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Heisenberg Antiferromagnet on Triangulated Trees

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Abstract. We study the spin-1/2 antiferromagnetic Heisenberg Hamiltonian with nearest-neighbour pair interaction on some graphs with an odd number of vertices, called triangulated trees. The model has valence bond ground states in which there is a localized spinon (1/2 spin) on some site. The space of ground states for a tree of L triangles is $2(L+1)$ dimensional. In the limit of infinite volume we find an infinity of pure ground states. To any (infinite) branch of the tree there belongs a spin-Peierls ground state without spinon. There is an infinite family of equivalent pure ground states differing only locally, in the position of the spinon. In any finite-volume ground state the pair correlations decay exponentially. For the chain of triangles we show the existence of a gap in the energy spectrum above the ground state.

Table of Contents

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1. Introduction

In this paper we study the antiferromagnetic fully anisotropic Heisenberg Hamiltonian for 1/2 spins with purely nearest neighbour pair interactions on triangulated trees. These are graphs such that the only cycles are triangles and every bond belongs to a unique triangle (Fig. 2.1.a). Our aim is to provide a "soluble" model in which spinons (spin-1/2 excitations) appear separately and their role in the ground state properties is easy to demonstrate.

Spin systems with antiferromagnetic interactions exhibit a large variety of ground states. This holds already for Ising models which offer three typical examples: (i) On a bipartite lattice the nearest-neighbour antiferromagnetic Hamiltonian has two antiferromagnetically ordered ground states (Néel states). (ii) Defining the same Hamiltonian on the triangular lattice one finds that every triangle is frustrated in the sense that it is impossible to minimize simultaneously the energy of all the three interactions. This leads to a large ground state degerancy: the number of ground state configurations grows exponentially with the number of sites [Wa]. Six of these configurations (those with alternating up and down spins in the three principal lattice directions) are isolated, i.e., any local transformation increases their energy, the others can be modified locally without any energy cost. One can introduce an equivalence relation between ground state configurations by defining them equivalent if they differ only locally. Configurations in the same class form a *ground phase.* Thus, in six of these phases there is a unique configuration and long-range order. In the other phases the pair correlations fall off as $\langle S_k S_l \rangle \sim |k-l|^{-1/2}$ [St]. These phases are analogs of what is called a spin liquid in quantum systems, although this term is sometimes reserved for "incompressible" spin liquids: unique massive ground states with exponentially decaying correlations [La]. (iii) The nearest-neighbour antiferromagnetic Ising model on the Kagomé lattice is superfrustrated [Sü81]: apart from a trivial essential singularity, the free energy can be continued analytically to $T = 0$ and $|\langle S_k S_l \rangle| < 4(3/4)^{|k-l|}$ independently of the temperature. The number of ground state configurations grows exponentially with the volume and none of them is isolated (this can be seen easily on the contour representation, a point which was overlooked in [Sü81b]). According to the former terminology, any ground phase can be considered as a spin liquid phase. In the Ising limit, the models we are going to study in this paper are also superfrustrated [Sü81b], so the above said are valid for them as well.

For quantum antiferromagnets the analogs of the above examples exist and the variety of possibilities is further increased by quantum effects but much less is known for sure. In n-sublattice structures the ground states may be n-sublattice Néel states if zS, the coordination number times the length of the individual spins is large enough. The spin 1/2 isotropic Heisenberg antiferromagnet on the square lattice still probably has Néel order in the ground state. By a Goldstone theorem this would imply that there is no gap in the spectrum above the ground state energy [Wr]. In fact, finite size scaling predicts no gap to the multiplet excitations but a gap to singlet excitations [SüFa].

In one dimension quantum fluctuations seem to destroy the antiferromagnetic order in the ground state, let the spin be however large. According to Haldane [Ha], for integer spins the isotropic Heisenberg antiferromagnets should have a unique ground state with exponentially decaying correlations and a gap in the spectrum above the ground state. A rigorous example with $S = 1$ is given by Affleck et al [AKLT]. The spin 1/2 chain can be solved with Bethe Ansatz [Be]. There is no gap to the excitations [Be],[Hu],[dCP] and the pair correlations fall off according to a power law [Fo]. The ground state in infinite volume is generally believed to be unique but Faddeev and Takhtajan [FaTa] claim that the two limits obtained via sequences of chains with, respectively, an even or odd number of sites are different.

On the triangular lattice the quantum antiferromagnetic Heisenberg-Ising model for $S \geq 1$ spins probably has three-sublattice Néel ground states. For spin 1/2, Anderson [An73] suggested that the ground state should be unique, not breaking thereby the translational and, in the isotropic case, the rotational symmetry of the Hamiltonian. That such a state is favorable in comparison with the Néel state was shown later by a perturbational calculation for the anisotropic Hamiltonian around the Ising limit [FaAn]. In the subsequent literature the ground state of the isotropic model was often called a spin liquid although its uniqueness was not proved, finite size scaling showed the absence of a gap to the excitations [SüFa], and one could suspect that the correlations follow a power law decay as in the Ising limit. In fact, more recent works performed with different techniques provide arguments that the ground state is Néel ordered with three sublattices, as for $S \geq 1$ ([HuEl], [JlG], [DEMW], [YoMi]), and the same conclusion seems to hold for the anisotropic model with one easy axis (the perturbed XY model) [KMFF]. When next-nearest-neighbour antiferromagnetic pair interactions are added to the isotropic nearest neighbour Hamiltonian, an approximate calculation [Ba] shows a phase transition at positive temperatures with two chirally ordered translationally invariant low temperature phases, which implies also the existence of two ground states with similar properties.

In chiral spin states the expectation values of the operators $\mathbf{S}_j\ (\mathbf{S}_k \wedge \mathbf{S}_l)$ are non-vanishing for a macroscopic set of triangles (jkl). Non-vanishing homogeneous ground state chirality is thought to be possible for Heisenberg antiferromagnets and consistent (apart from uniqueness) with spin liquid properties. Spin liquids in general ([An87],[La]) and chiral spin liquids in particular [WWZ] play an important role in the fascinating theory of high-temperature superconductivity (see [CaJo] and [BEMS] for recent reviews). However, in the only soluble model [WWZ] with chiral ground states known for us, the Hamiltonian is not translationally invariant and neither are so the chiral ground states which therefore cannot be spin liquids.

Notions like Néel state or spin liquid refer to infinite systems. In finite volumes it is more appropriate to use different terms. For an even number N of 1/2 spins the ground state of isotropic antiferromagnets is usually nondegenerated and thus a singlet state (this may be true even for anisotropic Hamiltonians, as in the model discussed in this paper).

Any N-spin singlet can be expanded in the nonorthogonal basis of the so-called valence bond (VB) states which are tensor products of $N/2$ two-spin singlets (dimers). If there is an exponentially large (in N) number of linearly independent terms with roughly equal weights, one speaks about a resonating valence bond (RVB) state. It is conjectured [Ta] that RVB states on bipartite lattices with nearest neighbour dimers may produce long-range antiferromagnetic correlations if both the dimension and the coordination number are sufficiently large. However, in Heisenberg antiferromagnets which have Néel order in infinite volume the ground state in finite volume must be some particular RVB state with dimers bridging over the whole system. The limit of such states is probably the uniform mixture of pure (Néel) states with all possible orientations of the sublattice-magnetization. Spin liquids, on the other hand, may be limits of RVB states with short dimers. The distinction in finite volume is not perfectly clear-cut. Short dimers rarely mean dimers of bounded length, certainly not in the triangular or Kagomé lattice antiferromagnets. Instead, the coefficients of VB states with increasing dimer length may show some specific, so far unknown, decay properties.

Valence bond states themselves can be ground states of some antiferromagnetic Hamiltonians. The infinite volume limit is a state with a localized array of singlet pairs; if it breaks the translational symmetry, it is called a spin-Peierls state. Examples of Hamiltonians can be found with purely pair interactions ([MaGh],[ShSu] and the present work) or with interactions involving more than two spins ([Kl],[CCK]). The example of [CCK] is particularly interesting because the number of valence bond ground states grows exponentially with the size of the system. In the infinite volume limit the resulting ground states which differ only locally form an equivalence class which may carry the properties of a spin liquid phase, similarly to the ground phases of superfrustrated Ising Hamiltonians.

In the models we investigate in the present work something like this occurs due to a spinon, an unpaired spin, in the ground state.

After the identification of triplet ([Hu],[dCP]) and singlet ([Ov],[FaSü]) excitation branches in the isotropic Heisenberg chain of even length, it was recognised [FaTa] that these and other multiplet excitations can be decomposed in an even number of elementary excitations: quasiparticles with spin 1/2 and given momentum and energy. Later and in the context of itinerant electron systems neutral objects with spin 1/2 which can somehow be indentified as entities in infinite-volume equilibrium states were called "spinons" (by Anderson, [La]). The above-mentioned quasiparticles, bound in the excited states of the Heisenberg chain, are considered as an example. Free spinons most often are visualized as real-space spin 1/2-particles which remain unpaired due to a dimerization defect [KRS]. Such a defect can be provoked at least in two different ways: by choosing, in finite volume, the number of spin 1/2-particles to be odd or, by exciting a singlet pair into a triplet state and carrying "half of the excitation" to the sample boundary [La]. Making evident the existence of spinons in equilibrium states is an exciting challenge: In two dimensions these are particles which obey fractional statistics [KaLa] and, according to a scenario advanced by Laughlin, they

are the key objects of the "anyon superconductivity" ([La],[CaJo],[BEMS]).

According to what we know about its Ising limit, the best candidate for a model possessing a genuine spin-liquid ground state is the spin-1/2 isotropic Heisenberg antiferromagnet on the Kagomé lattice. Unfortunately, we were unable to treat this model rigorously. The best we could do is to study the model on triangulated trees. Let us summarize our principal findings. Triangulated trees contain an odd number of sites. Completing with an additional site on a "dangling bond" or removing a site of a triangle (Fig. 2.1.d), the system has a unique valence bond ground state which corresponds to the unique dimerization of the graph with nearest-neighbour dimers (Proposition 3.4). On a tree of L triangles the subspace of ground states is $2(L+1)$-dimensional. There is a basis of VB ground states with nearest-neighbour dimerization and a localized spinon (also called free spin) on a site (Theorem 3.3). In any ground state the pair correlations decay exponentially (Theorem 6.3). For a chain of L triangles we show, by adapting Affleck et al's proof [AKLT], the existence of an L-independent gap in the spectrum above the ground state energy (Theorem 7.1). The pure ground states in the infinite volume — or at least some of them — can also be given. For the infinite chain of triangles there exists a countable number of pure spin-Peierls ground states (Proposition 8.2). With the exception of two of them, they contain a localized free spin at some site. These states are equivalent in the sense that the corresponding GNS representations of the algebra of quasi-local observables are unitarily equivalent (Theorem 8.3). The Hilbert spaces supporting the different representations can be considered coinciding. This single Hilbert space contains then countably many ground state vectors, any one of which can be transformed into any other one by a finite displacement of the ground state spinon. The whole family forms a *ground phase* which is translationally invariant and does not break parity (reflexion with respect to axes in the plane of the triangles and perpendicular to the chain) either. Moreover, there is a gap above the energy of the ground state (Theorem 8.5). Therefore this ground phase can be considered as a curious realization of a spin liquid. The remaining two ground states are isolated: not equivalent with any other ground state. They correspond to the spinon being at $+$ or $-\infty$, respectively. They are invariant under translation (this can, in fact be seen as an "accident" due to the peculiar geometry of the chain of triangles) and are transformed into each other by reflexion. The existence of a gap (Theorem 8.4) can be proved by using the results obtained for finite chains of triangles and shows that these two states are isolated. The clustering of the correlations follows from the product structure of the ground states.

Similar results are obtained for infinite trees of triangles (Section 8.2). There exists an isolated spin-Peierls state, with spinon at infinity, for each branch of triangles, and an equivalence class of only locally different ground states with a single spinon at some site. This latter, again, realizes something like a spin liquid phase, but we cannot prove the existence of a gap, in spite of the exponential decrease of pair correlations.

A short version of the present paper has recently been published [MoSü].

2. Notion of △ Tree and the Model

A *triangulated tree* or △ *tree* (delta tree) is a connected graph such that any bond belongs to a unique triangle and the only cycles are triangles. A special case is the △ *chain* which is a chain of triangles.
A $\triangle^+$ *tree* is either two △ trees connected by an extra bond or a △ tree with an additional site connected to it by a single bond.

In a △ tree any two triangles are connected via a *unique* △ chain. See Figure 2.1.

Definition. A triangle will be called a *boundary triangle* of a △ tree Λ, if it is connected to the rest of Λ through a unique site. A *boundary site* of Λ is a site of a boundary triangle which is not shared with other triangles.

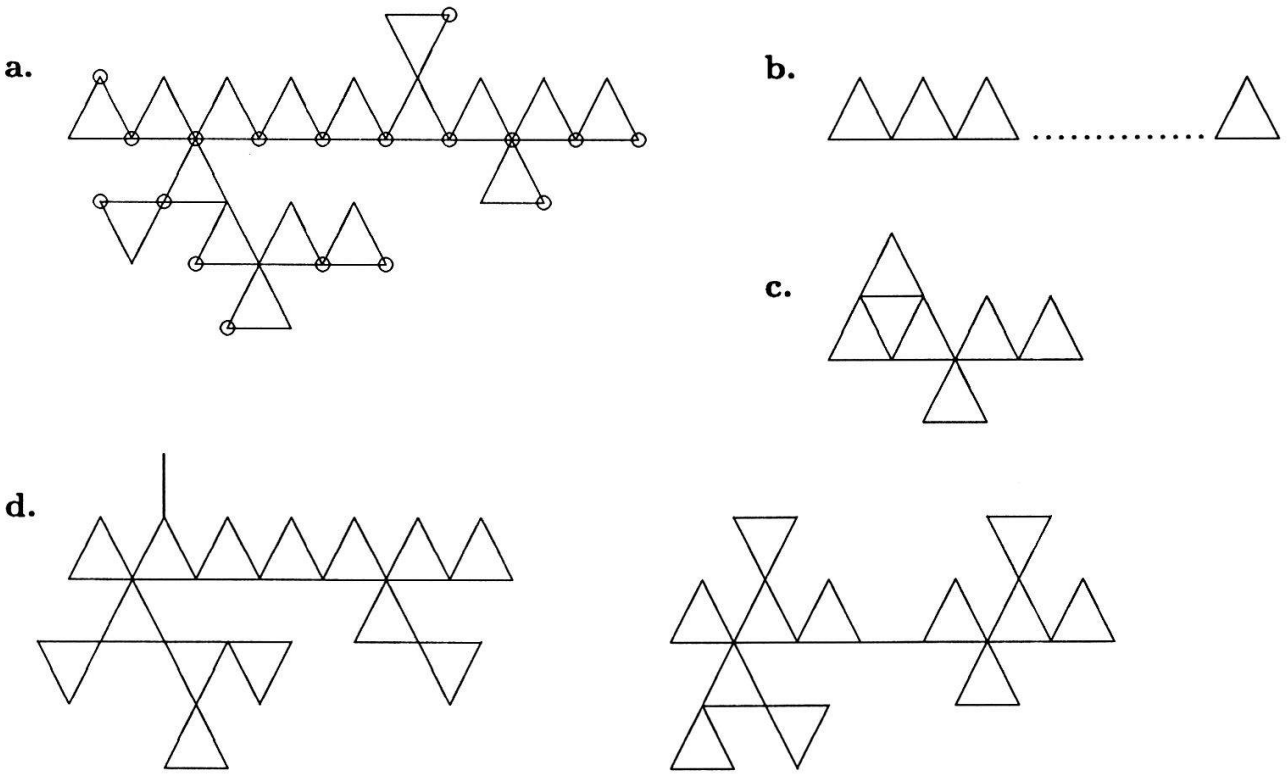

Fig. 2.1. **a.** *A* △ *tree. The circles indicate a possible choice for the construction of a basis in the space of ground states.* **b.** *The* △ *chain.* **c.** *A graph which is not a* △ *tree (some bonds belong to more than one triangle).* **d.** *The two kinds of* $\triangle^+$ *trees.*

The model considered in this work is a system of 1/2 quantum spins on a △ tree Λ with the anisotropic Heisenberg Hamiltonian H^Λ

$$H^\Lambda = \sum_a H_a \tag{2.1a}$$

With the exception of the last section, we deal with finite sets. The index a labels the triangles of Λ and the one-triangle Hamiltonian H_a is given by

$$H_a = \lambda_a + \nu_a + \mu_a + \sum_{\substack{k,m \in \{1,2,3\} \\ k<m}} \{\lambda_a \sigma_x(a,k)\sigma_x(a,m) + \nu_a \sigma_y(a,k)\sigma_y(a,m) + \mu_a \sigma_z(a,k)\sigma_z(a,m)\} \tag{2.1b}$$

Here $\{1,2,3\}$ number the sites on the ath triangle, and $\sigma_\alpha(p)$ represents the action of the Pauli matrix σ_α, α in $\{x,y,z\}$, on the pth site and the unit matrix on any other site.

The parameters λ_a, ν_a and μ_a are chosen to satisfy

$$0 < \lambda_a + \nu_a + \mu_a < \infty \quad \text{and} \quad 0 < \lambda_a\nu_a + \nu_a\mu_a + \mu_a\lambda_a < \infty \qquad \text{for all } a. \tag{2.1c}$$

Hence the interaction is purely or dominantly antiferromagnetic.

To simplify the notations, we will consider the homogeneous case, i.e., $\lambda_a = \lambda$, $\nu_a = \nu$ and $\mu_a = \mu$ for all a, but the results are valid for the inhomogeneous Hamiltonian.

In the sequel S_z will denote the z component of the total spin, i.e.,

$$S_z = \frac{1}{2}\sum_{p\in\Lambda} \sigma_z(p)$$

The operators above act on the Hilbert space $\mathfrak{H}^\Lambda$ spanned by the vectors $\gamma_1 \otimes \gamma_2 \otimes \cdots \otimes \gamma_{|\Lambda|}$ where

$$\gamma_j = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \uparrow \quad \text{or} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \downarrow.$$

We will omit the sign $\otimes$ if there is no place for confusion.
If p and q are two sites then the unique singlet state formed by the corresponding spins is

$$[p|q] = \{\uparrow(p)\downarrow(q) - \downarrow(p)\uparrow(q)\}/\sqrt{2}$$

If p and q are nearest neighbours, this state is called a *dimer*. In figures, it will be represented by $p \rightarrowtail q$ or, if the sign is not interesting, by p—q or p=q.
We will denote the usual graph distance between the sites p and q by $d(p,q)$.

3. Valence Bond States as Ground States

On any Δ tree Λ, there are states obtained by taking the tensor product of dimers and a single one-site $\uparrow$ or $\downarrow$ state, such that each triangle carries a dimer on one of its bonds. These states are called the valence bond (VB) states. Thus, a VB state has the form $[p_1|p_2]\ldots[p_{N-2}|p_{N-1}]\gamma(p_N)$ where $N = |\Lambda|$, p_{2n-1} and p_{2n} are nearest neighbours and $\gamma = \uparrow$ or $\downarrow$. The unique site of Λ which does not belong to any dimer is referred to as a free site and the spin on it as the free spin. Any point of Λ can be a free site. If the free site

and S_z are fixed then the associated VB state is completely determined: On each triangle the dimer is carried by those two sites at an equal distance to the free site. It follows also that two VB states differ only on the $\triangle$ chain which connects their free sites (Fig. 3.1). As any $\triangle$ tree of L triangles has $2L+1$ sites, there are exactly $2(2L+1)$ different VB states, the factor 2 coming from the two possible orientations of the free spin. Each such VB state is an eigenvector of S_z with eigenvalue $\pm 1/2$ depending on the orientation of the free spin.

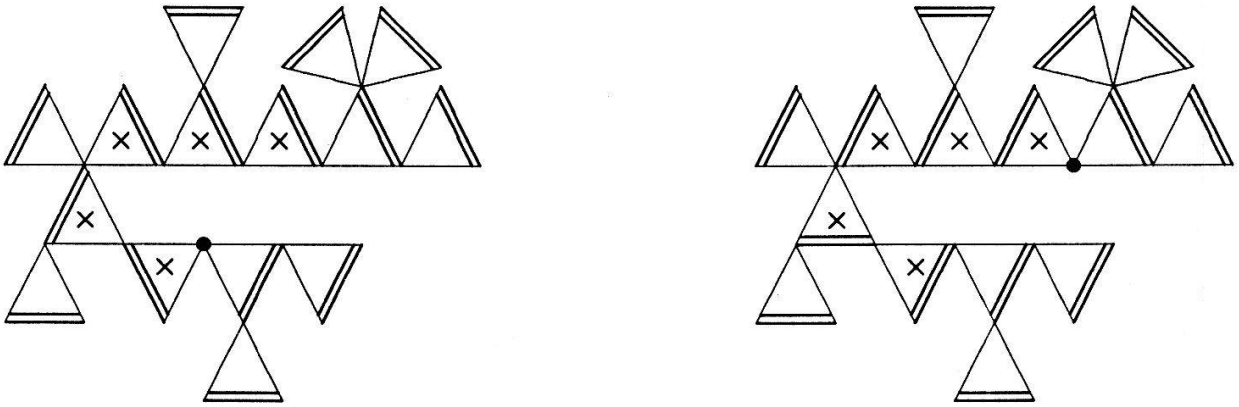

Fig. 3.1. *Two valence bond states on a $\triangle$ tree. Notice that they differ only on the $\triangle$ chain connecting their respective free sites.*

Proposition 3.1. *The valence bond states on any $\triangle$ tree Λ are ground states of H^Λ with ground state energy 0.*

Proof. If Λ has more than one triangle, the restriction of a VB state to any triangle a is a VB state or the linear combination of two VB states on a. So it suffices to show that the VB states on a are ground states of H_a and their energy is 0. Now

$$\begin{aligned} H_a \downarrow\otimes\downarrow\otimes\downarrow &= (\lambda+\nu+4\mu)\downarrow\otimes\downarrow\otimes\downarrow + (\lambda-\nu)\{\uparrow\otimes\uparrow\otimes\downarrow+\uparrow\otimes\downarrow\otimes\uparrow+\downarrow\otimes\uparrow\otimes\uparrow\} \\ H_a \uparrow\otimes\uparrow\otimes\downarrow &= H_a\uparrow\otimes\downarrow\otimes\uparrow = H_a\downarrow\otimes\uparrow\otimes\uparrow = \\ &= (\lambda+\nu)\{\uparrow\otimes\uparrow\otimes\downarrow+\uparrow\otimes\downarrow\otimes\uparrow+\downarrow\otimes\uparrow\otimes\uparrow\} + (\lambda-\nu)\downarrow\otimes\downarrow\otimes\downarrow \end{aligned} \tag{3.1}$$

Similar equations are valid with all spins reversed. From these equalities one finds that the VB states $\gamma(p)\,[p'|p'']$ are eigenstates of H_a with eigenvalue 0 which has multiplicity 4. The other eigenvalues are $2\left(\Sigma \pm \sqrt{\Sigma^2-3\Xi}\right)$ with multiplicity 2, where $\Sigma = \lambda+\nu+\mu$, and $\Xi = \lambda\nu+\nu\mu+\mu\lambda$. It is easy to check that the VB states are (the only) ground states of H_a if and only if $\Sigma > 0$ and $\Xi > 0$. □

Lemma 3.2. *Let Λ be a $\triangle$ tree, and p a site in Λ. A state $\phi = \phi'\gamma(p)$ with $\gamma = \uparrow, \downarrow$ and ϕ' in $\mathfrak{H}^{\Lambda\backslash\{p\}}$ is a ground state of H^Λ if and only if ϕ is the valence bond state with free spin γ on site p, i.e., ϕ' is the tensor product of dimers on $\Lambda\backslash\{p\}$.*

Proof. The part "if" was shown in Proposition 3.1. By this proposition any ground state ϕ has to minimize H_a for all a (because the VB states do so). Then, in particular, ϕ has to minimize each H_b where b is a triangle containing the site p. The spin $\gamma(p)$ being fixed, ϕ must contain a dimer on the remaining two sites of b. By developing this dimer, we get two mutually orthogonal states, so the above argument can be repeated. See Figure 3.2. □

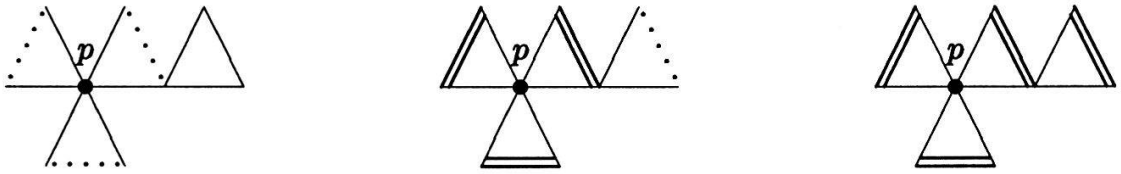

Fig. 3.2. *The inductive construction of the unique ground state on a Δ tree with a fixed spin value on site p.*

Theorem 3.3. *Let Λ be a Δ tree with L triangles. There are $L+1$ valence bond states which form a basis for the space of ground states with given S_z $(= 1/2$ or $-1/2)$.*

This basis can be choosen in the following way: Fix a set of $L+1$ sites of Λ such that it contains exactly one boundary site of any boundary triangle and one or two sites of any triangle (Fig. 2.1a). Then choose the $L+1$ valence bond states with the free sites taken from this set and with suitable spin on the free site.

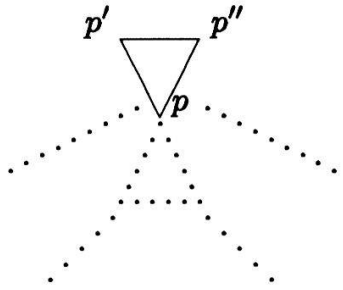

Fig. 3.3. *The triangle added for the induction step in the proof of Theorem 3.3.*

Proof. i) Each ground state is a linear combination of VB states.

This holds for a single triangle. The general case is proved by induction in the number of triangles. Suppose that for any Δ tree Λ_L of L triangles each ground state is a linear combination of VB states. Fix Λ_L, and consider Λ_{L+1} obtained by completing a site p in Λ_L into a new triangle. Let p' and p'' be the two added sites. (The new triangle is necessarily a boundary triangle. See Figure 3.3.) We have $H^{\Lambda_{L+1}} = H^{\Lambda_L} + H_{L+1}$.

Let ϕ be a ground state of $H^{\Lambda_{L+1}}$. Then $H^{\Lambda_L}\phi = H_{L+1}\phi = 0$, i.e., ϕ is a ground state for H^{Λ_L} and H_{L+1}. Expand ϕ as

$$\phi = \phi_\uparrow \tau_\uparrow + \phi_\downarrow \tau_\downarrow + \phi_0 \tau_0 + \phi_d \tau_d \equiv \phi' + \phi_d \tau_d \tag{3.2}$$

where ϕ_s ($s = \uparrow, \downarrow, 0$ or d) is in $\mathfrak{H}^{\Lambda_L}$ and $\tau_\uparrow = \uparrow(p')\uparrow(p'')$, $\tau_\downarrow = \downarrow(p')\downarrow(p'')$, $\tau_0 = \{\uparrow(p')\downarrow(p'') + \downarrow(p')\uparrow(p'')\}/\sqrt{2}$ and $\tau_d = [p'|p'']$. Due to the orthogonality of the states τ_s, each nonvanishing ϕ_s must be a ground state for H^{Λ_L}. Hence by the induction hypothesis each ϕ_s is a linear combination of VB states in $\mathfrak{H}^{\Lambda_L}$. Moreover, $\phi_d\,[p'|p'']$ is a linear combination of VB states in $\mathfrak{H}^{\Lambda_{L+1}}$, and is therefore by Proposition 3.1 a ground state of $H^{\Lambda_{L+1}}$. Then ϕ' is also a ground state. By expanding ϕ_s for $s = 0, \uparrow, \downarrow$ in the basis $\{\uparrow(p), \downarrow(p)\}$ we obtain

$$\phi' = \left(\chi_\uparrow^{(+)}\uparrow(p) + \chi_\downarrow^{(+)}\downarrow(p)\right)\tau_\uparrow + \left(\chi_\uparrow^{(0)}\uparrow(p) + \chi_\downarrow^{(0)}\downarrow(p)\right)\tau_0 + \left(\chi_\uparrow^{(-)}\uparrow(p) + \chi_\downarrow^{(-)}\downarrow(p)\right)\tau_\downarrow \quad (3.3)$$

Using Eq. (3.1),

$$\begin{aligned}
0 = H_{L+1}\,\phi' = \\
&= \left((\lambda+\nu+4\mu)\chi_\uparrow^{(+)} + (\lambda-\nu)(\chi_\uparrow^{(-)} + \sqrt{2}\,\chi_\downarrow^{(0)})\right)\{\uparrow\uparrow\uparrow\}\,(p,p',p'') \\
&+ \left((\lambda-\nu)\chi_\downarrow^{(-)} + (\lambda+\nu)(\chi_\downarrow^{(+)} + \sqrt{2}\,\chi_\uparrow^{(0)})\right)\{\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow\}\,(p,p',p'') \\
&+ \left((\lambda-\nu)\chi_\uparrow^{(+)} + (\lambda+\nu)(\chi_\uparrow^{(-)} + \sqrt{2}\,\chi_\downarrow^{(0)})\right)\{\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow\}\,(p,p',p'') \\
&+ \left((\lambda+\nu+4\mu)\chi_\downarrow^{(-)} + (\lambda-\nu)(\chi_\downarrow^{(+)} + \sqrt{2}\,\chi_\uparrow^{(0)})\right)\{\downarrow\downarrow\downarrow\}\,(p,p',p'')
\end{aligned}$$

The states on $\{p,p',p''\}$ are orthogonal, therefore we have to solve for A and B the following system of equations:

$$\begin{cases} (\lambda+\nu+4\mu)\,A + (\lambda-\nu)\,B = 0 \\ (\lambda-\nu)\,A + (\lambda+\nu)\,B = 0 \end{cases}$$

Since the determinant of the system is $4\Xi > 0$, there is only a trivial solution, $A = B = 0$. This means

$$\chi_\uparrow^{(+)} = \chi_\downarrow^{(-)} = 0, \qquad -\chi_\uparrow^{(0)} = \frac{1}{\sqrt{2}}\chi_\downarrow^{(+)} = \phi^{(+)} \qquad \text{and} \qquad -\chi_\downarrow^{(0)} = \frac{1}{\sqrt{2}}\chi_\uparrow^{(-)} = \phi^{(-)}.$$

Applying these equalities in Eq. (3.3) we find

$$\begin{aligned}
\phi' &= \sqrt{2}\,\phi^{(+)}\downarrow(p)\,\tau_\uparrow - \{\phi^{(+)}\uparrow(p) + \phi^{(-)}\downarrow(p)\}\,\tau_0 + \sqrt{2}\,\phi^{(-)}\uparrow(p)\,\tau_\downarrow \\
&= \phi^{(+)}\,[p''|p]\uparrow(p') + \phi^{(+)}\,[p'|p]\uparrow(p'') + \phi^{(-)}\,[p|p']\downarrow(p'') + \phi^{(-)}\,[p|p'']\downarrow(p')
\end{aligned}$$

Comparison with Eq. (3.2) shows that

$$\phi^{(+)}\downarrow(p) = \frac{1}{\sqrt{2}}\phi_\uparrow \qquad \text{and} \qquad \phi^{(-)}\uparrow(p) = \frac{1}{\sqrt{2}}\phi_\downarrow.$$

These are therefore ground states (if $\not\equiv 0$) of H^{Λ_L}. By Lemma 3.2, $\phi^{(+)}$ and $\phi^{(-)}$ are some multiples of the product of dimers in $\mathfrak{H}^{\Lambda_L\setminus\{p\}}$. So ϕ' is a linear combination of VB states.

ii) For fixed S_z there are at most $L+1$ linearly independent VB states.

Let, e.g., $S_z = +1/2$. The proof is done by induction in the number of triangles. For $L = 1$ the claim holds true (c.f. Fig. 3.4). Suppose that for any $\triangle$ tree Λ_L made of L triangles there are at most $L+1$ linearly independent VB states in $\mathfrak{H}^{\Lambda_L}$.

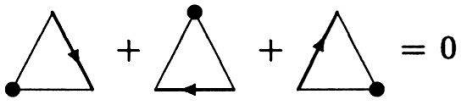

Fig. 3.4. *Linear dependence of the valence bond states on a triangle. The free spin is fixed.*

Let $\psi^L(q)$ be the VB state in $\mathfrak{H}^{\Lambda_L}$ with free spin $\uparrow$ on site q and $\mathcal{F}_L = \{\psi^L(q)\}_{q\in I_L}$, with $I_L \subset \Lambda_L$ and $|I_L| = L+1$, a maximal set of linearly independent VB states. Here I_L is chosen as described in the Theorem. Consider the $\triangle$ tree Λ_{L+1} obtained from Λ_L by completing a site p in Λ_L into a new triangle with boundary sites p' and p''. Let $\psi^{L+1}(q) = \psi^L(q)\,[p'|p'']$ if q is in I_L and $\psi^{L+1}(p') = \phi^L\,[p''|p]\uparrow(p')$ (where ϕ^L is the product of dimers on $\Lambda_L\backslash\{p\}$). Choose $I_{L+1} = I_L \cup \{p'\}$ and set $\mathcal{F}_{L+1} = \{\psi^{L+1}(q)\}_{q\in I_{L+1}}$.

If q' belongs to Λ_L then $\psi^{L+1}(q') = \pm\psi^L(q')\,[p'|p'']$ is by the induction hypothesis a linear combination of $\{\psi^L(q)\,[p'|p'']\}_{q\in I_L} \subset \mathcal{F}_{L+1}$. Also, $\psi^{L+1}(p'') = \phi^L\,[p|p']\uparrow(p'')$ is a linear combination of $\psi^{L+1}(p)$ and $\psi^{L+1}(p')$ (see Fig. 3.4) and hence of the elements of $\mathcal{F}_{L+1}$.

It remains to prove that $\mathcal{F}_{L+1}$ is linearly independent. Consider

$$
\begin{aligned}
0 &= \sum_{q\in I_{L+1}} C_q\,\psi^{L+1}(q) \\
&= \left\{C_{p'}\,\phi^L\,[p''|p] + \sum_{q\in I_L}\frac{C_q}{\sqrt{2}}\,\psi^L(q)\downarrow(p'')\right\}\uparrow(p') - \left\{\sum_{q\in I_L}\frac{C_q}{\sqrt{2}}\,\psi^L(q)\uparrow(p'')\right\}\downarrow(p')
\end{aligned}
$$

The two sums being orthogonal on p', we get $\sum_{q\in I_L} C_q\,\psi^L(q) = 0$ and by the induction hypothesis $C_q = 0$ for all q in I_L. The first term reduces to $C_{p'}\,\psi^{L+1}(p') = 0$ hence $C_{p'} = 0$ as well. This finishes the proof. □

In the case of the Δ^+ tree there is a unique VB state which, with suitably chosen numbering, has the form $[1|2]\,[3|4]\ldots[N-1|N]$.

Proposition 3.4. *Let Λ be a Δ^+ tree. Then H^Λ has a unique ground state which is the valence bond state on Λ.*

Proof. The Hamiltonian H^Λ can be written as $H^\Lambda = H^+ + H^{\Lambda'}$ where H^+ is the term for the extra bond:

$$H^+ = \lambda\,(\sigma_x(p)\sigma_x(p') + 1) + \nu\,(\sigma_y(p)\sigma_y(p') + 1) + \mu\,(\sigma_z(p)\sigma_z(p') + 1)$$

The VB state ψ on Λ is the product of dimers with a dimer $[p|p']$ on the extra bond. Hence ψ minimizes each term of H^Λ.

If ϕ is any ground state of H^Λ then it has to minimize H^+ and so $\phi = \phi'\,[p|p']$. By Lemma 3.2, ϕ' is the product of dimers on $\Lambda\backslash\{p,p'\}$, therefore ϕ is the VB state on Λ. □

4. Some Excited States

In the case when $\lambda = \nu$, on a Δ tree Λ, some excited states of H^Λ can easily be constructed.

If c is a triangle of Λ, let $(c,1)$, $(c,2)$ and $(c,3)$ denote the three sites on it. If Λ_0 is a Δ tree within Λ, then $\Lambda\backslash\Lambda_0$ is the union of Δ^+ trees. Let $\phi(\Lambda\backslash\Lambda_0)$ denote the unique VB state on $\Lambda\backslash\Lambda_0$.

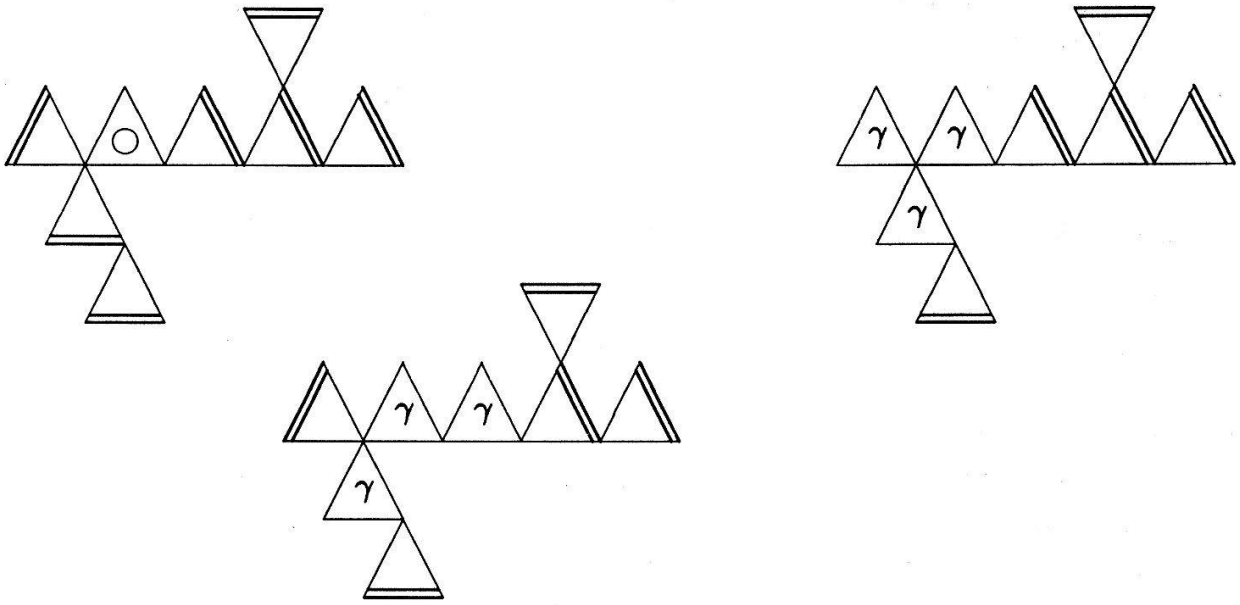

Fig. 4.1. *Example of excited states on a Δ tree.*  *and*  *are the states described in Equation (4.1).*

Proposition 4.1. *Let Λ be any Δ tree and H^Λ the Heisenberg Hamiltonian (2.1) on it. Suppose in addition that $\lambda = \nu$. Then*

1. *For any triangle c of Λ, the states*

$$\phi(\Lambda\backslash c)\,\{\uparrow(c,1)\uparrow(c,2)\downarrow(c,3) + \uparrow(c,1)\downarrow(c,2)\uparrow(c,3) + \downarrow(c,1)\uparrow(c,2)\uparrow(c,3)\}/\sqrt{3}$$

$$\phi(\Lambda\backslash c)\,\{\downarrow(c,1)\downarrow(c,2)\uparrow(c,3) + \downarrow(c,1)\uparrow(c,2)\downarrow(c,3) + \uparrow(c,1)\downarrow(c,2)\downarrow(c,3)\}/\sqrt{3}$$

   are excited states of H^Λ with energy 6λ.

2. *Let Λ_0 be a (connected) Δ tree contained in Λ, then*

$$\phi(\Lambda\backslash\Lambda_0) \bigotimes_{p\in\Lambda_0} \uparrow(p) \qquad \text{and} \qquad \phi(\Lambda\backslash\Lambda_0) \bigotimes_{p\in\Lambda_0} \downarrow(p)$$

   are excited states of H^Λ with energy $2M_0\,(2\mu+\lambda)$, where M_0 is the number of triangles in Λ_0.

Proof. Setting $\lambda = \nu$ in Eq. (3.1), we find for a single triangle

$$H_a \triangle^{\gamma} = (4\mu + 2\lambda) \triangle^{\gamma}$$
$$H_a \triangle^{\circ} = 6\lambda \triangle^{\circ} \tag{4.1}$$
$$H_a \triangle^{\prime} = 0$$

where $\gamma = \uparrow$ or $\downarrow$, $\triangle^{\gamma} = \gamma \otimes \gamma \otimes \gamma$, $\triangle^{\circ} = \{\uparrow \otimes \uparrow \otimes \downarrow + \uparrow \otimes \downarrow \otimes \uparrow + \downarrow \otimes \uparrow \otimes \uparrow\}/\sqrt{3}$, or the state with all spins reversed, and $\triangle^{\prime}$ represents any VB state i.e. $\gamma(p)\,[p'|p'']$ if p, p' and p'' are the sites of the triangle.
The result is a simple consequence of Lemma 3.2 and of Equation (4.1) after a suitable decomposition of H^{Λ}. □

5. Overlap Between Valence Bond States. Orthonormal Basis for Ground States

To compute the overlap between VB states on a $\triangle$ tree or to construct an orthonormal basis, one may fix the value of $S_z = +1/2$. Obviously, two VB states with different orientation of the free spins are orthogonal. We first consider a $\triangle$ chain of L triangles which we denote by Λ_L. The triangles will be numbered from left to right. For $1 \le a \le L-1$, $a_+ = a+1_-$ label the common site of the triangles a and $a+1$, 1_- and L_+ are the left- and rightmost sites, respectively, and for $1 \le a \le L$, a_0 denotes the third site of a, different from $a_\pm$. As a basis for the space of ground states of $H^L = H^{\Lambda_L}$ we use

$$\psi^{1,L}(a_-, \gamma) = [1_-|1_0] \ldots [a-1_-|a-1_0]\gamma(a_-)\,[a_0|a_+] \ldots [L_0|L_+],$$

for $a = 1, \ldots, L$ and $\gamma = \uparrow, \downarrow$, and (5.1a)

$$\psi^{1,L}(L_+, \gamma) = [1_-|1_0] \ldots [L_-|L_0]\gamma(L_+)$$

There are L other VB states given by

$$\begin{aligned}\psi^{1,L}(a_0, \gamma) &= [1_-|1_0] \ldots [a-1_-|a-1_0]\,[a_+|a_-]\gamma(a_0)\,[a+1_0|a+1_+] \ldots [L_0|L_+] \\ &= -\psi^{1,L}(a_+, \gamma) - \psi^{1,L}(a_-, \gamma)\end{aligned} \tag{5.1b}$$

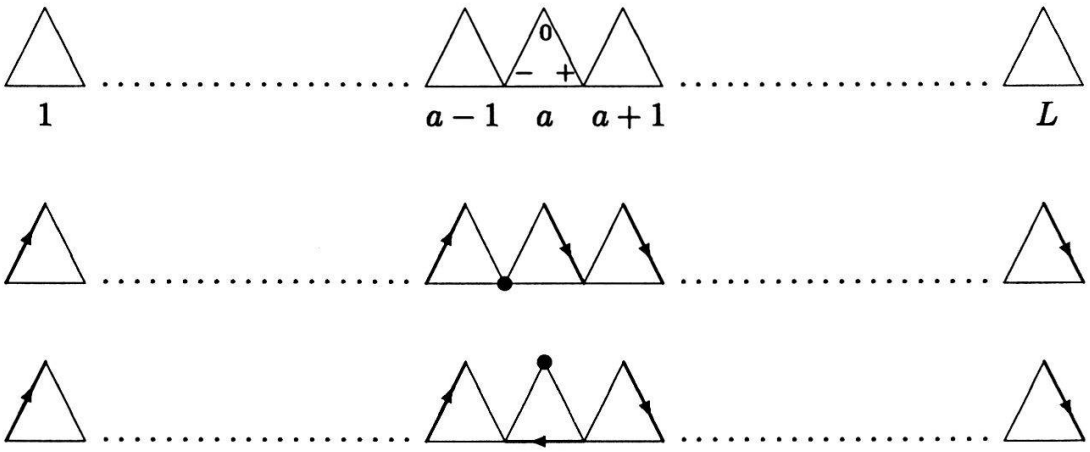

Fig. 5.1. *The numbering of the sites on the $\triangle$ chain and the valence bond states of Eq. (5.1).*

In this way, $\psi^{1,L}(q,\gamma)$ is defined for any q in Λ_L. A graphic representation of these states is given in Fig. 5.1. We will denote by $\Lambda_{b,c}$ the Δ chain formed by the triangles $b, b+1, \ldots, c$. The VB state $\psi^{b,c}(a_s,\gamma)$ where $s = 0, \pm$ is defined as the shift of the state $\psi^{1,c-b+1}(a-b+1_s,\gamma)$. Below we use the simplified notations

$$\psi(q) = \psi^{1,L}(q,\uparrow), \qquad \psi^{a,b}(q) = \psi^{a,b}(q,\uparrow). \tag{5.2}$$

Lemma 5.1. *For a Δ chain we have*

$$\big(\psi(p)\,\big|\,\psi(q)\big) = (-2)^{-d(p,q)}$$

Proof. Recall that two VB states differ only on the Δ chain connecting their free sites (see Figure 5.2a or Equation 5.1) hence the part outside this Δ chain contributes a factor 1 to the overlap. Therefore $\big(\psi(a_-)\,\big|\,\psi(b_+)\big) = \big(\psi^{a,b}(a_-)\,\big|\,\psi^{a,b}(b_+)\big)$. We first prove $\big(\psi(1_-)\,\big|\,\psi(b_+)\big) = (-2)^{-b}$ by induction. It holds for $b = 0$ $(0_+ = 1_-)$. If it holds up to b then by expanding the dimers on the triangle $b+1$ and using the orthogonality of VB states with different free spin, we get

$$\begin{aligned}\big(\psi(1_-)\,\big|\,\psi(b+1_+)\big) &= \frac{1}{\sqrt{2}}\,\big(\psi^{1,b}(1_-)\,\big|\,\psi^{1,b}(b_+)\big)\,\big([b+1_0|b+1_+]\,\big|\,\downarrow(b+1_0)\uparrow(b+1_+)\big)\\ &= -\frac{1}{2}\,\big(\psi^{1,b}(1_-)\,\big|\,\psi^{1,b}(b_+)\big) = -\frac{1}{2}\,(-2)^{-b} = (-2)^{-b-1}.\end{aligned}$$

The other cases are obtained by using the relation (5.1b). □

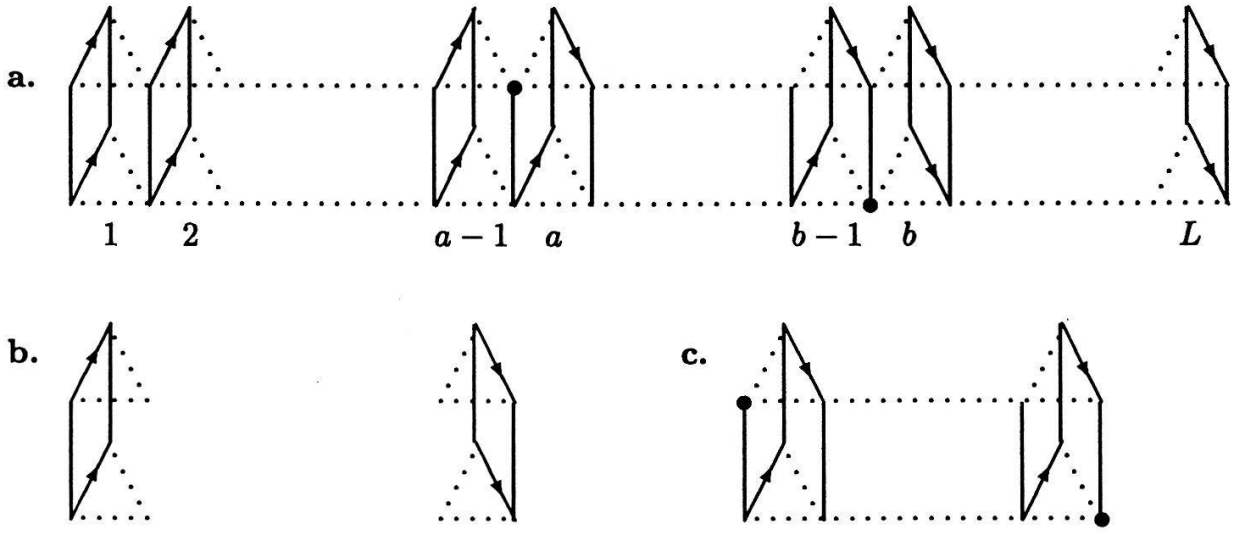

Fig. 5.2. a. *Overlap between two valence bond states on the Δ chain.* b. *These diagrams contribute a factor one to the overlap.* c. *The diagram which gives the $(-2)^{-d(p,q)}$ contribution.*

In the sequel, for a Δ tree Λ and q in Λ, $\psi(q)$ denotes the VB state with free site q and spin $+1/2$. For Δ chains the sign of $\psi(q)$ is fixed in such way that we get (5.2).

Lemma 5.2. *Let p and q be two sites of a $\triangle$ tree. Then*

$$\left|\left(\psi(p)\,\middle|\,\psi(q)\right)\right| = 2^{-d(p,q)}$$

Proof. Again, two VB states are identical (up to dimer orientations) outside the $\triangle$ chain connecting their respective free sites (see Figure 3.1). Therefore the preceeding Lemma applies. □

We are now able to introduce an orthonormal basis for the space of ground states on $\triangle$ trees. As before, we will first consider the case of a $\triangle$ chain.

Proposition 5.3. *Let Λ_L be a $\triangle$ chain with L triangles and q any site in Λ_L. Let b in $\{1,\ldots,L+1\}$ be determined by the equality $q = b_-$ or b_0. Then the following set of vectors is an orthonormal basis for the ground states of H^L ($\gamma = \uparrow, \downarrow$):*

$$\varphi_q^{1,L}(0,\gamma) = \psi^{1,L}(q,\gamma)$$

$$\varphi_q^{1,L}(a,\gamma) = \begin{cases} \frac{1}{\sqrt{3}}\left\{\psi^{1,L}(a_+,\gamma) + 2\psi^{1,L}(a_-,\gamma)\right\} & \text{if } 1 \le a < b \\ \frac{1}{\sqrt{3}}\left\{\psi^{1,L}(q,\gamma) + 2\psi^{1,L}(a_+,\gamma)\right\} & \text{if } a = b \\ \frac{1}{\sqrt{3}}\left\{\psi^{1,L}(a_-,\gamma) + 2\psi^{1,L}(a_+,\gamma)\right\} & \text{if } b < a \le L. \end{cases}$$

By analogy, we can also define on $\Lambda_{c,d}$ the states $\varphi_q^{c,d}(a,\gamma)$ for any site q in $\Lambda_{c,d}$.
The freedom in choosing the distinguished site q arbitrarily will be exploited later in the computation of the correlation functions.

Proof. Fix $S_z = +1/2$ and write $\varphi_q(a)$ for $\varphi_q^{1,L}(a,\uparrow)$ $a = 0,\ldots L$. For $1 \le a < b$ and p in $\Lambda_{a+1,L}$,

$$\left(\varphi_q(a)\,\middle|\,\psi(p)\right) = \frac{1}{\sqrt{3}}\left(\psi(a_+) + 2\psi(a_-)\,\middle|\,\psi(p)\right) = \frac{1}{\sqrt{3}}(-2)^{-d(a_+,p)} + \frac{2}{\sqrt{3}}(-2)^{-d(a_-,p)} = 0$$

because $d(a_+,p) = d(a_-,p) - 1$ for such p. Hence $\left(\varphi_q(a)\,\middle|\,\varphi_q(c)\right) = 0$ if $1 \le a < b$ and $c = 0$ or $c > a$. The same kind of argument gives $\left(\varphi_q(a)\,\middle|\,\varphi_q(c)\right) = 0$ if $a > b$ and $c = 0$ or $c > a$.

It remains to show that the states $\varphi_q(a)$ are normalized. This is the case for $\varphi_q(0) = \psi(q)$. If $\varphi_q(a) = \frac{1}{\sqrt{3}}\left\{\psi(p) + 2\psi(p')\right\}$ notice that $d(p,p') = 1$; hence $\left(\varphi_q(a)\,\middle|\,\varphi_q(a)\right) = \frac{1}{3}\left\{1 + 2(-\frac{1}{2}) + 2(-\frac{1}{2}) + 4\right\} = 1$. □

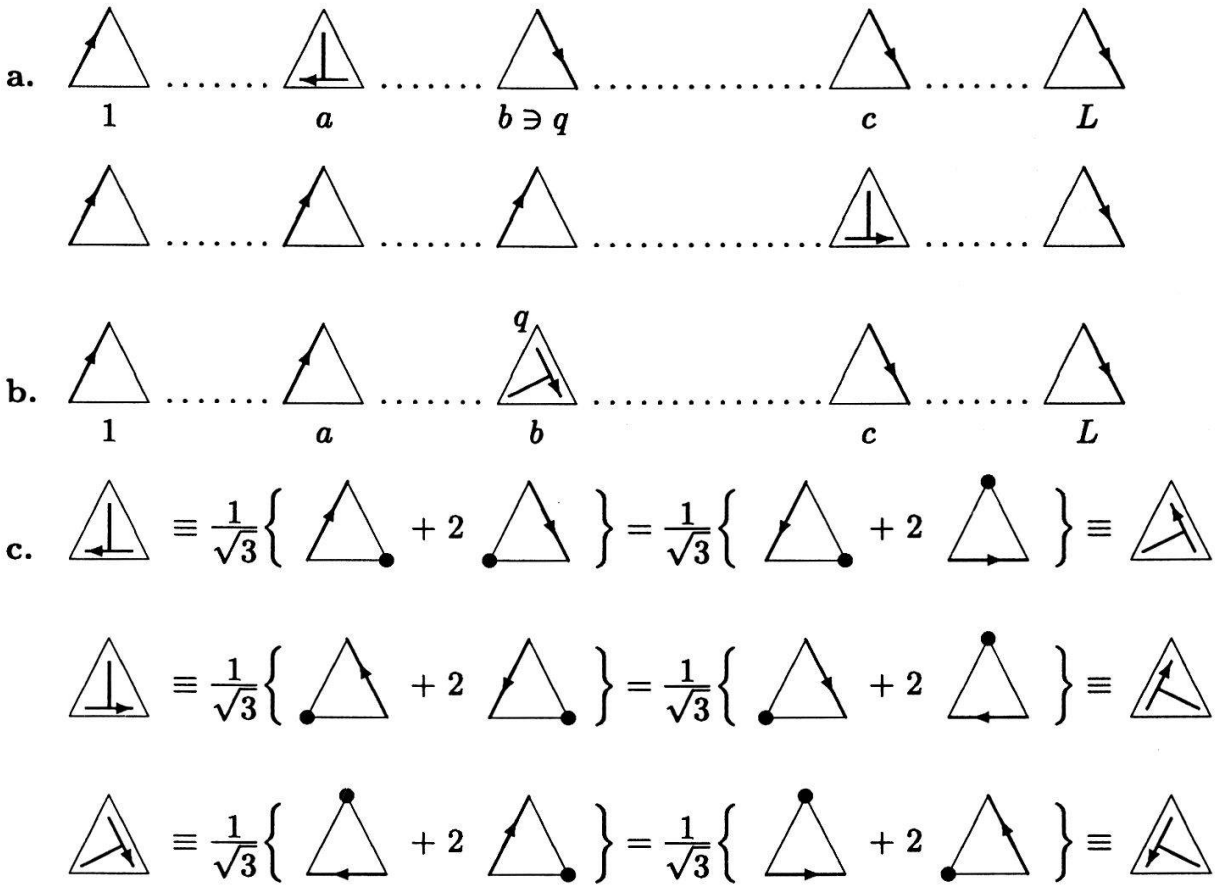

Fig. 5.3. a. *Elements of the orthonormal basis for ground states on the* $\triangle$ *chain.* b. *The state* $\varphi_q(b)$ *when* $q = b_0$. c. *Meaning of the symbol* .

Proposition 5.4. *Let* Λ *be a* $\triangle$ *tree and* q *a site in* Λ. *Let* V_0 *be the set of triangles containing the site* q. *A basis* $\{\psi(p)\}$ *of valence bond states on* Λ *can be chosen so that an orthonormal basis for the ground states of* H^Λ *in the subspace* $S_z = +1/2$ *is given by*

$$\varphi_q(0) = \psi(q)$$

$$\varphi_q(c) = \frac{1}{\sqrt{3}} \{\psi(q) + 2\psi(c_s)\} \qquad \text{if } c \text{ is in } V_0$$

where c_s *is a suitably chosen site on the triangle* c *different from* q, *and*

$$\varphi_q(c) = \frac{1}{\sqrt{3}} \{\psi(c_t) + 2\psi(c_s)\} \qquad \text{if } c \text{ is not in } V_0$$

where c_s *and* c_t *are suitably chosen sites on the triangle* c *such that* $d(c_s, q) = d(c_t, q) + 1$.

Proof. Recall that for any p in Λ there is a VB state $\psi(p)$ which is unique up to a sign. First choose the sign of $\psi(q)$ arbitrarily by fixing the orientation of its dimers. Let $\varphi_q(0) = \psi(q)$.

For any site p in $\Lambda \backslash \{q\}$ the sign of $\psi(p)$ is fixed in the following way. Outside the $\triangle$ chain connecting p and q the dimers are oriented as in $\psi(q)$. Let $b = \{p_1, p_2, p_3\}$ be a triangle

on the Δ chain connecting p and q and $\psi(q) = [p_1|p_2]\phi$. If in $\psi(p)$, p_1 and p_3 form a dimer then choose $[p_3|p_1]$; if p_2 and p_3 form a dimer then choose $[p_2|p_3]$.

For a triangle b, we denote by p the site in b such that $d(b,q) = d(p,q)$ and by p' and p'' the two other sites of b such that the dimer of $\psi(q)$ on b is $[p'|p'']$. Then $\varphi_q(b) = \{\psi(p) + 2\psi(p')\}/\sqrt{3}$.

The set of states $\{\varphi_q(b)\}$ obtained by the previous construction has the following properties:
1) The two VB states entering the construction of $\varphi_q(b)$ are equal on $\Lambda \backslash b$. The restriction of $\varphi_q(b)$ to b is one of the states defined in Figure 5.3c.
2) A VB state $\psi(p)$ and a state $\varphi_q(b)$ (resp. two states $\varphi_q(b)$ and $\varphi_q(c)$) are indentical outside the Δ chain connecting the site p and the triangle b (resp. the triangles b and c).

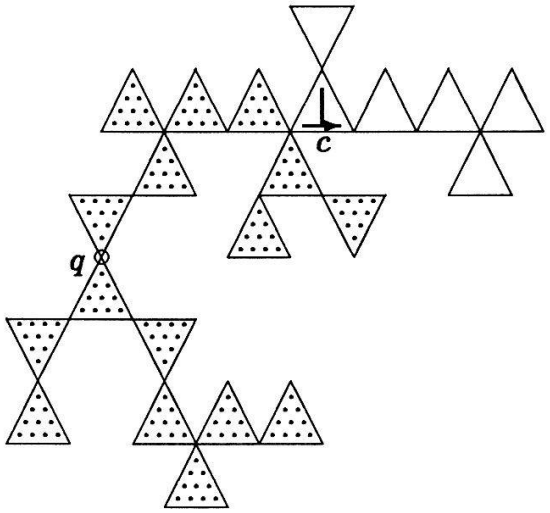

Fig. 5.4. *A state $\varphi_q(c)$ on a Δ tree. Any valence bond state having its free site on the shaded triangles is orthogonal to $\varphi_q(c)$.*

Take any state $\varphi_q(c)$ with $c \neq 0$, and consider the largest connected Δ tree $\widetilde{\Lambda}_c$ in Λ which contains q and is neighboring with the triangle c. (See Figure 5.4 where $\widetilde{\Lambda}_c$ is marked by dots). Then $\varphi_q(c)$ is orthogonal to any VB state $\psi(q')$ with free site q' in $\widetilde{\Lambda}_c$. To see this, notice that when computing $\big(\psi(q') \,\big|\, \varphi_q(c)\big)$ we are brought back to the case of the Δ chain connecting q' and c. We can then use Proposition 5.3 and Figure 5.3c. (Notice that if $\varphi_q(c) = \{\psi(p) + 2\psi(p')\}/\sqrt{3}$, then $d(p',q') = d(p,q') + 1$).

The orthogonality of $\varphi_q(b)$ and $\varphi_q(c)$ for $b \neq c$ follows from the fact that either b is in $\widetilde{\Lambda}_c$ or c is in $\widetilde{\Lambda}_b$. $\square$

6. Exponential Decay of the Two Point Correlation Functions in the Ground States

We begin by proving two lemmas for Λ_L, the $\triangle$ chain with L triangles. Let $\psi(q,\gamma)$ denote the valence bond state on Λ_L with free spin $\gamma = \uparrow$ or $\downarrow$ on site q. The dimers are oriented as in Eq. (5.1). Let $\Lambda^{q,p}$ denote the $\triangle$ chain connecting the sites q and p (i.e. $\Lambda^{q,p}$ is the minimal connected $\triangle$ chain which contains q and p).

Lemma 6.1. *Let q and p be two sites of the $\triangle$ chain Λ_L, such that $d(q,p) \geq 2$. Then*

$$\big(\psi(r,\eta)\,\big|\,\sigma_\alpha(q)\sigma_\beta(p)\psi(s,\gamma)\big) = \kappa_\gamma\,(-2)^{-d(r,s)}\,\big|\big(\eta\,\big|\,\sigma_\beta\sigma_\alpha\,\gamma\big)\big|$$

if p and q belong to $\Lambda^{s,r}$, and

$$\big(\psi(r,\eta)\,\big|\,\sigma_\alpha(q)\sigma_\beta(p)\psi(s,\gamma)\big) = 0$$

in all the other cases. Here κ_γ is a complex unit, independent of s and r.

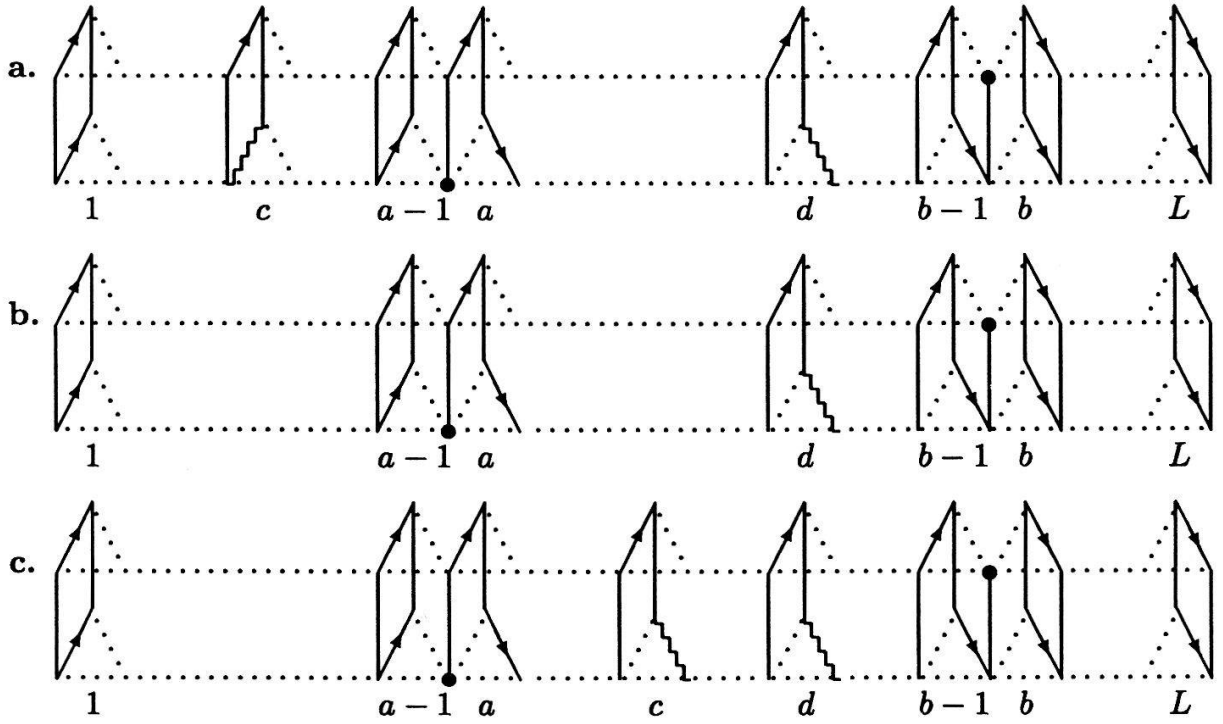

Fig. 6.1. *Diagrams for the off-diagonal correlation functions appearing in Lemma 6.1.* a. *A vanishing correlation.* b. *A nonzero one point off-diagonal correlation.* c. *A nonzero two point off-diagonal correlation.*

Proof. Suppose that $s \leq r$.
(i) Recall that $\sigma_\alpha(p)\,[p|q]$ is a triplet state and thus it is orthogonal to $[p|q]$ ($\sigma_\alpha(p)$ is a Pauli matrix). Let $C_\alpha(\gamma)$ denote the complex number with modulus 1 given by $\sigma_\alpha\,\gamma = C_\alpha(\gamma)\,\gamma'$ ($\gamma,\gamma' = \uparrow$ or $\downarrow$).

(ii) Compute $J(q) = \big(\psi(r,\eta)\,\big|\,\sigma_\alpha(q)\,\psi(s,\gamma)\big)$ for q in $\Lambda^{r,s}$, see figure 6.1b. If $q = s$ then

$$J(q) = \big(\psi(r,\eta)\,\big|\,\psi(s,\sigma_\alpha\gamma)\big) = C_\alpha(\gamma)\,2^{-d(s,r)}\,\big|\big(\eta\,\big|\,\sigma_\alpha\gamma\big)\big|$$

using Lemma 5.1. If $q \neq s$ we may suppose that $q = c_0$ or c_+. Expanding the dimers of $\psi(r,\eta)$ and $\psi(s,\gamma)$ which contain c_0 and using Lemma 5.1 we get

$$\begin{aligned} -J(c_0) = J(c_+) &= -\frac{1}{2}\big(\psi^{1,c-1}(c_-,\uparrow)\,\big|\,\psi^{1,c-1}(s,\gamma)\big)\,\big(\psi^{c+1,L}(r,\eta)\,\big|\,\psi^{c+1,L}(c_+,\sigma_\alpha\downarrow)\big) \\ &\quad -\frac{1}{2}\big(\psi^{1,c-1}(c_-,\downarrow)\,\big|\,\psi^{1,c-1}(s,\gamma)\big)\,\big(\psi^{c+1,L}(r,\eta)\,\big|\,\psi^{c+1,L}(c_+,\sigma_\alpha\uparrow)\big) \\ &= C_\alpha(\gamma)\,(-2)^{-d(s,r)}\,\big|\big(\eta\,\big|\,\sigma_\alpha\gamma\big)\big| \end{aligned}$$

Hence $J(q) = (-)\,C_\alpha(\gamma)\,(-2)^{-d(s,r)}\,\big|\big(\eta\,\big|\,\sigma_\alpha\gamma\big)\big|$. The minus sign is present only if $q = c_0$.

(iii) Let $J(p,q) = \big(\psi(r,\eta)\,\big|\,\sigma_\alpha(q)\sigma_\beta(p)\,\psi(s,\gamma)\big)$. If q is outside $\Lambda^{r,s}$, the $\triangle$ chain between r and s, then q belongs to the same dimer in $\psi(r,\eta)$ and $\psi(s,\gamma)$. Hence $J(q,p) = 0$ due to *(i)*. If both p and q are in $\Lambda^{r,s}$ then suppose $q < p = b_t$ with $t = 0$ or $+$. Notice that because of $d(q,p) \geq 2$, $\sigma_\alpha(q)$ and $\sigma_\beta(p)$ do not act on the same dimer. Expanding the dimers of $\psi(r,\eta)$ and $\psi(s,\gamma)$ which contain b_0 and using *(ii)* we get

$$\begin{aligned} J(q,b_0) &= J(q,b_+) \\ &= -\frac{1}{2}\big(\psi^{1,b-1}(b_-,\uparrow)\,\big|\,\sigma_\alpha(q)\psi^{1,b-1}(s,\gamma)\big)\,\big(\psi^{b+1,L}(r,\eta)\,\big|\,\psi^{b+1,L}(b_+,\sigma_\beta\uparrow)\big) \\ &\quad -\frac{1}{2}\big(\psi^{1,b-1}(b_-,\downarrow)\,\big|\,\sigma_\alpha(q)\psi^{1,b-1}(s,\gamma)\big)\,\big(\psi^{b+1,L}(r,\eta)\,\big|\,\psi^{b+1,L}(b_+,\sigma_\beta\downarrow)\big) \\ &= (-)\,C_\alpha(\gamma)\,C_\beta(\sigma_\alpha\gamma)\,\big|\big(\eta\,\big|\,\sigma_\beta\sigma_\alpha\gamma\big)\big|\,(-2)^{-d(s,r)} \end{aligned}$$

The minus sign appears only if $q = c_0$.

In summary,

$$J(q,p) = \kappa_\gamma\,(-2)^{-d(s,r)}\,\big|\big(\eta\,\big|\,\sigma_\beta\sigma_\alpha\gamma\big)\big|$$

where $\kappa_\gamma = (-)\,C_\alpha(\gamma)\,C_\beta(\sigma_\alpha\gamma)$ is independent of r and s, $|\kappa_\gamma| = 1$ (the minus sign appears if $q = c_0$ and $p = b_+$ or if $q = c_-$ and $p = b_0$). □

In order to compute the correlations in any ground state between the sites q and p of a $\triangle$ chain, we use the orthonormal basis with distinguished site q constructed in Proposition 5.3.

Lemma 6.2. *Let q and p be two sites of the $\triangle$ chain Λ_L, such that $d(q,p) \geq 2$. Then,*

$$\Big|\big(\varphi_q(b,\eta)\,\big|\,\sigma_\alpha(q)\sigma_\beta(p)\,\varphi_q(a,\gamma)\big)\Big| \leq \frac{2}{\sqrt{3}}\,2^{-d(q,p)}\,\big|\big(\eta\,\big|\,\sigma_\alpha\sigma_\beta\gamma\big)\big|$$

if $a = 0$ and b labels the boundary triangle of $\Lambda^{q,p}$ containing p, or if a, b label the two boundary triangles of $\Lambda^{q,p}$, and

$$\Big|\big(\varphi_q(b,\eta)\,\big|\,\sigma_\alpha(q)\sigma_\beta(p)\,\varphi_q(a,\gamma)\big)\Big| = 0$$

in all the other cases.

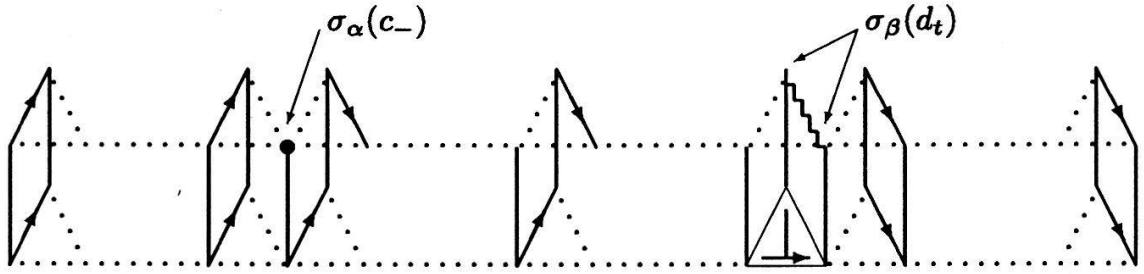

Fig. 6.2. *One of the nonzero off-diagonal correlation functions in Lemma 6.2.*

Proof. Let $I(a,b) = \big(\varphi_q(b,\eta)\,\big|\,\sigma_\alpha(q)\,\sigma_\beta(p)\,\varphi_q(a,\gamma)\big)$. If p or q is outside the Δ chain which connects a_- to b_+ then $I(a,b)=0$ by Lemma 6.1. We may suppose that $q = c_0$ or c_- and $p = d_0$ or d_+ with $c < d$. If $1 \le a < c < d \le b \le L$ then with the notations of Lemma 6.1,

$$\begin{aligned} I(a,b) &= \frac{1}{3}\big(\psi(b_-,\eta) + 2\psi(b_+,\eta)\,\big|\,\sigma_\alpha(q)\sigma_\beta(p)\,[\psi(a_+,\gamma) + 2\psi(a_-,\gamma)]\big) \\ &= \frac{1}{3}\kappa_\gamma\Big\{(-2)^{-d(a_+,b_-)} + 2(-2)^{-d(a_-,b_-)}\Big\}\big|\big(\eta\,\big|\,\sigma_\alpha\sigma_\beta\gamma\big)\big| \\ &\quad + \frac{2}{3}\kappa_\gamma\Big\{(-2)^{-d(a_+,b_+)} + 2(-2)^{-d(a_-,b_+)}\Big\}\big|\big(\eta\,\big|\,\sigma_\alpha\sigma_\beta\gamma\big)\big| \end{aligned}$$

This holds because $d(a_-,b_-) = d(a_+,b_-)+1$ and $d(a_-,b_+) = d(a_+,b_+)+1$. If $a=0$ or $a=c$ then by definition $\varphi_q(a,\gamma) = C^q\psi(q,\gamma) + C^+\psi(c_+,\gamma)$ with $C^q = 1$ or $1/\sqrt{3}$ and $C^+ = 0$ or $2/\sqrt{3}$ respectively. In both cases, since q is not in Λ^{c_+,b_-} we get by Lemma 6.1

$$\begin{aligned} I(a,b) &= C^q\big(\varphi_q(b,\eta)\,\big|\,\sigma_\alpha(q)\sigma_\beta(p)\,\psi(q,\gamma)\big) \\ &= \begin{cases} 0 & \text{if } p \neq b_0, b_+ \\ \dfrac{C^q}{\sqrt{3}}\kappa_\gamma\, 2\,(-2)^{-d(q,b_+)}\,\big|\big(\eta\,\big|\,\sigma_\alpha\sigma_\beta\gamma\big)\big| & \text{if } p = b_0, b_+ \end{cases} \end{aligned}$$

Now $d(q,b_+) = d(q,b_0)$, therefore in the second case we get

$$|I(a,b)| \le \frac{2}{\sqrt{3}}\,2^{-d(q,p)}\,\big|\big(\eta\,\big|\,\sigma_\alpha\sigma_\beta\gamma\big)\big|. \quad \square$$

Theorem 6.3. *Let Λ be any Δ tree (containing more than one triangle), let q and p be two sites of Λ such that their distance $d(q,p) \ge 2$. Then for any normalized ground state Ψ of H^Λ,*

$$\big|\big(\Psi\,\big|\,\sigma_\alpha(q)\sigma_\beta(p)\Psi\big)\big| \le \frac{16}{\sqrt{3}}\,2^{-d(q,p)}$$

Proof. Let $\Psi = \sum_{a,\gamma}\Psi_{a,\gamma}\,\varphi_q(a,\gamma)$, then

$$\big|\big(\Psi\,\big|\,\sigma_\alpha(q)\sigma_\beta(p)\,\Psi\big)\big| \le \sum_{\substack{a,b\\ \gamma,\eta}}\big|\big(\varphi_q(b,\eta)\,\big|\,\sigma_\alpha(q)\sigma_\beta(p)\,\varphi_q(a,\gamma)\big)\big|.$$

By Lemma 6.2 there are only four couples (a,b) for which we have a nonvanishing term. Moreover, for any α,β, $|(\eta\,|\,\sigma_\alpha\sigma_\beta\gamma)|=1$ for two different (γ,η). Therefore

$$|(\Psi\,|\,\sigma_\alpha(q)\sigma_\beta(p)\,\Psi)|\le\frac{16}{\sqrt{3}}\,2^{-d(q,p)}.\quad\square$$

The correlation functions in the case of a Δ^+ tree are trivial. If the spin operators do not act on the same dimer then the correlation function is 0 in the ground state.

7. Energy Gap in the Δ Chain

By energy gap we understand the separation between the lowest eigenvalue of H^Λ and the rest of its spectrum. The main result of this section is that for finite Δ chains the gaps for H^L have an L-independent positive lower bound.

Theorem 7.1. *Consider the anisotropic Heisenberg Hamiltonian H^L on the Δ chain Λ_L. There exists a positive constant ε, independent of L, such that*

$$(\Phi\,|\,H^L\Phi)\ge\varepsilon(\Phi\,|\,\Phi)\tag{7.1}$$

for any vector Φ orthogonal to the space of ground states of H^L.

Following Section 2.3 of Affleck et al [AKLT], for $n\le L$, let H^n denote the Hamiltonian for the triangles $1,\dots,n$, Q^n the orthogonal projection in $\mathfrak{H}^{\Lambda_L}=\mathfrak{H}^L$ onto the space of ground states of H^n, and $P^n=\mathbb{1}-Q^n$. If $H^n\psi=0$ then $H^{n-1}\psi=0$. It follows that $Q^{n-1}\ge Q^n$ and so $P^{n-1}\le P^n$. In fact, $Q^{n-1}-Q^n$ is a projection.

Let N_n be the dimension of the range of $Q^{n-1}-Q^n$ when restricted to $\mathfrak{H}^n$ and $\{\xi^n_{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{A}_n}$ an orthonormal basis in $\mathfrak{H}^n$ for the range of this projection. Here $\mathbb{A}_n$ is an index set of N_n elements, to be specified below. Hence each $\xi^n_{\mathbf{a}}$ is a ground state of H^{n-1} and is orthogonal to the ground states of H^n. Notice that $N_n=4\cdot 2n-2(n+1)=6n-2$.
For $0\le l<n$, denote by $H^{n-l,n}$ the Hamiltonian for the union of triangles $n-l,\dots,n$, and $Q^{n-l,n}$ the orthogonal projection in $\mathfrak{H}^n$ onto the space of the ground states of $H^{n-l,n}$. Define

$$\zeta^{n,l}_{\mathbf{a}}=\begin{cases}Q^{n-l,n}\,\xi^n_{\mathbf{a}}\,\Big/\,\big\|Q^{n-l,n}\,\xi^n_{\mathbf{a}}\big\| & \text{if } Q^{n-l,n}\,\xi^n_{\mathbf{a}}\neq 0\\ 0 & \text{otherwise}\end{cases}$$

and

$$\varepsilon(l)=\sup_{n>l}\sup_{\mathbf{a}}\big\|Q^{n-l,n}\,\xi^n_{\mathbf{a}}\big\|^2$$

Finally let $P(\zeta^{n,l}_{\mathbf{a}})$ be the orthogonal projection in $\mathfrak{H}^L$ onto the subspace $\zeta^{n,l}_{\mathbf{a}}\otimes\mathfrak{H}^{\Lambda_L\setminus\Lambda_n}$ if $\zeta^{n,l}_{\mathbf{a}}\neq 0$, and $P(\zeta^{n,l}_{\mathbf{a}})=0$ if $\zeta^{n,l}_{\mathbf{a}}=0$, and let $P(\zeta^{n,l})$ denote the smallest orthogonal projection in $\mathfrak{H}^L$ which is greater than or equal to $P(\zeta^{n,l}_{\mathbf{a}})$ for all $\mathbf{a}$: $P(\zeta^{n,l})=\sup_{\mathbf{a}\in\mathbb{A}_n}P(\zeta^{n,l}_{\mathbf{a}})$.

The proof of Theorem 7.1 relies on several lemmas. Denote the triplet states on the sites n_0 and n_+ by

$$\tau^n_\gamma = \gamma(n_0)\gamma(n_+) \qquad \gamma = \uparrow, \downarrow$$
$$\tau^n_0 = \{\uparrow(n_0)\downarrow(n_+) + \downarrow(n_0)\uparrow(n_+)\}/\sqrt{2}$$

and introduce the multiindices $\mathbf{a} = (a, \gamma|s)$ where $0 \le a < n$, $\gamma = \uparrow$ or $\downarrow$ and $s = \uparrow, \downarrow$ or 0. Excluding $(0, \uparrow|\downarrow)$ and $(0, \downarrow|\uparrow)$, these latter form the set A_n. We have

Lemma 7.2. *An orthonormal basis for the range of $Q^{n-1} - Q^n$ restricted to $\mathfrak{H}^n$ is*

$$\xi^n_{(a,\gamma|s)} = \varphi_{n\_}(a,\gamma)\,\tau^n_s \qquad \text{for } a = 1, \ldots, n-1 \text{ and } s = \uparrow, \downarrow, 0$$
$$\xi^n_{(0,\gamma|\gamma)} = \varphi_{n\_}(0,\gamma)\,\tau^n_\gamma \qquad \text{for } \gamma = \uparrow, \downarrow$$

and

$$\xi^n_{(0,\uparrow|0)} = \frac{1}{\sqrt{3}}\left\{\sqrt{2}\,\varphi_{n\_}(0,\uparrow)\,\tau^n_0 + \varphi_{n\_}(0,\downarrow)\,\tau^n_\uparrow\right\}$$
$$\xi^n_{(0,\downarrow|0)} = \frac{1}{\sqrt{3}}\left\{\sqrt{2}\,\varphi_{n\_}(0,\downarrow)\,\tau^n_0 + \varphi_{n\_}(0,\uparrow)\,\tau^n_\downarrow\right\}$$

Here $\{\varphi_{n\_}(a,\gamma)\}$ is the orthonormal basis for the ground states of H^{n-1}, defined in Proposition 5.3.

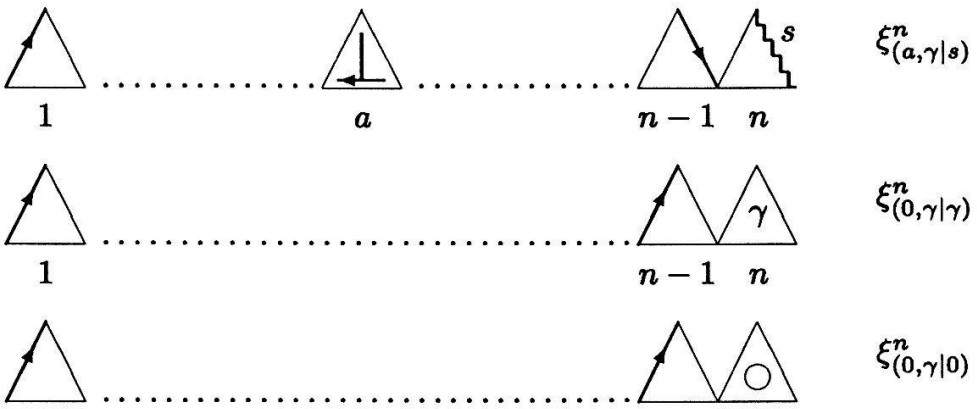

Fig. 7.1. *The states in Lemma 7.2 for an orthonormal basis of the range of $Q^{n-1} - Q^n$ restricted to $\mathfrak{H}^n$. The wavy line marks a triplet state.*

Proof. The states $\xi^n_{\mathbf{a}}$ given in the Lemma are by construction mutually orthogonal ground states of H^{n-1} and their number is $6n - 2$.

It remains to show that they are orthogonal to the ground states of H^n. First notice that the states $\xi^n_{(0,\gamma|s)}$ when considered on the last triangle are orthogonal to the VB states on that triangle. So take $\xi^n_{(a,\gamma|s)}$ with $a \neq 0$. If $\psi(q,\gamma)$ is a VB state in $\mathfrak{H}^n$ with $q \neq n_0, n_+$ then $\psi(q,\gamma)$ has a dimer between the sites n_0 and n_+, and $\left([n_0|n_+]\,\middle|\,\tau^n_s\right) = 0$ for any s. If $q = n_0, n_+$ then $\psi(q,\gamma) = \varphi_{n\_}(0,\uparrow) \otimes \kappa(n_0, n_+) + \varphi_{n\_}(0,\downarrow) \otimes \varsigma(n_0, n_+)$ which is orthogonal to the states $\xi^n_{(a,\gamma|s)}$ with $a \neq 0$. □

Using the basis defined in Lemma 7.2 one can prove the following.

Lemma 7.3. *For* $\mathbf{a} = (a, \gamma|s)$ *in* $\mathbb{A}_n$ *and* $0 \le l < n$,

i) $Q^{n-l,n}\,\xi^n_{\mathbf{a}} = 0$ *if* $n - l \le a < n$ *or* $a = 0$, *hence* $P(\zeta^{n,l}_{\mathbf{a}}) = 0$.

ii) For each l, $\quad \varepsilon(l) \le 2^{-l}$.

iii) If $a \ne b$ *then* $\left(\zeta^{n,l}_{(a,\gamma|s)} \,\middle|\, \zeta^{n,l}_{(b,\eta|t)}\right) = 0$. *As a consequence* $\sum_{\mathbf{a}\in\mathbb{A}_n} P(\zeta^{n,l}_{\mathbf{a}}) \le 6\,P(\zeta^{n,l})$.

Proof. To prove *i)* and *ii)* we compute the overlaps between $\xi^n_{\mathbf{a}}$ and $\chi \otimes \psi^{n-l,n}(q,\eta)$ where χ is a normalized state in $\mathfrak{H}^{n-l-1}$ and $\psi^{n-l,n}(q,\eta)$ is a VB state (cf. Eq. 5.2).

i) This result is obtained in the same way as Lemma 7.2.

ii) For $\mathbf{a} = (a, \gamma|s)$ with $1 \le a \le n - l - 1$, we can write

$$\xi^n_{\mathbf{a}} = \left\{\theta_{a,\uparrow} \otimes \psi^{n-l,n-1}(n - l_-, \uparrow) + \theta_{a,\downarrow} \otimes \psi^{n-l,n-1}(n - l_-, \downarrow)\right\} \tau^n_s$$

where $\theta_{a,\uparrow}$ and $\theta_{a,\downarrow}$ are states in $\mathfrak{H}^{n-l-1}$, see Figure 7.2, with $\|\theta_{a,\uparrow}\|, \|\theta_{a,\downarrow}\| \le \sqrt{2}$ obtained by the expansion of the dimer $[n - l - 1_0|n - l_-]$.

Following the proof of Lemma 7.2, we have $\left(\chi \otimes \psi^{n-l,n}(q,\eta) \,\middle|\, \xi^n_{\mathbf{a}}\right) = 0$ if $q \ne n_0, n_+$. If $q = n_0$ or n_+, we write $\psi^{n-l,n}(q,\eta) = \psi^{n-l,n-1}(n_-,\uparrow)\otimes\kappa(n_0,n_+) - \psi^{n-l,n-1}(n_-,\downarrow)\otimes\varsigma(n_0,n_+)$. Here $\kappa(n_0,n_+) = \pm\downarrow(p)\eta(q)/\sqrt{2}$ and $\varsigma(n_0,n_+) = \pm\uparrow(p)\eta(q)/\sqrt{2}$ for $\{p,q\} = \{n_0,n_+\}$. Using the overlap formula of Lemma 5.1,

$$\begin{aligned}
|\left(\chi \otimes \psi^{n-l,n}(q,\eta) \,\middle|\, \xi^n_{\mathbf{a}}\right)| \le \\
|\left(\chi \,\middle|\, \theta_{a,\uparrow}\right)|\,|\left(\kappa(n_0,n_+) \,\middle|\, \tau^n_s\right)|\,|\left(\psi^{n-l,n-1}(n_-,\uparrow) \,\middle|\, \psi^{n-l,n-1}(n - l_-,\uparrow)\right)| \\
+|\left(\chi \,\middle|\, \theta_{a,\downarrow}\right)|\,|\left(\varsigma(n_0,n_+) \,\middle|\, \tau^n_s\right)|\,|\left(\psi^{n-l,n-1}(n_-,\downarrow) \,\middle|\, \psi^{n-l,n-1}(n - l_-,\downarrow)\right)| \\
\le |\left(\psi^{n-l,n-1}(n_-,\eta) \,\middle|\, \psi^{n-l,n-1}(n - l_-,\eta)\right)| = 2^{-l}.
\end{aligned}$$

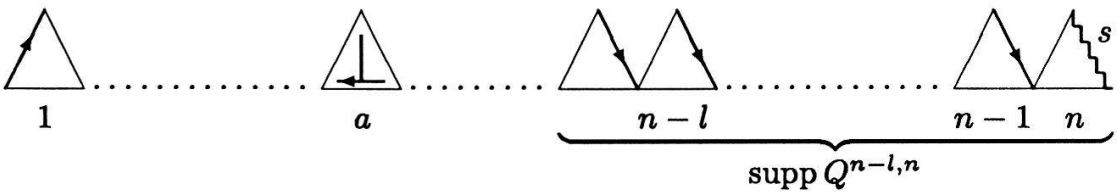

Fig. 7.2. *A state* $\xi^n_{\mathbf{a}}$ *where* $\mathbf{a} = (a, \gamma|s)$ *with* $1 \le a \le n - l - 1$ *and the support of* $Q^{n-l,n}$. $\zeta^{n,l}_{\mathbf{a}} \ne 0$ *in this case.*

iii) The orthogonality of $\zeta^{n,l}_{(a,\gamma|s)}$ and $\zeta^{n,l}_{(b,\eta|t)}$ for $1 \le a, b < n - l - 1$ (when these states are nonvanishing) and $a \ne b$ follows from the orthogonality of $\xi^n_{\mathbf{a}}$ and $\xi^n_{\mathbf{b}}$, because $Q^{n-l,n}$ acts like the identity in $\mathfrak{H}^{\Lambda_{1,n-l-1}\setminus\{n-l-1_+\}}$. The case $1 \le a < b = n - l - 1$ is treated by expanding the states $\xi^n_{\mathbf{a}}$ and $\xi^n_{\mathbf{b}}$ on site $n - l - 1_0$. and using the argument above.

Thus, for a given $\mathbf{a} = (a, \gamma|s)$, $1 \le a \le n - l - 1$ there are at most 6 different $\mathbf{b} = (a, \eta|t)$ such that $\left(\zeta^{n,l}_{\mathbf{a}} \,\middle|\, \zeta^{n,l}_{\mathbf{b}}\right) \ne 0$. □

The following two lemmas do not refer to a special choice of the basis $\xi^n_{\mathbf{a}}$.

Lemma 7.4. *If $|n-m| \geq l+2$, for each* $\mathbf{a}, \mathbf{b}$ *the projections* $P(\zeta_{\mathbf{a}}^{n,l})$ *and* $P(\zeta_{\mathbf{b}}^{m,l})$ *are orthogonal, i.e.,* $P(\zeta_{\mathbf{a}}^{n,l})P(\zeta_{\mathbf{b}}^{m,l}) = 0$.

Proof. One can follow the proof of Lemma 2.3 of [AKLT]. The bound $l+1$ therein is replaced by $l+2$. This is due to the fact that in the present model the ground states minimize the Hamiltonian by triangles; two adjacent triangles having a common site, the associated projections will not commute, in general. □

Corollary. *If $|n-m| \geq l+2$ then* $P(\zeta^{n,l})\, P(\zeta^{m,l}) = 0$.

Lemma 7.5. *For all n and $0 \leq l < n$,*

$$Q^{n-1} - Q^n \leq 2\,\varepsilon(l) \sum_{\mathbf{a}\in\mathbb{A}_n} P(\zeta_{\mathbf{a}}^{n,l}) + \frac{2}{e_{l+1}}\, H^{n-l,n}$$

where e_{l+1} is the smallest nonzero eigenvalue of $H^{n-l,n}$.

Proof. This is identical to that of Lemma 2.2 in [AKLT]. The sum $\sum_{\mathbf{a}\in\mathbb{A}_n}$ replaces $\sum_{i=1}^{8}$ therein. □

Proof of Theorem 7.1. The proof follows closely that of Theorem 2.1 in [AKLT] and is given here for the reader's convenience. Recall that P^L is the orthogonal projection onto the complement of the space of grounds states of H^L.

Write P^L as a sum of mutually orthogonal projections,

$$P^L = \sum_{n=l+2}^{L} (P^n - P^{n-1}) + P^{l+1} = \sum_{n=l+2}^{L} (Q^{n-1} - Q^n) + P^{l+1}$$

The integer $l < L$ will be chosen later. By the Lemmas 7.5 and 7.3 we have

$$\sum_{n=l+2}^{L} (Q^{n-1} - Q^n) \leq 12\,\varepsilon(l) \sum_{n=l+2}^{L} P(\zeta^{n,l}) + \frac{2}{e_{l+1}} \sum_{n=l+2}^{L} H^{n-l,n} \tag{7.2}$$

The second term in (7.2) can be bounded by using

$$\sum_{n=l+2}^{L} H^{n-l,n} \leq (l+1)\, H^L \tag{7.3}$$

By using the Corollary of Lemma 7.4, the first term in (7.2) can be estimated as

$$\sum_{n=l+2}^{L} P(\zeta^{n,l}) \leq (l+2)\,\mathbf{1} \tag{7.4}$$

Inserting (7.3) and (7.4) into (7.2),

$$\sum_{n=l+2}^{L} (Q^{n-1} - Q^n) \leq 12\,(l+2)\,\varepsilon(l)\,\mathbf{1} + \frac{2\,(l+1)}{e_{l+1}}\,H^L$$

Moreover, from the spectral decomposition of H^{l+1}, $P^{l+1} \leq \frac{1}{e_{l+1}}\,H^{l+1} \leq \frac{1}{e_{l+1}}\,H^L$ thus

$$P^L \leq 12\,(l+2)\,\varepsilon(l)\,\mathbf{1} + \frac{2\,l+3}{e_{l+1}}\,H^L$$

If Φ is a vector orthogonal to the ground states of H^L then $P^L\,\Phi = \Phi$. Hence the above equation can be written as

$$\big(1 - 12(l+2)\,\varepsilon(l)\big)\,\big(\,\Phi\,\big|\,\Phi\,\big) \leq \frac{2\,l+3}{e_{l+1}}\big(\,\Phi\,\big|\,H^L\,\Phi\,\big).$$

The theorem is proven if we can choose an l such that $1 - 12(l+2)\,\varepsilon(l)$ is positive. But this is the case for any $l \geq 7$ because by Lemma 7.3, $\varepsilon(l) \leq 2^{-l}$. □

Remark. If the Hamiltonian H^L is inhomogeneous with parameters λ_a, ν_a and μ_a, Theorem 7.1 remains true if

$$e_{l+1} = \inf_n e_{l+1}(n) > 0$$

where $e_{l+1}(n)$ is the lowest nonzero eigenvalue of $H^{n-l,n}$. This is always the case if there exists a constant $\epsilon > 0$, such that $\lambda_a + \nu_a + \mu_a \geq \epsilon$ and $\lambda_a\nu_a + \nu_a\mu_a + \mu_a\lambda_a \geq \epsilon$.

Proposition 7.6. *Let Λ_L^+ be the $\triangle^+$ chain with L triangles. The gap ε_L of $H^{\Lambda_L^+}$ is separated from zero as $L \to \infty$, i.e., $\liminf \varepsilon_L > 0$.*

This follows from the proof given in [AKLT] and the one above, noticing that

$$N_n = \dim \mathrm{Ran}(Q^{n-1} - Q^n) = 3$$

when restricted to $\mathfrak{H}^{\Lambda_n^+}$.

8. The Infinite Volume Limit

In this section we consider the infinite $\triangle$ chain and the infinite $\triangle$ Cayley tree.

For the infinite volume systems we use the following

Definitions. *i)* A *local observable* is any polynomial of the Pauli matrices $\sigma_\alpha(p)$. An *observable* is any element in the norm closure $\mathfrak{A}$ of the algebra of local observables.

ii) The *support* $\mathrm{supp}\,A$ of a (local) observable A is the set of *triangles* on which the observable does not act as the identity operator $\mathbf{1}$.

iii) A *state* ϱ is a positive linear normalized functional on the algebra of observables $\mathfrak{A}$, i.e., $\varrho(\mathbb{1}) = 1$, $\varrho(\alpha A + \beta B) = \alpha\,\varrho(A) + \beta\varrho(B)$ and $\varrho(A^*A) \geq 0$ for all A, B in $\mathfrak{A}$ and α, β in $\mathbb{C}$, where A^* denotes the adjoint of A. A state ϱ is *pure* if it cannot be decomposed in a convex combination of two other states, i.e., $\varrho = \alpha\varrho_1 + (1-\alpha)\varrho_2$ for $0 < \alpha < 1$ implies $\varrho = \varrho_1 = \varrho_2$.

*iv)*As in the paper of Affleck et al [AKLT] the most natural way to define a *ground state* ω for this model is to ask that the ground states minimize the energy of each triangle, i.e. $\omega(H_a) = 0$ for each triangle a.

8.1. The Infinite $\triangle$ Chain

Let $\mathbb{L}$ denote the doubly infinite $\triangle$ chain. In this subsection we study some proprieties in the limit when a sequence of finite $\triangle$ chains tends to $\mathbb{L}$.

Lemma 8.1. *For a given q in $\mathbb{L}$, let Λ_q be the smallest $\triangle$ chain containing 0 and q. Take any $\triangle$ chain $\Lambda \supset \Lambda_q$ and consider $\psi(q,\gamma)$, the valence bond state in Λ with free spin γ on q (cf. Eq. 5.1). There exists a unitary operator $\vartheta^{q,\gamma}$, independent of Λ, with support in Λ_q such that*

$$\psi(q,\gamma) = \vartheta^{q,\gamma}\,\psi(0_-,\downarrow).$$

Remark. This result remains true for any $\triangle$ tree.

Proof. Notice first that $\sigma_x(q)\,\psi(q,\downarrow) = \psi(q,\sigma_x\downarrow) = \psi(q,\uparrow)$, and that $\sigma_x(q)$ is a unitary operator. It is sufficient to prove that for any q in $\mathbb{L}$, there exists a unitary operator ϑ^q such that $\vartheta^q\,\psi(0,\downarrow) = \psi(q,\downarrow)$. Then $\vartheta^{q,\downarrow} = \vartheta^q$ and $\vartheta^{q,\uparrow} = \sigma_x(q)\,\vartheta^q$. Consider the operator

$$\vartheta_{c_t,c_-} = \vartheta_{c_-,c_t} = -\frac{1}{2}\left\{\mathbf{1} + \sum_{\alpha=x,y,z} \sigma_\alpha(c_-)\,\sigma_\alpha(c_t)\right\} \qquad \text{for } t = 0,+\,,$$

and let $\vartheta_{c_-,c_-} = \mathbf{1}$. Then, $\vartheta^*_{c_t,c_-} = \vartheta_{c_t,c_-}$, $\vartheta^2_{c_t,c_-} = \mathbf{1}$ and $\vartheta_{c_t,c_-}\psi(c_-,\downarrow) = \psi(c_t,\downarrow)$.

Now if $q = a_0$ or a_-, define

$$\vartheta^q = \begin{cases} \vartheta_{q,a_-}\,\vartheta_{a_-,a_+}\cdots\vartheta_{-1_-,-1_+} & \text{if } a < 0 \\ \vartheta_{q,0_-} & \text{if } a = 0 \\ \vartheta_{q,a_-}\,\vartheta_{a-1_+,a-1_-}\cdots\vartheta_{0_+,0_-} & \text{if } a > 0 \end{cases}$$

Then $(\vartheta^q)^*\,\vartheta^q = \mathbf{1}$ and $\psi(q,\downarrow) = \vartheta^q\,\psi(0,\downarrow)$. □

For the integers $a \leq b$, let

$$\phi^{a,b}_- = [a_-|a_0]\,[a+1_-|a+1_0]\ldots[b_-|b_0]$$
$$\phi^{a,b}_+ = [a_0|a_+]\,[a+1_0|a+1_+]\ldots[b_0|b_+]$$

Recall that for $a \leq b$ and $a \leq c \leq b+1$, $\psi^{a,b}(c_-,\gamma)$ denotes the VB state on $\Lambda_{a,b}$, the $\triangle$ chain of the triangles $a, a+1, \ldots, b$, with free spin γ on the site c_- (cf. Eq. 5.1).

Proposition 8.2. *Let A be any local observable and q any site, then for all $a \leq b$ such that q and the support of A is contained in the triangles $a, a+1, \ldots, b$ and for $\gamma = \uparrow, \downarrow$, the limits below exist and define pure ground states of $\mathbb{L}$ (see Fig. 8.1):*

$$\omega_+(A) \equiv \lim_{L\to\infty} \left(\psi^{-L,L}(-L_-,\gamma) \,|\, A\,\psi^{-L,L}(-L_-,\gamma)\right) = \left(\phi_+^{a,b} \,|\, A\,\phi_+^{a,b}\right)$$

$$\omega_-(A) \equiv \lim_{L\to\infty} \left(\psi^{-L,L}(L_+,\gamma) \,|\, A\,\psi^{-L,L}(L_+,\gamma)\right) = \left(\phi_-^{a,b} \,|\, A\,\phi_-^{a,b}\right)$$

$$\omega_{q\gamma}(A) \equiv \lim_{L\to\infty} \left(\psi^{-L,L}(q,\gamma) \,|\, A\,\psi^{-L,L}(q,\gamma)\right) = \left(\psi^{a,b}(q,\gamma) \,|\, A\,\psi^{a,b}(q,\gamma)\right).$$

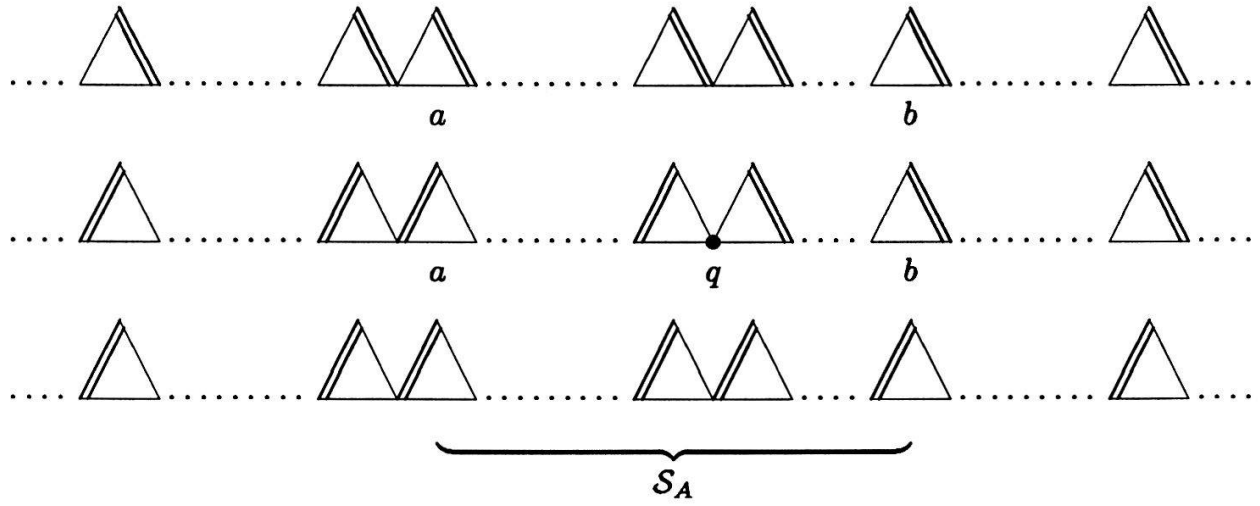

Fig. 8.1. *The finite volume valence bond states used to construct the ground states in Proposition 8.2 for ω_+, $\omega_{q\gamma}$ and ω_-, respectively. S_A denotes the maximal set of sites on which a local observable A with support in $\{a, \ldots, b\}$ is not equal to one.*

Proof. If L is large enough, we can write

$$\psi^{-L,L}(-L_-,\gamma) = \psi^{-L,a-1}(-L_-,\gamma)\,\phi_+^{a,b}\,\phi_+^{b+1,L}$$

hence if $\operatorname{supp} A \subset \{a, \ldots, b\}$,

$$\begin{aligned}\omega_+(A) &= \lim_{L\to\infty} \left(\psi^{-L,a-1}(-L_-,\gamma) \,|\, \psi^{-L,a-1}(-L_-,\gamma)\right) \left(\phi_+^{a,b} \,|\, A\,\phi_+^{a,b}\right) \left(\phi_+^{b+1,L} \,|\, \phi_+^{b+1,L}\right) \\ &= \left(\phi_+^{a,b} \,|\, A\,\phi_+^{a,b}\right)\end{aligned}$$

In particular, for any triangle c,

$$\omega_+(H_c) = \left(\phi_+^{c-1,c} \,|\, H_c\,\phi_+^{c-1,c}\right) = 0$$

So that ω_+ is a ground state. It remains to show that it is a pure state. To see this notice that we can consider $\mathbb{L}$ as the disjoint union of finite sets Λ_j, $\mathbb{L} = \bigsqcup_j \Lambda_j$. To each Λ_j,

associate the full algebra $\mathfrak{A}_{\Lambda_j}$ of $2^{|\Lambda_j|} \times 2^{|\Lambda_j|}$ matrices acting on $\mathfrak{H}^{\Lambda_j}$. Then $\mathfrak{A} = \bigotimes_j \mathfrak{A}_{\Lambda_j}$ and the local observables can be written as polynomials in the elements of the $\mathfrak{A}_{\Lambda_j}$.

Choosing the partition $\mathbb{L} = \bigsqcup_a \Lambda_a$ where $\Lambda_a = \{a_0, a_+\}$, ω_+ is seen to be a product state in the sense of Proposition A.1: if $A = \bigotimes_a A_a$, a finite product with A_a in $\mathfrak{A}_{\Lambda_a}$, then

$$\omega_+(A) = \prod_a \left(\phi_+^{a,a} \,|\, A_a \, \phi_+^{a,a} \right)$$

By Proposition A.1, ω_+ is a pure state because all the $\left(\phi_+^{a,a} \,|\, \cdot \, \phi_+^{a,a} \right)$ are pure states of $\mathfrak{A}_{\Lambda_a}$. The proof for ω_- and $\omega_{q\gamma}$ can be done in the same way. □

If B is any (local) observable, the state defined by

$$\omega_B(A) \equiv \frac{\omega(B^* A B)}{\omega(B^* B)} \tag{8.1}$$

for all observables A is called a (local) perturbation of the state ω. In particular, if $\omega = \omega_{0\downarrow} \equiv \omega_0$ and B is an observable with support in $\Lambda_{-K,K}$ such that $\phi = B\,\psi^{-K,K}(0_-, \downarrow)$ is a ground state of $H^{\Lambda_{-K,K}}$, the corresponding perturbed state will be called a *local ground pertubation* of ω_0. Thus, $\omega_{q\gamma}$ are local ground perturbations of ω_0 (in this case B is the operator $\vartheta^{q,\gamma}$ defined in Lemma 8.1). Also, choosing any linear combination of VB states with free spin in $\Lambda_{-K,K}$ (for $K < L$), and defining the corresponding limit state, gives a local ground perturbation of ω_0. Obviously, the local ground perturbations are pure ground states of the infinite Δ chain, being product of pure states with a suitable choice of partition. We believe that these are all the pure ground states in the sense $\omega(H_a) \equiv 0$ but we cannot prove this.

Next we are interested in the GNS representations associated with ground states of the model, and more precisely with their unitary equivalence.

Definition. Let ϱ and τ be two states. If the GNS representations of $\mathfrak{A}$ associated with ϱ and τ are unitarily equivalent, the states ϱ and τ are said to be *equivalent states*. This leads to an equivalence relation between states.

Theorem 8.3. *For the infinite Δ chain,*

1. *Any local ground perturbation of ω_0 is equivalent to ω_0.*

2. *The states ω_+, ω_- and ω_0 are nonequivalent.*

Proof. (i) If ω_B is a local ground perturbation of ω_0 with supp $B \subset \Lambda_{a,b}$ and $\mathfrak{A}_{a,b}^c$ denotes the set of observables commuting with any local observable with support in $\Lambda_{a,b}$, then $(\omega_0 - \omega_B)|\mathfrak{A}_{a,b}^c = 0$. Applying the first result of Proposition A.2, as ω_0 and ω_B are pure states, they are equivalent.

(ii) We can also use Proposition A.2 to prove that the states ω_0 and ω_+ are nonequivalent. For any L, the unitary local observable $A_+(b) = \sigma_z(b_0)\,\sigma_z(b_+)$ belongs to $\mathfrak{A}^c_{-L,L}$ if $b < -L$, and $\omega_0(A_+(b)) = 0$ while $\omega_+(A_+(b)) = -1$ (this follows from the different orientation of the dimers in the definition of these states if $b < 0$, see Figure 8.1). Then,

$$|\omega_0(A_+(b)) - \omega_+(A_+(b))| = 1 = \|A_+(b)\|$$

i.e. $\|(\omega_0 - \omega_+)|\mathfrak{A}^c_{-L,L}\|$ does not tend to 0 when L tends to ∞. Hence ω_0 and ω_+ are not equivalent by the second result of Proposition A.2. The same kind of arguments show that the state ω_- is not equivalent to ω_0 and ω_+. □

This theorem shows the existence of three different equivalence classes of ground states. These can be interpreted as different *ground phases* of the infinite Δ chain. This situation is similar to that encountered in the one dimensional ferromagnetic Ising chain in the absence of an external magnetic field (see [BrRo] or [ArMa]). The above ground phases have the full translational symmetry of $\mathbb{L}$, while $\omega_\pm$ break a discrete symetry (reflexion) which is not broken by the phase of ω_0. The class of ω_0 is, hence, invariant under the space group of $\mathbb{L}$ and also under rotations in spin space: For every local observable A, $\Lambda \supset \operatorname{supp} A$ and α in $\{x, y, z\}$,

$$\{\omega(\Pi^\Lambda_\alpha\, A\, \Pi^\Lambda_\alpha) \mid \omega \text{ in the class of } \omega_0\} = \{\omega(A) \mid \omega \text{ in the class of } \omega_0\}$$

where $\Pi^\Lambda_\alpha = \bigotimes_{p\in\Lambda} \sigma_\alpha(p)$. Therefore this class is a particular kind of spin liquid phase with a zero energy spinon mode.

In what follows, we turn our attention to the states $\omega_\pm$ and ω_0.

Notice at first that by the product structure of these states, if A and B are two local observables with disjoint supports then

$$\omega_s(AB) = \omega_s(A)\,\omega_s(B) \qquad \text{for } s = -, 0, +.$$

Hence all the local correlation functions are asymptotically clustering.

Now we study the gap proprieties of the above states.

Definition. Let ω be a ground state. We say that ω has a gap $\varepsilon > 0$, if for any local observable A such that $\omega(A) = 0$,

$$\omega(A^*[H, A]) \equiv \lim_{L\to\infty} \omega(A^*[H^{-L,L}, A]) \geq \varepsilon\,\omega(A^*A)$$

The above limit exists because if $\operatorname{supp} A$ is on the triangles $a, a+1, \ldots, b$, then

$$[H, A] = [H^{-L,L}, A] = [H^{a,b}, A]$$

for any $L \geq \max\{|a|, b\}$.

Theorem 8.4. *The ground states ω_+ and ω_- have a gap.*

Proof. The proof for $\omega_\pm$ differing only in notations, we treat the case of ω_+. Let A be a local observable such that supp A is on the triangles $a, a+1, \ldots, b$ and $\omega_+(A) = 0$. Choose $L \geq \max\{|a|, b\}$. If $c \leq a$ we get

$$\begin{aligned}\left(\psi^{-L,L}(c_-,\eta) \,\middle|\, A\psi^{-L,L}(-L_-,\gamma)\right) &= \left(\psi^{-L,a-1}(c_-,\eta) \,\middle|\, \psi^{-L,a-1}(-L_-,\gamma)\right)\left(\phi_+^{a,b} \,\middle|\, A\phi_+^{a,b}\right) \\ &= 0\end{aligned}$$

If $c > a$,

$$\begin{aligned}\left|\left(\psi^{-L,L}(c_-,\eta) \,\middle|\, A\psi^{-L,L}(-L_-,\gamma)\right)\right| &\leq \frac{1}{\sqrt{2}}\|A\|\left|\left(\psi^{-L,a-1}(a_-,\eta) \,\middle|\, \psi^{-L,a-1}(-L_-,\gamma)\right)\right| \\ &\leq \frac{\|A\|}{\sqrt{2}}\, 2^{-|L+a|} \longrightarrow 0 \qquad (L \to \infty)\end{aligned}$$

By Theorem 7.1, $H^{\Lambda_{-L,L}}$ has a gap ε independent of L, so that

$$\begin{aligned}\omega_+(A^*[H,A]) &= \left(A\psi^{-L,L}(-L_-,\gamma) \,\middle|\, [H^{-L,L},A]\,\psi^{-L,L}(-L_-,\gamma)\right) \\ &= \left(A\psi^{-L,L}(-L_-,\gamma) \,\middle|\, H^{-L,L}A\,\psi^{-L,L}(-L_-,\gamma)\right) \\ &\geq (\varepsilon - \varsigma_L)\left(A\psi^{-L,L}(-L_-,\gamma) \,\middle|\, A\,\psi^{-L,L}(-L_-,\gamma)\right) \\ &= (\varepsilon - \varsigma_L)\,\omega_+(A^*A),\end{aligned}$$

where $\varsigma_L \to 0$ as L tends to infinity. Therefore $\omega_+(A^*[H,A]) \geq \varepsilon\,\omega_+(A^*A)$, which proves the theorem. □

The state ω_0 has no gap in the above sense, due to the existence of local ground perturbations. For example, taking $A = \sigma_x(0_-)$, we have $\omega_0(A) = 0$ but $\omega_0(A^*[H,A]) = 0$ also, because ω_A is a local ground perturbation. Nevertheless, ω_0 is separated by a gap from the local perturbations which are "orthogonal" to the whole phase associated with ω_0:

Theorem 8.5. *The ground state ω_0 has a gap in the following weak sense:*
For any local observable A such that $\omega_0((\vartheta^{q,\eta})^*\,A) = 0$ for $\eta = \downarrow, \uparrow$ and all q in $\mathbb{L}$,

$$\omega_0(A^*[H,A]) \geq \varepsilon\,\omega_0(A^*A)$$

with some $\varepsilon > 0$. Here $\vartheta^{q,\gamma}$ is the operator introduced in Lemma 8.1.

Proof. Notice that

$$\left(\psi^{-L,L}(q,\eta) \,\middle|\, A\,\psi^{-L,L}(0_-,\downarrow)\right) = \left(\vartheta^{q,\eta}\psi^{-L,L}(0_-,\downarrow) \,\middle|\, A\,\psi^{-L,L}(0_-,\downarrow)\right) = \omega_0((\vartheta^{q,\eta})^*A) = 0$$

if supp A and $\Lambda_q \subset \Lambda_{-L,L}$. Therefore the proof in Theorem 8.4 applies without the ς_L term. □

8.2. The Infinite △ Cayley Tree

The infinite △ Cayley tree is the infinite limit of a sequence of finite △ Cayley trees Γ_L, such that the Lth level Γ_L is obtained from Γ_{L-1} by completing each boundary site of Γ_{L-1} into a triangle. (Fig. 8.2.) Any boundary triangle is connected to the first level triangle by a △ chain of L triangles.

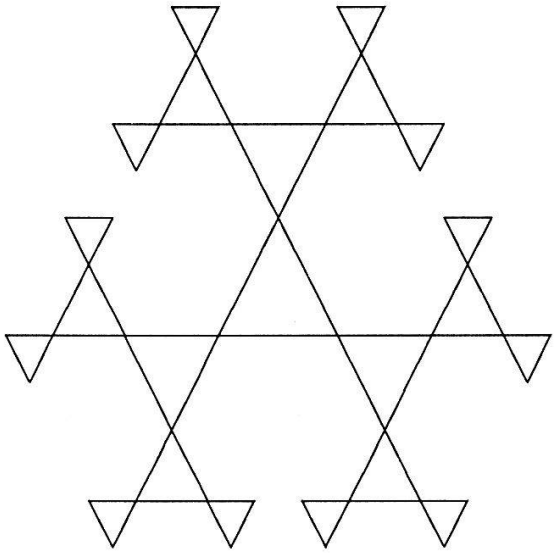

Fig. 8.2. *The fourth level Γ_4 in the construction of the infinite △ Cayley tree.*

There is a continuous set of pure ground states obtained by "sending" the free spin to infinity. More precisely:

Proposition 8.6. *Let $\{\Gamma_L\}_{L=1}^{\infty}$ be the sequence of finite △ trees tending to the infinite △ Cayley tree $\mathbb{\Gamma}$. Choose a sequence $Q = \{q_L\}_{L=1}^{\infty}$ of sites in $\mathbb{\Gamma}$ such that q_L is a boundary site of Γ_L and $d(q_L, q_{L-1}) = 1$. Let $\psi^L(q_L, \downarrow)$ be the valence bond state on Γ_L with free spin $\downarrow$ on site q_L. Then the limit*

$$\omega_Q(A) = \lim_{L\to\infty} \left(\psi^L(q_L, \downarrow) \,|\, A\,\psi^L(q_L, \downarrow)\right)$$

exists for any local observable A and defines a pure ground state.
Similarly, let 0 label a site of Γ_1. The limit

$$\omega_0(A) = \lim_{L\to\infty} \left(\psi^L(0, \downarrow) \,|\, A\,\psi^L(0, \downarrow)\right)$$

exists for all local element A in $\mathfrak{A}$ and defines a pure ground state. (Figure 8.3.)

Proof. The proof is the same as in Proposition 8.2 by noticing that if $\operatorname{supp} A \subset \Gamma_M$, then

$$\omega_Q(A) = \left(\psi^M(q_M, \downarrow) \,|\, A\,\psi^M(q_M, \downarrow)\right)$$

and

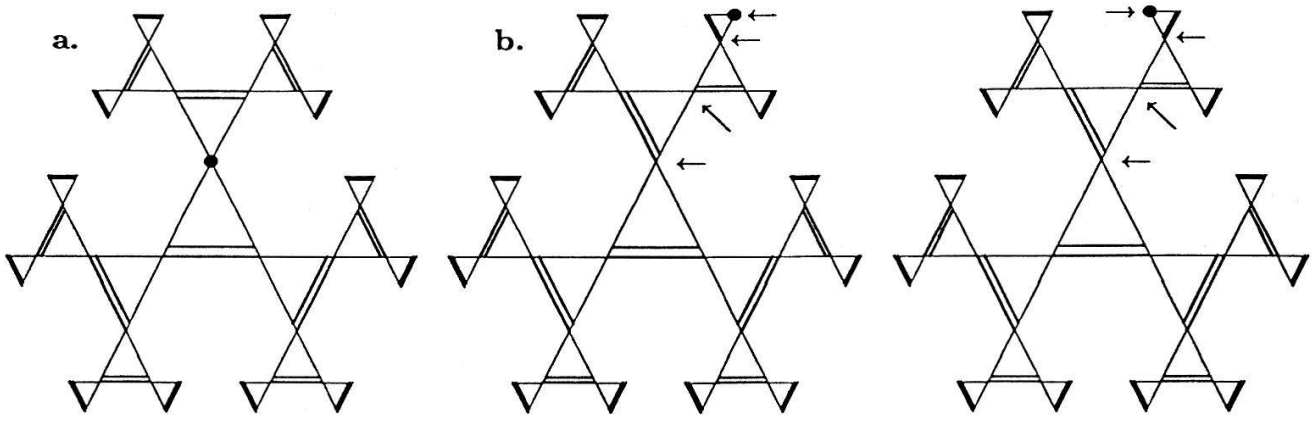

Fig. 8.3. a. and b. *show the valence bond states on* Γ_4 *used to construct the ground states of Proposition 8.6 for* ω_0 *and two states* ω_Q *and* $\omega_{Q'}$*, respectively. The arrows indicate the sites of* Q *and* Q'.

$$\omega_0(A) = \left(\psi^M(0,\downarrow) \,\middle|\, A\psi^M(0,\downarrow)\right). \quad \square$$

Notice that ω_Q and $\omega_{Q'}$ differ only on the infinite Δ chain containing the symmetric difference of the sets Q and Q' (a property which is inherited from the finite Δ trees). One could also define the state ω_Q by choosing a branch in $I\!\Gamma$: There is a one to one correspondence between Q and the branch to which the sites of Q belong.

Again, there exist local ground perturbations of ω_0 which can be obtained as weak limits of linear combinations of a finite number of VB states, the positions of the free sites being fixed during the limit. We believe that these are all the pure ground states. The same argument as in the case of the infinite Δ chain leads to

Theorem 8.7. *Consider the infinite* Δ *Cayley tree and the ground states* $\{\omega_Q\}$, ω_0 *on it.*

1. *Any local ground perturbation of* ω_0 *is a pure ground state equivalent to* ω_0.
2. *Any state* ω_Q *is not equivalent to* ω_0.
3. *Any two states* ω_Q *and* $\omega_{Q'}$ *are nonequivalent.*

Again, the class of ω_0 corresponds to the notion of a spin liquid phase.

As in the case of the infinite Δ chain, if A and B are local observables with disjoint supports then $\omega(AB) = \omega(A)\omega(B)$. Here $\omega = \omega_0$ or ω_Q for any sequence Q. This follows again from the product structure of these states (Proposition A.1).

Although there are many similarities between finite Δ chains and finite Δ Cayley trees, the proof of the existence of an L-independent gap for the Δ chains Λ_L cannot be transfered to the Δ Cayley trees Γ_L. The reason is the exponential increase with L of the number of sites in the latter case. This may cause the vanishing of the gap in the infinite volume limit.

The structure of the space of ground states of an infinite Δ Cayley tree is typical of any infinite Δ tree with branches obtained as the limit of an increasing sequence of finite Δ trees $\{\Gamma_L\}$ such that the boundary triangles of Γ_L are linked to the first level Γ_1 by a Δ chain of L triangles. To each infinite branch there corresponds a pure ground state with "free site at the infinity" on that branch. There is also a phase of equivalent pure ground states. These are local ground perturbations of a state obtained as the limit of finite volume valence bond states with a fixed free spin on the first level.

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A. Appendix. Some Theorems for the Infinite Volume Limit.

Proposition A.1. *Let $\{\mathfrak{A}_a\}_{a\in\mathbb{A}}$, be a family of C^*-algebras, and $\mathfrak{A} = \bigotimes_{a\in\mathbb{A}} \mathfrak{A}_a$.*
If, for each a in $\mathbb{A}$, ϱ_a is a state of $\mathfrak{A}_a$, then the associated product state *ϱ of $\mathfrak{A}$ is (uniquely) defined by*

$$\varrho(A_1 A_2 \dots A_n) = \varrho_{a_1}(A_1)\,\varrho_{a_2}(A_2)\dots\varrho_{a_n}(A_n)$$

whenever $a_1, \dots, a_n$ are distinct elements of $\mathbb{A}$ and A_j is in $\mathfrak{A}_{a_j}$. (Here A_j is canonically identified with an element of $\mathfrak{A}$.)
Moreover the state ϱ of $\mathfrak{A}$ is pure if and only if each ϱ_a is a pure state of $\mathfrak{A}_a$.

Proof. The proof is given in [KaRi]. (See Propositions 11.4.6 and 11.4.7 therein.) Notice that a state ϱ_a of $\mathfrak{A}_a$ is naturally associated with a state $\widehat{\varrho}_a$ of the canonical image of $\mathfrak{A}_a$ in $\mathfrak{A}$ and reciprocally. □

Let $\{\mathfrak{A}_j\}$ be a sequence of C^*-algebras such that,

— $\mathfrak{A}_j$ is *isomorphic to the algebra of $n_j \times n_j$ complex matrices, and

— $\mathbb{1} \in \mathfrak{A}_1 \subsetneq \mathfrak{A}_2 \subsetneq \dots \subsetneq \mathfrak{A}_n \subsetneq \dots$

Then the norm closure $\mathfrak{A}$ of $\bigcup \mathfrak{A}_j$ is referred to as a *uniformly matricial* C^*-algebra or *uniformly hyperfinite (UHF) algebra* with generating nest $\{\mathfrak{A}_j\}$.

Proposition A.2. *Let $\mathfrak{A}$ be a uniformly matricial C^*-algebra with generating nest $\{\mathfrak{A}_n\}$.*

Denote by $\mathfrak{A}_n^c$ the subset of $\mathfrak{A}$ commuting with all the elements of $\mathfrak{A}_n$.

If ϱ_1 and ϱ_2 are pure states of $\mathfrak{A}$, then their respective GNS representations are equivalent if there is some n such that $\| (\varrho_1 - \varrho_2)|\mathfrak{A}_n^c \| < 2$ and only if $\| (\varrho_1 - \varrho_2)|\mathfrak{A}_n^c \|$ tends to 0 when $n \to \infty$.

Proof. This follows from Proposition 12.4.3 in [KaRi]. Notice that pure states are primary states in the sense of [KaRi], and that for pure states quasi-equivalence implies equivalence [KaRi]. □