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Frustrated XY Model: Continuity of the Ground State Energy in Function of the Frustration

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Abstract. The frustrated XY model can describe Josephson-junction arrays in transverse magnetic fields. It is shown that the ground state energy is uniformly continuous when the frustration runs through all the real numbers. A consequence of this fact is that the critical current of the Josephson-junction array is non-zero at $T = 0$ for any frustration.

Introduction

This contribution concerns the uniformly frustrated XY model (illustrated by the case of an infinite square lattice). Since the periodicity of the ground state configurations for a rational frustration $f = p/q$ depends essentially on the denominator q, one could have doubts about the continuity (over all the real numbers) of the ground state energy $E(f)$ (energy per bond) in function of the frustration f. Indeed one can read in several papers[1,2] that $E(f)$ is not continuous. This question is relevant to study several quantities, particularily at $T = 0$, such as the helicity modulus or the critical current of a Josephson-junction array. The behaviour of $E(f)$ is also relevant for the discussion of the stability of commensurate flux phase in the tJ model[3].

The model

The hamiltonian of the system in site variables θ_r is: $H = -\sum_{<rr'>} \cos(\theta_r - \theta_{r'} - A_{rr'})$ such that: $\theta_r \in (-\pi, \pi]$ and $\sum_{\square R} A_{rr'} = 2\pi f$ where $\sum_{\square R}$ means that the sum is taken in clockwise direction over the bonds surrounding the plaquette R.

This system is exactly equivalent to the following, written in the bond variables $\phi_{rr'}$: $H = -\sum_{<rr'>} \cos(\phi_{rr'})$ with the constraints: $\phi_{rr'}=-\phi_{r'r} \in (-\pi, \pi]$ and $\sum_{\square R} \phi_{rr'} = -2\pi f + 2\pi m_R$ for each plaquette where m_R is any integer corresponding to a topological charge.

Uniform continuity of $E(f)$

Definition: $E(f)$ is uniformly continuous in the real numbers if $\forall \varepsilon > 0$, $\exists\ \delta(\varepsilon)$ such that for all real $f, \bar{f}$, one has: $| \bar{f} - f | < \delta \Rightarrow | E(\bar{f}) - E(f) | < \varepsilon$.

From a ground state $\Phi_f \equiv \{\phi_{rr'}\}$ corresponding to the frustration f, we are going to construct some states $\bar{\Phi}_{fn} \equiv \{\bar{\phi}_{rr'}\}$ for a frustration $\bar{f}$ of energy $E_{fn}(\bar{f})$ close to the energy $E(\bar{f})$:

(a): Choosing a positive integer n, one cuts out the lattice into vertical strips containing horizontal line of $2n + 1$ sites (Fig.1).

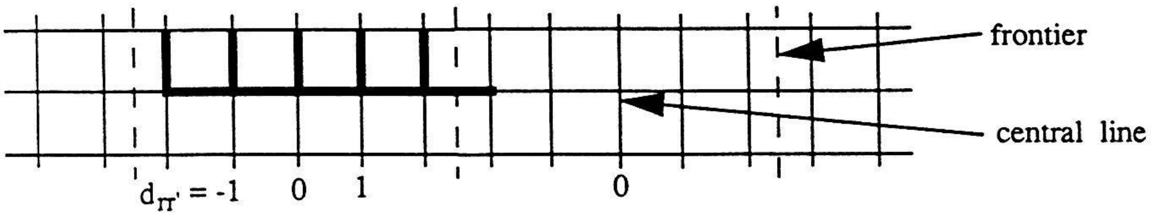

Figure 1: The infinite lattice is cut out into vertical strips. We call a plaquette (and a bond) "joint plaquette" (and "joint bond") if a frontier crosses it and "internal plaquette (and "internal bond") if it doesn't.

(b): To the internal bonds, the following transformation is applied:

$$\phi_{rr'} \mapsto \bar{\phi}_{rr'} = \begin{cases} \phi_{rr'} & \text{on the horizontal bonds} \\ \phi_{rr'} + 2\pi(f - \bar{f})d_{rr'} & \text{on the vertical bonds} \end{cases} \tag{1}$$

where the vertical bond rr' is considered such that r is under r' and $d_{rr'}$ is the algebraic distance between the bond and the central vertical line of the strip belonging to it. So all internal plaquette constraints are satisfied for $\bar{f}$.
(c): The $\phi_{rr'}$ of joint bonds crossing one frontier are determinated by the plaquette constraints after choosing freely one of them. Then this choice is done such that the average of $\cos\bar{\phi}_{rr'}$ crossing each frontier is positive (it is possible by adding π to each $\bar{\phi}_{rr'}$ if the average would be negative). One notes $E_{fn}(\bar{f})$ the energy of the state $\bar{\Phi}_{fn} \equiv \{\bar{\phi}_{rr'}\}$ thus constructed.

Now the mean variation over any set of bonds corresponding to $2n+1$ sites of one strip in a horizontal line (see bold bonds in Fig.1) is:

$$\frac{1}{2(2n+1)} \sum_{<rr'>} [-cos(\bar{\phi}_{rr'}) + cos(\phi_{rr'})] \leq \frac{1}{2(2n+1)} \left[\sum_{vert.bds} 2\pi \mid \bar{f} - f \mid\mid d_{rr'} \mid +2 \right] \quad (2)$$

The sum on right hand side puts an upper bound on the variation on the vertical bonds and the variation due to the horizontal joint bond is bounded by 2. Moreover the mean of the variation on the joint bonds over a frontier is bounded by 1 (according to (c)). So one can write:

$$E_{fn}(\bar{f}) - E(f) \leq \frac{1}{2(2n+1)} \left[\sum_{vert.bds} 2\pi \mid \bar{f} - f \mid\mid d_{rr'} \mid +1 \right]$$
$$= \frac{1}{2(2n+1)} \left[n(n+1)2\pi \mid \bar{f} - f \mid +1 \right]. \quad (3)$$

Let ε such that $0 < \varepsilon < 1/2$. One can choose $n(\varepsilon) \in [\frac{1}{2\varepsilon} - 1, \frac{1}{2\varepsilon}]$ and $\delta(\varepsilon) = 2\varepsilon^2/\pi$, so the right hand side of (Eq.3) will be less than ε (when $\mid \bar{f} - f \mid < \delta(\varepsilon)$). Then one has found $\delta(\varepsilon)$ such that, for a certain $n(\varepsilon)$, one has $\mid \bar{f} - f \mid < \delta(\varepsilon) \Rightarrow E_{fn}(\bar{f}) - E(f) < \varepsilon$. As, by definition, $E(\bar{f}) \leq E_{fn}(\bar{f})$, one has: $E(\bar{f}) - E(f) < \varepsilon$.
Since this statement depends only on $\mid f - \bar{f} \mid$ one has also by exchanging f and $\bar{f}$: $E(f) - E(\bar{f}) < \varepsilon$. Therefore $\forall\varepsilon > 0, \exists\, \delta$ such that for all real numbers $f, \bar{f}$, one has : $\mid \bar{f} - f \mid < \delta \Rightarrow \mid E(\bar{f}) - E(f) \mid < \varepsilon$.

Thus the ground state energy is uniformly continuous when f runs through the real numbers. Moreover by knowing $E(0) = -1$ and $E(1/2) = -1/\sqrt{2}$, and using the dependence $\delta(\varepsilon) = 2\varepsilon^2/\pi$, it is easy to see that $E(f)$ is always strictly negative.

Critical current of Josephson-junction array

Following Teitel and Jayaprakash[4], the $T = 0$ critical current of the array can be defined by:

$$i_c(f) = \max_{\delta}\{(i_{c0}/N) \sum_{vert.bds} \sin(\phi_{rr'} + \delta)\} \quad (4)$$

where i_{c0} is the single junction critical current, N is the number of sites and $\{\phi_{rr'}\}$ is a ground state configuration. Since $\sin x = cos(x - \pi/2)$, we have:

$$i_c(f) = \max_{\delta}\{(i_{c0}/N) \sum_{vert.bds} \cos(\phi_{rr'} + \delta - \pi/2)\} \quad (5)$$

The sum of cosines reaches its maximum, the absolute value of the ground state energy $(-NE(f))$, at $\delta{=}\pi/2$. So $i_c(f) = -i_{c0}E(f)$ and $i_c \neq 0$ even if f is irrational.

References

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