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Perturbative Renormalization and Effective Lagrangians in Φ_4^4

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Abstract : Polchinski's proof of the perturbative renormalizability of massive Euclidean Φ_4^4 is considerably simplified, in some respects clarified and extended to general renormalization conditions and Green's functions with arbitrary external momenta. Φ_3^4 and Φ_2^4 are also dealt with. Moreover we show that adding e.g. $\Phi^{\geq 5}$ type interactions to the bare Lagrangian, with coupling constants vanishing at least as some inverse power of the UV-cutoff, does not alter the Green's functions in the limit where the UV-cutoff is removed. Establishing the validity of the action principle in this formalism has not yet been possible, but some partial results are obtained.

1. Introduction

The complete proofs of perturbative renormalizability of (a corresponding class of) relativistic or Euclidean quantum field theories which are currently available [1-5] are complicated and long. Motivated by Wilson's renormalization group Polchinski [6] has initiated a search for more digestible proofs which, in particular, would do without analyzing Zimmermann forests [2] or Gallavotti-Nicolò trees [4].

Polchinski [6] indeed found an elementary (and, as compared to e.g. BPHZ, evidently simplified) method to prove the perturbative renormalizability of massive Euclidean Φ_4^4. However the existence of the Green's functions is proved for small external momenta only and the renormalization conditions are imposed at an unphysical scale. Filling in the remaining gaps (as well as clarifying some of the technicalities), then, would be a necessary step to establish this method as a valid way of treating the renormalizability problem. Doing precisely this we present in this paper a largely improved version of Polchinski's proof of the perturbative renormalizability of massive Euclidean Φ_4^4.

When we had finished our work we learned[1] about ref.[8] where a combination of the continuous renormalization group [6] and of the tree-expansion [4] methods is used to arrive at a fairly simple proof that the renormalized Green's functions of the massive Euclidean Φ_4^4 are bounded as the UV-cutoff is removed. However, convergence is not shown; moreover, in our opinion our proof of perturbative renormalizability is less complicated.

The contents is organized as follows. Section 2 is devoted to a convenient definition of the flow of effective Lagrangians L^Λ, $0 \le \Lambda \le \Lambda_0 < \infty$, where Λ_0 is an UV-cutoff and where L^{Λ_0} serves essentially as the bare Lagrangian for the Euclidean massive Φ_4^4. Some relevant properties of the effective Lagrangians are investigated in detail. In particular we show that L^Λ at scale $\Lambda = 0$ is the generating functional of the (regularized) connected amputated Green's functions which immediately makes it possible to impose standard renormalization conditions. The differential form [6] of the flow equation (which is a first order differential equation for L^Λ with respect to Λ) as well as a first order differential equation for L^Λ with respect to Λ_0 (which is not present in [6]) are derived and estimated using suitable norms on L^Λ. Inspired by [6] we want to prove renormalizability by investigating the

[1]We are grateful to D.Brydges for bringing ref.[8] to our attention.

estimated differential equations. This is done in section 3. First, an (as compared to [6]) improved induction hypothesis as applied to the estimated flow equation leads easily to the boundedness of the norm of $L^{\Lambda=0}$ as $\Lambda_0 \to \infty$. Repeating now the analogous procedure on the estimated Λ_0-differential equation directly yields the convergence of $L^{\Lambda=0}$ in the limit $\Lambda_0 \to \infty$. Compared to [6] our proof constitutes a major short-cut.

In section 4 we continue the investigation of the structure of Φ_4^4. The first question we address is to what extent L^{Λ_0} may be changed (e.g. by the addition of irrelevant terms) in order that it still defines the same field theory in the limit where the cutoff is removed. After this we prove that if the Green's functions are differentiable with respect to some parameters for finite Λ_0 then they stay so as $\Lambda_0 \to \infty$ (this property of the Green's functions is crucial if one wants to prove the validity of the action principle [7]). It is clear that a priori there is no guarantee that the flow equation method works as well if e.g. the space-time dimension is not 4. In order to dispel such doubts we briefly indicate what kind of induction hypotheses (to investigate the estimated differential equations) turn out to be successful for Euclidean massive Φ_3^4 and Φ_2^4.

2. Flow equation for effective Lagrangians

2.1. Let Λ, $0 \le \Lambda \le \Lambda_0 < \infty$, be a scale parameter, where Λ_0 is meant to play the rôle of an UV-cutoff which ultimately will be sent to infinity. Let $m^2 > 0$. For $x, y \in \mathbb{R}^4$ [2] we define the regularized covariance, $C_\Lambda^{\Lambda_0}$, by

$$C_\Lambda^{\Lambda_0}(x,y) := \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)}}{p^2+m^2} \left(R(\Lambda_0,p) - R(\Lambda,p)\right) \quad . \tag{2.1}$$

There is a large amount of arbitrariness in the choice of the regularizing function, R. However, it seems as if for our purposes it is most economical to require that for $\Lambda > 0$

$$R(\Lambda,p) := \left(1 - e^{-(\Lambda/m)^4}\right) \cdot K(\tfrac{p^2}{\Lambda^2}) \quad , \tag{2.2}$$

[2] The notation is as follows: If $x \in \mathbb{R}^4$ then its components are x_μ, $\mu = 1, \ldots, 4$; and $x_{i,\mu} \equiv (x_i)_\mu$, $x_i x_j \equiv \sum_{\mu=1}^4 x_{i,\mu} x_{j,\mu}$, $x_i^2 \equiv x_i x_i$, $\Box \equiv \sum_{\mu=1}^4 \frac{\partial^2}{\partial x_\mu^2}$. Also $\partial_\Lambda := \frac{\partial}{\partial \Lambda}$, $\partial_{\Lambda_0} := \frac{\partial}{\partial \Lambda_0}$, $\partial_{p_{i,\mu}} := \frac{\partial}{\partial p_{i,\mu}}$.

where $K : [0,\infty) \to [0,1]$ is a compactly supported C^∞ function with

$$K(a) := \begin{cases} 1 & , \quad 0 \le a \le 1 \\ \text{smooth} & , \quad 1 \le a \le 4 \\ 0 & , \quad 4 \le a \quad . \end{cases} \tag{2.3}$$

Obviously $R(\Lambda = 0, p) := \lim_{\Lambda \to 0} R(\Lambda, p) = 0$, so $R(\Lambda, p) \in C^\infty(\mathbb{R}^4)$ for all Λ; and $\partial_\Lambda R(\Lambda, p)$ converges uniformly in p as $\Lambda \to 0$. Clearly $C_\Lambda^{\Lambda_0}(x,y)$ is, among other things, real, Euclidean invariant and $C^\infty(\mathbb{R}^4)$ in x and y.

We introduce an intermediate volume cutoff, V, e.g. $V = V(l) := [-l, l]^4 \subset \mathbb{R}^4$, $0 < l < \infty$. Let $\phi \in \mathcal{S}(\mathbb{R}^4)$ (the Schwartz test function space over $\mathbb{R}^4$) and define the functional Laplace operator, $\Delta(\Lambda, \Lambda_0)$, by

$$\Delta(\Lambda, \Lambda_0) := \tfrac{1}{2} \int d^4x \, d^4y \, \frac{\delta}{\delta\phi(x)} \, C_\Lambda^{\Lambda_0}(x,y) \, \frac{\delta}{\delta\phi(y)} \quad , \tag{2.4}$$

which acts on sufficiently regular functionals of ϕ by the usual rules; e.g. if $f \in C(\mathbb{R}^8)$ then

$$\Delta(\Lambda, \Lambda_0) \int_{V^2} d^4x_1 \, d^4x_2 \, f(x_1, x_2) \Box\phi(x_1) \Box\phi(x_2) :=$$
$$\int_{V^2} d^4x_1 \, d^4x_2 \, f(x_1, x_2) \Box^2 C_\Lambda^{\Lambda_0}(x_1, x_2) \quad .$$

2.2. Let g be a formal variable. The effective Lagrangian at scale Λ (and regularized by the volume cutoff V), L_V^Λ, is defined as follows. First of all L_V^Λ is taken to be a formal power series (fps) in g, hence $L_V^\Lambda := \sum_{r=0}^\infty g^r \, L_{V,r}^\Lambda$. Now, if $\Lambda = \Lambda_0$ we set

$$L_{V,r}^{\Lambda_0} := \begin{cases} 0 & , \quad r = 0 \\ \int_V d^4x \left(a_r^{\Lambda_0} \cdot \phi^2(x) - b_r^{\Lambda_0} \cdot \phi(x) \Box\phi(x) + c_r^{\Lambda_0} \cdot \phi^4(x) \right) & , \quad r \ge 1 \quad , \end{cases} \tag{2.5}$$

where $a_r^{\Lambda_0}$, $b_r^{\Lambda_0}$ and $c_r^{\Lambda_0}$ are some constants which will become uniquely determined functions of Λ_0, r, m^2, R and of the renormalization conditions once the latter are specified (see (2.17)). On the other hand, if $\Lambda \in [0, \Lambda_0]$ then we require that in the sense of fps

$$e^{-L_V^\Lambda - I_V^\Lambda} := e^{\Delta(\Lambda, \Lambda_0)} e^{-L_V^{\Lambda_0}} \quad , \tag{2.6}$$

where the fps I_V^Λ, i.e. $I_V^\Lambda := \sum_{r=0}^\infty g^r \, I_{V,r}^\Lambda$, in (2.6) is supposed to collect precisely the ϕ-independent pieces of the r.h.s. of (2.6); in other words we impose that $\frac{\delta}{\delta\phi} I_{V,r}^\Lambda \equiv 0$ and

$\frac{\delta}{\delta\phi}L^{\Lambda}_{V,r} \not\equiv 0$. Expanding both sides of (2.6) as fps in g one notes that $L^{\Lambda}_{V,r=0} = 0$, that there is no loss of generality in defining $I^{\Lambda}_{V,r=0} := 0$ (hence $I^{\Lambda_0}_{V} = 0$) and that we obtain a recursive formula for the nontrivial expansion coefficients of L^{Λ}_{V} and I^{Λ}_{V}:

$$L^{\Lambda}_{V,1} + I^{\Lambda}_{V,1} = e^{\Delta(\Lambda,\Lambda_0)} L^{\Lambda_0}_{V,1} \quad , \tag{2.7a}$$

and if $r > 1$ then

$$\begin{aligned} L^{\Lambda}_{V,r} + I^{\Lambda}_{V,r} &= e^{\Delta(\Lambda,\Lambda_0)} L^{\Lambda_0}_{V,r} + \sum_{k=2}^{r} \frac{(-1)^k}{k!} \\ &\sum_{r_1,\ldots,r_k : \sum_{i=1}^{k} r_i = r} \Big(\prod_{i=1}^{k} (L^{\Lambda}_{V,r_i} + I^{\Lambda}_{V,r_i}) - e^{\Delta(\Lambda,\Lambda_0)} \prod_{i=1}^{k} L^{\Lambda_0}_{V,r_i} \Big) \quad . \end{aligned} \tag{2.7b}$$

These formulae show by induction that the definition (2.6) has been legitimate in the sense that $L^{\Lambda}_{V,r}$ and $I^{\Lambda}_{V,r}$ are finite quantities and also that $L^{\Lambda}_{V,r}$ is an even polynomial in ϕ of degree $\leq 4r$.

2.3. The physical significance of the flow equation (2.6) is expressed in the following Proposition.

PROPOSITION 1. *L^{Λ}_{V} is the generating functional of the order $r \geq 1$ perturbative amputated connected Green's functions of the Euclidean quantum field theory defined by the propagator $C^{\Lambda_0}_{\Lambda}$ and by the vertices $\{L^{\Lambda_0}_{V,s} \,:\, s \geq 1\}$.*

PROOF: Let $< f, g >:= \int d^4x\, \overline{f}(x) g(x)$, assume that $J \in \mathcal{S}(\mathbb{R}^4)$ and define $\delta_J(x) := \frac{\delta}{\delta J(x)}$. Notice that due to (2.1)-(2.3) $C^{\Lambda_0}_{\Lambda}$ maps $\mathcal{S}(\mathbb{R}^4)$ into itself. The basic combinatoric identity to be proved forms the contents of the following Lemma.

LEMMA 2. *Let $r_i \geq 1$, $1 \leq i \leq k$. With the above notation we find*

$$\begin{aligned} & e^{\Delta(\Lambda,\Lambda_0)} \Big(L^{\Lambda_0}_{V,r_1}(\phi) \cdots L^{\Lambda_0}_{V,r_k}(\phi) \Big) \Big|_{\phi = C^{\Lambda_0}_{\Lambda} J} \\ &\quad = e^{-\frac{1}{2}<J, C^{\Lambda_0}_{\Lambda} J>} \Big(L^{\Lambda_0}_{V,r_1}(\delta_J) \cdots L^{\Lambda_0}_{V,r_k}(\delta_J) \Big) e^{\frac{1}{2}<J, C^{\Lambda_0}_{\Lambda} J>} \quad . \end{aligned} \tag{2.8}$$

PROOF OF LEMMA 2: Notice, first, that it is sufficient to prove (2.8) in the case where each of the $L^{\Lambda_0}_{V,r_i}$, $1 \le i \le k$, is a monomial in the fields. Therefore $\prod_{i=1}^k L^{\Lambda_0}_{V,r_i}(\phi)$ may be viewed as a monomial in ϕ and $\Box\phi$. Perhaps the most elegant method to verify (2.8) is to do so by induction in the degree of $\prod_{i=1}^k L^{\Lambda_0}_{V,r_i}(\phi)$. We wish to illustrate this procedure by considering a baby version of (2.8); its extension to (2.8) is straightforward (remember that because of (2.1), (2.2) one has the symmetry $C^{\Lambda_0}_{\Lambda}(x,y) = C^{\Lambda_0}_{\Lambda}(y,x)$).

So let us replace ϕ and $\Box\phi$ by x, $x \in \mathbb{R}$, and J by y, $y \in \mathbb{R}$, and show by induction in n, $n \in \mathbb{N}_0$, that $e^{\frac{1}{2}\frac{d^2}{dx^2}}\, x^n = e^{-\frac{1}{2}y^2}\, \frac{d^n}{dy^n}\, e^{\frac{1}{2}y^2}$ at $y = x$. Clearly this holds for $n = 0$; and now assume that its validity has been checked for $n-1$. Because $\left[e^{\frac{1}{2}\frac{d^2}{dx^2}}, x\right] = \frac{d}{dx}\, e^{\frac{1}{2}\frac{d^2}{dx^2}}$, we find that $e^{\frac{1}{2}\frac{d^2}{dx^2}}\, x^n = \left[e^{\frac{1}{2}\frac{d^2}{dx^2}}, x\right] x^{n-1} + x\, e^{\frac{1}{2}\frac{d^2}{dx^2}}\, x^{n-1} = \left(\frac{d}{dx} + x\right) e^{\frac{1}{2}\frac{d^2}{dx^2}}\, x^{n-1}$. Inserting the induction hypothesis immediately yields the required result for n. ■

PROOF OF PROPOSITION 1 CONTINUED: We write $e^{-L^{\Lambda}_V - I^{\Lambda}_V} = \sum_{r\ge 0} g^r \left(e^{-L^{\Lambda}_V - I^{\Lambda}_V}\right)_r$, with $\left(e^{-L^{\Lambda}_V - I^{\Lambda}_V}\right)_0 = 1$, and for $r \ge 1$

$$\left(e^{-L^{\Lambda}_V - I^{\Lambda}_V}\right)_r = \sum_{k=1}^{r} \frac{(-1)^k}{k!} \sum_{r_1,\dots,r_k : \sum_{i=1}^k r_i = r} e^{\Delta(\Lambda,\Lambda_0)}\left(L^{\Lambda_0}_{V,r_1}(\phi)\cdots L^{\Lambda_0}_{V,r_k}(\phi)\right) \quad .$$

Hence (2.8) permits a rewriting of $e^{-L^{\Lambda}_V - I^{\Lambda}_V}$ as follows:

$$e^{-L^{\Lambda}_V - I^{\Lambda}_V}\Big|_{\phi = C^{\Lambda_0}_{\Lambda} J} = e^{-\frac{1}{2}<J, C^{\Lambda_0}_{\Lambda} J>}\, e^{-L^{\Lambda_0}_V(\delta_J)}\, e^{\frac{1}{2}<J, C^{\Lambda_0}_{\Lambda} J>} \quad .$$

The generating functional, $Z^{\Lambda_0}_{\Lambda,V}(J)$, of the perturbative, regularized and unamputated Green's functions is given by

$$\begin{aligned} Z^{\Lambda_0}_{\Lambda,V}(J) &= \int d\mu_{C^{\Lambda_0}_{\Lambda}}(\Phi)\; e^{-L^{\Lambda_0}_V(\Phi) + <J,\Phi>} \\ &= e^{-L^{\Lambda_0}_V(\delta_J)}\, e^{\frac{1}{2}<J, C^{\Lambda_0}_{\Lambda} J>} \quad , \end{aligned}$$

where $d\mu_{C^{\Lambda_0}_{\Lambda}}(\Phi)$ denotes the Gaussian measure with covariance $C^{\Lambda_0}_{\Lambda}$ which formally is given by $d\mu_{C^{\Lambda_0}_{\Lambda}}(\Phi) \sim d\Phi\, e^{-\frac{1}{2}<\Phi, \left(C^{\Lambda_0}_{\Lambda}\right)^{-1}\Phi>}$. Thus we infer that $L^{\Lambda}_V + I^{\Lambda}_V - \frac{1}{2} < J, C^{\Lambda_0}_{\Lambda} J >$ at $\phi = C^{\Lambda_0}_{\Lambda} J$ generates the corresponding connected Green's functions. Since $\frac{\delta}{\delta\phi} I^{\Lambda}_V = 0$ and because $< J, C^{\Lambda_0}_{\Lambda} J >$ is 0^{th} order in g the connected, unamputated Green's functions of order $r \ge 1$ are generated by $L^{\Lambda}_V(\phi)$ at $\phi = C^{\Lambda_0}_{\Lambda} J$ and so the claim follows. ■

2.4. Therefore the fps L^Λ, defined as

$$L^\Lambda := \lim_{V \uparrow \mathbb{R}^4} L_V^\Lambda \quad , \tag{2.9}$$

exists and has expansion coefficients L_r^Λ which obey

$$L_r^\Lambda = \sum_{n=1}^{4r} \int \prod_{i=1}^{n-1} \tfrac{d^4 p_i}{(2\pi)^4}\, \widehat{\phi}(p_1) \cdots \widehat{\phi}(p_{n-1}) \widehat{\phi}(-\sum_{j=1}^{n-1} p_j)\; \mathcal{L}_{r,n}^\Lambda(p_1, \ldots, p_{n-1}) \quad . \tag{2.10}$$

The $\mathcal{L}_{r,n}^\Lambda$ enjoy (among others) the following properties:

a) $\mathcal{L}_{r,n}^\Lambda = 0$, $n \notin 2\mathbb{N}$ (due to the $\mathbb{Z}_2$-symmetry $\phi \mapsto -\phi$).

b) $\mathcal{L}_{r,n}^\Lambda = 0$, $n > 2r+2$ (only connected Φ^4-graphs contribute). (2.11)

c) Only the totally symmetric part of $\mathcal{L}_{r,n}^\Lambda$ contributes, so henceforth we will assume that $\mathcal{L}_{r,n}^\Lambda(p_{\pi(1)}, \ldots, p_{\pi(n-1)}) = \mathcal{L}_{r,n}^\Lambda(p_1, \ldots, p_{n-1})$, $\forall \pi \in \mathrm{S}_n$, where we have put $p_n := -\sum_{j=1}^{n-1} p_j$.

d) $\partial_\Lambda \mathcal{L}_{r,n}^\Lambda$, $\partial_{\Lambda_0} \mathcal{L}_{r,n}^\Lambda$ and $\mathcal{L}_{r,n}^\Lambda$ are $C^\infty(\mathbb{R}^{4(n-1)})$. By Proposition 1 $\mathcal{L}_{r,n}^{\Lambda=0}$ is the amputated connected n-point Green's function of order r (with UV-cutoff Λ_0).

e) $\mathcal{L}_{r,n}^\Lambda$ is invariant under the orthogonal group; in particular $\mathcal{L}_{r,2}^\Lambda(p)$ depends on p only through p^2. Therefore

$$\partial_{p_\mu} \mathcal{L}_{r,2}^\Lambda(p=0) = 0 \quad , \tag{2.12a}$$

and there are functions $l_{(1)r}^\Lambda$, $l_{(2)r}^\Lambda$ such that

$$\partial_{p_\mu} \partial_{p_\nu} \mathcal{L}_{r,2}^\Lambda = \delta_{\mu,\nu} \cdot l_{(1)r}^\Lambda(p^2) + p_\mu p_\nu \cdot l_{(2)r}^\Lambda(p^2) \quad . \tag{2.12b}$$

As regards notation, we will set $\partial_{p_\mu} \partial_{p_\nu} \mathcal{L}_{r,2}^\Lambda \big|_{\delta_{\mu,\nu}} := \delta_{\mu,\nu} \cdot l_{(1)r}^\Lambda(p^2)$.

2.5. Because $\left[\Delta(\Lambda, \Lambda_0), \partial_\Lambda \Delta(\Lambda, \Lambda_0)\right] = 0$ and $\frac{\delta}{\delta\phi} I_{V,r}^\Lambda = 0$ taking the derivative of (2.6) with respect to Λ yields the differential form of the flow equation written sloppily as

$$\partial_\Lambda \left(L_V^\Lambda + I_V^\Lambda\right) = \left(\partial_\Lambda \Delta(\Lambda, \Lambda_0)\right)\, L_V^\Lambda - \tfrac{1}{2} < \delta_\phi L_V^\Lambda, \left(\partial_\Lambda C_\Lambda^{\Lambda_0}\right) \delta_\phi L_V^\Lambda > \quad . \tag{2.13}$$

Equation (2.13) amounts to (at least) two separate differential equations, namely one for the ϕ-dependent and one for the ϕ-independent parts of (2.13). Discarding the latter we may safely take the limit $V \uparrow \mathbb{R}^4$ and apply the expansion (2.10). Considering arbitrary variations $\widehat{\phi} \mapsto \lambda \cdot \widehat{\phi}$, $\lambda \in \mathbb{R}$, and $\widehat{\phi} \mapsto \widehat{\phi} + \varepsilon$, $\varepsilon \in \mathcal{S}(\mathbb{R}^4)$, taking into account the symmetry and the continuity of $\mathcal{L}^{\Lambda}_{r,n}$, one arrives at [6]

$$\begin{aligned}\partial_\Lambda \mathcal{L}^{\Lambda}_{r,n}(p_1, \dots, p_{n-1}) = -\binom{n+2}{2} \int \frac{d^4p}{(2\pi)^4} \frac{\partial_\Lambda R(\Lambda, p)}{p^2+m^2} \mathcal{L}^{\Lambda}_{r,n+2}(p, -p, p_1, \dots, p_{n-1}) \\ + \sum_{a=1}^{r-1} \sum_{b=2}^{n} \frac{b(n+2-b)}{2} \Big[\frac{\partial_\Lambda R(\Lambda, q)}{q^2+m^2} \\ \cdot \mathcal{L}^{\Lambda}_{a,b}(p_1, \dots, p_{b-1}) \mathcal{L}^{\Lambda}_{r-a,n+2-b}(-q, p_b, \dots, p_{n-1}) \Big]_{\text{symm.}} \end{aligned} \quad (2.14)$$

where $q := -\sum_{i=1}^{b-1} p_i$, and $[\cdots]_{\text{symm.}}$ indicates the symmetrization operation

$$[f(p_1, \dots, p_{n-1})]_{\text{symm.}} := \frac{1}{n!} \sum_{\pi \in S_n} f(p_{\pi(1)}, \dots, p_{\pi(n-1)}) \quad ,$$

where again $p_n := -\sum_{j=1}^{n-1} p_j$.

2.6. The boundary conditions (b.c.) imposed on the flow of $\{\mathcal{L}^{\Lambda}_{r,n}\}$ are of mixed type. Those at $\Lambda = \Lambda_0$ have been given in (2.5); using the convention that ∂^w_p stands for any w-th order momentum derivative (including all mixed partial derivatives of order w if it acts on a function whose arguments are several independent momenta) they read:

$$\begin{aligned} \underline{\Lambda = \Lambda_0} :\ &\text{a) } \mathcal{L}^{\Lambda_0}_{r,n} = 0 \quad , \quad n = \text{odd or } n \geq 6, \\ &\text{b) } \mathcal{L}^{\Lambda_0}_{r,4}(p_1, p_2, p_3) = c^{\Lambda_0}_r, \\ &\text{c) } \mathcal{L}^{\Lambda_0}_{r,2}(p) = b^{\Lambda_0}_r \cdot p^2 + a^{\Lambda_0}_r, \end{aligned} \quad (2.15)$$

implying that

$$\partial^w_p \mathcal{L}^{\Lambda_0}_{r,n} = 0 \quad , \quad n + w \geq 5 \quad . \quad (2.16)$$

We wish to remind the reader that so far the values of the parameters $a^{\Lambda_0}_r$, $b^{\Lambda_0}_r$ and $c^{\Lambda_0}_r$ have not been prescribed yet. However, these values will be fixed shortly (see the remarks which follow (2.17)).

In contrast to (2.15) the b.c. at $\Lambda = 0$, i.e. the renormalization conditions, determine the values of most of the relevant terms, where we call relevant those with $n + w \leq 4$:

$$\underline{\Lambda = 0}: \begin{array}{l} \text{a) } \mathcal{L}^0_{r,4}(p_1 = P_1, p_2 = P_2, p_3 = P_3) = c^R_r, \\ \text{b) } \partial_{p_\mu}\partial_{p_\nu}\mathcal{L}^0_{r,2}(p = P_4)\big|_{\delta_{\mu,\nu}} = 2\delta_{\mu,\nu}\, b^R_r, \\ \text{c) } \mathcal{L}^0_{r,2}(p = P_5) = a^R_r, \end{array} \tag{2.17}$$

where P_i, $1 \le i \le 5$, specify the renormalization points, and a^R_r, b^R_r and c^R_r are the Λ_0-independent renormalization constants.[3] The freedom in choosing a^R_r, b^R_r and c^R_r is the freedom of fixing the renormalization scheme; e.g. $P_1 = P_2 = \ldots = P_5 = 0$ and $c^R_r = \delta_{r,1}$, $a^R_r = b^R_r = 0$ gives the BPHZ scheme at zero external momentum. Because

$$L^\Lambda_r = \sum_{k=1}^{r} \frac{(-1)^{k+1}}{k!} \sum_{r_1,\ldots,r_k : \sum_{i=1}^k r_i = r} \left[e^{\Delta(\Lambda,\Lambda_0)}\, L^{\Lambda_0}_{r_1} \cdots L^{\Lambda_0}_{r_k} \right]_{c,\phi} \quad ,$$

where $[\cdots]_{c,\phi}$ indicates that (before the infinite volume limit is taken) the connected and ϕ-dependent contributions must be extracted, we see, inductively in r, that the parameters $a^{\Lambda_0}_r$, $b^{\Lambda_0}_r$ and $c^{\Lambda_0}_r$ are indeed uniquely determined by Λ_0, r, m^2, R and $\{a^R_s, b^R_s, c^R_s : 1 \le s \le r\}$. Note that it would have been inconsistent to impose b.c. at $\Lambda = 0$ also on the remaining relevant terms as long as we stick to (2.15). Nevertheless, these remaining relevant terms will be seen to be well under control because of the Euclidean symmetry of the theory (cf. (2.12)).

2.7. We are primarily interested in showing that $\lim_{\Lambda_0\to\infty} \mathcal{L}^0_{r,n}$ exists if the b.c. (2.15), (2.17) are imposed. For technical reasons it is more convenient to consider a fixed $\Lambda = \Lambda_1$, $0 < \Lambda_1 < \Lambda_0$, and prove that $\lim_{\Lambda_0\to\infty} \mathcal{L}^{\Lambda_1}_{r,n}$ exists. Because of the 1-parameter group property of $e^{\Delta(\Lambda,\Lambda')}$ we see that in particular

$$L^0_r = \sum_{k=1}^{r} \frac{(-1)^{k+1}}{k!} \sum_{r_1,\ldots,r_k : \sum_{i=1}^k r_i = r} \left[e^{\Delta(0,\Lambda_1)}\, L^{\Lambda_1}_{r_1} \cdots L^{\Lambda_1}_{r_k} \right]_{c,\phi} \quad ,$$

and so the limit $\lim_{\Lambda_0\to\infty} \mathcal{L}^0_{r,n}$ exists if and only if $\lim_{\Lambda_0\to\infty} \mathcal{L}^{\Lambda_1}_{r,n}$ exists as well. The above equation also relates the b.c. at $\Lambda = 0$ to those at $\Lambda = \Lambda_1$, and therefore the system which

[3] Another option would have been to replace b) in (2.17) by the requirement $\mathcal{L}^0_{r,2}(p = P_4) = b'^R_r$, assuming that $P_4^2 \ne P_5^2$. But in this case some minor details in the proof of Propositions 4 and 5 (in section 3) need to be adjusted.

will be investigated is the following one: The differential equation (2.14) for $\Lambda \in [\Lambda_1, \Lambda_0]$ with b.c. at $\Lambda = \Lambda_0$ given by (2.15) whereas at $\Lambda = \Lambda_1$ we have:

$$\begin{aligned} \underline{\Lambda = \Lambda_1}:\ & \text{a) } \mathcal{L}^{\Lambda_1}_{r,4}(p_1 = P_1, p_2 = P_2, p_3 = P_3) = c^R_r + O(r), \\ & \text{b) } \partial_{p_\mu}\partial_{p_\nu}\mathcal{L}^{\Lambda_1}_{r,2}(p = P_4)\big|_{\delta_{\mu,\nu}} = 2\delta_{\mu,\nu}\left(b^R_r + O(r)\right), \\ & \text{c) } \mathcal{L}^{\Lambda_1}_{r,2}(p = P_5) = a^R_r + O(r), \end{aligned} \tag{2.18}$$

where $O(r)$ stands for the contribution of a finite number of connected amputated Feynman graphs of order r whose external momenta are P_1, P_2, P_3 or P_4 or P_5, whose propagators are $\frac{R(\Lambda_1,p)}{p^2+m^2}$, and which have either one vertex (which is $\mathcal{L}^{\Lambda_1}_{r,n}$, $n > 4$ (for a) in (2.18)) or $n > 2$ (for b) and c) in (2.18))) or more than one vertex (which, then, are of the type $\mathcal{L}^{\Lambda_1}_{s,n}$, $1 \le s < r$).

2.8. Let $f : \mathbb{R}^{4n} \to \mathbb{C}$ be sufficiently regular such that the following definition of norms makes sense:

$$\|\partial^w f\|_{(a,b)} := \max_{\{i_1,\dots,i_w\}} \max_{\{\mu_1,\dots,\mu_w\}} \sup_{\{p_1,\dots,p_n :\, |p_i|<\max\{a,b\},\, 1\le i\le n\}} \left|\partial_{p_{i_1,\mu_1}} \cdots \partial_{p_{i_w,\mu_w}} f(p_1,\dots,p_n)\right| \ . \tag{2.19}$$

We set

$$f_n(p) := (p^2 + m^2)^n \quad , \quad n \ge 0 \quad , \tag{2.20}$$

and estimate these norms for $R(\Lambda, p)$ (cf. (2.2), (2.3)) and get

$$\left\|\frac{\partial^w \partial_\Lambda R(\Lambda,\cdot)}{f_n}\right\|_{(a,\infty)} \le c \cdot \Lambda^{-w-2n-1} \quad , \tag{2.21}$$

where c does not depend on Λ. It turns out to be convenient to consider the dimensionless functions $\mathcal{A}^{\Lambda}_{r,n}$ defined as

$$\mathcal{A}^{\Lambda}_{r,n}(p_1,\dots,p_{n-1}) := \Lambda^{n-4} \cdot \mathcal{L}^{\Lambda}_{r,n}(p_1,\dots,p_{n-1}) \quad . \tag{2.22}$$

Now we act with w-th order momentum derivatives on the flow equation (2.14), perform estimates using (2.19)-(2.22) and end up with a slightly improved version of one of the inequalities which were of greatest importance in [6]:

$$\|\partial_\Lambda\big(\Lambda^{4-n}\partial^w \mathcal{A}^\Lambda_{r,n}\big)\|_{(2\Lambda,\eta)} \le c_{w,n,r}\cdot\Lambda^{3-n}\cdot\Big(\|\partial^w\mathcal{A}^\Lambda_{r,n+2}\|_{(2\Lambda,\eta)} + \sum_{a=1}^{r-1}\sum_{b=2}^{n} \sum_{w_1,w_2,w_3\,:\,w_1+w_2+w_3=w} \Lambda^{-w_1}\cdot\|\partial^{w_2}\mathcal{A}^\Lambda_{a,b}\|_{(2\Lambda,\eta)}\cdot\|\partial^{w_3}\mathcal{A}^\Lambda_{r-a,n+2-b}\|_{(2\Lambda,\eta)}\Big)\quad, \tag{2.23}$$

where $\eta \in \mathbb{R}_+$ is arbitrary, and $c_{w,n,r}$ is independent of Λ and η. Equation (2.23) will be seen to be sufficient to prove the boundedness of the norms $\|\mathcal{A}^{\Lambda_1}_{r,n}\|_{(2\Lambda,\eta)}$ as $\Lambda_0\to\infty$ (cf. section 3). In order to be able to prove (in section 3) the convergence of $\mathcal{A}^{\Lambda_1}_{r,n}$ as well we are going to derive two inequalities by estimating equations involving $\partial_{\Lambda_0}\partial^w_p\mathcal{A}^\Lambda_{r,n}$.

Apply ∂^w_p on (2.14), integrate it now from Λ, $\Lambda<\Lambda_0$, up to Λ_0, assume that $n+w\ge 5$, remember the b.c. (2.16), apply now ∂_{Λ_0} and estimate the resulting equation. One obtains

$$\begin{aligned}\|\Lambda^{4-n}\partial_{\Lambda_0}\partial^w\mathcal{A}^\Lambda_{r,n}\|_{(2\Lambda,\eta)} \le\ & c'_{w,n,r}\cdot\Lambda_0^{3-n}\cdot\Big(\|\partial^w\mathcal{A}^{\Lambda_0}_{r,n+2}\|_{(2\Lambda_0,\eta)} + \sum_{a=1}^{r-1}\sum_{b=2}^{n} \sum_{w_1,w_2,w_3\,:\,w_1+w_2+w_3=w} \Lambda_0^{-w_1}\cdot\|\partial^{w_2}\mathcal{A}^{\Lambda_0}_{a,b}\|_{(2\Lambda_0,\eta)}\cdot\|\partial^{w_3}\mathcal{A}^{\Lambda_0}_{r-a,n+2-b}\|_{(2\Lambda_0,\eta)}\Big)\\ &+ c''_{w,n,r}\int_\Lambda^{\Lambda_0} ds\ s^{3-n}\cdot\Big(\|\partial_{\Lambda_0}\partial^w\mathcal{A}^s_{r,n+2}\|_{(2s,\eta)} + \sum_{a=1}^{r-1}\sum_{b=2}^{n} \sum_{w_1,w_2,w_3\,:\,w_1+w_2+w_3=w} s^{-w_1}\cdot\|\partial_{\Lambda_0}\partial^{w_2}\mathcal{A}^s_{a,b}\|_{(2s,\eta)}\cdot\|\partial^{w_3}\mathcal{A}^s_{r-a,n+2-b}\|_{(2s,\eta)}\Big)\quad,\end{aligned} \tag{2.24}$$

for $\Lambda<\Lambda_0$ and $n+w\ge 5$, and where $c'_{w,n,r}$ and $c''_{w,n,r}$ are independent of Λ_0, Λ and η. On the other hand one may integrate (2.14) from Λ_1 up to some Λ, $\Lambda<\Lambda_0$, apply ∂^w_p and ∂_{Λ_0} which yields the estimate

$$\begin{aligned}|\Lambda^{4-n}\partial_{\Lambda_0}\partial^w_p\mathcal{A}^\Lambda_{r,n}(p_1,\dots,p_{n-1})| \le\ & |\Lambda_1^{4-n}\partial_{\Lambda_0}\partial^w_p\mathcal{A}^{\Lambda_1}_{r,n}(p_1,\dots,p_{n-1})|\\ &+ c''_{w,n,r}\int_{\Lambda_1}^{\Lambda} ds\ s^{3-n}\cdot\Big(\|\partial_{\Lambda_0}\partial^w\mathcal{A}^s_{r,n+2}\|_{(2s,M)} + \sum_{a=1}^{r-1}\sum_{b=2}^{n} \sum_{w_1,w_2,w_3\,:\,w_1+w_2+w_3=w} s^{-w_1}\cdot\|\partial_{\Lambda_0}\partial^{w_2}\mathcal{A}^s_{a,b}\|_{(2s,M)}\cdot\|\partial^{w_3}\mathcal{A}^s_{r-a,n+2-b}\|_{(2s,M)}\Big)\quad,\end{aligned} \tag{2.25}$$

for $\Lambda_1\le\Lambda<\Lambda_0$ and $M\ge\max\{|p_i|\ :\ 1\le i\le n-1\}$.

3. Proof of perturbative renormalizability

We have now prepared enough information to prove the

THEOREM 3. *The Euclidean massive Φ_4^4 theory, defined by (2.6), (2.15) and (2.17), is perturbatively renormalizable, i.e. the limit* $\lim_{\Lambda_0\to\infty}\mathcal{L}_{r,n}^{\Lambda=0}(p_1,\cdots,p_{n-1})$ *exists for all* r,n. *Moreover,* $\lim_{\Lambda_0\to\infty}\mathcal{L}_{r,n}^{\Lambda=0}(p_1,\cdots,p_{n-1})$ *is a* C^∞ *and polynomially bounded function of* $p_1,\cdots,p_{n-1}$.

The proof of this well known fact can be deduced from the statements of the three Propositions below.

Let us add one word about notation. The symbol $Plog(z)$, $z\in\{\frac{\Lambda}{\Lambda_1},\frac{\Lambda_0}{\Lambda_1}:\Lambda\in[\Lambda_1,\Lambda_0]\}$ and thus $z\in[1,\infty)$, will be used quite frequently in what follows. Each time it appears it stands for some possibly new polynomial in $\log(z)$ whose coefficients depend neither on Λ nor on Λ_0 and which are taken to be nonnegative whenever $Plog(z)$ takes part in an inequality.

PROPOSITION 4. *(Boundedness) For any fixed* $\eta\geq 0$ *and for* $\Lambda\in[\Lambda_1,\Lambda_0]$ *we have*

$$\|\partial^w\mathcal{A}_{r,n}^{\Lambda}\|_{(2\Lambda,\eta)}\leq\Lambda^{-w}\cdot\left\{Plog(\tfrac{\Lambda}{\Lambda_1})+\tfrac{\Lambda}{\Lambda_0}\cdot Plog(\tfrac{\Lambda_0}{\Lambda_1})\right\}\quad. \tag{3.1}$$

PROOF: We proceed along the natural induction scheme which was set up in ref. [6].

The induction hypothesis is that (3.1) holds true for $\{(r,n):((r<r_0)\wedge(n\in\mathbb{N}))\vee((r=r_0)\wedge(n>n_0))\}$, for all $w\geq 0$. The induction step consists in proving (3.1) for $\partial_p^w\mathcal{A}_{r_0,n_0}^{\Lambda}$, $w\geq 0$. In other words, for fixed r_0 we move inductively downwards in n_0. Once one has dealt with $(r_0,n_0=1)$ the induction hypothesis automatically becomes true for the pair $(r_0':=r_0+1,n_0':=2r_0'+2)$, because $\mathcal{L}_{r_0',u}^{\Lambda}=0$ if $u>2r_0'+2$ (cf. (2.11)), and in this way one reaches any $\mathcal{L}_{r_0,n_0}^{\Lambda}$ starting from $\mathcal{L}_{1,4}^{\Lambda}$ (remember that $\mathcal{L}_{0,n}^{\Lambda}=0$, $n\in\mathbb{N}$, and $\mathcal{L}_{1,n}^{\Lambda}=0$, $n\geq 5$). Due to the $\mathbb{Z}_2$-invariance (3.1) comes for free for $n=$ odd, and so we only will worry about the even n.

Given n_0, we show first how to prove (3.1) for all w obeying $w + n_0 \geq 5$. Using these results we investigate the case $w + n_0 \leq 4$ later on.

In order to deal with the irrelevant vertices, i.e. with $\partial_p^w \mathcal{A}^{\Lambda}_{r_0,n_0}$ with $w + n_0 \geq 5$, we employ the b.c. at Λ_0 (cf. (2.16)) to write

$$
\begin{aligned}
\|\Lambda^{4-n_0}\partial^w \mathcal{A}^{\Lambda}_{r_0,n_0}\|_{(2\Lambda,\eta)} &= \|\Big(\Lambda^{4-n_0}\partial^w \mathcal{A}^{\Lambda}_{r_0,n_0} - \Lambda_0^{4-n_0}\partial^w \mathcal{A}^{\Lambda_0}_{r_0,n_0}\Big)\|_{(2\Lambda,\eta)} \\
&= \|\Big(\int_{\Lambda}^{\Lambda_0} ds\ \partial_s(s^{4-n_0}\partial^w \mathcal{A}^{s}_{r_0,n_0})\Big)\|_{(2\Lambda,\eta)} \\
&\leq \int_{\Lambda}^{\Lambda_0} ds\ \|\partial_s(s^{4-n_0}\partial^w \mathcal{A}^{s}_{r_0,n_0})\|_{(2\Lambda,\eta)} \\
&\leq \int_{\Lambda}^{\Lambda_0} ds\ \|\partial_s(s^{4-n_0}\partial^w\ \mathcal{A}^{s}_{r_0,n_0})\|_{(2s,\eta)} \quad .
\end{aligned}
$$

Using (2.23) and the induction hypothesis to express the r.h.s. of (2.23) in terms of (3.1) we obtain

$$
\begin{aligned}
&\|\Lambda^{4-n_0}\partial^w \mathcal{A}^{\Lambda}_{r_0,n_0}\|_{(2\Lambda,\eta)} \leq \\
&\qquad \int_{\Lambda}^{\Lambda_0} ds\ s^{3-n_0-w} \cdot \Big\{Plog(\tfrac{s}{\Lambda_1}) + \tfrac{s}{\Lambda_0} \cdot Plog(\tfrac{\Lambda_0}{\Lambda_1})\Big\} \quad .
\end{aligned}
\tag{3.2}
$$

Since $\int_{\Lambda}^{\Lambda_0} ds\ s^{-a} \cdot Plog(\frac{s}{\Lambda_1}) = \Lambda^{-a+1} \cdot \Big\{Plog(\frac{\Lambda}{\Lambda_1}) + \sum_{b=1}^{a-1} \left(\frac{\Lambda}{\Lambda_0}\right)^b \cdot P_b log(\frac{\Lambda_0}{\Lambda_1})\Big\}$, if $(a \geq 2) \wedge (a \in \mathbb{N})$, we can easily compute the integral on the r.h.s. of (3.2), estimate the result and arrive at (3.1).

We are left with verifying (3.1) for the relevant vertices, i.e. for w with $w + n_0 \leq 4$. On three of these vertices we have imposed "by hand" a "finiteness condition", i.e. the b.c. (2.18) at Λ_1 and at an arbitrarily selected set of momenta. What regards the remaining two relevant vertices one observes that, due to the form of L^{Λ_0} and the Euclidean invariance of the regularized theory, they are constrained by the "implicit finiteness condition (2.12) at $p = 0$", for all Λ. We can profit from this situation in the sense that integrating (2.23) now over the interval $[\Lambda_1, \Lambda]$ and using power series expansions with respect to the momenta will lead to (3.1) for the relevant terms. Let us start with $n_0 = 4$.

Obviously

$$
\begin{aligned}
\Big|(\mathcal{A}^{\Lambda}_{r_0,4} - \mathcal{A}^{\Lambda_1}_{r_0,4})(P_1,P_2,P_3)\Big| &\leq \int_{\Lambda_1}^{\Lambda} ds\ \Big|\partial_s\big(\mathcal{A}^{s}_{r_0,4}(P_1,P_2,P_3)\big)\Big| \\
&\leq \int_{\Lambda_1}^{\Lambda} ds\ \|\partial_s \mathcal{A}^{s}_{r_0,4}\|_{(2s,M)} \quad ,
\end{aligned}
$$

if $M \geq \max\{|P_i| \,:\, 1 \leq i \leq 3\}$. The induction hypothesis states that (3.1) holds in particular for $\{\mathcal{A}^{\Lambda_1}_{r_0,n} \,:\, n \geq 6\}$ and for $\{\mathcal{A}^{\Lambda_1}_{r,n} \,:\, (r < r_0) \wedge (n \in \mathbb{N})\}$. Applying this knowledge to bound the $O(r)$ term in part a) of (2.18) one arrives at the conclusion that there is a Λ_0-independent constant, c, such that $\left|\mathcal{A}^{\Lambda_1}_{r_0,4}(P_1,P_2,P_3)\right| \leq c$. Therefore $\left|\mathcal{A}^{\Lambda}_{r_0,4}(P_1,P_2,P_3)\right| \leq c + \int_{\Lambda_1}^{\Lambda} ds\ \|\partial_s \mathcal{A}^{s}_{r_0,4}\|_{(2s,M)}$. (2.23) and the induction hypothesis now give $\left|\mathcal{A}^{\Lambda}_{r_0,4}(P_1,P_2,P_3)\right| \leq Plog(\frac{\Lambda}{\Lambda_1}) + \frac{\Lambda}{\Lambda_0} \cdot Plog(\frac{\Lambda_0}{\Lambda_1})$. Using Taylor's formula

$$\begin{aligned} f(p_1,\dots,p_m) &= f(q_1,\dots,q_m) + \sum_{i=1}^{m}\sum_{\mu=1}^{4}(p-q)_{i,\mu}\int_0^1 d\lambda\ \partial_{k_{i,\mu}} f(k_1,\dots,k_m) \\ k_j &:= q_j + \lambda(p_j - q_j) \quad , \quad 1 \leq j \leq m \quad , \end{aligned} \tag{3.3}$$

for $f = \mathcal{A}^{\Lambda}_{r_0,4}$, $m = 3$ and $q_j = P_j$, and the bound (3.1) for $\partial^1_p \mathcal{A}^{\Lambda}_{r_0,4}$, one finally gets (3.1) for $\|\mathcal{A}^{\Lambda}_{r_0,4}\|_{(2\Lambda,\eta)}$ with $\eta = M$, and hence for all $\eta \geq 0$.

If $n_0 = 2$ we begin by repeating the analogous steps for $\partial^2_p \mathcal{A}^{\Lambda}_{r_0,2}$: One shows that $\left|\partial^2_p \mathcal{A}^{\Lambda_1}_{r_0,2}(P_4)\,|_{\delta_{\mu,\nu}}\right| \leq c'$, and (3.3) for $f = \partial^2_p \mathcal{A}^{\Lambda_1}_{r_0,2}$, $m = 1$, $p = P_4$, $q = 0$, together with (2.12b) and $\|\partial^3 \mathcal{A}^{\Lambda_1}_{r_0,2}\|_{(2\Lambda_1,P_4)}$ now yields $\left|\partial^2_p \mathcal{A}^{\Lambda_1}_{r_0,2}(P_4)\right| \leq c''$, where $\partial_{\Lambda_0} c'' = 0$. Therefore $\left|\Lambda^2 \partial^2_p \mathcal{A}^{\Lambda}_{r_0,2}(P_4)\right| \leq \Lambda_1^2 \cdot c'' + \int_{\Lambda_1}^{\Lambda} ds\ \|\partial_s(s^2 \partial^2 \mathcal{A}^{s}_{r_0,2})\|_{(2s,M)}$, $M \geq |P_4|$, and so one obtains the $Plog$-type bound for $\left|\partial^2_p \mathcal{A}^{\Lambda}_{r_0,2}(P_4)\right|$. Applying (3.3) once more, with $f = \partial^2_p \mathcal{A}^{\Lambda}_{r_0,2}$, and using the known bound for $\|\partial^3 \mathcal{A}^{\Lambda}_{r_0,2}\|_{(2\Lambda,\eta)}$, we arrive at the desired bound for $\|\partial^2 \mathcal{A}^{\Lambda}_{r_0,2}\|_{(2\Lambda,\eta)}$.

Because of (2.12a) we may consider (3.3) for $f = \partial^1_p \mathcal{A}^{\Lambda}_{r_0,2}$, $m = 1$, $q = 0$ and use $\|\partial^2 \mathcal{A}^{\Lambda}_{r_0,2}\|_{(2\Lambda,\eta)}$ to bound $\|\partial^1 \mathcal{A}^{\Lambda}_{r_0,2}\|_{(2\Lambda,\eta)}$.

The case $n_0 = 2$, $w = 0$, finally, can be treated in complete analogy to $n_0 = 4$, $w = 0$.

∎

So far we have not said anything about the degrees, as well as about the η-dependence of the coefficients, of the polynomials $Plog(z)$ which appear on the r.h.s. of (3.1). What concerns the η-dependence of the coefficients, the extensive use which is made of the Taylor formula to prove (3.1) suggests that it should be possible to establish polynomial bounds; and one should not expect to get better bounds with this method. Let $P\eta$ resp. $P'log(z)$ be polynomials in $\frac{\eta}{\Lambda_1}$ resp. $\log(z)$ with coefficients which are nonnegative whenever required and which depend neither on η nor on Λ nor on Λ_0. It is not difficult to prove the following facts by tracing the degrees of the polynomials which are involved through our inductive proof of Proposition 4.

PROPOSITION 4'. *Instead of (3.1) one can prove the bounds*

$$\|\partial^w \mathcal{A}^{\Lambda}_{r,n}\|_{(2\Lambda,\eta)} \leq \Lambda^{-w} \cdot P\eta \cdot \Big\{P'log(\tfrac{\Lambda}{\Lambda_1}) + \tfrac{\Lambda}{\Lambda_0} \cdot P'log(\tfrac{\Lambda_0}{\Lambda_1})\Big\} \quad . \tag{3.4}$$

The degrees of the polynomials in (3.4) obey:

$$\deg\Big(P'log(z)\Big) \leq \begin{cases} r+1-\frac{n}{2} & , \quad n \geq 4 \\ r-\frac{n}{2} & , \quad n=2 \end{cases} ,$$

$$\deg\Big(P\eta\Big) \leq \begin{cases} 6r-4-\frac{n}{2} & , \quad n \geq 4 \\ 6r-2-\frac{n}{2} & , \quad n=2 \end{cases} .$$

In the literature where (discrete or continuous) renormalization group methods are applied to prove the perturbative renormalizability it is not uncommon [5,8] to consider Theorem 3 to be proved once the boundedness of the norms is shown, because it seems to be thought trivial to prove the convergence of $\mathcal{L}^{\Lambda}_{r,n}$, $\Lambda_0 \to \infty$, as well. Nevertheless we also present explicitly a convergence proof. One of its virtues is that it is much more direct and simple than the one given in [6].

PROPOSITION 5. *(Convergence) Assume that for $\eta \geq 0$ and $\Lambda \in [\Lambda_1, \Lambda_0]$*

$$\|\partial^w \mathcal{A}^{\Lambda}_{r,n}\|_{(2\Lambda,\eta)} \leq \Lambda^{-w} \cdot Plog(\tfrac{\Lambda_0}{\Lambda_1}) \quad . \tag{3.5}$$

Then one finds the following bounds:

$$\|\partial_{\Lambda_0} \partial^w \mathcal{A}^{\Lambda}_{r,n}\|_{(2\Lambda,\eta)} \leq \Lambda_0^{-2} \cdot \Lambda^{-w+1} \cdot Plog(\tfrac{\Lambda_0}{\Lambda_1}) \quad . \tag{3.6}$$

PROOF: The method is induction, and the induction scheme is precisely as in the proof of Proposition 4; it works because obviously (3.6) is true for $\{(r,n) : ((r=0) \wedge (n \in \mathbb{N})) \vee ((r \geq 1) \wedge (n > 2r+2))\}$, for all $w \geq 0$.

So we wish to prove the validity of (3.6) for $\partial_{\Lambda_0}\partial_p^w \mathcal{A}^{\Lambda}_{r_0,n_0}$ under the assumption that it has been proven for $\{(r,n) : ((r<r_0)\wedge(n\in \mathbb{N}))\vee((r=r_0)\wedge(n>n_0))\}$, for all $w\geq 0$.

Given n_0, and considering separately the cases $w+n_0\geq 6$ and $w+n_0=5$, the bound (3.6) follows easily from applying (3.5) and (3.6) to the r.h.s. of (2.24), for all $\Lambda\in[\Lambda_1,\Lambda_0)$, thus by continuity also if $\Lambda\to\Lambda_0$.

Turning our attention to the relevant vertices one notes that (2.12b), (2.18) and (3.6) imply that $\left|\partial_{\Lambda_0}\mathcal{A}^{\Lambda_1}_{r_0,4}(P_1,P_2,P_3)\right|$, $\left|\partial_{\Lambda_0}\partial_p^2\mathcal{A}^{\Lambda_1}_{r_0,2}(P_4)\right|$ and $\left|\partial_{\Lambda_0}\mathcal{A}^{\Lambda_1}_{r_0,2}(P_5)\right|$ do not exceed $\Lambda_0^{-2}\cdot Plog(\frac{\Lambda_0}{\Lambda_1})$, because $\partial_{\Lambda_0}a_r^R=\partial_{\Lambda_0}b_r^R=\partial_{\Lambda_0}c_r^R=0$. Equation (2.25) now provides the bounds for $\left|\partial_{\Lambda_0}\mathcal{A}^{\Lambda}_{r_0,4}(P_1,P_2,P_3)\right|$, $\left|\partial_{\Lambda_0}\partial_p^2\mathcal{A}^{\Lambda}_{r_0,2}(P_4)\right|$ and $\left|\partial_{\Lambda_0}\mathcal{A}^{\Lambda}_{r_0,2}(P_5)\right|$. A (repeated, if $n_0=2$) application of (3.3) together with $\|\partial_{\Lambda_0}\partial^1\mathcal{A}^{\Lambda}_{r_0,4}\|_{(2\Lambda,\eta)}$, $\|\partial_{\Lambda_0}\partial^3\mathcal{A}^{\Lambda}_{r_0,2}\|_{(2\Lambda,\eta)}$ and (2.12a) yields (3.6) for $n_0=4$, $w=0$ and $n_0=2$, $w=2$, now also for $n_0=2$, $w=1$ which in turn gives the bound (3.6) for $n_0=2$, $w=0$. ■

4. Further results

4.1. It is of some interest to know whether the b.c. (2.15) are stable in the sense that shifting them slightly would not affect the field theory in the limit $\Lambda_0\to\infty$.

In order to investigate this question we introduce two sets of functions, $\{\mathcal{L}^{(i)\Lambda}_{r,n} : i=1,2;\, n\geq 1,\, r\geq 0;\, \Lambda\in[\Lambda_1,\Lambda_0]\}$, each of which obeys the flow equation (2.6) in the interval $[\Lambda_1,\Lambda_0]$ with boundary conditions specified below. In particular, the boundary conditions will be regular enough to assert the existence and sufficient regularity of the functions $\mathcal{L}^{(i)\Lambda}_{r,n}$ and again it follows that both sets of functions satisfy the differential flow equation (2.14); moreover these b.c. will ensure that $\mathcal{L}^{(i)\Lambda}_{r,n}=0$ if n is larger than some finite $N(r,i)$ or if $r=0$.

The b.c. imposed on $\{\mathcal{L}^{(1)\Lambda}_{r,n}\}$ are as follows (cf. also (2.15), (2.18)):

$$
\begin{aligned}
\text{a)}\quad & \mathcal{L}^{(1)\Lambda_0}_{r,n}=0 \quad , \quad r=0 \text{ or } n=\text{odd or } n\geq 6\\
& \mathcal{L}^{(1)\Lambda_0}_{r,4}(p_1,p_2,p_3)=c_r^{(1)\Lambda_0} \qquad\qquad (4.1)\\
& \mathcal{L}^{(1)\Lambda_0}_{r,2}(p)=b_r^{(1)\Lambda_0}\cdot p^2+a_r^{(1)\Lambda_0} \quad ,
\end{aligned}
$$

where, for $r \geq 1$, $a_r^{(1)\Lambda_0}$, $b_r^{(1)\Lambda_0}$ and $c_r^{(1)\Lambda_0}$ are uniquely determined by

$$\begin{aligned} \text{b)}\ & \mathcal{L}_{r,4}^{(1)\Lambda_1}(P_1,P_2,P_3) = c_r \\ & \partial_{p_\mu}\partial_{p_\nu}\mathcal{L}_{r,2}^{(1)\Lambda_1}(P_4)\Big|_{\delta_{\mu,\nu}} = 2\delta_{\mu,\nu}\, b_r \\ & \mathcal{L}_{r,2}^{(1)\Lambda_1}(P_5) = a_r \quad . \end{aligned} \tag{4.2}$$

Here the parameters a_r, b_r and c_r are only supposed to be sufficiently well-behaved such that the following bounds hold:

$$\text{c)}\ \|\partial^w \mathcal{A}_{r,n}^{(1)\Lambda}\|_{(2\Lambda,\eta)} \leq \Lambda^{-w} \cdot Plog(\tfrac{\Lambda_0}{\Lambda_1}) \quad , \quad \eta \geq 0, w \geq 0 \quad . \tag{4.3}$$

The b.c. for $\{\mathcal{L}_{r,n}^{(2)\Lambda}\}$ are small variations of (4.1), (4.2):

$$\begin{aligned} \text{a)}\ & \mathcal{L}_{r,n}^{(2)\Lambda_0} = 0 \quad , \quad r = 0 \text{ or } n = \text{odd or } n \geq n'(r) \\ & \mathcal{L}_{r,n}^{(2)\Lambda_0}(p_1,\dots,p_{n-1}) = f_{r,n}^{\Lambda_0}(p_1,\dots,p_{n-1}) \quad , \quad n \geq 6 \\ & \mathcal{L}_{r,4}^{(2)\Lambda_0}(p_1,p_2,p_3) = c_r^{(2)\Lambda_0} + f_{r,4}^{\Lambda_0}(p_1,p_2,p_3) \\ & \mathcal{L}_{r,2}^{(2)\Lambda_0}(p) = b_r^{(2)\Lambda_0}\cdot p^2 + a_r^{(2)\Lambda_0} + f_{r,2}^{\Lambda_0}(p) \quad , \end{aligned} \tag{4.4}$$

where $1 \leq n'(r) < \infty$, and $\{f_{r,n}^{\Lambda_0}(p_1,\dots,p_{n-1})\}$ is a set of sufficiently smooth (e.g. C^∞), polynomially bounded, S_n-symmetric functions invariant under the orthogonal group with

$$\text{b)}\ \|\partial^w f_{r,n}^{\Lambda_0}\|_{(2\Lambda_0,\eta)} \leq \Lambda_0^{4-n-w} \cdot Plog(\tfrac{\Lambda_0}{\Lambda_1}) \quad , \quad \eta \geq 0, n+w \geq 5 \quad . \tag{4.5}$$

The "renormalization conditions" for $\mathcal{L}_{r,n}^{(2)\Lambda}$, $r \geq 1$, read:

$$\begin{aligned} \text{c)}\ & \mathcal{L}_{r,4}^{(2)\Lambda_1}(P_1,P_2,P_3) = c_r + C_r \\ & \partial_{p_\mu}\partial_{p_\nu}\mathcal{L}_{r,2}^{(2)\Lambda_1}(P_4)\Big|_{\delta_{\mu,\nu}} = 2\delta_{\mu,\nu}\,(b_r + B_r) \\ & \mathcal{L}_{r,2}^{(2)\Lambda_1}(P_5) = a_r + A_r \quad , \end{aligned} \tag{4.6}$$

assuming that

d) $|A_r| \leq \Lambda_0^{-1} \cdot \Lambda_1^3 \cdot Plog(\frac{\Lambda_0}{\Lambda_1})$

$$|B_r| \leq \Lambda_0^{-1} \cdot \Lambda_1 \cdot Plog(\tfrac{\Lambda_0}{\Lambda_1}) \tag{4.7}$$

$|C_r| \leq \Lambda_0^{-1} \cdot \Lambda_1 \cdot Plog(\frac{\Lambda_0}{\Lambda_1})$.

Proposition 6. *Under the hypotheses (4.1)-(4.7) we find that, for any fixed $\eta \geq 0$ and for $\Lambda \in [\Lambda_1, \Lambda_0]$,*

$$\|\partial^w (\mathcal{A}_{r,n}^{(1)\Lambda} - \mathcal{A}_{r,n}^{(2)\Lambda})\|_{(2\Lambda,\eta)} \leq \Lambda_0^{-1} \cdot \Lambda^{-w+1} \cdot Plog(\tfrac{\Lambda_0}{\Lambda_1}) \quad . \tag{4.8}$$

Proof: Define $\mathcal{D}_{r,n}^{\Lambda} := \mathcal{A}_{r,n}^{(1)\Lambda} - \mathcal{A}_{r,n}^{(2)\Lambda}$. Subtracting the differential equation (2.14) for $\mathcal{L}_{r,n}^{(1)\Lambda}$ from the one for $\mathcal{L}_{r,n}^{(2)\Lambda}$ and performing estimates one easily gets (in analogy to the inequality (2.23))

$$\begin{aligned}\|\partial_\Lambda (\Lambda^{4-n} \partial^w \mathcal{D}_{r,n}^{\Lambda})\|_{(2\Lambda,\eta)} \leq const \cdot \Lambda^{3-n} \cdot \Big(\|\partial^w \mathcal{D}_{r,n+2}^{\Lambda}\|_{(2\Lambda,\eta)} + \sum_{a=1}^{r-1} \sum_{b=2}^{n} \\ \sum_{w_1,w_2,w_3 :\, w_1+w_2+w_3=w} \Lambda^{-w_1} \cdot \|\partial^{w_2} \mathcal{D}_{a,b}^{\Lambda}\|_{(2\Lambda,\eta)} \\ \cdot \big\{ \|\partial^{w_3} \mathcal{D}_{r-a,n+2-b}^{\Lambda}\|_{(2\Lambda,\eta)} + \|\partial^{w_3} \mathcal{A}_{r-a,n+2-b}^{(1)\Lambda}\|_{(2\Lambda,\eta)} \big\} \Big) \quad .\end{aligned} \tag{4.9}$$

Because for each r there is a finite $N(r)$ such that $\mathcal{A}_{r,n}^{(1)\Lambda} = \mathcal{D}_{r,n}^{\Lambda} = 0$, for $n > N(r)$, the by now standard induction procedure may be employed.

$n_0 + w \geq 5$: Due to the obvious equality

$$\|(\Lambda^{4-n_0} \partial^w \mathcal{D}_{r_0,n_0}^{\Lambda} - \Lambda_0^{4-n_0} \partial^w \mathcal{D}_{r_0,n_0}^{\Lambda_0})\|_{(2\Lambda,\eta)} = \|\Big(\int_{\Lambda}^{\Lambda_0} ds\ \partial_s (s^{4-n_0} \partial^w \mathcal{D}_{r_0,n_0}^{s}) \Big)\|_{(2\Lambda,\eta)}$$

and because $\|\partial^w \mathcal{D}_{r,n}^{\Lambda_0}\|_{(2\Lambda,\eta)} \leq \|\partial^w \mathcal{D}_{r,n}^{\Lambda_0}\|_{(2\Lambda_0,\eta)}$ we find that

$$\|\Lambda^{4-n_0} \partial^w \mathcal{D}_{r_0,n_0}^{\Lambda}\|_{(2\Lambda,\eta)} \leq \|\Lambda_0^{4-n_0} \partial^w \mathcal{D}_{r_0,n_0}^{\Lambda_0}\|_{(2\Lambda_0,\eta)} + \int_{\Lambda}^{\Lambda_0} ds\ \|\partial_s (s^{4-n_0} \partial^w \mathcal{D}_{r_0,n_0}^{s})\|_{(2s,\eta)} \quad .$$

Use (4.1), (4.4) and (4.5) to bound $\partial_p^w \mathcal{D}_{r_0,n_0}^{\Lambda_0}$, and insert (4.3), (4.8) into (4.9) to verify (4.8) for $\|\partial^w \mathcal{D}_{r_0,n_0}^{\Lambda}\|_{(2\Lambda,\eta)}$.

$n_0 + w \leq 4$: If $n_0 = 4$ we plug (4.2), (4.6) and (4.3), (4.8), (4.9) into

$$\left|\mathcal{D}^{\Lambda}_{r_0,4}(P_1, P_2, P_3)\right| \leq \left|\mathcal{D}^{\Lambda_1}_{r_0,4}(P_1, P_2, P_3)\right| + \int_{\Lambda_1}^{\Lambda} ds\ \|\partial_s \mathcal{D}^{s}_{r_0,4}\|_{(2s,M)}$$

in order to check (4.8) with the help of (3.3). If $n_0 = 2$ one bounds successively $w = 2$, $w = 1$ and $w = 0$, just as in the proof of Proposition 4. ∎

4.2. The action principle [7] plays a prominent rôle in the derivation of some characteristic properties (such as the Callan-Symanzik- and renormalization group-equations) of renormalized Lagrangian quantum field theories. It would, therefore, be desirable to prove its validity also within our approach to renormalization. The following analysis of the behaviour of partial derivatives of the Euclidean Green's functions is one step in this direction.

Assume that the mass, m, as well as the renormalization constants a^R_r, b^R_r and c^R_r (cf. (2.17)) are C^k functions of parameters $\lambda_i \in \mathbb{R}$, $1 \leq i \leq N$; the regularizing function $K(\,\cdot\,)$ (cf. (2.2)) is supposed to be independent of the λ_i. It is not difficult to see that this means that also $\mathcal{L}^{\Lambda}_{r,n}$ is C^k, at least as long as $\Lambda_0 < \infty$. We will write (as usual) $\partial_{\lambda_i} := \frac{\partial}{\partial \lambda_i}$.

PROPOSITION 7. *Also* $\lim_{\Lambda_0 \to \infty} \mathcal{L}^{\Lambda}_{r,n}$ *is a* C^k *function of* $\lambda_1, \ldots, \lambda_N$.

PROOF: Apply ∂_{λ_i} on (2.14) and repeat the procedure which was employed to prove the validity of Propositions 4 and 5. Using $\|\partial^w \mathcal{A}^{\Lambda}_{r,n}\|_{(2\Lambda,\eta)} \leq \Lambda^{-w} \cdot Plog(\frac{\Lambda_0}{\Lambda_1})$ we find uniform convergence on compact subsets of $(\lambda_1, \ldots, \lambda_N)$-space:

$$\|\partial^w \partial_{\lambda_i} \mathcal{A}^{\Lambda}_{r,n}\|_{(2\Lambda,\eta)} \leq \Lambda^{-w} \cdot Plog(\tfrac{\Lambda_0}{\Lambda_1}) \tag{4.10}$$

$$\|\partial_{\Lambda_0} \partial^w \partial_{\lambda_i} \mathcal{A}^{\Lambda}_{r,n}\|_{(2\Lambda,\eta)} \leq \Lambda_0^{-2} \cdot \Lambda^{-w+1} \cdot Plog(\tfrac{\Lambda_0}{\Lambda_1}) \quad . \tag{4.11}$$

Instead of the sufficient bound (4.10) also a (3.1)-type bound could have been proved. ∎

4.3. We have found simple Ansätze (namely (3.1) and (3.6)) for the norms of $\partial^w_p \mathcal{A}^{\Lambda}_{r,n}$ resp. of $\partial_{\Lambda_0} \partial^w_p \mathcal{A}^{\Lambda}_{r,n}$ exhibiting the satisfactory property that they are preserved by the induction procedure and that they enable us to prove convergence as $\Lambda_0 \to \infty$. This might be a fortunate coincidence happening in $D = 4$ only, and if so the flow equation method would loose some of its undeniable attractivity. In order to help to clarify the situation we have investigated also the perturbative Φ_D^4 in $D = 2, 3$; the results confirm the hope that $D = 4$ is not an exceptional case. To be explicit, define $\mathcal{A}_{r,n}^{\Lambda} := \Lambda^{n(\frac{D}{2}-1)-D} \cdot \mathcal{L}_{r,n}^{\Lambda}$ and derive the analogues of (2.23)-(2.25) for $D = 2, 3$. The b.c. are imposed as follows, for $D = 2, 3$:

$$\begin{aligned} \underline{\Lambda = \Lambda_0} : \ & \mathcal{L}_{r,n}^{\Lambda_0} = 0 \quad , \quad n = \text{odd or } n \geq 6, \\ & \mathcal{L}_{r,4}^{\Lambda_0}(p_1, p_2, p_3) = c_r^{\Lambda_0}, \\ & \mathcal{L}_{r,2}^{\Lambda_0}(p) = a_r^{\Lambda_0} \quad . \end{aligned} \tag{4.12}$$

$$\begin{aligned} \underline{\Lambda = 0} : \ & \mathcal{L}_{r,4}^{0}(p_1 = P_1, p_2 = P_2, p_3 = P_3) = c_r^R, \\ & \mathcal{L}_{r,2}^{0}(p = P_5) = a_r^R \quad . \end{aligned} \tag{4.13}$$

It can be shown that the usual inductive proof works with the Ansätze described below.

Proposition 8. *For the Euclidean massive Φ_3^4 we find, for $\eta \geq 0$ and $\Lambda \in [\Lambda_1, \Lambda_0]$, that*

$$\|\partial^w \mathcal{A}_{r,n}^{\Lambda}\|_{(2\Lambda,\eta)} \leq \begin{cases} \Lambda^{-w} \cdot \left(\frac{\Lambda_1}{\Lambda}\right)^2 \cdot C & , \quad n \geq 6 \\ \Lambda^{-w} \cdot \left(\frac{\Lambda_1}{\Lambda}\right) \cdot C' & , \quad n \leq 4 \quad , \end{cases} \tag{4.14}$$

$$\|\partial_{\Lambda_0} \partial^w \mathcal{A}_{r,n}^{\Lambda}\|_{(2\Lambda,\eta)} \leq \begin{cases} \Lambda_0^{-2} \cdot \Lambda^{-w+1} \cdot \left(\frac{\Lambda_1}{\Lambda}\right)^2 \cdot C'' & , \quad n \geq 6 \\ \Lambda_0^{-2} \cdot \Lambda^{-w+1} \cdot \left(\frac{\Lambda_1}{\Lambda}\right) \cdot C''' & , \quad n \leq 4 \quad , \end{cases} \tag{4.15}$$

where C, C', C'' and C''' depend neither on Λ nor on Λ_0. In particular we see that $\lim_{\Lambda_0 \to \infty} \mathcal{L}_{r,4}^{\Lambda_0}$ is finite, for all r.

Proposition 9. *For the Euclidean massive Φ_2^4 one obtains, for $\eta \geq 0$ and $\Lambda \in [\Lambda_1, \Lambda_0]$,*

$$\|\partial^w \mathcal{A}_{r,n}^{\Lambda}\|_{(2\Lambda,\eta)} \leq \begin{cases} \Lambda^{-w} \cdot \left(\frac{\Lambda_1}{\Lambda}\right)^4 \cdot C & , \quad n \geq 6 \\ \Lambda^{-w} \cdot \left(\frac{\Lambda_1}{\Lambda}\right)^2 \cdot C' & , \quad n = 4 \\ \Lambda^{-w} \cdot \left(\frac{\Lambda_1}{\Lambda}\right)^2 \cdot \left(1 + \log\left(\frac{\Lambda}{\Lambda_1}\right)\right) \cdot C'' & , \quad n = 2 \quad , \end{cases} \tag{4.16}$$

$$\|\partial_{\Lambda_0}\partial^w \mathcal{A}^{\Lambda}_{r,n}\|_{(2\Lambda,\eta)} \le \begin{cases} \Lambda_0^{-2}\cdot\Lambda^{-w+1}\cdot\left(\frac{\Lambda_1}{\Lambda}\right)^4\cdot C''' & , \quad n\ge 6 \\ \Lambda_0^{-2}\cdot\Lambda^{-w+1}\cdot\left(\frac{\Lambda_1}{\Lambda}\right)^2\cdot C'''' & , \quad n=4 \\ \Lambda_0^{-2}\cdot\Lambda^{-w+1}\cdot\left(\frac{\Lambda_1}{\Lambda}\right)\cdot C''''' & , \quad n=2 \quad , \end{cases} \tag{4.17}$$

and $C, C', \ldots$ depend neither on Λ nor on Λ_0; again $\lim_{\Lambda_0\to\infty}\mathcal{L}^{\Lambda_0}_{r,4}$ is finite, for all r.

We think that it is quite remarkable that for $D = 2,3$ and 4 the bounds (4.16), (4.14) and (3.1) for $\mathcal{L}^{\Lambda_0}_{r,n}$ predicted by this relatively simple method actually agree with the true behaviour of $\mathcal{L}^{\Lambda_0}_{r,n}$ when $\Lambda_0\to\infty$.

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