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# Complete sets of compatible non selfadjoint observables

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**Abstract.** Compatibility for observables understood as symmetric but not selfadjoint operators is defined in terms of commutation of the isometry semigroups that they generate. Compatible observables are shown to admit a joint probability distribution. The completeness of a set of commuting observables is discussed by means of the generated von Neumann algebra.

## 1. Introduction

In the usual formalism of Quantum mechanics, the observables are represented by selfadjoint operators to which is univocally associated a projection-valued (PV) spectral decomposition. This one-to-one correspondence between spectral families and selfadjoint observables allows us to answer in rather simple way, not only to specific problems concerning the probability distribution of an observable  $A$  in a state  $\psi$ , i.e.  $\langle E(\lambda)\psi, \psi \rangle$  where  $E(\lambda)$  is the spectral family of  $A$ , but gives also a way to a fully satisfactory solution of the following two questions: (i) when are two observables  $A_1$  and  $A_2$  compatible? (ii) when does a set  $(A_1, \dots, A_n)$  of compatible observables give the maximal informations (in Dirac's sense) on the physical system?

As is known the answer to the first question is that the spectral families  $E_i(\lambda)$  of  $A_i$  ( $i = 1, 2$ ) commute for any  $\lambda$  (in this case the  $A_i$ 's are said to *commute strongly*).

The solution of the second problem, due to Jauch [1], makes use of the von Neumann algebra  $\mathfrak{N}$  generated by  $(A_1, \dots, A_n)$ . Since  $(A_1, \dots, A_n)$  are strongly commuting,  $\mathfrak{N}$  is abelian (i.e.,  $\mathfrak{N} \subseteq \mathfrak{N}'$ , the commutant of  $\mathfrak{N}$ ). Then  $(A_1, \dots, A_n)$  is *complete*, i.e. it gives the maximal amount of informations on the system, if  $\mathfrak{N} = \mathfrak{N}'$  [in this case  $\mathfrak{N}$  is said to be *maximal abelian* and the set  $(A_1, \dots, A_n)$  is said to be a *complete set of compatible observables* (CSCO)].

In [2], Antoine and two of us proved that a description of CSCO's directly in terms of unbounded operators can be given. This can be done by considering the  $SV^*$ -algebra generated by  $(A_1, \dots, A_n)$ . An  $SV^*$ -algebra is, in some sense, the unbounded analog of a von Neumann algebra [3]. What was proved in that paper is essentially that a family  $(A_1, \dots, A_n)$  of strongly commuting self-adjoint operators generates an abelian  $SV^*$ -algebra  $\mathcal{A}$  on an appropriate domain  $\mathcal{D}$  and the completeness of the set  $(A_1, \dots, A_n)$  is then equivalent to the "maximality" of  $\mathcal{A}$  and also to the existence of a cyclic vector in the domain  $\mathcal{D}$ .

Of course, nothing changes in Jauch's approach or, to some extent, in the formulation developed in [2], if we define an observable directly as its associated spectral measure, due to the one-to-one correspondence mentioned above; many authors, in fact, call *observable* either a self-adjoint operator or a PV-measure. Nevertheless the first choice appears to be much more intuitive, since for systems having a classical analogous, one can directly connect a classical observable to a selfadjoint operator.

This one-to-one correspondence between measures and observables is, however, lost in the recent approach to the quantum theory of measurement based on positive operator-valued (POV)-measures [Ref. 4–6 or 7, for a review].

The motivation for the choice of POV-measures lies in the fact that the quantum mechanical axiom of repeatability, which forces to represent observables as PV-measures, has been the object of a large criticism, supported by the existence of quantum mechanical experiments which are unrepeatable for their own nature [7].

In contrast with what happens for PV-measures, to a POV-measure, which is not PV, does not correspond necessarily an operator; but when it does, this operator is symmetric but not self-adjoint.

On the other hand, any symmetric operator gives rise, via Naimark's theorem [see, e.g. Ref. 8, appendix] to, at least, one POV-measure.

These facts make evident that to represent an observable as a symmetric operator is more restrictive than to represent it by a POV-measure.

In this paper we will ask ourselves once again the questions (i) and (ii) posed above in this different framework.

Moving in the spirit of that we called before an intuitive approach, we confine ourselves to consider observables to be represented by symmetric operators and to begin with we will consider only the case where they are maximal (a typical example of this kind is the momentum operator on the half-line). This choice presents two advantages: (a) a maximal symmetric operator admits a unique POV-measure and (b) a maximal symmetric operator is the generator of a one-parameter semigroup of isometries (which we will consider as generalized symmetries).

In Section 2 we discuss what we mean when we say that two observables of this kind are compatible. The main result is that the existence of a joint probability distribution for two maximal symmetric operators is equivalent to the fact that their generated semigroups of isometries commute. Several possible definitions of compatibility for, say, *generalized* observables have been discussed by one of us in [9]; however, we follow here a different approach.

In Section 3 we define the *completeness* of such a system of observables in terms of the generated von Neumann algebra. This latter is in this case non-abelian but it splits into two abelian parts (which are not von Neumann algebras) for which a concept of maximality can also be given.

## 2. Compatibility of non-selfadjoint observables

As discussed in the Introduction throughout this paper we will consider observables which are represented by a maximal symmetric (not necessarily

selfadjoint) operators. The first thing we have to do is to define the *compatibility* of two observables of this kind. In order to do this we need some mathematical preliminaries.

**Lemma 2.1.** *Let  $T(t)$ ,  $t \geq 0$ , be a strongly continuous one parameter semigroup of contractions in Hilbert space  $\mathfrak{h}$  and let  $A$  be its generator. Then  $T(t)$  consists of isometries, if, and only if,  $A = iS$  where  $S$  is a maximal symmetric operator.*

*Proof.* Assume that  $T(t)$  is an isometry for all  $t \geq 0$ . Let  $f \in D(A)$  and  $g \in D(A)$ , then we have

$$\begin{aligned} \langle Af, g \rangle &= \lim_{\varepsilon, 0} \frac{1}{\varepsilon} \langle (T(\varepsilon) - 1)f, g \rangle = \lim_{\varepsilon, 0} \frac{1}{\varepsilon} \langle (T(\varepsilon) - T^*(\varepsilon)T(\varepsilon))f, g \rangle \\ &= \lim_{\varepsilon, 0} \frac{1}{\varepsilon} \langle T(\varepsilon)f, (1 - T(\varepsilon))g \rangle = \langle f, -Ag \rangle. \end{aligned}$$

Therefore  $-A \subseteq A^*$  and thus  $A = iS$  for  $S \subseteq S^*$ . The maximality of  $S$  follows from the fact that the resolvent of  $S$  is non-empty, because of the Hille–Yosida theorem [Ref. 8, n. 143].

On the other hand, let  $S$  be maximal symmetric; then  $A = iS$  has non-empty resolvent and thus generates a semigroup of contractions  $T(t)$ . Let  $T = (A + 1)(A - 1)^{-1}$  be the cogenerator of  $T(t)$  [10, III.8]. As is easy to check  $T$  coincides with the Cayley transform of  $S$ , which is an isometric operator. Therefore  $T$  is an isometry and hence  $T(t)$  consists of isometries [10, Ch. III, Prop. 9.2].

**Proposition 2.2.** *For every strongly continuous semigroup  $V(t)$  of isometries (generated by the maximal symmetric operator  $S$ ) in Hilbert space  $\mathfrak{h}$ , there exists a unique subspace  $\mathfrak{h}_0 \subseteq \mathfrak{h}$  such that  $V_0(t) = V(t) \upharpoonright \mathfrak{h}_0$  induces a group of unitary operators; the generator  $S_0$  of  $V_0(t)$  is a selfadjoint restriction of  $S$ .*

*Proof.* As shown in [10], Proposition 8.3  $V(t)$  can be decomposed as  $V(t) = V_0(t) \oplus V_1(t)$ , where  $V_0(t)$  is a unitary semigroup and  $V_1(t)$  is completely non-unitary (in the sense that there exist no invariant subspaces where it acts as a unitary operator). Now, defining  $V_0(-t) = V_0^*(t)$ , we get a unitary group whose generator is, as is known, self-adjoint. If  $T_i$  ( $i = 0, 1$ ) are the cogenerators of  $V_i(t)$  we get  $T = T_0 \oplus T_1$  where  $T$  is the cogenerator of  $V(t)$ . From this it follows that  $S_0 \subseteq S$ .

**Remark.** If we call *generalized symmetries* those generated by isometries and *proper symmetries* those generated by unitaries, the previous proposition reads in the following way: each semigroup of generalized symmetries on the physical space  $\mathfrak{h}$  admits a subspace where they are proper symmetries. For a physical discussion of generalized symmetries, see [9].

As is known [8], a semigroup of isometries can be extended to a unitary group on a larger space as well as its generator can be extended to a selfadjoint operator

in a (in principle) different larger Hilbert space. The next Proposition shows that the spaces where these extensions live can be chosen to be the same.

Before going forth, let us recall that a symmetric operator (resp., a semigroup of contractions) admits a *minimal* extension to a selfadjoint operator (resp., to a unitary group) in a larger Hilbert space  $\mathfrak{H}$  and this minimal extension is unique (up to unitary equivalence) [8, 10].

**Proposition 2.3.** *Let  $T(t)$  be a semigroup of isometries in Hilbert space  $\mathfrak{h}$  and  $S$  the maximal symmetric operator such that  $A = iS$  generates  $T(t)$ . Let  $U(t)$  be the minimal unitary extension of  $T(t)$  to a larger Hilbert space  $\mathfrak{H} \supseteq \mathfrak{h}$  and  $B$  the selfadjoint operator in  $\mathfrak{H}$  which generates  $U(t)$ . Let  $S'$  be the minimal selfadjoint extension of  $S$  to a larger Hilbert space  $\mathfrak{H}$ . Then one can choose  $\mathfrak{H} = \mathfrak{H}'$  and  $U(t)$  is generated by  $S'$  (i.e.  $B = S'$ , up to unitary transformations).*

*Conversely, the unitary group  $V(t)$  generated by  $S'$  is a minimal extension of  $T(t)$  and  $V(t)$  and  $U(t)$  coincide (up to a unitary transformation).*

*Proof.* As is known [8]  $T(t)$  admits an extension  $U(t)$  to be a larger Hilbert space  $\mathfrak{H}$  with the property  $T(t) \subseteq U(t)$   $t \geq 0$ . Let  $B$  be the self-adjoint operator in  $\mathfrak{H}$  generating  $U(t)$  and  $S'$  the minimal self-adjoint extension of  $S$  to a larger Hilbert space  $\mathfrak{H}'$ . If  $f \in D(S) \subseteq \mathfrak{h}$  then

$$\lim_{\varepsilon, 0} \frac{1}{\varepsilon} (T(\varepsilon) - 1)f$$

exists in  $\mathfrak{H}$ ; but  $1/\varepsilon(T(\varepsilon) - 1)f = 1/\varepsilon(U(\varepsilon) - 1)f \forall f \in \mathfrak{h}$ . This implies that  $f \in D(B) \cap \mathfrak{h}$  and  $Sf = Bf \forall f \in D(S)$ . Therefore  $B$  is an extension of  $S$ . Let us now assume that  $B$  is not a minimal extension of  $S$  and let  $E(\lambda)$  be the spectral family of  $B$ . Then the linear span  $\mathfrak{D}_0$  of the set  $E(\lambda)\mathfrak{h}$ ,  $\lambda \in \mathbf{R}$  is not dense in  $\mathfrak{H}$ . Let  $\mathfrak{H}_0$  be its closure. From the definition itself  $B \upharpoonright \mathfrak{D}_0$  is essentially self-adjoint. Let us call  $B_0$  its closure. The unitary group  $U_0(t)$  generated by  $B_0$  coincides with the restriction of  $U(t)$  to  $\mathfrak{H}_0$ . Then we have  $U(t)\mathfrak{h} \subseteq U_0(t)\mathfrak{H}_0 = \mathfrak{H}_0$  and this contradicts the minimality of  $U(t)$ .

Conversely, let  $E'(\lambda)$  be the spectral resolution of  $S'$ , the minimal extension of  $S$  to the larger space  $\mathfrak{H}'$ , and  $V(t) = \int e^{i\lambda t} dE'(\lambda)$  the unitary group generated by  $S'$ . Then we have

$$\overline{\text{span} \{E'(\lambda)\mathfrak{h}, \lambda \in \mathbf{R}\}} = \overline{\text{span} \{V(t)\mathfrak{h}, t \geq 0\}} = \mathfrak{H}'$$

The first equality follows from the fact that the Stieltjes integral which gives  $V(t)f$  can be approximated by finite linear combinations of elements of the form  $E'(\lambda)f$ . Since minimal extensions are unique (up to unitary equivalence), the statement is completely proved.

**Remark.** If  $\mathfrak{h}$ ,  $\mathfrak{H}$ ,  $T(t)$ ,  $U(t)$ ,  $S$  and  $B$  are defined as above and  $T$  and  $U$  are, respectively, the cogenerators of  $T(t)$  and  $U(t)$  then from the previous proposition and from Eq. 9.6 Ch. III of [10] we get the equivalence of the following statements

- (i)  $U(t)$  is a minimal extension of  $T(t)$
- (ii)  $B$  is a minimal extension of  $S$



- (iii)  $U$  is a minimal extension of  $T$  (in the sense that the linear span of  $U^n \mathfrak{h}$ ,  $n \in \mathbf{N}$  is dense in  $\mathfrak{H}$ ).

We have now at our disposal enough information for discussing the compatibility of two generalized observables.

**Definition 2.4.** Two maximal symmetric operators are said to *commute strongly* if the isometry semigroups that they, respectively, generate commute.

There is another possibility of defining the compatibility of two maximal symmetric operators which has a more direct physical meaning. It is, in fact, natural to call compatible two observables which admit a joint probability distribution. The following definition, which is due to Davies [4], has this sense (we remark that it can be given also for observables defined as POV-measures)

**Definition 2.5.** Let  $S_1$  and  $S_2$  be two maximal symmetric operators and  $B_1$  and  $B_2$  their respective generalized spectral families.  $S_1$  and  $S_2$  are said to be *operationally compatible* if there exists a positive operator valued measure  $B$  on the Borel sets of the plane such that for every pair  $\Delta, \Delta'$  of Borel sets on the line one has

$$B_1(\Delta) = B(\Delta \times \mathbf{R})$$

$$B_2(\Delta') = B(\mathbf{R} \times \Delta')$$

**Proposition 2.6.** Let  $S_1$  and  $S_2$  be two operationally compatible maximal symmetric operators. Then  $S_1$  and  $S_2$  admit self-adjoint extension  $\mathcal{S}_1$  and  $\mathcal{S}_2$  in the same larger Hilbert space  $\mathfrak{H}$ . Moreover  $\mathcal{S}_1$  and  $\mathcal{S}_2$  commute strongly.

*Proof.* Let  $B(\Theta)$ ,  $\Theta \in \mathcal{B}(\mathbf{R}^2)$  (the family of Borel sets in the plane) be the positive operator valued measure of Definition 2. Then [8, App. 2] there exists in a larger Hilbert space  $\mathfrak{H}$  a family  $E(\Theta)$  of projection operators with the properties ( $P$  denotes the projection of  $\mathfrak{H}$  onto  $\mathfrak{h}$ )

$$(i) \quad B(\Theta) = PE(\Theta) \upharpoonright \mathfrak{h} \quad \forall \Theta \in \mathcal{B}(\mathbf{R}^2)$$

$$(ii) \quad E(\phi) = 0; \quad E(\mathbf{R}^2) = 1$$

$$(iii) \quad E(\Theta_1 \cap \Theta_2) = E(\Theta_1)E(\Theta_2) \quad \forall \Theta_1, \Theta_2 \in \mathcal{B}(\mathbf{R}^2)$$

$$(iv) \quad E(\Theta_1 \cup \Theta_2) = E(\Theta_1) + E(\Theta_2) \quad \forall \Theta_1, \Theta_2 \in \mathcal{B}(\mathbf{R}^2) \text{ with } \Theta_1 \cap \Theta_2 = \phi$$

Set  $E_1(\Delta) = E(\Delta \times \mathbf{R})$ ;  $E_2(\Delta') = E(\mathbf{R} \times \Delta')$  then we get  $B_1 = PE_1 \upharpoonright \mathfrak{h}$  and  $B_2 = PE_2 \upharpoonright \mathfrak{h}$ . Let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be the self-adjoint operators defined respectively by  $E_1$  and  $E_2$  on  $\mathfrak{H}$ . As is easy to see,  $\mathcal{S}_1$  and  $\mathcal{S}_2$  extend, respectively,  $S_1$  and  $S_2$ . By theorem 2.1 of [4],  $E_1$  and  $E_2$  commute, i.e.  $\mathcal{S}_1$  and  $\mathcal{S}_2$  commute strongly.

We will now exploit the following generalization of theorem IV of [8], Appendix.

**Proposition 2.7.** *If  $T(t_1, \dots, t_n)$ ,  $t_i \geq 0$  is a weakly continuous,  $n$ -parameter contraction semigroup in Hilbert space  $\mathfrak{h}$ , then there exists in a larger Hilbert space  $\mathfrak{H}$  a  $n$ -parameter unitary group  $U(t_1, \dots, t_n)$ ,  $t_i \in \mathbf{R}$  such that*

$$T(t_1, \dots, t_n) = PU(t_1, \dots, t_n)$$

where  $P$  is the projection of  $\mathfrak{H}$  onto  $\mathfrak{h}$ . If  $T(t_1, \dots, t_n)$  is, for every  $(t_1, \dots, t_n) \in \mathbf{R}^n$  an isometry then we get, in particular

$$T(t_1, \dots, t_n) \subseteq U(t_1, \dots, t_n) \upharpoonright \mathfrak{h}$$

*Proof.* The first part is proved in [10], Ch. I, Proposition 6.2. The second part can be shown exactly as in [8], Appendix, Remark 4.

**Proposition 2.8.** *Let  $S_1, S_2$  be strongly commuting maximal symmetric operators. Then there exist selfadjoint extensions  $A_1, A_2$  of, respectively,  $S_1$  and  $S_2$ , acting on the same larger Hilbert space  $\mathfrak{H}$  which commute strongly*

*Proof.* Let  $V_1(t), V_2(s)$  be the semigroups generated respectively by  $S_1, S_2$ . Set

$$V(t, s) = V_1(t)V_2(s)$$

Since  $V_1(t), V_2(s)$  commute, then  $V(t, s)$  is a semigroup of isometries; by proposition 6, there exists a unitary group  $U(t, s)$  in a larger Hilbert space  $\mathfrak{H}$  such that

$$V(t, s) \subseteq U(t, s)$$

Clearly,  $V_1(t) = U(t, 0) \upharpoonright \mathfrak{h}$  and  $V_2(s) = U(0, s) \upharpoonright \mathfrak{h}$ . Let  $A_1, A_2$  be the generators of  $U(t, 0)$  and  $U(0, s)$ , respectively. By Proposition 2.3,  $A_1, A_2$  extend  $S_1$  and  $S_2$ , respectively. By the construction,  $A_1, A_2$  act on the same Hilbert space and commute strongly.

We conclude now this Section with the following

**Proposition 2.9.** *Let  $S_1$  and  $S_2$  be maximal symmetric operators. Then the following statements are equivalent*

- (i)  $S_1$  and  $S_2$  commute strongly (i.e. the generated semigroups commute)
- (ii)  $S_1$  and  $S_2$  are operationally compatible.

*Proof.* (i)  $\Rightarrow$  (ii) From Proposition 2.8 there exist self-adjoint extensions  $A_1, A_2$  of, respectively,  $S_1, S_2$  acting in the same Hilbert space which commute strongly. Let  $E_1(\cdot), E_2(\cdot)$  be, respectively, the spectral families of  $A_1$  and  $A_2$ . Since  $E_1(\cdot)$  and  $E_2(\cdot)$  commute  $E(\Delta \times \Delta') = E_1(\Delta)E_2(\Delta')$  defines a spectral family on the Borel sets of the plane. Set

$$B(\Delta \times \Delta') = PE(\Delta \times \Delta')$$

then the marginal distributions defined by  $B(\Delta \times \Delta')$ ,  $B_1(\Delta) = PE(\Delta \times \mathbf{R})$  and  $B_2(\Delta') = PE(\mathbf{R} \times \Delta')$  are, as is easily seen, generalized spectral families of  $S_1, S_2$  respectively.

(ii)  $\Rightarrow$  (i) If  $S_1$  and  $S_2$  are operationally compatible they admit a joint generalized spectral family  $B(\Delta \times \Delta')$  whose marginal distributions exactly coincide with the generalized spectral families  $B_1(\cdot)$  and  $B_2(\cdot)$  of, respectively,  $S_1$  and  $S_2$ . Proposition 2.6 then implies that there exist self-adjoint extensions  $A_1, A_2$  of, respectively,  $S_1, S_2$  acting in the same Hilbert space which commute strongly. Let  $U(t, s) = \exp(itA_1) \exp(isA_2)$ ; then, by application of Proposition 2.3, for the semigroups of isometries  $V_1(t)$  and  $V_2(s)$  generated, respectively, by  $S_1, S_2$  we get  $V_1(t) \subseteq U(t, 0)$  and  $V_2(s) \subseteq U(0, s)$ . Therefore  $V_1(t)$  and  $V_2(s)$  commute.

**Remark.** Of course, the equivalence stated in the previous proposition does not imply that the generalized spectral families of two strongly commuting maximal symmetric operators commute.

### 3. Complete sets of compatible observables

After the discussion in the previous section, it is now clear what we mean when we say that two *observables are compatible*: either they admit a joint generalized spectral family or, equivalently, the generated semigroups of isometries commute. Now, following Jauch's approach [1], we will try to discuss the notion of *completeness* of a system of observables having a look at the von Neumann algebra generated by them.

In the usual approach the situation is quite clear: if two self-adjoint observables are compatible, the von Neumann algebra generated by them (i.e. generated by their commuting spectral families) is abelian and then the notion of completeness of a set of a self-adjoint observables can be discussed in terms of *maximality* of the generated von Neumann algebra.

For non self-adjoint observables the situation changes drastically. Let us first sketch what happens for the von Neumann algebra generated by one maximal symmetric operator. This question has been discussed in detail by one of us in [11].

If  $S$  is a maximal symmetric operator we can consider, at least, three objects: the von Neumann algebra  $\{B(\lambda), \lambda \in \mathbf{R}\}''$  generated by the unique generalized spectral family  $B(\cdot)$  of  $S$ ; the von Neumann algebra  $\{V(t), V^*(t), t \geq 0\}''$  generated by the contraction semigroup  $V(t)$  generated by  $S$ ; finally, the smallest von Neumann algebra to which  $S$  is affiliated, i.e.  $\{U, U^*, E(\lambda)\}''$  where  $U$  is the partial isometry appearing in the polar decomposition  $S = UH$  of  $S$  and  $E(\cdot)$  is the spectral family of the self-adjoint operator  $H$ . As shown in Ref. 11 the two latter von Neumann algebras are the same as well as in the case of a self-adjoint operator. So, we define  $\{V(t), V^*(t) \ t \geq 0\}''$  to be the *von Neumann algebra generated by  $S$* .

**Definition 3.1.** Let  $\mathfrak{M}$  be a von Neumann algebra; we say that  $\mathfrak{M}$  is *semi-abelian* if  $\mathfrak{M}$  contains two subalgebras  $\mathfrak{A}$  and  $\mathfrak{B}$  with the following properties

- (a)  $\mathfrak{A}$  and  $\mathfrak{B}$  are abelian
- (b)  $\mathfrak{A} \cup \mathfrak{B}$  generates  $\mathfrak{M}$
- (c)  $\mathfrak{A} = \mathfrak{B}^*$

We shall call  $\mathfrak{A}$  and  $\mathfrak{B}$  the two component algebras of  $\mathfrak{M}$ .



Notice that the above decomposition is not unique. In fact if  $U \in \mathfrak{M}$  is a unitary operator, also  $U\mathfrak{A}U^{-1}$  and  $U\mathfrak{B}U^{-1}$  are component algebras. We leave open the problem whether any two decompositions of  $\mathfrak{M}$  are unitarily equivalent.

Now if  $S$  is a maximal symmetric operator with generated isometry semigroup  $V(t)$ , the above discussion leads immediately to the following conclusion

**Proposition 3.2.** *The von Neumann algebra generated by a maximal symmetric operator  $S$  is semi-abelian.*

*Proof.* Let  $V(t)$ ,  $t \geq 0$ , be the isometry semigroup generated by  $S$ . Let  $\mathfrak{A}$  denote the algebra generated by  $V(t)$ ,  $t \geq 0$  and  $\mathfrak{B}$  the algebra generated by the adjoints  $V^*(t)$ ,  $t \geq 0$ . It is evident that both  $\mathfrak{A}$  and  $\mathfrak{B}$  are abelian algebras generating the whole von Neumann algebra  $\{V(t), V^*(t); t \geq 0\}''$ .

A completely analogous reasoning shows that if  $S_1$  and  $S_2$  are strongly commuting maximal symmetric operators, the von Neumann algebra generated by them, i.e. the von Neumann algebra generated by the isometry semigroups they, respectively, generate, is also semi-abelian. The contrary is, however, not true, in general: two maximal symmetric operators affiliated with a semi-abelian von Neumann algebra might be not compatible.

**Remark.** As a simple example of the situation described above, let us consider a physical system consisting of a particle constrained to move in the half-space  $x \geq 0$ ,  $-\infty < y, z < +\infty$ . In this case the Hilbert space of states is  $L^2(0, \infty) \times L^2(\mathbf{R}^2)$ . The component  $p_1$  of the momentum operator  $\mathbf{p}$  is then (represented by) a maximal symmetric, but not self-adjoint, operator whereas  $p_2$  and  $p_3$  are self-adjoint. The von Neumann algebra generated by  $\{V_i(t_i), V_i^*(t_i); i = 1, 2, 3 \ t_1 \geq 0, t_2, t_3 \in \mathbf{R}\}$ , where  $V_1(t_1)$ ,  $t_1 \geq 0$ , is the semigroup of isometries generated by  $p_1$  and  $V_2(t_2)$ ,  $V_3(t_3)$  are, respectively the unitary groups generated by  $p_2$  and  $p_3$ , is semi-abelian. Its components are

$$\mathfrak{A} = \text{alg} \{V_1(t_1), V_2(t_2), V_3(t_3) \ t_1 \geq 0, t_2, t_3 \in \mathbf{R}\}$$

$$\mathfrak{B} = \text{alg} \{V_1^*(t_1), V_2(t_2), V_3(t_3) \ t_1 \geq 0, t_2, t_3 \in \mathbf{R}\}$$

If  $\mathfrak{M}$  is a semi-abelian von Neumann algebra and  $\mathfrak{A}$  and  $\mathfrak{B}$  are its component algebras then  $\mathfrak{A} \subseteq \mathfrak{A}'$  (notice that  $\mathfrak{A}'$  is *not* a von Neumann algebra, but it is weakly-closed) and also for the weak-closure  $[\mathfrak{A}]_w$  we have  $[\mathfrak{A}]_w \subseteq \mathfrak{A}'$ . The same holds true, evidently, for  $\mathfrak{B}$ . These facts suggest the following

**Definition 3.3.** Let  $\mathfrak{M}$  be a semi-abelian von Neumann algebra and denote with  $\mathfrak{A}$  anyone of its component algebras. We say that  $\mathfrak{M}$  is a maximal semi-abelian von Neumann algebra if  $[\mathfrak{A}]_w = \mathfrak{A}'$ .

**Proposition 3.4.** *Let  $\mathfrak{M}$  be a maximal semi-abelian von Neumann algebra then  $\mathfrak{M}'$  is an abelian von Neumann algebra, and  $\mathfrak{M}$  is discrete and semi-finite.*

*Proof.* Let  $\mathfrak{A}$  be any component algebra of  $\mathfrak{M}$ . By assumption,  $\mathfrak{M}' \subseteq \mathfrak{A}' = [\mathfrak{A}]_w \subseteq \mathfrak{M}$ . Therefore  $\mathfrak{M}'$  is abelian. For the remaining part of the statement see, e.g., [12, Ch. I §8].

If  $\mathfrak{M}$  is the von Neumann algebra generated by a set of maximal symmetric operators,  $\mathfrak{M}$  cannot be finite, because, otherwise, these operators would be self-adjoint by Segal's theorem [13].

The next proposition shows that to call maximal a semi-abelian von Neumann algebra fulfilling the requirements of Definition 3.3 is reasonable.

**Proposition 3.5.** *A semi-abelian von Neumann algebra  $\mathfrak{M}$  is maximal if, and only if,  $\mathfrak{M}$  is not properly contained in any other semi-abelian von Neumann algebra.*

*Proof.* Let us assume that  $\mathfrak{M} \subseteq \mathfrak{N}$ , where  $\mathfrak{N}$  is a semi-abelian von Neumann algebra and let  $\mathfrak{A}$  and  $\mathfrak{B}$  be component algebras of  $\mathfrak{N}$ ; then  $\mathfrak{A} \cap \mathfrak{M}$  and  $\mathfrak{B} \cap \mathfrak{M}$  are components of  $\mathfrak{M}$ . Hence  $[\mathfrak{A} \cap \mathfrak{M}]_w = \mathfrak{A}'$  and  $[\mathfrak{B} \cap \mathfrak{M}]_w = \mathfrak{B}'$ . This implies that  $\mathfrak{M} = \mathfrak{N}$ .

Conversely, let us assume that  $\mathfrak{M}$  is not properly contained in any other semi-abelian von Neumann algebra and suppose that for one component algebra  $\mathfrak{A}$  of  $\mathfrak{M}$ , we have  $[\mathfrak{A}]_w \subset \mathfrak{A}'$  and let  $X$  be a non-normal element of  $\mathfrak{A}'$  not belonging to  $[\mathfrak{A}]_w$  (such elements do necessarily exist). Set  $\mathfrak{A}_1 = [\mathfrak{A}, X]_w$  be the weak closure of the algebra generated by  $\mathfrak{A}$  and  $X$ ,  $\mathfrak{B}_1$  be the set of its adjoints. Then these two algebras are the components of a semi-abelian von Neumann algebra  $\mathfrak{N}$  containing properly  $\mathfrak{M}$ . This is a contradiction.

**Definition 3.6.** Let  $\mathfrak{R} = \{S_1, \dots, S_n\}$  be a set of strongly commuting maximal symmetric operators. We say that  $\mathfrak{R}$  is a *complete set of commuting operators* (CSCO) if the semi-abelian von Neumann algebra generated by  $\mathfrak{R}$  is maximal.

**Remark.** It is known that, in a separable Hilbert space, maximal abelian von Neumann algebras are characterized by the existence of a cyclic vector. For maximal semi-abelian von Neumann algebras we can say that they admit a cyclic vector (this follows from the fact that if  $\mathfrak{M}$  is maximal semi-abelian, then the commutant  $\mathfrak{M}'$  is abelian, therefore it has a separating vector, which is cyclic for  $\mathfrak{M}$ ). The converse is, however, not true.

In [1] §6, Jauch proved that any CSCO, in the usual sense, described via a maximal abelian von Neumann algebra  $\mathfrak{M}$ , gives rise to a direct integral decomposition of the Hilbert space into *coherent* subspaces where the whole algebra  $\mathfrak{N}$  of observables of the system acts in *irreducible* way. This is the meaning that can be given to Dirac's statement that a CSCO provides the maximal possible amount of informations about the system.

Jauch's argument relies on the following two facts

- (a)  $\mathfrak{M}'$  is an abelian von Neumann algebra
- (b) the center  $\mathfrak{Z} = \mathfrak{N} \cap \mathfrak{N}'$  of  $\mathfrak{N}$  exactly coincides with  $\mathfrak{M}'$ .

An analogous statement holds in the case the CSCO consists of maximal symmetric operators.

In fact, let  $\mathfrak{R}$  denote the von Neumann algebra generated by all observables (supposed to be represented here by maximal symmetric operators) of the physical system and  $\mathfrak{R}$  denote a CSCO. Let  $\mathfrak{M}$  be the von Neumann algebra generated by  $\mathfrak{R}$ . By definition,  $\mathfrak{M}$  is a maximal semi-abelian von Neumann algebra; thus, as shown before,  $\mathfrak{M}'$  is abelian. This, in turn, implies that the center  $\mathfrak{Z} = \mathfrak{R} \cap \mathfrak{R}'$  of  $\mathfrak{R}$  coincides with  $\mathfrak{R}'$ .

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