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Fierz transformation and Dimensional Regularization

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Abstract. We discuss an extension of the Fierz transformation to a spinor space determined by dimensional regularization.

Local four fermion interactions define renormalizable theories in two dimension [1, 2]. Some of them exhibit asymptotic freedom and dynamical mass generation, which makes them an interesting field theoretical laboratory [2]. In four dimension a four fermion operator has an engineering dimension of (mass)^6 and is termed non-renormalizable by naive power counting. However, recent developments show that some of the four fermion theories are renormalizable even in four dimension [3, 4, 5]. Moreover, they can be equivalent to well known field theoretical models. So, for example, the generalized Nambu-Jona-Lasinio model is equivalent to a local fermion-scalar field theory [4], while certain fermion theories with vector four fermion coupling are identical to standard Yang-Mills gauge theories [6].

Four fermion interactions can be constructed out of any of the covariant bilinears. Not all of these densities are independent. The Fierz transformation [7, 8] describes the relation between them. The question arises how this transformation is influenced by the presence of a regularization. When gauge symmetries are involved [6] one needs a gauge invariant regularization like dimensional regularization [9, 10, 11]. In this context one has to address the problem of the Fierz transformation for arbitrary complex dimension d. This is the problem we discuss in this paper.

Let us look first at the Fierz transformation in four dimension [7, 8]. We can construct five basic scalar tensor products in the spinor space generated by four γ matrices. In Euclidean space they are

$$\begin{aligned} J_0 &= (\mathbf{1})_{a_1a_2}(\mathbf{1})_{a_3a_4}, \\ J_1 &= (\gamma^\mu)_{a_1a_2}(\gamma^\mu)_{a_3a_4}, \\ J_2 &= \tfrac{1}{2}(\sigma^{\mu\nu})_{a_1a_2}(\sigma^{\mu\nu})_{a_3a_4}, \\ J_3 &= (i\gamma_5\gamma^\mu)_{a_1a_2}(i\gamma_5\gamma^\mu)_{a_3a_4}, \\ J_4 &= (\gamma_5)_{a_1a_2}(\gamma_5)_{a_3a_4}. \end{aligned} \tag{1}$$

Here we have implied a summation over the indices μ, ν. The unit matrix in the spinor space is denoted by $\mathbf{1}$ and $\sigma^{\mu\nu}$, γ_5 are defined by

$$\sigma^{\mu\nu} \doteq \frac{i}{2}[\gamma^\mu, \gamma^\nu], \qquad \gamma_5 \doteq \gamma^1\gamma^2\gamma^3\gamma^4. \tag{2}$$

An exchange of the two matrix indices a_2 and a_4 in any one of the above tensor products can be expressed as a linear combination of the original ones:

$$J_i' = \frac{1}{4}\sum_{j=0}^{4} m_{ij} J_j, \quad 0 \leq i \leq 4. \tag{3}$$

The tensor products J_i' are identical to the products J_i but with the indices a_2 and a_4 exchanged. The m_{ij} are elements of the Fierz matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & 0 & 2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & 2 & 0 & -2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix} \tag{4}$$

The square of this matrix is proportional to the unit matrix: $(\frac{1}{4}\mathbf{M})^2 = \mathbf{I}_5$.

We want to extend this transformation to a spinor space defined by dimensional regularization. Since in dimensional regularization the vector index μ runs over all positive integer values [10, 11], there are infinitely many γ matrices in this regularized spinor space. They satisfy the anticommutation rule

$$\{\gamma^\mu, \gamma^\nu\} = 2\,\delta_{\mu\nu}\mathbf{1}. \tag{5}$$

The trace of the metric tensor $\delta_{\mu\nu}$ is defined to be

$$\sum_{\mu=1}^{\infty} \delta_{\mu\mu} \doteq d. \tag{6}$$

The trace operation in the regularized spinor space is cyclic and linear. We shall leave the trace of the unit matrix arbitrary: $\xi(d) \doteq \mathrm{tr}(\mathbf{1})$. With these definitions the trace of any product of γ matrices is completely determined. (For a thorough discussion of dimensional regularization based on an infinite dimensional space see [11].)

In this regularized spinor space the Fierz transformation assumes the following form:

$$\frac{\alpha_m^2}{m!}\sum_{\nu_i=1}^{\infty}(\Gamma^{\nu_1\ldots\nu_m})_{a_1a_4}(\Gamma^{\nu_1\ldots\nu_m})_{a_3a_2} = 2^{-\frac{d}{2}}\sum_{l=0}^{\infty} Q_l^m(d)\frac{\alpha_l^2}{l!}\sum_{\mu_i=1}^{\infty}(\Gamma^{\mu_1\ldots\mu_l})_{a_1a_2}(\Gamma^{\mu_1\ldots\mu_l})_{a_3a_4}, \tag{7}$$

where m can assume any integer value from zero to infinity. The Γ's are skew-symmetric matrices which are defined by [12]:

$$\Gamma^{\mu_1\ldots\mu_l} \doteq \frac{1}{l!}\sum_P (-1)^P \gamma^{\mu_{i_1}}\ldots\gamma^{\mu_{i_l}}, \tag{8}$$

where the sum runs over all permutations P of the indices $\mu_1\ldots\mu_l$ and $(-1)^P$ denotes the signature of a permutation. For example

$$\Gamma = \mathbf{1}, \quad \Gamma^\mu = \gamma^\mu, \quad \Gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu], \quad \ldots.$$

α_l^2 is related to the square of Γ matrices: For a non-vanishing Γ matrix the square is [13]:

$$(\Gamma^{\mu_1\ldots\mu_l})^2 = \alpha_l^2\mathbf{1}, \quad \alpha_l^2 \doteq (-1)^{\frac{l(l-1)}{2}}. \tag{9}$$

The Γ matrices are orthogonal by the trace [12]:

$$\mathrm{tr}(\Gamma^{\mu_1\ldots\mu_l}\Gamma^{\nu_1\ldots\nu_m}) = \xi\delta_{lm}\alpha_l^2\sum_P(-1)^P\delta_{\mu_1\nu_{i_1}}\ldots\delta_{\mu_l\nu_{i_l}}. \tag{10}$$

The $Q_l^m(d)$ are the equivalent of the coefficients m_{ij} in the regularized spinor space. They are analytic functions in d and evaluate to

$$Q_l^m(d) = \frac{(-1)^{ml}}{m!}\sum_{s=0}^{\min\{m,l\}}(-2)^s\binom{l}{s}\binom{m}{s}s!(d-s)(d-s-1)\ldots(d-m+1). \tag{11}$$

For $s=m$ the product $(d-s)\ldots(d-m+1)$ is defined to be one.

Since we can't compare individual elements of Γ matrices in eq. (7), we have to understand the equation as an algebraic identity, which can be used in trace expressions under regularized Feynman integrals. Hence, we have to check such identities by multiplying them with elements of Γ matrices and summing over all indices. We obtain a set of trace relations which by definition conclusively test the validity of the indexed identities. As a consequence of such trace equations the trace of the unit matrix is fixed:

$$\mathrm{tr}(\mathbf{1}) = 2^{\frac{d}{2}}. \tag{12}$$

We give here a brief outline of the steps that lead to the regularized Fierz transformation, i.e. eq. (7). We first discuss the Fierz transformation in a spinor space generated by d_0 γ matrices, where d_0 is an even integer. From group theoretical arguments [13, 14] we can show that the Fierz transformation for a tensor product of identity matrices is

$$(\mathbf{1})_{a_1a_4}(\mathbf{1})_{a_3a_2} = 2^{-\frac{d_0}{2}}\sum_{l=0}^{d_0}\frac{\alpha_l^2}{l!}\sum_{\mu_i=1}^{d_0}(\Gamma^{\mu_1\ldots\mu_l})_{a_1a_2}(\Gamma^{\mu_1\ldots\mu_l})_{a_3a_4}. \tag{13}$$

This equation gives us a hint how to form the regularized Fierz transformation: The indices μ_i run in the regularized spinor space from one to infinity and the length of the Γ matrices, i.e. the number of γ matrices that appear in a product in eq. (8), are not restricted. We are thus able to make the ansatz

$$(\mathbf{1})_{a_1a_4}(\mathbf{1})_{a_3a_2} = \frac{1}{c(d)}\sum_{l=0}^{\infty}\frac{\alpha_l^2}{l!}\sum_{\mu_i=1}^{\infty}(\Gamma^{\mu_1\ldots\mu_l})_{a_1a_2}(\Gamma^{\mu_1\ldots\mu_l})_{a_3a_4}, \tag{14}$$

where $c(d)$ is an arbitrary factor. As we have already mentioned these type of equations are understood as identities in trace relations. For instance, we can multiply the ansatz (14) by $(\Gamma^{\nu_1\ldots\nu_m})_{a_4a_1}(\Gamma^{\rho_1\ldots\rho_n})_{a_2a_3}$ or by $(\Gamma^{\nu_1\ldots\nu_m})_{a_4a_3}(\Gamma^{\rho_1\ldots\rho_n})_{a_2a_1}$ and sum over all indices a_i. As a matter of fact, all other ways of arranging the indices a_i lead to expressions containing the transpose of Γ matrices. Such expressions don't occur in physical calculations and we will leave them aside. Therefore we obtain two trace equations, which can be simplified using equations (8) to (10). In one of the trace equations a sum over a product of three Γ matrices occurs. We introduce a factor $Q_n^l(d)$ to evaluate this sum

$$\frac{\alpha_l^2}{l!}\sum_{\mu_i=1}^{\infty}\Gamma^{\mu_1\ldots\mu_l}\Gamma^{\rho_1\ldots\rho_n}\Gamma^{\mu_1\ldots\mu_l} = Q_n^l(d)\,\Gamma^{\rho_1\ldots\rho_n}. \tag{15}$$

We can obtain the explicit form of $Q_n^l(d)$ by commuting one of the γ^μ matrices that appears in the sum to the other one and thereby reducing the length of the whole product. In the resulting sums we can insert similar proportionality factors as $Q_n^l(d)$. This enables us to extract a recurrence relation for the factors, for which eq. (11) is the solution. The trace equations are only valid, if we not only fix the factor $c(d)$ but also the trace of the unit matrix: $c(d) = \xi(d) = 2^{\frac{d}{2}}$. We multiply the ansatz (14) by

$$\frac{\alpha_m^2}{m!}\sum_{\nu_i=1}^{\infty}(\Gamma^{\nu_1\ldots\nu_m})_{b_1a_1}(\Gamma^{\nu_1\ldots\nu_m})_{a_2b_2},$$

and sum over all indices that occur twice. Again we use eq. (15) to eliminate a product of three Γ matrices. We finally obtain eq. (7), i.e. the regularized Fierz transformation for any type of scalar tensor product. If we multiply eq. (7) by

$$\frac{\alpha_n^2}{n!}\sum_{\rho_i=1}^{\infty}(\Gamma^{\rho_1\ldots\rho_n})_{a_4a_1}(\Gamma^{\rho_1\ldots\rho_n})_{a_2a_3},$$

and sum over all indices a_i, we get

$$2^d\delta_{mn} = \sum_{l=0}^{\infty} Q_l^m(d)Q_n^l(d). \tag{16}$$

From eq. (7) and eq. (16) it is obvious that the coefficients $Q_l^m(d)$ are the equivalent of the coefficients m_{ij} in the regularized spinor space.

Let us finally quote, as an example, the form of the Fierz transformation for a vector-vector interaction. Eq. (7) gives for $m = 1$:

$$\sum_{\nu=1}^{\infty}(\gamma^\nu)_{a_1a_4}(\gamma^\nu)_{a_3a_2} = 2^{-\frac{d}{2}}\sum_{l=0}^{\infty}(-1)^{\frac{l(l+1)}{2}}\frac{(d-2l)}{l!}\sum_{\mu_i=1}^{\infty}(\Gamma^{\mu_1\ldots\mu_l})_{a_1a_2}(\Gamma^{\mu_1\ldots\mu_l})_{a_3a_4}. \tag{17}$$

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