzmann factor, given by "bare" coupling constants. In the second version, an explicit use of an ultraviolet cutoff was not necessary. Instead of "bare" coupling constants, we introduced polymer-dependent renormalization group flow functions. These renormalization group flow functions relate the corresponding coupling constants before and after one renormalization group step. Using these functions, we may reconstruct the "bare" coupling constants. The ultraviolet cutoff in this version is given by the finiteness of the polymers. The ultraviolet limit is equivalent to the corresponding thermodynamic

limit of the multigrid polymer system. In later sections, we used only the second version of a multigrid polymer system.

We have studied the use of renormalization conditions of polymer activities. The renormalization conditions determine the renormalization group flow functions (or some components of it). The renormalization conditions were formulated by requiring that the application of an operator $\mathcal{L}_{\lambda,X}$ to the irrelevant activities $R_\lambda(X|\Psi)$ gives zero. As examples of the operator $\mathcal{L}_{\lambda,X}$, we considered the Φ^4-model in 2 and 3 dimensions and the 2-dimensional nonlinear $O(N)$ σ-model.

We have studied the explicit and recursive construction of Moebius transforms of irrelevant activities. For this, a general cluster expansion formula (proposition 6.1) was helpful. This recursive calculation method is necessary for a proof of recursive bounds and ultimately, the proof of the existence of the thermodynamic limit. The necessity of renormalization conditions, i.e. the introduction of operators $\mathcal{L}$, becomes evident by a thorough investigation of the recursive calculation scheme for the Moebius transform of irrelevant activities.

In the renormalization theory of hierarchical models studied here in a general point of view, there are some items lacking, which are of importance. In the following, we shall present a list of these missing items.

a) Specification of running coupling constants $\lambda \in \mathcal{O} \subseteq \mathbf{R}^m$ and the region $\mathcal{O}$; global and local renormalization group flow; distinction of relevant and irrelevant coupling constants (operators)
b) Universality classes: In which way is the model uniquely specified by the "bare" partition functions $Z_\lambda: \mathbf{R}^N \to \mathbf{R}_+$?
c) Explicit definition of the normalization constants $c_0(X)$; definition of the vacuum
d) The problem of stating recursive bounds on polymer activities
e) Large field problem, stability bounds, positivity, etc.
f) Convergent multigrid expansions of Green functions

These foregoing items have to be studied, if one is considering concrete models. These items may also serve as starting points for further investigations on renormalization theory of hierarchical models.

The strategy for a renormalization theory of hierarchical models, suggested in this paper, can also be used as an outline for a renormalization theory of complete field theoretic models (without hierarchical approximation). The consideration of

hierarchical models is not very restrictive for an investigation of a general renormalization theory.

References

[1] W. Zimmermann, Convergence of Bogoljubov's method of renormalization in momentum space, Commun. Math. Phys. 15 (1969) 208

[2] G. Gallavotti, The Structure of Renormalization Theory: Renormalization, Form Factors and Resummation in Scalar Field Theory, Les Houches 1984, in *Critical Phenomena, Random Systems, Gauge Theories*

[3] G. Gallavotti, Renormalization Theory and Ultraviolet Stability for Scalar Fields via Renormalization Group Methods, Rev. Mod. Phys. 57 (1985) 471

[4] G. Gallavotti, F. Nicolò, Renormalization Theory in Four Dimensional Scalar Fields (I), Commun. Math. Phys. 100 (1985) 545

[5] G. Gallavotti, F. Nicolò, Renormalization Theory in Four Dimensional Scalar Fields (II), Commun. Math. Phys. 101 (1985) 247

[6] K.G. Wilson, Renormalization Group and Strong Interactions, Phys. Rev. *D*3 (1971) 1818

[7] J. Kogut, K.G. Wilson, The Renormalization Group and the ϵ-Expansion, Phys. Rep. *C*12 (1974) 75

[8] J. Glimm, A. Jaffe, The Positivity of the φ_3^4 Hamiltonian, Fortschr. Phys. 21 (1973) 327

[9] J. Glimm, A. Jaffe, Quantum Physics, second edition, Springer Verlag, Heidelberg 1987

[10] G.A. Battle, P. Federbush, A Phase Cell Cluster Expansion for Euclidean Field Theories, Ann. of Phys. 142 (1982) 95

[11] G.A. Battle, P. Federbush, A Phase Cell Cluster Expansion for a Hierarchical Φ_3^4 Model, Commun. Math. Phys. 88 (1983) 263

[12] J. Magnen, R. Sénéor, Phase Space Cell Expansion and Borel Summability of the Euclidean Φ_3^4-Theory, Commun. Math. Phys. 56 (1977) 237

[13] J. Feldman, J. Magnen, V. Rivasseau, R. Sénéor, Construction and Borel Summability of Infrared Φ_4^4 by a Phase Space Expansion, Commun. Math. Phys. 109 (1987) 137

[14] J. Gawędzki, A. Kupiainen, A Rigorous Block Spin Approach to Massless Lattice Theories, Commun. Math. Phys. 77 (1980) 31

[15] J. Gawędzki, A. Kupiainen, Block Spin Renormalization Group for Dipole Gas and $(\nabla\Phi)^4$, Ann. Phys. 147 (1983) 198

[16] J. Gawędzki, A. Kupiainen, Massless Lattice Φ_4^4 Theory: Rigorous Control of a Renormalizable Asymptotically Free Model , Commun. Math. Phys. 99 (1985) 197

[17] A. Pordt, Convergent Multigrid Polymer Expansions and Renormalization for Euclidean Field Theory, DESY 90-020 (1990)

[18] D. Brydges, H.Yau, Grad Φ Perturbations of massless Gaussian Fields, Commun. Math. Phys. 129 (1990) 351

[19] H. Kunz, Analyticity and Clustering Properties of Unbounded Spin Systems, Commun. Math. Phys. 59 (1978) 53

[20] C. Gruber, H. Kunz, General Properties of Polymer Systems, Commun. Math. Phys. 22 (1971) 133

[21] G. Mack, A. Pordt, Convergent Perturbation Expansions for Euclidean Quantum Field Theory, Commun. Math. Phys. 97 (1985) 267

[22] G. Mack, Multigrid Methods in Quantum Field Theory, Cargèse 1987, in *Nonperturbative Quantum Field Theory*, N.Y. 1988

[23] A. Pordt, Mayer Expansion for Euclidean Lattice Field Theory : Convergence Properties and Relation with Perturbation Theory, DESY 85-103 (1985)

[24] G. Mack, A. Pordt, Convergent Weak Coupling Expansions for Lattice Field Theories that look like Perturbation Series, Rev. in Math. Phys. 1 (1989) 47

[25] A. Pordt, Renormalization Theory for Use in Convergent Expansions for Euclidean Field Theory, Cargèse 1987, in *Nonperturbative Quantum Field Theory*, N.Y. 1988

[26] F.J. Dyson, Existence of a Phase Transition in a One-Dimensional Ising Ferromagnet, Commun. Math. Phys. 12 (1969) 91

[27] F.J. Dyson, Nonexistence of Spontaneous Magnetization in a One-Dimensional Ising Ferromagnet, Commun. Math. Phys. 12 (1969) 212

[28] G. Benfatto, M. Cassandro, G. Gallavotti, F. Nicolò, E. Olivieri, E.Presutti, E. Scacciatelli, On the Ultraviolet Stability in the Euclidean Scalar Field Theories, Commun. Math. Phys. 71 (1980) 343

[29] K. Gawędzki, A. Kupiainen, Continuum Limit of the Hierarchical O(N) Non Linear σ-Model, Commun. Math. Phys. 106 (1986) 533

[30] A. Pordt, T. Reiß , On the Renormalization Group Iteration of a Twodimensional Hierarchical Nonlinear O(N) σ- Model, Ann. de l'I.H.P. 55 (1991) 545

[31] K.R. Ito, Renormalization Group Flow of Two-Dimensional Hierarchical Heisenberg Model of Dyson-Wilson Type, Commun. Math. Phys. 139 (1991) 45