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# Lyapunov Exponents of the Schrödinger Equation with Quasi-periodic Potential on a Strip

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**Abstract.** We describe conditions under which all the nonnegative Lyapunov exponents of the quasi-periodic difference Schrödinger equation on a strip are strictly positive.

We shall study the Lyapunov exponents of the difference equation:

$$-y_{n+1} + Q_n y_n - y_{n-1} = 0 \quad -\infty < n < +\infty \quad (1)$$

where  $y_n \in \mathbb{R}^m$  and  $Q_n$  is a symmetric  $m \times m$  matrix whose off-diagonal elements do not depend on  $n$ , and the diagonal elements are quasi-periodic functions

$$q_{ni}(\theta) = \lambda f_i(e^{2\pi i(\theta + n\alpha)}) - E$$

with  $f_i(z)$  analytic on  $\mathcal{A} \equiv \{z \mid r < |z| < 1/r\}$ , taking values in  $[-1, 1]$  for  $|z| = 1$ ,  $\lambda$  is a (large) parameter called coupling constant,  $E$  is the energy, and  $\alpha$  is any irrational number. Without loss of generality we shall assume that  $\max_{1 \leq i \leq m} \sup_{|z|=1} f_i(z) = 1$  and  $\min_{1 \leq i \leq m} \inf_{|z|=1} f_i(z) = -1$ .

Equation (1) becomes equivalent to the finite-difference Schrödinger equation when the off-diagonal elements are chosen properly. For example, the case  $q_{ij} = -1$  for  $|i-j| = 1$  and  $q_{ij} = 0$  for  $|i-j| > 1$  corresponds to Schrödinger operator on the strip  $\mathbb{Z} \times \{1, \dots, m\}$ . Equation (1) can be written in the form

$$\begin{pmatrix} y_{n+1} \\ y_n \end{pmatrix} = \begin{pmatrix} Q_n & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} y_n \\ y_{n-1} \end{pmatrix} \equiv A_n \begin{pmatrix} y_n \\ y_{n-1} \end{pmatrix} \quad (2)$$

and, thus, the asymptotic behavior of the solutions of equation (1) is determined by the asymptotic behavior of the product  $S(n) := A_n \cdots A_1$ .

Various problems in solid-state physics give rise to different classes of matrices  $Q_n$ . These classes are characterized by the level of randomness. The case of independent random  $Q_n$  was studied in [GM], where it was shown that under certain algebraic conditions on the support of the corresponding measure in the space of symmetric matrices all the

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Lyapunov exponents are different and, therefore, the smallest non-negative exponent is, in fact, positive. When  $Q_n$ 's are non-deterministic at least some of the Lyapunov exponents are strictly positive [KS,S,K]. On the other hand, if  $Q_n$ 's form a periodic sequence, it is easy to show that the "interesting" Lyapunov exponents vanish.

The case of quasi-periodic potentials exhibits mixed behavior. If the coupling constant  $\lambda$  is small and  $\alpha$  is poorly approximated by rationals, it is known [BLT] that at least on part of the spectrum the Lyapunov exponent is zero when  $m = 1$ . When  $\lambda$  is large and  $m = 1$  the Lyapunov exponent is positive [Si,CS,FSW,SS].

Here we study the case of  $m > 1$  and large  $\lambda$ . We prove [GS] that all non-negative Lyapunov exponents are positive. The method we use is an extension of the one used in [SS], which, in turn, grew out of M. Herman's proof of positivity of the Lyapunov exponent in the case of  $m = 1$ ,  $f(z) = z + 1/z$ , and  $\lambda > 1$  [H].

To describe our result let us consider the following decomposition of  $S(n)$ :

$$S(n) = U(n)D(n)V(n)$$

where  $U, V \in O(n)$ , and  $D(n) = \text{diag}(d_1^{(n)}, \dots, d_{2m}^{(n)})$  with  $d_1 \geq d_2 \geq \dots \geq d_{2m} \geq 0$ . Since  $A_n \in Sp(m, \mathbb{R})$  for all  $n$ , we have  $d_k = d_{2m-k+1}^{-1}$ . The  $k$ -th Lyapunov exponent  $\gamma_k$  is defined by

$$\gamma_k := \lim_{n \rightarrow \infty} \frac{1}{n} \log d_k(n). \quad (3)$$

Clearly,  $\gamma_k = -\gamma_{2m-k+1}$ , and  $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_m \geq 0$ . Existence of the limit in (3) for almost every  $\theta$  and its independence of  $\theta$  are guaranteed by the Subadditive Ergodic Theorem [Ki] and ergodicity of the underlying dynamical system  $\theta \mapsto \theta + \alpha$ .

Our main result is the following

**Theorem.** *There exists  $\lambda_0$  such that for all  $\lambda > \lambda_0$  and all  $E$ , there exists a set  $\Omega(E) \subset [0, 1]$  of Lebesgue measure 1, such that*

$$\gamma_m(E) = \gamma_m(E, \theta) > 0 \quad \forall \theta \in \Omega(E).$$

This, together with Oseledec's Multiplicative Ergodic Theorem [O,GM] implies existence of  $m$  solutions of equation (1), which decay exponentially as  $n \rightarrow +\infty$  and grow exponentially as  $n \rightarrow -\infty$ , and  $m$  other solutions of equation (1), which decay exponentially as  $n \rightarrow -\infty$  and grow exponentially as  $n \rightarrow +\infty$ . These  $2m$  solutions are linearly independent and form a basis of all solutions of equation (1). There are two important consequences of this fact.

- 1) The Green's function of  $H$  decays exponentially for almost every energy  $E$ .
- 2) The spectrum of the operator  $H$  defined by the left hand side of equation (1) is singular.

**Remarks.** We should point out that the spectrum of  $H$  can actually be purely singular continuous if  $\alpha$  is a Liouville number [CFKS].

The requirement that  $\lambda$  is large cannot be avoided, for when  $\lambda$  is small KAM theory is applicable and there is absolutely continuous spectrum [BLT].

We strongly use the non-triviality of  $f_i$ 's in our proof. This condition is necessary, for there are examples in which the presence of constant  $f_i$ 's leads to appearance of zero Lyapunov exponents.

## References

- [BLT] Bellisard J., Lima R., Testard D.: *A Metal-Insulator Transition for the Almost Mathieu Model*, Comm. Math. Phys., **88**, 207–234 (1983)
- [CFKS] Cycon H., Froese, R., Kirsch W., Simon B.: *Schrödinger operators, with application to quantum mechanics and global geometry*. Berlin, Heidelberg, New York: Springer 1987
- [CS] Chulaevsky V., Sinai Ya.: *Anderson localization for multi-frequency quasi-periodic potentials in one dimension*, Comm. Math. Phys., **125**, 91–112 (1989)
- [FSW] Fröhlich J., Spencer T., Wittwer P.: *Localization for a class of one-dimensional quasi-periodic potentials*, Comm. Math. Phys., **132**, 5–25 (1990)
- [GS] Goldsheid I.Ya., Sorets E.: *Lyapunov exponents of the Schrödinger equation with quasi-periodic potential on a strip*, to appear in Comm. Math. Phys. (1991)
- [GM] Goldsheid I.Ya., Margulis G.A.: *Lyapunov indices of products of random matrices*, Uspekhi Mat. Nauk 44:5, 13–60 (1989). Translation in Russian Math. Surveys 44:5, 11–71 (1989)
- [H] Herman M.: *Une méthode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractère local d'un Théorème d'Arnold et de Moser sur le tore en dimension 2*, Commun. Math. Helv., **58**, 453–502 (1983)
- [Ki] Kingman J.F.C.: Subadditive processes. Springer Lecture Notes in Math. **539** 1976
- [K] Kotani S.: *Lyapunov indices determine absolutely continuous spectrum of stationary random one-dimensional Schrödinger operators*, Proc. of Taniguchi Symp. SA. Kata-ta, 225–247 (1982)
- [KS] Kotani S., Simon, B.: *Stochastic Schrödinger Operators and Jacobi matrices on the strip*, Comm. Math. Phys. **119**, 403–429 (1988).
- [O] Oseledec V.I.: *A multiplicative ergodic theorem, Characteristic Lyapunov exponents of dynamical systems*, Trudy Moskov. Mat. Obshch. **19**, 179–210 (1968) translation in Trans. Moscow Math. soc. **19**, 197–231 (1968)
- [S] Simon B.: *Kotani theory for one-dimensional stochastic Jacobi matrices*, Commun. Math. Phys. **89**, 227–234 (1983).
- [Si] Sinai Ya.: *Anderson localization for one-dimensional difference Schrödinger operator with quasi-periodic potential*, J. Stat. Phys., **46**, 861–918 (1987)
- [SS] Sorets E., Spencer T. : *Positive Lyapunov exponents for Schrödinger operator with quasi-periodic potentials*, to appear in Comm. Math. Phys. (1991).