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Unitary time evolution for irregular external field problems

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Abstract

For second quantized fermionic systems with static first quantized hamiltonian we proof that existence of a Schrödinger picture is not equivalent to the condition defining the regular external field problem.

Fermionic many particle systems may be investigated efficiently within the C^*-algebraic framework [1,2]. The primary concept is the CAR-algebra $\mathcal{A}$ [3,4], associated with the one particle Hilbert space $\mathcal{H}$ and the canonical anticommutation relations. $\mathcal{A}$ is generated by the image of $\mathcal{H}$ under an antilinear injection a of $\mathcal{H}$ into $\mathcal{A}$. The anticommutation relations in $\mathcal{A}$ state that for any f, g in $\mathcal{H}$ holds:

$$\{a(f), a(g)^*\} = (f,g)e\ , \qquad \{a(f), a(g)\} = 0\ .$$

Here e denotes the unit of $\mathcal{A}$ and $(\ \cdot\ ,\ \cdot\ )$ is the scalar product in $\mathcal{H}$. An orthogonal decomposition of $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ induces a quasi-free, pure, gauge invariant state ω_P on $\mathcal{A}$:

$$\omega_P(a(f_n)\ldots a(f_1)a(g_1)^*\ldots a(g_m)^*) \ := \ \delta_{nm}\det\left((f_i, P\,g_j)\right)\ .$$

Here P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}^+$. The GNS-construction associates with ω_P an up to isometrical equivalence unique representation $\Pi_P : \mathcal{A} \longrightarrow \mathcal{L}(\mathcal{F}_P)$ of $\mathcal{A}$ by linear bounded operators on a representation space $\mathcal{F}_P$. We denote $\Psi_P(f) := \Pi_P(a(f))$. $\Psi_P(f)$ is the tested time zero quantum field. Inspection of ω_{uPu^*}, with u being a unitary operator on $\mathcal{H}$, shows that the representation $\Pi_P^u : a(f) \longmapsto \Psi_P(u^*f)$ is isometrically equivalent with Π_{uPu^*}, i.e. there exists an isometric isomorphism $\Gamma : \mathcal{F}_P \longrightarrow \mathcal{F}_{uPu^*}$ such that $\Gamma\Psi_P(u^*f) = \Psi_{uPu^*}(f)\Gamma$ holds for all f in $\mathcal{H}$.

Two representations Ψ_{P_1} and Ψ_{P_2} are isometrically equivalent if and only if (iff) $P_1 - P_2 \,\epsilon\, HS$ [5]. Here HS is the algebra of Hilbert Schmidt operators on $\mathcal{H}$. We denote the associated equivalence relation as $P_1 \sim P_2$ ($\Longleftrightarrow$ $P_1 - P_2 \,\epsilon HS$). Now, due to the isometrical equivalence between the representations Π_P^u and Π_{uPu^*}, the C^*-automorphism $a(f) \longmapsto a(u^*f)$ is unitarily implemented in a representation Π_P iff $P \sim uPu^*$ holds. Unitary implementability means that there exists a unitary $\Gamma_P(u)$ on $\mathcal{F}_P$ such that the intertwining relation $\Psi_P(u^*f) = \Gamma_P(u)^*\Psi_P(f)\Gamma_P(u)$ holds for all f in $\mathcal{H}$.

This unitary implementability criterion can be employed in order to see, whether an algebraic time evolution automorphism $a(f) \longmapsto a(u(t)^*f)$ has an associated Schrödinger picture in a chosen representation: The algebraic time evolution is unitarily implemented in a representation Π_P iff the first quantized dynamics $\{u(t)\,|\,t\epsilon\mathbf{R}\}$ obeys

$$P \sim u(t)Pu(t)^* \qquad \text{for all } t \text{ in } \mathbf{R}. \tag{1}$$

For the case of a static dynamics, i.e. $u(t) := e^{-ith}$ with h being self-adjoint in $\mathcal{H}$, it has been claimed [6] that the condition (1) is equivalent with

$$P \sim \Theta(h). \tag{2}$$

Here $\Theta(h)$ denotes the positive spectral projection of the hamiltonian h. Klaus and Scharf [7] have pointed out that there was no proof known for this statement. While being able to infer condition (1) from (2), these authors left it open whether (2) indeed follows from (1), though they considered it as very likely to be true. Condition (2) is taken in [7] as defining the regular external field problem, when P is identified with the postive spectral projection of the free Dirac hamiltonian.

Since the question still seems unsettled, we should like to demonstrate in this little note that (2) is stronger than (1) and not equivalent to it. We do so by constructing examples, which obey (1), but violate (2).

A trivial example is this: Let $h := h_0 + c\mathbf{1}$, with h_0 being the free Dirac hamiltonian on the Hilbert space of Cauchy data to the Dirac equation with mass m and a real constant $c > 2m$. Thus the spectrum of h is purely continuous and is given by

$$\mathrm{spec}(h) = (-\infty, -m + c] \cup [m + c, \infty).$$

Let $P := \Theta(h_0)$. Obviously condition (1) is obeyed, since $u(t)$ commutes with P. On the other hand, since $\Theta(h)\Theta(h_0) = \Theta(h_0)$, we obtain $\Theta(h) - \Theta(h_0) = \Theta(h)\,(\mathbf{1} - \Theta(h_0)) = \Theta(h)\Theta(-h_0)$. This operator, however, projects onto the subspace spanned by the improper eigenvectors of h_0 with eigenvalue E in the interval $(m - c, -m)$. Since the Hilbert Schmidt norm of an orthogonal projection is given by the dimension of its range, $\Theta(h)\Theta(h_0)$ has divergent HS norm. Thus condition (2) is not realized while (1) is. This demonstrates that (2) does not follow from (1).

More interesting examples can be easily constructed for the case of chiral (zero mass) Dirac fermions in two-dimensional space-time, which are exposed to an external static electromagnetic potential. This amounts to choosing the one particle space $\mathcal{H} := \mathrm{L}^2(\mathbf{R})$ with the usual scalar product and as hamiltonian in $\mathcal{H}$:

$$h(A) := -i\frac{d}{dx} - A(Q).$$

Here $A\epsilon\mathcal{C}_0^\infty(\mathbf{R}:\mathbf{R})$ (compact support) is assumed. Q denotes the multiplication operator. Observe that with $h_0 := h(0)$ and $\alpha(x) := \int_0^x d\xi A(\xi)$ holds:

$$h(A) = e^{i\alpha(Q)}h_0 e^{-i\alpha(Q)}.$$

Thus the time evolution operators read with $\gamma(Q,t) := \alpha(Q) - \alpha(Q - t\mathbf{1})$

$$u(t) := e^{-ith(A)} = e^{i\gamma(Q,t)}e^{-ith_0}.$$

In order to see, whether a certain A defines a regular external field problem, we have to check $\Theta(h(A)) \sim \Theta(h_0)$. According to a theorem by Hermaszewski and Streater [8], the equivalence $\Theta(h_0) \sim e^{i\alpha(Q)}\Theta(h_0)e^{-i\alpha(Q)}$ holds, iff $\alpha(\infty) - \alpha(-\infty) \,\epsilon 2\pi\mathbf{Z}$. The theorem's assumption $\frac{d\alpha}{dx} \epsilon \mathcal{C}_0^\infty(\mathbf{R}:\mathbf{R})$ is realized due to $A \epsilon \mathcal{C}_0^\infty(\mathbf{R}:\mathbf{R})$. Thus the condition (2), with $P := \Theta(h_0)$, is valid, iff A obeys

$$\int_{-\infty}^{\infty} dx A(x) \,\epsilon\, 2\pi\mathbf{Z}.$$

In contrast to this, condition (1) holds for any $A \epsilon \mathcal{C}_0^\infty(\mathbf{R}:\mathbf{R})$. This can be seen as follows: $u(t)Pu(t)^* = e^{i\gamma(Q,t)}Pe^{-i\gamma(Q,t)}$ since e^{ith_0} commutes with P. Now $\gamma(\,\cdot\,,t)\,\epsilon\,\mathcal{C}_0^\infty(\mathbf{R}:\mathbf{R})$ for any t. Thus, according to the criterion by Hermaszewski and Streater, $P \sim u(t)Pu(t)^*$ is trivially fulfilled.

Thus we have shown that the conditions (1) and (2) are inequivalent on the very general set of all pairs (P,h) of arbitrary projections P and self-adjoint hamiltonians h (on a given Hilbert space). Obviously, this result does not rule out a possible equivalence between (1) and (2) on smaller sets of pairs (P,h). For instance, the following subset of pairs is of physical interest. Assume $P := \Theta(h_0)$ with h_0 being the free mass m Dirac hamiltonian. Let $h := h_0 + V(Q)$ with $V(Q)$ being a self-adjoint multiplication operator, which is bounded relative to h_0. Now let H_m denote the set of all such pairs (P,h) with m being kept fixed. For $m > 0$ we do not know whether (1) and (2) are equivalent on H_m. For $m = 0$ and two-dimensional space-time our chiral example rules out such an equivalence, since $A(Q)$ is bounded for $A \,\epsilon\, \mathcal{C}_0^\infty(\mathbf{R}:\mathbf{R})$.

References

[1] P. J. M. Bongaarts, Ann. Phys. **56**, 108 (1970)

[2] P. Falkensteiner, H. Grosse, Nucl. Phys. B**305**, 126 (1988)

[3] H. Araki, W. Wyss, Helv. Phys. Acta **37**, 136 (1964)

[4] H. Araki, Contemp. Math. **62**, 23 (1987)

[5] R. T. Powers, E. Stormer, Commun. math. Phys. **16**, 1 (1970)

[6] G. Labonté, Can. J. Phys. **53**, 1533 (1975)

[7] M. Klaus, G. Scharf, Helv. Phys. Acta **50**, 779 (1977)

[8] Z. J. Hermaszewski, R. F. Streater, J. Phys. A**16**, 2801 (1983)