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SUBORDINACY AND SPECTRAL THEORY FOR INFINITE MATRICES

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Abstract

The notion of subordinacy, previously used as a tool in the spectral theory of ordinary differential operators, is extended and applied to solutions of three–term recurrence relations. The resulting asymptotic analysis yields a complete description of the spectral behaviour of infinite tridiagonal matrices, and provides a unified approach to the spectral theory of such operators, without detailed special assumptions.

1. Introduction

This paper is concerned with the spectral analysis of infinite, real, Hermitean tridiagonal matrices. In particular, we shall show how the notion of subordinacy, borrowed and adapted from the spectral theory of ordinary differential operators, may be applied to this class of infinite matrices to yield a complete description of their spectral properties.

There is a considerable body of literature treating the theory and applications of infinite tridiagonal matrices. Such matrices can give rise to discrete, absolutely continuous, or singular continuous spectrum ([1],[2],[3],[4], [5]), or indeed to a combination of all three. Various special classes of Jacobi matrices have been extensively studied ([1], [4],[6],[7],[8],[9],[10],[11]), particularly having regard to the stochastic properties of random matrices, and the localisation phenomena which are important in applications to solid state physics. The theory of tridiagonal matrices is closely connected with the study of 3 term recurrence relations ([12],[13],[14],[15]), of polynomials orthogonal with respect to a weight function, or more generally with respect to a measure, ([2],[3],[12],[13],[16],[17],[18],[19],[20]), and there are links with the analytic theory of continued fractions ([12],[21],[22],[23],[24], [25],[26]). Despite the accumulated body of results in these areas, we believe that the work presented here is the first to provide a unified approach to the general problem of spectral analysis of infinite tridiagonal matrices, without detailed special assumptions. In addition, it seems to us to be of value to extend into the discrete domain the method of subordinacy, which is being successfully applied to the analysis of differential operators ([27],[28],[29],[30]).

In order to understand the central rôle of infinite tridiagonal matrices in the study of general Hilbert space operators, let T be a self–adjoint operator acting in a separable Hilbert space $\mathcal{H}$, and let $f \in \bigcap_{k=1}^{\infty} D(T^k)$.

Here $D(T^k)$ denotes the domain of T^k, so that the vector f is assumed to be in the domain of the k'th power of T, for k arbitrarily large. (Such vectors f are dense in $\mathcal{H}$). Assuming that f is not a linear combination of finitely many eigenvectors of T, it follows that the sequence $\{f, Tf, T^2f, \cdots\}$ is linearly independent. We can therefore apply the Gram–Schmidt process to derive an orthonormal sequence of vectors $e_1, e_2, e_3, \cdots$ in which each e_n is of the form $e_n = P_{n-1}(T)f$, where P_{n-1} is a polynomial of degree n–1. The polynomials P_{n-1} can, moreover, be chosen to be real. They are uniquely determined up to a sign, and the coefficients of each polynomial may be expressed (up to a sign) in terms of the various matrix elements $< f, T^k f >$, $k = 0, 1, 2, \cdots$.

An infinite matrix having real components M_{ij}, i, j = 1, 2, 3, $\cdots$ may now be defined by

$$M_{ij} = < e_i, T e_j > = M_{ji} .$$

The matrix M may then be verified to be tridiagonal in the sense that

$$M_{ij} = 0 , \text{ for } |i-j| > 1 , \text{ whereas}$$

$$M_{ij} \neq 0 , \text{ for } |i-j| = 1 .$$

Thus, given a self–adjoint operator T and a general vector $f \in \bigcap_{k=1}^{\infty} D(T^k)$ we are led naturally to define a corresponding real, Hermitean, tridiagonal matrix M. If T is bounded, M may be regarded as representing the action of T on the closed linear subspace $\mathcal{H}_f$ spanned by $\{f, Tf, T^2f, \cdots\}$. If T is unbounded, complications can arise. Thus $\mathcal{H}_f$ need not then be an invariant subspace for T, and even if $\mathcal{H}_f$ <u>is</u> an invariant subspace, the linear operator T, restricted to finite linear combinations of the $T^k f$, need not be essentially self–adjoint as an operator in $\mathcal{H}_f$. Both of these problems are overcome if we choose f to be a vector of uniqueness for the operator T ; for the definition and main properties of vectors of uniqueness see [31], where a criterion is given for a given vector to be a vector of uniqueness. Vectors of uniqueness for T are again dense in $\mathcal{H}$.

In terms of the matrix M , the condition that the matrix defines uniquely the restriction of T to the subspace $\mathcal{H}_f$ is that we should have the limit point rather than the limit circle case. In the following section we shall consider more precisely the implications of this statement. For a discussion of limit point/limit circle for differential equations see [32],[33]; for the discrete case see [34]. In this paper we shall restrict our attention to the limit point case. In that case, the action of the self–adjoint operator T in $\mathcal{H}_f$ is given by $T\underline{x} = \underline{y}$, where $\underline{x} = \sum_1^\infty x_i e_i$, $\underline{y} = \sum_1^\infty y_i e_i$, and $y_i = \sum_k M_{ik} x_k$. Since M is tridiagonal, we have a finite sum on the r.h.s. The vector $\underline{x}$ will be in the domain of T provided $\sum_1^\infty | y_i |^2 < \infty$. Note also $\underline{x} \in \mathcal{H}_f \Rightarrow \sum_1^\infty | x_i |^2 < \infty$. We can think of the operation of T in $\mathcal{H}_f$ as equivalent to the action of a matrix operator M: $\underline{x} \to M\underline{x} = \underline{y}$ in the sequence space $l^{(2)}$, and the rest of this paper is devoted to the spectral analysis of such matrix operators. A self–adjoint operator T acting in a Hilbert space $\mathcal{H}$ may always be expressed as the direct sum of such matrix operators, each of which has purely simple spectrum. Indeed, one may show (cf. [35] p. 253 for the case of bounded self–adjoint operators) that by a suitable choice of the starting vector f every component of the spectrum of T is reflected in the spectral decomposition of the matrix operator M.

A key notion is that of subordinacy. For any complex number z, the tridiagonal matrix M defines an associated set of 3–term recurrence relations

$$\sum_k M_{ik} u_k = z u_i \qquad (i = 2, 3, 4, \cdots)$$

A solution $\{u_i\}$ of these recurrence relations is said to be subordinate if $\sum_1^N | u_i |^2$ is asymptotically vanishingly small, in the limit $N \to \infty$, compared with the corresponding sum for any other solution of the recurrence relations not a constant multiple of $\{u_i\}$. (For a more precise definition, see Section 2; for the notion of subordinacy for solutions of differential equations see [27],[28],[29]). We assume of course that $\{u_i\}$ is not the trivial solution $u_i = 0$ for all i.

If z is strictly complex (i.e. $\mathrm{Im} z \neq 0$) a subordinate (even $l^{(2)}$) solution always exists in the limit point case. On the other hand, if $z = \lambda$ is real, a subordinate solution of the recurrence relations may or may not exist, and the situation will be different, in general, for different λ. The main result of this paper, stated in Theorem 3, is a complete decomposition of the spectrum of M into its discrete, singular and absolutely continuous components, where each component of the spectrum is characterised by the existence or non–existence of subordinate solutions of the recurrence relations for real values of λ. Through this result, the nature of the spectrum of M will therefore depend on an analysis of the large i behaviour of solutions $\{u_i(\lambda)\}$ of the recurrence relations for real λ. The outcome of this approach is that spectral analysis and asymptotic analysis are indeed two aspects of the same thing – each determines the other.

Analogous results have been derived ([27],[28]) for differential equations. In the present context, though the work follows broadly similar lines to [28], additional difficulties are presented due to the discrete nature of the problem. The organisation of this paper is as follows.

In Section 2, we introduce the notation for infinite tridiagonal matrices and their associated set of recurrence relations. We define a function $m_\infty(z)$, analytic in the upper half plane with positive imaginary part, and show how $m_\infty(z)$ is related to the matrix elements of the resolvent for the infinite matrix operator T. We state a Lemma relating spectral properties of T to boundary values of the resolvent, and give a precise meaning to the notion of subordinacy for solutions of the recurrence relations.

In Section 3, we prove a number of results relating subordinacy to the boundary behaviour of $m_\infty(z)$. These results, presented in Theorems 1 and 2, are the core of the paper, and are the key to an understanding of the rôle of subordinacy in the spectral analysis of infinite matrices.

Finally, in Section 4, a complete spectral decomposition is carried out, based on the idea of subordinacy, and using the results of both of the two previous sections. Further details of this and related work is to be found in [36].

2. Spectral analysis and boundary behaviour

We consider the general infinite, real, Hermitean tridiagonal matrix, given by

$$M = \begin{bmatrix} a_1 & b_1 & 0 & 0 & . \\ b_1 & a_2 & b_2 & 0 & . \\ 0 & b_2 & a_3 & b_3 & . \\ . & . & . & . & . \end{bmatrix} \tag{1}$$

where the a's and b's are real constants and $b_i \neq 0$.

Associated with the matrix M are the 3 term recurrence relations

$$b_{n-1} u_{n-1} + (a_n - z)u_n + b_n u_{n+1} = 0, \tag{2}$$

$n = 2,3,4,\cdots$, where z is a complex number. A solution $\{u_n\}$ of (2) may be regarded as an infinite component column vector $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \cdot \\ \cdot \end{bmatrix}$, and may be denoted by $\underline{u}(z)$, or simply $\underline{u}$ whenever the complex parameter z is understood. Thus $\underline{u}(z)$ is a formal solution of the matrix equation $(M-z)\underline{u}(z) = 0$, except for the first component of this equation. The first component equation, namely

$$(a_1 - z)u_1 + b_1 u_2 = 0 \tag{2$'$}$$

need not in general be satisfied, but plays an important rôle in evaluating the resolvent matrix and in characterising the singular spectrum. For real values of z, in eq. (2), we shall often write $z = \lambda$ (λ real) and a solution of the recurrence relations will then be denoted by $\underline{u}(\lambda)$.

For fixed z, the solution set of the recurrence relations is a vector space of dimension 2, of which a convenient basis may be defined by solutions $\underline{\varphi}(z)$, $\underline{\psi}(z)$ subject to respective 'initial conditions'

$$\left.\begin{matrix} \varphi_1 = 1 \\ \varphi_2 = 0 \end{matrix}\right\} \qquad \left.\begin{matrix} \psi_1 = 0 \\ \psi_2 = 1 \end{matrix}\right\} \tag{3}$$

For Im$z \neq 0$, two possibilities can occur. Either we have the limit–circle case, in which both solutions $\underline{\varphi}(z)$, $\underline{\psi}(z)$ (and hence also every solution of the recurrence relations (2)) belong to the Hilbert space $l^{(2)}$, or we are in the limit point case, in which there is exactly one linearly independent solution of (2) belonging to $l^{(2)}$. We shall assume henceforth the limit point case, which is the only case for which M defines uniquely a self–adjoint operator T in $l^{(2)}$. For all $\underline{x} \in D(T)$ we then have $T\underline{x} = \underline{y}$, where $\underline{y} = M\underline{x}$ and $\| y \|^2 = \sum_1^\infty |y_k|^2 < \infty$. The limit point/limit circle theory follows closely the analogous theory for differential equations (see [32]). For details and further developments see [36]. In the limit point case, at most one linearly independent solution of the recurrence relations is square summable for each real z value.

For $\mathrm{Im} z \neq 0$, the solution $\underline{\psi}(z)$ is not itself square summable. We can therefore define uniquely a function $m_\infty(z)$ in the entire complex plane, apart from the real axis, by the condition that

$$\underline{\varphi}(z) + m_\infty(z)\underline{\psi}(z) \in l^{(2)} \tag{4}$$

The function $m_\infty(z)$ defined here is a discrete counterpart of the Titchmarsh–Weyl m–coefficient ([37],[38],[39]). The following Lemma summarises some of its principal properties.

<u>LEMMA 1</u>

The function $m_\infty(z)$ defined by (4) is analytic in the entire complex plane apart from the real axis. Moreover, $b_1 m_\infty(z)$ has positive imaginary part for $\mathrm{Im} z < 0$ and negative imaginary part for $\mathrm{Im} z > 0$.

The function $m_\infty(z)$ is related to the resolvent operator $(T-z)^{-1}$ by the formula

$$<\underline{e}_1, (T-z)^{-1}\underline{e}_1> = (a_1 - z + b_1 m_\infty(z))^{-1}, \tag{5}$$

where $\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$.

<u>PROOF</u>

For each $n \geq 2$, one can define a set C_n of complex numbers m by the condition that $\underline{u} \equiv \underline{\varphi} + m\underline{\psi}$ satisfies a relation of the form

$$(\sin\gamma)u_n - (\cos\gamma)u_{n+1} = 0,$$

for some real γ. Using standard methods from the corresponding theory for differential equations, one may verify that C_n is a circle, that the radius r_n of C_n decreases with n, and that, in the limit as $n \to \infty$, C_n shrinks to the single point m_∞. The equation of C_n may be expressed in the form

$$\sum_2^n |\varphi_j + m\psi_j|^2 + b_1\, \mathrm{Im}m/\mathrm{Im}z = 0.$$

Hence the interior of the circle is given by the condition

$$\sum_2^n |\varphi_j + m\psi_j|^2 < -b_1 \mathrm{Im}m/\mathrm{Im}z.$$

Since the limiting point m_∞ is itself interior to each circle, this inequality holds, for each $n\geq 2$, in the case $m = m_\infty$. Taking the limit as $n\to\infty$, we also have

$$\sum_2^\infty |\varphi_j+m_\infty\psi_j|^2 \leq -b_1 \mathrm{Im}\, m_\infty/\mathrm{Im}\, z \;, \tag{6}$$

from which the sign of $\mathrm{Im}(b_1 m_\infty)$ follows as in the statement of the Lemma.

From the form (1) of the matrix M, together with the recurrence relations (2) satisfied by

$$\underline{u} \equiv \underline{\varphi} + m_\infty \underline{\psi} \,,$$

we have $\underline{u} \in D(T)$, and $(T-z)\underline{u} = (M-z)\underline{u}$ is identically zero, apart from its first component, which is given by $[(T-z)\underline{u}]_1 = (a_1-z)u_1+b_1u_2$.

However, the initial conditions (3) imply

$$u_1 = 1 \;,\; u_2 = m_\infty \;.$$

Hence $(T-z)\underline{u} = (a_1-z+b_1m_\infty)\underline{e}_1$.

The first component of $(T-z)^{-1}\underline{e}_1$ is therefore $(a_1-z+b_1m_\infty)^{-1}$, again using the initial condition $u_1 = 1$. Thus eq. (5) is proved.

The stated analytic properties of m_∞ now follow immediately on using eq. (5) to express m_∞ in terms of the matrix element $<\underline{e}_1,(T-z)^{-1}\underline{e}_1>$ which is known to be analytic and non–vanishing in the cut plane, for arbitrary self–adjoint operators T. Eq. (5) is, in fact, a special case of the general resolvent formula

$$\begin{aligned} <\underline{e}_i, (T-z)^{-1}\underline{e}_j> &= (a_1-z+b_1m_\infty)^{-1}u_iw_j \quad (i\geq j) \\ &= (a_1-z+b_1m_\infty)^{-1}u_jw_i \quad (i<j) \,, \end{aligned}$$

where $\underline{u} = \underline{\varphi}+m_\infty\underline{\psi}$, $\underline{w}$ is the formal solution of $(M-z)\underline{w} = \underline{0}$, subject to the condition $w_1 = 1$, and $\{\underline{e}_i\}$, $i = 1,2,3, \cdots$ is the standard basis for $l^{(2)}$.

We shall henceforth assume, without loss of generality, that b_1 is negative; for $b_1>0$ consider $-M$ instead of M. In that case, we have $\mathrm{Im}\, m_\infty(z)>0$ for $\mathrm{Im}\, z>0$.

It is well known that spectral properties of a self–adjoint operator are related to the boundary behaviour of the resolvent operator for z close to the real axis. If $\{E_\lambda\}$ is the resolution of the identity for $T = \int\lambda dE_\lambda$, we have

$$<\underline{e}_1 , (T-z)^{-1}\underline{e}_1> = \int_{-\infty}^{\infty} \frac{d\rho(t)}{(t-z)} \,, \tag{7}$$

where $\rho(t) = \langle \underline{e}_1 , E_t\underline{e}_1 \rangle$ is non–decreasing, right continuous, and satisfies $\lim_{t\to-\infty} \rho(t) = 0$, $\lim_{t\to+\infty} \rho(t) = 1$. The measure μ generated by $\rho(t)$ (i.e. $\mu = d\rho$) is the spectral measure associated with the self–adjoint operator T acting in $l^{(2)}$. In the present context, where the entire Hilbert space $l^{(2)}$ is generated by $\{T^k\underline{e}_1\}$, $k = 0,1,2, \cdots$ the spectral properties of T are completely described by the measure μ.

In spectral analysis, we are concerned with the decomposition of the spectral measure μ into its respectively absolutely continuous and singular components, $\mu = \mu_{ac}+\mu_s$, and with the further decomposition $\mu_s = \mu_d+\mu_{sc}$ of μ_s into its discrete and singular continuous components. (See for example [40] for definitions of these measures and for the construction of the corresponding spectral subspaces of the underlying Hilbert space). In particular, we seek to characterise the respective spectral supports of the measures μ, μ_{ac}, μ_d and μ_{sc}. A first step in this direction is provided by the following Lemma. For proof of the Lemma see [27],[41],[42] and [43].

<u>Lemma 2</u>

Define $I(\lambda,\epsilon)$, for $\epsilon>0$, $\lambda\in\mathbb{R}$, by

$$I(\lambda,\epsilon) = \langle \underline{e}_1,(T-\lambda-i\epsilon)^{-1}\underline{e}_1 \rangle = \int \frac{d\rho(t)}{t-\lambda-i\epsilon}, \tag{8}$$

and let

$$I_+(\lambda) = \lim_{\epsilon\to0+} I(\lambda,\epsilon), \tag{9}$$

for all $\lambda\in\mathbb{R}$ such that this limit exists.

Then

(i) μ_{ac} is the restriction of μ to the set S consisting of those $\lambda\in\mathbb{R}$ for which $I_+(\lambda)$ exists and has strictly positive imaginary part. The density function for μ_{ac} is given Lebesgue almost everywhere by

$$f(\lambda) = \frac{1}{\pi}\chi_S(\lambda).\mathrm{Im}I_+(\lambda), \tag{10}$$

where χ_S is the characteristic function of the set S.

(ii) μ_s is the restriction of μ to the set S′ consisting of those $\lambda\in\mathbb{R}$ for which $\lim_{\epsilon\to0+} \mathrm{Im}I(\lambda,\epsilon) = \infty$.

Notice that standard results in the theory of boundary values of analytic functions imply that $I_+(\lambda)$ exists for Lebesgue almost all λ.

The results of Lemma 2, though fundamental to the spectral analysis of self–adjoint operators, need to be adapted in order to be applied successfully in the context of infinite matrices. In the first place, they apply to the boundary behaviour of $I(\lambda,\epsilon)$ rather than $m_\infty(\lambda+i\epsilon)$. However, we have already seen that these two functions are related by eq. (5). Moreover, Herglotz's Theorem ([33], p.218 of [35]) allows us to write down a representation for $m_\infty(z)$ very similar to eq (7).

Note first of all, from (6) and using initial conditions (3) for $\underline{\varphi}$ and $\underline{\psi}$ with $\mathrm{Im}z>0$,

$$\mathrm{Im}\left[{}^{-1}/m_\infty(z)\right] = \mathrm{Im}m_\infty/|m_\infty|^2 \geq |\mathrm{Im}z/b_1m_\infty{}^2|\sum_2^\infty|\varphi_j+m_\infty\psi_j|^2$$

$$\geq|\mathrm{Im}z/b_1m_\infty{}^2||m_\infty|^2 = \mathrm{Im}z/|b_1| \tag{11}$$

Thus $|1/m_\infty| \geq \mathrm{Im}z/|b_1|$ and we have the estimate

$$|m_\infty(z)| \leq |b_1|/\mathrm{Im}z$$

This bound on the asymptotic behaviour of $|m_\infty|$ for large Imz enables us to use Herglotz's Theorem in the special form

$$m_\infty(z) = \int_{-\infty}^{\infty} \frac{d\omega(t)}{t-z}, \tag{12}$$

which may be compared with eq (7). Here $\omega(t)$ is non– decreasing, right continuous, and satisfies $\lim_{t\to-\infty}\omega(t) = 0$, $\lim_{t\to+\infty}\omega(t) \leq |b_1|$. The latter inequality implies that the measure $\gamma = d\omega$ is finite, with $\gamma(\mathbb{R}) \leq |b_1|$. (In fact one has equality, so that $\gamma(\mathbb{R}) = |b_1|$).

The behaviour of $\mathrm{Im}m_\infty$ in the limit as $\epsilon\to0+$ is closely related to spectral properties. Here let us note, for use in Section 3, the following bounds, valid e.g. for $0<\epsilon\leq1$,

$$\mathrm{Im}\, m_\infty(\lambda+i\epsilon) \geq C(\lambda)\epsilon \tag{13}$$

$$\mathrm{Im}\left[{}^{-1}/m_\infty(\lambda+i\epsilon)\right] \geq \epsilon/|b_1| \tag{13$'$}$$

Here $C(\lambda)$ is a strictly positive function of λ, and is independent of ϵ. Inequality (13) follows easily from (12), since $\mathrm{Im}m_\infty(\lambda+i\epsilon) = \int_{-\infty}^{\infty}\frac{\epsilon d\omega(t)}{(t-\lambda)^2+\epsilon^2}$, and (13)' follows from (11) on setting $z = \lambda+i\epsilon$.

A further obstacle to the immediate application of Lemma 2 is that the Lemma requires detailed knowledge of the boundary behaviour in the complex plane of a function $m_\infty(z)$ which is difficult to estimate in all but the simplest cases. To circumvent this difficulty, we introduce the notion of subordinacy, which involves only an asymptotic analysis of solutions of the recurrence relations (2) at <u>real</u> values of z.

<u>Definition</u>

A (non–trivial) solution $\{u_n\}$, n=1,2,3, $\cdots$, of the recurrence relations

$$b_{n-1}\, u_{n-1} + (a_n-\lambda)u_n + b_n\, u_{n+1} = 0\,, \tag{14}$$

$n = 2,3,4, \cdots$, for a given $\lambda\in\mathbb{R}$, is said to be <u>subordinate</u> if and only if

$$\lim_{N\to\infty} \|\underline{u}\|_N \,/\, \|\underline{v}\|_N = 0 \tag{15}$$

for any solution $\{v_n\}$ of (14) not a constant multiple of $\{u_n\}$. Here $\|\underline{x}\|_N$ is defined by

$$\|\underline{x}\|_N = \sqrt{\sum_1^N |x_n|^2} \tag{16}$$

We shall write the subordinate solution as $\underline{u}(\lambda)$, or as $\underline{u}$ whenever the value of the parameter λ is understood.

<u>Remarks</u>

1. For each value of λ, there is at most one linearly independent subordinate solution, which may be taken to be real.

2. It is sufficient, for subordinacy, for eq (15) to hold for just <u>one</u> solution $\underline{v}$ which is not a multiple of $\underline{u}$.

3. Any (non–trivial) $l^{(2)}$ solution of (14) is subordinate. In particular, an extension of subordinacy to allow complex parameters z would admit $\underline{\varphi}(z)+m_\infty(z)\underline{\psi}(z)$ as a subordinate solution.

4. A <u>sufficient</u>, but not necessary, condition for subordinacy is to replace eq (15) by $\lim_{N\to\infty} u_N/v_N = 0$. This condition is of limited application in that it rules out many cases of oscillatory behaviour of solutions, which may nevertheless be subordinate. A special case which explicitly disallows oscillatory solutions is the following:

Suppose, for some $\lambda\in\mathbb{R}$, a (real) solution $\{w_n\}$ of (14) exists for which $b_n w_n w_{n+1}$ is of fixed sign. Then a subordinate solution exists for this value of λ.

<u>PROOF</u>: Let $\{v_n\}$ be a second real solution of (14), not a multiple of $\{w_n\}$.

Then $$\frac{v_n}{w_n} - \frac{v_{n+1}}{w_{n+1}} = \frac{W_n}{b_n w_n w_{n+1}}\,,$$

where $W_n = b_n(v_n w_{n+1} - w_n v_{n+1})$ is the 'Wronskian' of the two solutions $\underline{v}$ and $\underline{w}$.

W_n is independent of n, and non–zero. Hence $\frac{v_n}{w_n} - \frac{v_{n+1}}{w_{n+1}}$ is of fixed sign. It follows that $\lim_{N\to\infty} v_N/w_N = \ell$ exists as a finite or infinite limit. If ℓ is infinite (i.e $\ell = \pm\infty$) then $\{w_n\}$ is subordinate. If ℓ is finite then $\lim_{N\to\infty} \frac{v_N - \ell w_N}{w_N} = 0$ and $\{v_n - \ell w_n\}$ is subordinate.

3. Subordinacy and boundary behaviour

The principal method of this paper is the establishment of a link between subordinacy and the existence of a real boundary value for $m_\infty(z)$. The basic idea is to show that $\varphi(z)+m_\infty(z)\psi(z)$ is close in norm to $\varphi(\lambda)+m_\infty(\lambda)\psi(\lambda)$ for $z = \lambda+i\epsilon$ and ϵ small, where $m_\infty(\lambda)$ is suitably defined.

Since the first vector is in $l^{(2)}$ whereas in general the second vector is not, we shall use the norm $\|\cdot\|_N$ defined by eq (16), rather than the $l^{(2)}$ norm. Moreover, ϵ and N will have to be related in a very precise manner, to be stated more precisely in Lemma 3 below.

THEOREM 1

Suppose, for some $\lambda\in\mathbb{R}$, that

$$m_\infty(\lambda) \equiv \lim_{\epsilon\to 0+} m_\infty(\lambda+i\epsilon) \tag{17}$$

exists and is real. Then, for this value of λ, there exists a subordinate solution of the recurrence equations (14), namely $\varphi(\lambda)+m_\infty(\lambda)\psi(\lambda)$.

To prove this and other subsequent results, we need the following Lemma, the proof of which is independent of the assumptions of the Theorem.

LEMMA 3

Given any $\delta > 0$ there exists N_0 (depending on δ and λ) such that, for each $N \geq N_0$, the following equations have solutions for ϵ, in the range $0 < \epsilon < \delta$.

$$\text{(i)}\quad \epsilon = \frac{|b_1|\ [\mathrm{Im}\ m_\infty(\lambda+i\epsilon)]^{\frac{1}{2}}}{[\ \|\varphi(\lambda)\|_N\ \|\psi(\lambda)\|_N\ ]^{\frac{1}{2}}\ [\ \|\varphi(\lambda)\|_N + \|\psi(\lambda)\|_N\ ]} \tag{18}$$

$$\text{(ii)}\quad \epsilon = \frac{|b_1|\;[\mathrm{Im}(-1/m_\infty(\lambda+i\epsilon))]^{\frac{1}{2}}}{[\;\|\varphi(\lambda)\|_N\;\|\psi(\lambda)\|_N\;]^{\frac{1}{2}}\;[\;\|\varphi(\lambda)\|_N + \|\psi(\lambda)\|_N\;]} \tag{19}$$

$$\text{iii)}\quad \epsilon = \frac{|b_1|\;[\mathrm{Im}\,m_\infty(\lambda+i\epsilon)]^{\frac{1}{2}}}{\|\psi(\lambda)\|_N^{3/2}\;[1+\mathrm{Im}\,m_\infty(\lambda+i\epsilon)]^{\frac{1}{2}}\;[\;\|\varphi(\lambda)+m\psi(\lambda)\|_N]^{\frac{1}{2}}} \tag{20}$$

Here $\varphi(\lambda)$, $\psi(\lambda)$ are solutions of the recurrence relations subject to (3), and $m_\infty(z)$ is defined by (4). In (iii), m is any fixed, real number.

PROOF

(i) In the limit point case, $\varphi(\lambda)$ and $\psi(\lambda)$ cannot both belong to $l^{(2)}$. Hence $\|\varphi(\lambda)\|_N + \|\psi(\lambda)\|_N \to \infty$ as $N \to \infty$.

Given $\delta > 0$, choose N_0 such that

$$\delta > \frac{|b_1|\;[\mathrm{Im}\,m_\infty(\lambda+i\delta)]^{\frac{1}{2}}}{[\;\|\varphi(\lambda)\|_{N_0}\;\|\psi(\lambda)\|_{N_0}\;]^{\frac{1}{2}}\;[\;\|\varphi(\lambda)\|_{N_0} + \|\psi(\lambda)\|_{N_0}]}$$

Since $\|\cdot\|_N$ is increasing we then have, for $\epsilon = \delta$ and

$N \geq N_0$,

$$\epsilon > \frac{|b_1|\;[\mathrm{Im}\;m_\infty(\lambda+i\epsilon)]^{\frac{1}{2}}}{[\;\|\varphi(\lambda)\|_N\;\|\psi(\lambda)\|_N\;]^{\frac{1}{2}}\;[\|\varphi(\lambda)\|_N + \|\psi(\lambda)\|_N]} \tag{21}$$

Since, for given λ and fixed $N \geq N_0$, by (13) the r.h.s. is bounded below in the limit $\epsilon \to 0+$ by a strictly positive multiple of $\epsilon^{\frac{1}{2}}$, we must have

$$\epsilon < \frac{|b_1|\;[\mathrm{Im}\;m_\infty(\lambda+i\epsilon)]^{\frac{1}{2}}}{[\;\|\varphi(\lambda)\|_N\;\|\psi(\lambda)\|_N]^{\frac{1}{2}}\;[\;\|\varphi(\lambda)\|_N + \|\psi(\lambda)\|_N]} \tag{21$'$}$$

for ϵ small enough. Noting that each side of (21) and (21)$'$ depends continuously on ϵ for $\epsilon > 0$, we can compare (21), which holds for $\epsilon = \delta$, with (21)$'$, which holds for ϵ small enough, to deduce that equality is attained, in eq (18), for some ϵ in the range $0 < \epsilon < \delta$. Part (i) of the Lemma is now proved.

We have shown, inter alia, that eq (18) always has a solution for ϵ, provided N is large enough. Indeed, we can define $\epsilon = \epsilon(N)$ to be the smallest positive value of ϵ satisfying (18). (It is easily verified, using the continuity in ϵ of the r.h.s., that for large enough N such a smallest solution does exist.) Note we explicitly show the N dependence, although of course $\epsilon(N)$ depends also on λ. Since $\epsilon(N) < \delta$ for N large enough, where $\delta > 0$ is arbitrary, we have $\lim_{N\to\infty} \epsilon(N) = 0$.

The proofs of (ii) and (iii) of the Lemma follow similarly, where in (ii) we use the bound (13)′ rather than (13).

PROOF of THEOREM 1

Consider the equation

$$u_n(z) = \varphi_n(\lambda) + m_\infty(z)\psi_n(\lambda) - b_1^{-1}\varphi_n(\lambda)\sum_1^n \psi_j(\lambda)p_j$$
$$+ b_1^{-1}\psi_n(\lambda)\left[\sum_1^n\varphi_j(\lambda)p_j - p_1\right], \tag{22}$$

for $\lambda\in\mathbb{R}$, $\mathrm{Im}z>0$, and $n = 1, 2, 3, \cdots$

Eq (22) may be derived by the standard 'variation of constants' method, and gives the solution $\{u_n\}$ of the inhomogeneous recurrence relations

$$b_{n-1}u_{n-1} + (a_n-\lambda)u_n + b_n u_{n+1} = p_n, \tag{23}$$

$n = 2, 3, 4, \cdots$, subject to initial conditions

$$u_1 = 1 \;,\; u_2 = m_\infty(z) \;.$$

Setting $p_n = (z-\lambda)u_n$ in (23), we see that any solution $\{u_n(z)\}$ of (22) with p_n defined in this way must be a solution of the complex z recurrence relations (2). From the initial conditions for $\{u_n(z)\}$, we have then

$$\underline{u}(z) = \underline{\varphi}(z) + m_\infty(z)\underline{\psi}(z) \tag{24}$$

With $z = \lambda+i\epsilon$, eq (22) may be written in the form

$$\underline{u} = \underline{\varphi}(\lambda) + m_\infty(z)\underline{\psi}(\lambda) + L\underline{u}\;, \tag{25}$$

where the linear operator L is defined by

$$(L\underline{u})_k = -b_1^{-1}\varphi_k(\lambda)\sum_1^k i\epsilon\psi_j(\lambda)u_j$$
$$+ b_1^{-1}\psi_k(\lambda)\sum_1^k i\epsilon(\varphi_j(\lambda)u_j - u_1) \tag{26}$$

L may be regarded as acting on the finite dimensional N—component sequence space ℓ_N spanned by $\underline{e}_1, \underline{e}_2, \cdots \underline{e}_N$, with norm $\|\cdot\|_N$. We shall relate ϵ and N by $\epsilon = \epsilon(N)$ as in the proof of Lemma 3. For simplicity of notation, we suppress the dependence on N of L and ϵ.

In ℓ_N, we define an iterative sequence $\{\underline{u}^{(0)}, \underline{u}^{(1)}, \underline{u}^{(2)}, \cdots\}$ of approximate solutions to eq (25) by

$$\underline{u}^{(0)} = \underline{0}\ ,\ \underline{u}^{(i+1)} = \underline{\varphi}(\lambda) + m_{\infty}(z)\underline{\psi}(\lambda) + L\underline{u}^{(i)}\ .$$

To assess the convergence of this iteration, we need a norm estimate for L . Using Schwarz's inequality in (26) , we find

$$\|L\underline{u}\|_N \leq 2\epsilon|b_1|^{-1}\,\|\underline{\varphi}(\lambda)\|_N\,\|\underline{\psi}(\lambda)\|_N\,\|\underline{u}\|_N\ ,$$

so that

$$\|\,L\,\| \leq \gamma\ , \tag{27}$$

with

$$\gamma = 2\epsilon|b_1|^{-1}\,\|\underline{\varphi}(\lambda)\|_N\,\|\underline{\psi}(\lambda)\|_N \tag{28}$$

We shall later verify, with $\epsilon=\epsilon(N)$, that $\gamma\to 0$ as $N\to\infty$. Hence $\gamma<1$ for N sufficiently large, and standard methods allow us to infer the convergence of the iteration to the solution $\underline{u}(z)$ of (25), given by (24). The iteration also leads to the estimates

$$\begin{aligned}\|\underline{u}(z)\|_N &\leq \|\underline{\varphi}(\lambda) + m_{\infty}(z)\underline{\psi}(\lambda)\|_N\,(1+\gamma+\gamma^2+\cdot\cdot)\\ &= (1-\gamma)^{-1}\|\underline{\varphi}(\lambda) + m_{\infty}(z)\underline{\psi}(\lambda)\|_N\ ,\ \text{and}\\ \|\underline{u}(z)\|_N &\geq \|\underline{\varphi}(\lambda) + m_{\infty}(z)\underline{\psi}(\lambda)\|_N\,(1-\gamma-\gamma^2-\cdot\cdot)\\ &= (1-2\gamma)\,(1-\gamma)^{-1}\,\|\underline{\varphi}(\lambda) + m_{\infty}(z)\underline{\psi}(\lambda)\|_N.\end{aligned}$$

Since $\gamma\to 0$ as $N\to\infty$, these two inequalities imply, with eq. (24),

$$\lim_{N\to\infty}\frac{\|\underline{\varphi}(z) + m_{\infty}(z)\underline{\psi}(z)\|_N}{\|\underline{\varphi}(\lambda) + m_{\infty}(z)\underline{\psi}(\lambda)\|_N} = 1 \tag{29}$$

Now define, with $\epsilon = \epsilon(N)$ and $z = \lambda + i\epsilon$, quantities F_1, F_2, F_3 by

$$\left.\begin{aligned}F_1(N) &= \gamma = 2\epsilon|b_1|^{-1}\,\|\underline{\varphi}(\lambda)\|_N\,\|\underline{\psi}(\lambda)\|_N\ ,\\ F_2(N) &= \frac{\|\,\underline{\varphi}(z) + m_{\infty}(z)\underline{\psi}(z)\,\|_N}{\|\underline{\varphi}(\lambda)\|_N + \|\underline{\psi}(\lambda)\|_N}\ ,\\ F_3(N) &= \left[\frac{|b_1|\ \mathrm{Im}\ m_{\infty}(z)}{\epsilon}\right]^{\frac{1}{2}}\frac{1}{\|\underline{\varphi}(\lambda)\|_N + \|\underline{\psi}(\lambda)\|_N}\end{aligned}\right\} \tag{30}$$

The relationship $\epsilon = \epsilon(N)$ is motivated by the requirement that

$$F_1 = 2F_3^2 \tag{31}$$

Using the expressions (30) for F_1 and F_3 , eq. (31) leads precisely to the equation (18) to be satisfied by ϵ .

Substituting the r.h.s. of (18) for ϵ back into F_1, now yields

$$F_1(N) = \frac{2[\|\underline{\varphi}(\lambda)\|_N \ \|\underline{\psi}(\lambda)\|_N]^{\frac{1}{2}}}{[\|\underline{\varphi}(\lambda)\|_N + \|\underline{\psi}(\lambda)\|_N]} [\mathrm{Im}\, m_\infty(\lambda + i\epsilon)]^{\frac{1}{2}}$$

Hence $F_1(N) \leq [\mathrm{Im}\, m_\infty(\lambda + i\epsilon)]^{\frac{1}{2}} \to 0$ as $N \to \infty$ since $\epsilon(N) \to 0$ and m_∞ has a real boundary value, by the hypothesis of the Theorem.

By eq. (31), $F_1 \to 0 \Rightarrow F_3 \to 0$.

Again, using (6) we find, by the triangle inequality

$$\|\underline{\varphi}(z) + m_\infty(z)\, \underline{\psi}(z)\|_N \leq |\varphi_1(z) + m_\infty(z)\psi_1(z)|$$
$$+ [|b_1|\ \mathrm{Im}\, m_\infty(z)/\epsilon]^{\frac{1}{2}} = 1 + [|b_1|\ \mathrm{Im}\, m_\infty(z)/\epsilon]^{\frac{1}{2}}\text{, so that}$$

$F_2(N) \leq [\|\underline{\varphi}(\lambda)\|_N + \|\underline{\psi}(\lambda)\|_N]^{-1} + F_3(N)$,

implying $F_2 \to 0$ as $N \to \infty$.

Combining the expression (30) for F_2 with eq. (29) now yields,

$$\lim_{N\to\infty} \frac{\|\underline{\varphi}(\lambda) + m_\infty(z)\underline{\psi}(\lambda)\|_N}{\|\underline{\varphi}(\lambda)\|_N + \|\underline{\psi}(\lambda)\|_N} = 0$$

However, $\lim_{\epsilon\to 0+} m_\infty(\lambda + i\epsilon) = m_\infty(\lambda)$. Hence

$$\lim_{N\to\infty} \frac{\|\underline{\varphi}(\lambda) + m_\infty(\lambda)\underline{\psi}(\lambda)\|_N}{\|\underline{\varphi}(\lambda)\|_N + \|\underline{\psi}(\lambda)\|_N} = 0$$

Noting the inequality

$$\|\underline{\varphi}(\lambda)\|_N + \|\underline{\psi}(\lambda)\|_N \leq \|\underline{\varphi}(\lambda) + m_\infty(\lambda)\underline{\psi}(\lambda)\|_N$$
$$+ [1 + |m_\infty(\lambda)|]\, \|\underline{\psi}(\lambda)\|_N\,,$$

we may deduce that

$$\lim_{N\to\infty} \frac{\|\underline{\varphi}(\lambda) + m_\infty(\lambda)\underline{\psi}(\lambda)\|_N}{\|\underline{\varphi}(\lambda) + m_\infty(\lambda)\, \underline{\psi}(\lambda)\|_N + [1 + |m_\infty(\lambda)|]\, \|\underline{\psi}(\lambda)\|_N} = 0\,,$$

from which it follows easily that

$$\lim_{N\to\infty} \|\underline{\varphi}(\lambda) + m_\infty(\lambda)\underline{\psi}(\lambda)\|_N \,/\, \|\underline{\psi}(\lambda)\|_N = 0$$

Hence $\underline{\varphi}(\lambda) + m_\infty(\lambda)\underline{\psi}(\lambda)$ is subordinate, as in the statement of the Theorem.

<u>Corollary to Theorem 1</u>

Suppose, for some $\lambda \in \mathbb{R}$, that

$$\lim_{\epsilon\to 0+} |m_\infty(\lambda + i\epsilon)| = \infty \tag{32}$$

Then, for this value of λ , $\underline{\psi}(\lambda)$ is a subordinate solution of the recurrence relations (14).

PROOF

The proof follows the same argument as in the proof of the main Theorem, with minor modifications. Instead of (22), we consider here the equation

$$u_n(z) = \psi_n(\lambda) + \varphi_n(\lambda)/m_\infty(z) - b_1^{-1}\, \varphi_n(\lambda) \sum_1^n \psi_j(\lambda)p_j + b_1^{-1}\, \psi_n(\lambda) \left[\sum_1^n \varphi_j(\lambda)p_j - p_1\right],$$

which with $p_n = (z-\lambda)u_n$ is satisfied by

$$\underline{u}(z) = \underline{\psi}(z) + \underline{\varphi}(z)/m_\infty(z) .$$

In eqs. (30), the definitions of F_2 and F_3 are replaced, respectively, by

$$F_2(N) = \frac{\| \underline{\psi}(z) + \underline{\varphi}(z)/m_\infty(z) \|_N}{\| \underline{\varphi}(\lambda) \|_N + \| \underline{\psi}(\lambda) \|_N}, \text{ and}$$

$$F_3(N) = \left[\frac{|b_1|}{\epsilon} \operatorname{Im} \frac{-1}{m_\infty(\lambda + i\epsilon)}\right]^{\frac{1}{2}} \cdot \frac{1}{\| \underline{\varphi}(\lambda) \|_N + \| \underline{\psi}(\lambda) \|_N}$$

With $\epsilon = \epsilon(N)$ now chosen according to (ii) of Lemma 3, eq. (31) remains valid and we can follow previous arguments to deduce that

$$\lim_{N\to\infty} \frac{\| \underline{\psi}(\lambda) \|_N}{\| \underline{\varphi}(\lambda) \|_N + \| \underline{\psi}(\lambda) \|_N} = 0$$

from which subordinacy of $\underline{\psi}(\lambda)$ may be deduced.

We are now ready to give a partial converse to Theorem 1 and its Corollary.

THEOREM 2

For some $\lambda \in \mathbb{R}$, suppose that a subordinate solution of the recurrence relations (14) exists. This solution is either a constant multiple of $\underline{\varphi}(\lambda) + m\underline{\psi}(\lambda)$, for some $m \in \mathbb{R}$, or a constant multiple of $\underline{\psi}(\lambda)$. Then a positive, decreasing sequence $\{\epsilon_j\}$ can be found, converging to zero, such that either

(i) $\lim_{j\to\infty} m_\infty(\lambda + i\epsilon_j) = m$, if $\underline{\varphi}(\lambda) + m\underline{\psi}(\lambda)$ is subordinate

or

(ii) $\lim_{j\to\infty} |m_\infty(\lambda + i\epsilon_j)| = \infty$, if $\underline{\psi}(\lambda)$ is subordinate.

PROOF

(i) Let $\underline{\varphi}(\lambda) + m\underline{\psi}(\lambda)$ be subordinate. (Note m must be real, since otherwise m and $\overline{m}$ would give rise to two linearly independent subordinate solutions.) We again follow the outlines of the proof of Theorem 1, starting in this case from the equation

$$u_n(z) = \varphi_n(\lambda) + m_\infty(z)\psi_n(\lambda) - b_1^{-1}(\varphi_n(\lambda) + m\psi_n(\lambda)) \sum_1^n \psi_j(\lambda)p_j + b_1^{-1}\psi_n(\lambda)\Big[\sum_1^n (\varphi_j(\lambda) + m\psi_j(\lambda))p_j - p_1\Big],$$

which with $p_n = (z-\lambda)u_n$ is satisfied by

$$\underline{u}(z) = \underline{\varphi}(z) + m_\infty(z)\underline{\psi}(z).$$

To replace eqs. (30), we have in this case

$$\left.\begin{aligned} F_1(N) &= 2\epsilon|b_1|^{-1}\|\underline{\varphi}(\lambda) + m\underline{\psi}(\lambda)\|_N\,\|\underline{\psi}(\lambda)\|_N \\ F_2(N) &= \frac{\|\underline{\varphi}(z) + m_\infty(z)\underline{\psi}(z)\|_N}{\|\underline{\psi}(\lambda)\|_N} \\ F_3(N) &= \left[\frac{|b_1|\ \mathrm{Im}\ m_\infty(\lambda+i\epsilon)}{\epsilon(1+\mathrm{Im}\ m_\infty(\lambda+i\epsilon))}\right]^{\frac{1}{2}} \frac{1}{\|\underline{\psi}(\lambda)\|_N} \end{aligned}\right]$$

Choosing $\epsilon = \epsilon(N)$ according to (iii) of Lemma 3, with F_1 and F_3 related by (31), we find

$$F_1(N) = 2\left[\frac{\mathrm{Im}\ m_\infty(\lambda+i\epsilon)\ \|\underline{\varphi}(\lambda) + m\underline{\psi}(\lambda)\|_N}{1+\mathrm{Im}\ m_\infty(\lambda+i\epsilon)}\right]^{\frac{1}{2}} \frac{1}{\|\underline{\psi}(\lambda)\|_N^{\frac{1}{2}}}$$

$\to 0$ as $N\to\infty$ by the subordinacy of $\underline{\varphi}(\lambda) + m\underline{\psi}(\lambda)$

Comparing $F_2(N)$ asymptotically with $F_3(N)$ and using eq (29), we may deduce in this case

$$\lim_{N\to\infty} \frac{\|\underline{\varphi}(\lambda) + m_\infty(z)\,\underline{\psi}(\lambda)\|_N}{\|\underline{\psi}(\lambda)\|_N\ (1+\mathrm{Im}\ m_\infty(\lambda+i\epsilon))} = 0 \tag{33}$$

Since, by the triangle inequality,

$$|m_\infty(z) - m| = \frac{\|(m_\infty(z) - m)\ \underline{\psi}(\lambda)\|_N}{\|\underline{\psi}(\lambda)\|_N}$$

$$\leq \frac{\|\underline{\varphi}(\lambda) + m_\infty(z)\ \underline{\psi}(\lambda)\|_N}{\|\underline{\psi}(\lambda)\|_N} + \frac{\|\underline{\varphi}(\lambda) + m\ \underline{\psi}(\lambda)\|_N}{\|\underline{\psi}(\lambda)\|_N},$$

eq (33) and the subordinacy of $\underline{\varphi}(\lambda) + m\underline{\psi}(\lambda)$ together imply

$$\lim_{N\to\infty} \frac{|\ m_\infty(z) - m\ |}{1 + \operatorname{Im} m_\infty(z)} = 0\ , \text{ with } z = \lambda + i\epsilon(N)\ .$$

Hence $\lim\limits_{N\to\infty} m_\infty(\lambda + i\epsilon(N)) = m$, and (i) of the Theorem now follows on taking $\{\epsilon_j\}$ to be a decreasing subsequence of $\{\epsilon(N)\}$.

The proof of (ii) is similar, and uses the same equations for $\underline{u}(z)$ and for F_1, F_2, F_3 as in the proof of the Corollary to Theorem 1, except that in eq (19) and in the equation for $F_3(N)$ we replace $\operatorname{Im} - \frac{1}{m_\infty(\lambda + i\epsilon)}$ by

$\operatorname{Im} - \frac{1}{m_\infty(\lambda + i\epsilon)} \,/\, (1 + \operatorname{Im} - \frac{1}{m_\infty(\lambda + i\epsilon)})$. As in case (i), subordinacy allows us to

deduce $F_1(N) \to 0$, and $F_3(N) \to 0$ follows from eq (31). By (6) and (13) , we also have

$$\frac{F_2(N)}{1 + \operatorname{Im}(-\frac{1}{m_\infty(\lambda + i\epsilon)})} \leq \text{const } F_3(N)$$

, and proceeding as before, subordinacy for $\underline{\psi}(\lambda)$ leads to

$$\lim_{N\to\infty} \frac{|\ ^1/m_\infty(z)\ |}{1 + \operatorname{Im}(-\frac{1}{m_\infty(z)})} = 0\ ,$$

implying $\lim\limits_{N\to\infty} |\ m_\infty(\lambda + i\epsilon(N))\ | = \infty$. This proves case (ii) of the Theorem.

<u>Remark</u>

The conclusion and proof of the Theorem remain valid if, in the hypotheses (i) and (ii), we replace subordinacy by <u>sequential</u> <u>subordinacy</u>. Thus $\underline{u}(\lambda)$ is said to be sequentially subordinate if there exists an increasing sequence $N_1, N_2, N_3 \cdots$ such that $N_j \to \infty$ and

$$\lim_{N_j\to\infty} \|\underline{u}(\lambda)\|_{N_j} \,/\, \|\underline{v}(\lambda)\|_{N_j} = 0$$

for every solution $\underline{v}(\lambda)$ of the recurrence relations not a constant multiple of $\underline{u}(\lambda)$.

4. Spectral analysis and subordinacy

The main result of this paper, to follow, establishes the link between spectral properties for the operator T defined by the infinite tridiagonal matrix M of eq (1) , and the existence of subordinate solutions to the recurrence relations (23). In characterising the support of the singular measure, as well as the complement of the support of the total measure μ, we shall also need to refer separately to the 'initial condition'

$$(a_1 - \lambda)u_1 + b_1 u_2 = 0 , \tag{34}$$

which corresponds to eq $(2)'$ in the case where $z = \lambda$ is real. Note that any solution of the recurrence relations (23) together with initial condition (34) will be a formal solution of the matrix equation $(M - \lambda)\underline{u}(\lambda) = \underline{0}$.

In Theorem 3, $\mu\restriction \mathscr{S}$ denotes the restriction of the measure μ to a subset $\mathscr{S}$ of $\mathbb{R}$.

THEOREM 3

Let T be the self–adjoint operator defined in $l^{(2)}$ by the matrix M, and let $\mu = d\rho$ be the spectral measure of T, as in eq. (7). Define disjoint subsets $\mathscr{S}_{ac}$, $\mathscr{S}_s$ and $\mathscr{S}_0$ of $\mathbb{R}$, such that $\mathbb{R} = \mathscr{S}_{ac} \cup \mathscr{S}_s \cup \mathscr{S}_0$, by

$\mathscr{S}_{ac} = \{ \lambda$; no subordinate solution of the recurrence relations (23) exists$\}$,

$\mathscr{S}_s = \{ \lambda$; a subordinate solution of the recurrence relations (23)exists, and satisfies initial condition (34)$\}$,

$\mathscr{S}_0 = \{ \lambda$; a subordinate solution of the recurrence relations (23) exists, but does not satisfy initial condition (34)$\}$.

Then $\mu(\mathscr{S}_0) = 0$, and the decomposition of μ into its respective absolutely continuous and singular components is given by

$$\left.\begin{aligned} \mu_{ac} &= \mu \restriction \mathscr{S}_{ac} \\ \mu_s &= \mu \restriction \mathscr{S}_s \end{aligned}\right\} \tag{36}$$

Moreover, the sets $\mathscr{S}_0$, $\mathscr{S}_{ac}$ and $\mathscr{S}_s$ are optimal with respect to Lebesgue measure, in the sense that

(i) $\mathscr{S}_0' \supseteq \mathscr{S}_0$ and $\mu(\mathscr{S}_0') = 0 \Rightarrow |\mathscr{S}_0' \setminus \mathscr{S}_0| = 0$,

(ii) $\mathscr{S}_{ac}' \subseteq \mathscr{S}_{ac}$ and $\mu_{ac} = \mu\restriction \mathscr{S}_{ac}' \Rightarrow |\mathscr{S}_{ac} \setminus \mathscr{S}_{ac}'| = 0$,

(iii) $|\mathscr{S}_s| = 0$,

where, in (i)–(iii), $|\cdot|$ denotes Lebesgue measure.

PROOF

Define $I(\lambda,\epsilon)$, S, S$'$ as in Lemma 2.

For $\lambda \in S$, the resolvent matrix element $< \underline{e}_1,(T - z)^{-1}\underline{e}_1 >$ has strictly complex boundary value, as z approaches λ from the upper half plane. Hence, by eq (5), $m_\infty(z)$ has strictly complex boundary value. By Theorem 2, a subordinate solution cannot exist for

$\lambda \epsilon S$, since this would imply $m_\infty(z)$ had a real or infinite boundary value for some subsequence $z_j = \lambda + i\epsilon_j$.
Hence $S \subseteq \mathscr{S}_{ac}$.

For $\lambda \epsilon S'$ $< \underline{e}_1, (T-z)^{-1}\underline{e}_1 >$ has infinite boundary value. Hence, by eq. (5), $m_\infty(z)$ has then a real boundary value $m_\infty(\lambda)$, satisfying the equation

$$a_1 - \lambda + b_1 m_\infty(\lambda) = 0 . \qquad (37)$$

For $\lambda \epsilon S'$, Theorem 1 now implies that $\underline{u}(\lambda) = \underline{\varphi}(\lambda) + m_\infty(\lambda)\underline{\psi}(\lambda)$. We can, moreover, use eqs. (3) and (37) to verify that the initial condition (34) is satisfied in this case. Hence $S' \subseteq \mathscr{S}_s$.

By Lemma 2, $S \subseteq \mathscr{S}_{ac}$ and $S' \subseteq \mathscr{S}_s$, with $\mathscr{S}_{ac} \cap \mathscr{S}_s = \phi$, implying eqs. (36) for μ_{ac} and μ_s .

We shall prove (i) – (iii) of the Theorem in reverse order.

The function $m_\infty(z)$ is analytic, with positive imaginary part, in the upper half plane, and hence has finite boundary value at (Lebesgue) almost all $\lambda \epsilon \mathbb{R}$. For $\lambda \epsilon \mathscr{S}_s$, Theorem 2 implies that the boundary value $m_\infty(\lambda)$ is real and satisfies eq (37). Hence, at almost all $\lambda \epsilon \mathscr{S}_s$ the function $z - a_1 - b_1 m_\infty(z) = -1/< \underline{e}_1, (T-z)^{-1}\underline{e}_1 >$, which again has positive imaginary part in the upper half plane, has zero boundary value. By the analytic theory of boundary values, the set of all $\lambda \epsilon \mathbb{R}$ with zero boundary value is of Lebesgue measure zero. (We discount the possibility $z - a_1 - b_1 m_\infty(z) \equiv 0$, which cannot occur in this context.) Hence $| \mathscr{S}_s | = 0$ as in (iii) .

Similarly, for almost all $\lambda \epsilon \mathscr{S}_{ac}$, $< \underline{e}_1, (T-z)^{-1}\underline{e}_1 >$ has strictly complex boundary value, and by eq (10) of Lemma 2, almost everywhere on $\mathscr{S}_{ac}$ we have strictly positive density function for μ_{ac} . For $\mathscr{S}_{ac}' \subseteq \mathscr{S}_{ac}$ with $| \mathscr{S}_{ac} \setminus \mathscr{S}_{ac}' | = 0$, this implies $\mu_{ac}(\mathscr{S}_{ac} \setminus \mathscr{S}_{ac}') \neq 0$. In that case $\mu \restriction \mathscr{S}_{ac}' \neq \mu_{ac}$, and (ii) follows by contradiction.

Finally, suppose $\mathscr{S}_0' \supseteq \mathscr{S}_0$ and $\mu(\mathscr{S}_0') = 0$. We can then apply (ii) of the Theorem to the set $\mathscr{S}_{ac}' = \mathscr{S}_{ac} \cap (\mathscr{S}_0' \setminus \mathscr{S}_0)^c$, where $\mathscr{S}^c$ denotes the complement, in $\mathbb{R}$, of a set $\mathscr{S}$.
From (ii) , we have $| \mathscr{S}_{ac} \cap (\mathscr{S}_0' \setminus \mathscr{S}_0) | = 0$.
From (iii) , we have $| \mathscr{S}_s \cap (\mathscr{S}_0' \setminus \mathscr{S}_0) | = 0$.

However, $\mathscr{S}_0' \setminus \mathscr{S}_0 \subseteq \mathscr{S}_{ac} \cup \mathscr{S}_s$, so that these two results together give $| \mathscr{S}_0' \setminus \mathscr{S}_0 | = 0$, and (i) is proved.

Remarks

1. In the language of [27], $\mathscr{S}_{ac}$ and $\mathscr{S}_s$ are essential supports for the measures μ_{ac}, μ_s respectively.

2. An alternative characterisation of an essential support for μ_{ac} is as follows. Define the set M_{ac} by

$$M_{ac} = \{\lambda \; ; \; \liminf_{N\to\infty} \|\underline{u}(\lambda)\|_N / \|\underline{v}(\lambda)\|_N > 0 \text{ for any pair of}$$

non–trivial solutions $\underline{u}(\lambda), \underline{v}(\lambda)$ of the recurrence relations$\}$. Using the remark following the proof of Theorem 2, one may verify $S \subseteq M_{ac} \subseteq \mathscr{S}_{ac}$. Hence M_{ac} is an essential support for μ_{ac}.

3. Note $\mathscr{S}_s = \{\lambda$; a subordinate solution of the recurrence relations (23) exists, and is a formal solution of $(M - \lambda)\underline{u}(\lambda) = \underline{0}\}$; the set $\mathscr{S}_s$ may be further decomposed into supports $\mathscr{S}_d$, $\mathscr{S}_{sc}$ for the discrete and singular continuous spectra respectively. For $\lambda \in \mathscr{S}_d$, an $l^{(2)}$ solution of $(M - \lambda)\underline{u}(\lambda)$ exists, whereas for $\lambda \in \mathscr{S}_c$ we have $(M - \lambda)\underline{u}(\lambda) = \underline{0}$ for some non–$l^{(2)}$ subordinate solution.

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