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Facts and Fantasies in FQHE Theory

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Abstract. The quantum Hall effect occurs when an electronic system has chemical potential discontinuities (incompressibilities) at magnetic-field dependent densities. In the fractional quantum Hall effect (FQHE) the incompressibilities occur because of interactions among electrons in macroscopically degenerate Landau levels. It has been the task of theory to explain why the incompressibilities occur at certain Landau level filling factors and to describe the low-lying excitations of FQHE systems at nearby filling factors. We briefly review some of the novel ideas which have been developed in response to this challenge. We distinguish the ideas developed to explain the FQHE near filling factor $\nu = 1/m$, which are soundly based on microscopic theory, from the more speculative ideas used to extrapolate toward the FQHE physics at 'hierarcy' filling factors.

I. INTRODUCTION

The quantum Hall[1,2] effect occurs in two-dimensional electronic systems (2DES) in strong perpendicular magnetic fields (B). The effect is characterized by dissipationless current flow and quantized Hall conductivity over a finite range of magnetic field. The effect occurs only when the chemical potential of the system has discontinuities[3] at densities which depend on magnetic field. (The Hall conductivity is proportional to the magnetic-field dependence of the density at which the chemical potential gap occurs.) For free 2D electrons gaps occur at densities proportional to the magnetic field strength because of the quantization of the kinetic energy. (The set of states with a given value of the kinetic energy is known as a Landau level and the Landau level degeneracy per unit area, $(2\pi\ell^2)^{-1}$, is proportional to the magnetic field strength. Here $\ell = \hbar c/eB$ is the magnetic length.) Gaps at magnetic-field-dependent densities can also occur within a Landau level because of a periodic external potential[4] or because of electron-electron interactions. It is the gaps due to electron-electron interactions which are responsible for the fractional[5] quantum Hall effect (FQHE). In this article we will briefly review what is known about the physical origin of these gaps and about the unique fractionally charged excitations which are associated with their existence. The essential aspects of much of what is discussed below were first described in a series of pioneering papers by Laughlin[6], Haldane[7] and Halperin[8]. We will confine our attention completely to the simplest case in which the magnetic field is sufficiently strong to align all electronic spins and to accommodate all electrons in the Landau level of minimum kinetic energy. We begin by discussing the FQHE physics for $\nu \sim 1/m$ which is well understood (m is a small odd integer) and then turn our attention to a discussion of some of the related but more speculative ideas which are used to describe the FQHE physics near other filling factors.

II. FACTS

In this section we discuss a number of mathematical properties of many-body wavefunctions when all electrons are confined to the lowest Landau level of a two-dimensional electron gas. These *facts* explain the fractional quantum Hall effect for $\nu = 1/m$. For our discussion we will employ a disk geometry where the single-particle kinetic energy eigenstates are given by

$$\phi_m(x,y) = \frac{(x-iy)^m \exp(-\frac{(x^2+y^2)}{4\ell^2})}{(2\pi\ell^2 2^m m!)^{1/2}}. \tag{1}$$

These wavefunctions have a common Gaussian factor and are otherwise powers of $z = x - iy$. The Gaussian is a form factor corresponding to cyclotron orbit radius of an electron with the minimum allowed kinetic energy. The rest of the wavefunction depends only on z and not on x and y independently. This property is the quantum mechanical expression of the fact that the only degree of freedom available for the electrons once their kinetic energy is fixed is the center of their cyclotron orbits. It can be exploited to prove many general properties of many-body wavefunctions of electrons in the restricted Hilbert space of the lowest Landau level. Single-particle wavefunctions in which the electrons have angular momentum m are localized close to circles which enclose $m+1$ flux quanta:

$$B\pi\langle\phi_m|x^2+y^2|\phi_m\rangle = (m+1)\Phi_0 \tag{2}$$

where $\Phi_0 = hc/e$ is the electronic flux quantum. The discussion below will be for the case of electrons confined to a finite disk in which N_L single-particle states with angular momentum from 0 to $N_L - 1$ are available are available within a Landau level. Note that N_L flux quanta go through the disk. An entirely equivalent discussion can be given for the case of electrons on the surface of a sphere[9].

Any many-electron wavefunction formed entirely within the lowest Landau level must be a sum of products of one-electron orbitals for each coordinate which are of the form given be Eq. (1). It follows that the many electron wavefunction must take the form

$$\Psi[z] = \left(\prod_{k=1}^{N} \exp(-|z_k|^2/4)\right) P[z] \tag{3}$$

where $P[z]$ is a polynomial in each of the z_k's[10] and N is the number of electrons. Since $P[z]$ is antisymmetric under the interchange of particle indices only odd powers appear can appear when $P[z]$ is expanded in the relative coordinates of any pair of particles. It follows that we can always write

$$P[z] = Q[z] P_V[z] \tag{4}$$

where $Q[z]$ is a symmetric polynomial and $P_V[z] = \prod_{i<j\leq N}(z_i - z_j)$ is the Vandermonde determinant. The highest power to which each z_i appears in $P_V[z]$ is $N-1$ so that the maximum power to which a z_i can appear in $Q[z]$ in Eq. (4) is $N_L - N$. For a $N = N_L$, $Q[z]$ must be a constant so that $P_V[z]$ is the wavefunction for a full Landau level.

The states in the lowest Landau level can be described either in terms of which single-particle states are occupied , i.e. in terms of electrons, or in terms of which single-particle states are empty, i.e. in terms of holes. The total number of fermion many-particle states is

$$g = \frac{N_L!}{N!N_h!} \tag{5}$$

where $N_h = N_L - N$ is the number of holes in the system. The Landau level filling factor $\nu \equiv N/N_L$. An N-particle system is equivalent to an N-hole system[11]. Moreover each antisymmetric polynomial representing a many-fermion state is uniquely related by Eq. (4) to a symmetric polynomial representing a many-boson state. Since multiplying by a Vandermonde determinant increases the maximum power to which a coordinate appears by $N-1$ it follows from $Eq.(4)$ that an N fermion system which encloses N_L flux quanta is equivalent to a N boson system which encloses $N_L - N$ flux quanta. A system in which $N_L = N + N_h$ can be described as a system of N Fermi particles enclosing $N + N_h$ flux-quanta, or under particle-hole conjugation as a system of N_h Fermi particles enclosing $N + N_h$ flux-quanta. Using the fermion to boson mapping the same system can also be described as a system of N Bose particles enclosing N_h flux quanta or as a system of N_h Bose particles enclosing N flux-quanta. If the number of holes is smaller than the number of electrons, i.e. if the filling factor is close to one, it is more economical to describe the system in terms of hole degrees of freedom. From the above we see that in this language the particles see the original electrons, and in the fermion case also the other particles, as sources of magnetic flux.

With these preliminaries established we turn to a discussion of the origin of the chemical potential discontinuity at $\nu = 1/m$. When all electrons are confined to the lowest Landau level the only relevant term in the Hamiltonian is the interaction term. We assume here that the particles interact pairwise so that

$$H = \sum_{i<j} \sum_{l} V_l P_{ij}^{(l)} \tag{6}$$

where $P_{ij}^{(l)}$ projects particles i and j onto a state of relative angular momentum l and we have noted that for any pair of particles there exists only one state of each relative angular momentum (RAM):

$$\psi_l(\vec{r}_i - \vec{r}_j) = \phi_l([\vec{r}_i - \vec{r}_j]/\sqrt{2})/\sqrt{2}, \tag{7}$$

In Eq. (6) l is odd since any pair of Fermi particles can be found only in states of odd RAM and V_l is the energy[12] of a pair which has RAM l:

$$V_l = \int d^2\vec{r}|\psi_l(\vec{r})|^2 V(|\vec{r}|). \tag{8}$$

The V_l parameters were first introduced by Haldane[9]. The fractional quantum Hall effect occurs for sufficiently short-range repulsive interactions for which $V_1 > V_3 > V_5$.... The basic physics is usefully discussed in terms of the hard-core model for which only V_1 is non-zero. Zero energy eigenstates of this Hamiltonian must have wavefunctions which vanish at least as fast as $(z_i - z_j)^3$ as particles i and j approach each other. This requires that $P[z] \sim P_V[z]^3$ and hence that $N_L \geq N_m \equiv mN - m + 1$. Thus in the thermodynamic limit it is possible to form states where no pair of electrons are ever in a state of RAM one only when $\nu \leq 1/3$. As the electron density is increased at fixed magnetic field so that ν crosses 1/3 an added electron has[13,14] a finite probability for being in a state of relative angular momentum with some other electron and the chemical potential jumps, for the hard core model, from zero to $\sim V_1$. The chemical potential gap which occurs for the hard-core model cannot disappear abruptly as the electron-electron interactions is smoothly altered and there will be a class of interaction Hamiltonians for which the quantum Hall effect at $\nu = 1/m$ will occur. Finite-size numerical exact diagonalization studies[15] have been extremely effective in determining whether or not the FQHE will occur at some particular filling factor for some particular electron-electron interaction and clearly establish that it should occur at $\nu = 1/3$ and at $\nu = 1/5$ for realistic interactions.

An important aspect of the fractional quantum Hall effect concerns the description of the low-energy degrees of freedom for filling factors close to ones at which an incompressibility occurs. In the simplest case of ν near $1/m$ the low-energy degrees of freedom are fractionally charged quasipartilces and quasiholes which are believed to behave similarly to the non-interacting electrons. For example, our picture of the way in which disorder is responsible for creating a fractional Hall plateau which persists over a finite range of magnetic field even in the thermodynamic limit depends on an analogy to the localization of electrons and holes which occurs in the integer QHE. Some parts of this picture can be established with some rigor for $\nu < 1/m$. In that case the low-energy part of the Hilbert space is that in which RAM less than $m - 1$ are avoided and in which $P_V[z]^m$ is a factor of the wavefunction. The polynomial parts of all low energy states can be expressed in the form

$$P[z] = Q[z]P_V[z]^3. \tag{9}$$

For $N_L = N_m$ there is only one state in the low-energy Hilbert space by Laughlin[6] in his seminal article on the FQHE. This state has $Q[z]$ equal to a constant and is[16] the non-degenerate ground state of the hard-core model at $N_L = N_m$. For $N_L = N_m + N_{qh}$ the number of states in the low-energy part of the Hilbert space is equal to the number of distinct symmetric polynomials in N coordinates in which no particle appears to a power larger than N_{qh}. These states describe the quasihole degrees of freedom and all have an energy per particle $\sim V_m$. Quasihole states at $\nu < 1/3$ can placed in one-to-one correspondence with states for holes in a full Landau level since they can be characterized by the same symmetric polynomials in Eq. (4) for holes and in Eq. (9) for quasiholes. The quasihole system is therefore equivalent both to a system of N_{qh} Fermi particles in a system enclosing $N + N_{qh}$ flux quanta or to a system of N_{qh} Bose particles in a system enclosing N flux quanta. Just like the holes in a full Landau level the quasiholes see electrons, and if they are treated as fermions also other quasiholes, as sources of magnetic flux. Similary, for both holes and quasiholes edge excitations[17] are represented by symmetric polynomials of degree ~ 1 while single hole or quasihole excitations in the bulk are represented by symmetric polynomials with total degree $\sim N$ and states at different filling factors are represented by symmetric polynomials with total degree $\sim N^2$. However, it is easy to see using the plasma analogy[6] that if $Q[z]P_V[z]$ represents a state in which one or more holes are localized at various points in the system then $Q[z]P_V[z]^3$ represents states with $1/m$ of an electron charge localized at the same points in the system.

The above discussion shows that a number of aspects of the fractional quantum Hall effect can be simply understood in terms of the analytic many-body wavefunctions for electrons in the lowest Landau level and the separation of energy scales provided by the different RAM channels for interaction between electrons. The chemical potential has a discontinuous jump from $\sim V_m$ to $\sim V_{m-2}$ when N_m exceeds N_L. For $N_L = N_m + N_{qh}$ the low energy Hilbert space can be mapped to one for N_{qh} particles, the quasiholes, which see the electrons as sources of magnetic flux. In the next section we turn to a discussion of aspects of the FQHE which are not as clearly understood. In particular the effect occurs at a sequence of filling factors $\nu \neq 1/m$, $\nu \neq 1 - 1/m$. These effects, often called hierarchy FQHE's, are not explained by the above discussion. We will focus our discussion on the case $\nu = 2/5$ which is the prototype of the hierarchy FQHE's and discuss some of the arguments which have been advanced to describe this case.

FANTASIES

Experimentally the strongest fractional quantum Hall effects occur for $\nu = \nu_n = n/(2n+1)$ and for $\nu = 1 - \nu_n$. There is strong numerical evidence[14] that at each of these filling factors there is a discontinuity in the minimum probability for finding a pair of electrons in a state of RAM one. Thus it seems that all FQHE's have their origin in changes in short-distance electron-electron correlations. For $n \neq 1$

however, we know of no analytic argument in support of the above statement[18]. There exist several, apparently quite different arguments, which purport to explain the $\nu = \nu_n$ FQHE for $n \neq 1$. None of these *fantasies* provides a completely satisfactory explanation, although it seems likely that each captures a part of the truth and might be made more satisfactory with further progress. We briefly discuss three of these arguments in the following paragraphs. It is not possible to do justice to any of the pictures in the following and we endeavor only to give a flavor of how different the three pictures really are and how each is deficient. We will address all of our discussion to the case of $\nu = 2/5$.

The first picture for the $\nu = 2/5$ picture is the hierarchy picture[7,8,19]. In this picture the low-energy states with $\nu > 1/3$ are assumed to be describable in terms of quasielectron degrees of freedom in the same way as the low-energy states for $\nu < 1/3$ are describable in terms of quasihole degrees of freedom. It certainly seems clear that for $N_L = N_1 - 1$ that the lowest energy state has a localized defect (the quasielectron) which must contain an excess of 1/3 of an electron to allow the electron system to enclose one fewer flux quanta. (Recall that $N_1 = 3N - 2$.) For $N_L = N_1 - N_{qe}$ it is likely that the system can be considered as a gas of quasielectrons as long as they are sufficiently dilute[20]. In the hierarchy picture the quasielectrons are entirely equivalent to quasiholes except for their charge. Thus when the quasielectrons are treated as fermions they have non-degenerate ground states and incompressibilities whenever the quasielectron system can no longer avoid a RAM channel, i.e, whenever

$$N_m(N_{qe}) = mN_{qe} - m + 1 = N + N_{qe}. \tag{10}$$

At this point the quasielectrons of the $\nu = 1/3$ Laughlin state have formed a $\nu = 1/m$ state. In terms of the physical flux quanta these first level 'hierarchy' incompressibilities occur at

$$N_L = (3 - 1/(m-1))N - 3. \tag{11}$$

For example if $m = 3$, the incompressibility occurs at $\nu = N/N_L = 2/5$ in the thermodynamic limit. An entirely equivalent argument can be advanced to explain the $\nu = 2/7$ FQHE as being due to the formation of a $\nu = 1/3$ Laughlin state in quasiholes. The difference between the quasihole and quasielectron cases is that a clear physical basis can be established for the identification of a quasiparticle energy scale in the former case, as discussed in the previous section, but not in the latter case. The quasielectron interactions are as strong as the electron-electron interactions and once their separation is comparable to the electron-electron separation it is not clear why we should still be able to describe low energy states in terms of only quasielectron degrees of freedom. Moreover the hierarchy picture would naively predict a FQHE for $m = 5$ in Eq. (11) ($\nu = 4/11$) where none is observed. At higher levels this failure of the naive hierarchy picture is repeated and the dominance of the $\nu = \nu_n$ effects does not have appear to have a natural explanation. On the other hand the quasiparticle picture does impressively predict the finite size corrections to the N_L

values at which incompressible state occur for a fnite number of electrons. Moreover many aspects of the hierarchy picture are supported by numerical studies and recent work by Beran and Morf[21] may be able to explain systematics in the effective quasiparticle-quasiparticle interaction which lead to the dominance of the $\nu = \nu_n$ FQHE's.

Another explanation for the hierarchy FQHE's has been proposed in work by Jain[22]. The essential idea of this work is that the FQHE at $\nu = \nu_n$ is related to the integer quantum Hall effect at $\nu = n$. In Jain's picture the non-degenerate incompressible state at $\nu = 2/5$ for example is related to the product of the incompressible state which occurs for non-interacting electrons at $\nu = 2$ and $P_V[z]^2$. He is able to show that electrons in this state reside primarily in the lowest Landau level. He argues that the $\nu = 2$ state is incompressible and strongly correlated and that this incompressibility has echoes for inverse filling factors differing from $1/2$ by even integers. This 'echo' corresponds to multiplying the wavefunction by $P_V[z]^2$ is identical to the way in which Laughlin states are obtained starting from the full Landau level state. In this construction the states the dominance ν_n filling factors arises naturally. The difficulty with this approach is that it seems to require the introduction of higher Landau levels and there is ample evidence that the FQHE is essentially a phenomena of interactions among electrons sharing the same quantized kinetic energy. Numerical evidence[23] suggests that the ground state wavefunctions obtained by the Jain construction are good approximations to the true groud state. However, as we have emphasized, the essential requirement for the FQHE is the existence of a chemical potential jump at a fixed fractional filling factor. It is not obvious from the Jain construction why such a jump should occur since the fundamental gap on which the construction is based is associated with the Landau level separation which should not play any essential role in the strong magnetic field limit. The construction nevertheless provides a successful recipe for constructing variation wavefunctions.

The final class of explanations[24,25] for the hierarchy FQHE's which we will mention is based on the singular gauge treatement of electron statistics in two-dimensions. In this approach the statistics of particles can be altered by attaching flux localized at the positions of the particles. When two quanta of magnetic field are attached to electrons the statistics repeat so that a system of fermions is equivalent to a system of fermions with two flux quanta attached to each particle. The simplest approximation in dealing with the attached fluxes is to replace them with an average uniform flux density which does not fluctuate as the particles move about. In this average field approximation it is easy to verify that the the inverse filling factor of the system is either increased or decreased by two units.

$$\nu^{-}1 \rightarrow \nu^{-1} + 2 \tag{12}$$

It is then argued that adding the fluctuations in the attached fluxes does not change the situation qualitatively so that systems with inverse filling factors differing by two do not differ qualitatively. Thus the FQHE at $\nu = \nu_n$, is argued to be fundamentally associated with the integer QHE at $\nu = n$. This approach toward explaining the FQHE has a number of difficulties. First of all, no essential role is played by electron-electron interactions while it is clear from the microscopic theory that no effect occurs without interactions. It can be argued that electron-electron interactions are responsible for the gap which allows the $\nu = n$ and $\nu = \nu_n$ ground states to be adiabatically connected. In this case the argument is circular since it requires a gap, which itself gives the FQHE, to establish its validity. On a phenomenological level the process of changing statistic from Fermi continuously back to Fermi produces the same change in filling factor as multiplying a wavefunction by $P_V[z]^2$. It remains to be seen whether this explanation simply mimics the physics associated with changing the relative angular momenta channels which are important in the low-energy states, or there is some deeper significance.

IV. FUTURE

We have attempted here to give a brief sketch of what is understood absolutely clearly about the FQHE and of what ideas exist for explaining some more subtle aspects of this interesting phenomena. In particular the physical origin of the incompressibilities which occur at the hierarchy filling factors still does not have an absolutely clear explanation. We may hope that in the future the three different explanations of the hierarchy FQHE outlined above can be seen to be different facets of the same truth. At present it seems that the original hierarchy picture is the most satisfactory attempt at a full explanation.

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