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Chern-Simons Theory in the Axial Gauge: Manifold with Boundary[1]

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The relation of the three-dimensional topological Chern-Simons theory with a two-dimensional chiral theory of the Wess-Zumino-Witten type is analysed from the point of view of local quantum field theory. The theory is hereby defined in three-dimensional space-time with a two-dimensional plane boundary. An axial gauge fixing procedure, well adapted to this geometrical setup, is used. A regularization free procedure is followed.

1 Introduction

Topological field theories[2], in particular the three-dimensional Chern-Simons theory [2, 3] which we shall deal with, are known to be devoid of local observables in the space-time manifold in which they are defined. However, in the case of a manifold with boundary, local observables exist on the boundary [3] - [8]. For the three-dimensional Chern-Simons theory these observables are [3, 4] two-dimensional conserved chiral currents generating the Kac-Moody algebra [9] of the Wess-Zumino-Witten model [10]. This algebra of observables has been discussed and shown to exist in the Landau gauge by the authors of Ref. [11] from the point of view of (perturbative) local quantum field theory. They consider a plane boundary. In their approach the Green functions are defined in all of space-time, the properties of the boundary being described by (regularized) boundary terms in the action, as suggested by Symanzik [12]. They show how the chiral algebra follows from a Slavnov identity which takes into account the effects of the boundary. They conclude to the existence of an algebra of operators with a definite helicity on one side of the boundary, and of an algebra of operators of opposite helicity on the other side.

The aim of the present work is to derive the chiral algebra on the plane boundary in the spirit of Symanzik, too, but with two main differences with respect to the work of [11].

First we avoid the use of a boundary action and thus the problem of regularizing it.

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[2]See [1] for a review

Instead of this we choose to stay on a more general ground. The theory is specified by two requirements:

(a) Away of the boundary the field equations of the theory without boundary hold ("locality condition").

(b) Correlations between points separated by the boundary are suppressed ("decoupling condition").

We then show that these assumptions uniquely fix the form of the boundary contributions to the field equations, up to a discrete parity transformation.

The second difference with the approach of [11] is the use of an axial gauge instead of a covariant one. Since full Poincaré invariance is anyway lost due to the presence of the boundary, the choice of an axial gauge with axis normal to the boundary looks quite natural. Moreover it has the great advantage of allowing very explicit calculations and of making obvious the ultraviolet finiteness of the theory, just because Green functions are essentially made here of tree graphs only [13]. This latter point also makes evident the scale invariance of the theory, already proved in [14, 15] for the Landau gauge.

A bad aspect of the axial gauge is the occurrence of long distance ambiguities. Such ambiguities are related to the fact that this gauge is not a complete gauge fixing [16]. It turns out in our case that these ambiguities are solved by imposing a Ward identity expressing the invariance of the theory under the gauge transformations which preserve the axial gauge condition. The remarkable fact is that just this Ward identity, if restricted to the boundary, generates the chiral current algebra.

The plan of the paper is the following. After stating general facts on Chern-Simons theory in the axial gauge, we introduce a boundary in the form of a plane and compute the propagators obeying the decoupling condition. We write then the field equations and the Ward identities in presence of the boundary, in particular the Ward identity leading to the chiral current algebra. We finally show that the ambiguous solution of the field equations is made unique by the Ward identity. At the end we draw some conclusions.

2 Chern-Simons theory in the axial gauge: $\mathbb{R}^3$ space-time without boundary

Let us first describe the theory in unbounded $\mathbb{R}^3$ space-time. The classsical Chern-Simons action reads [2, 3]

$$\Sigma_{CS}(A_\mu) = -\frac{k}{4\pi}\mathrm{Tr}\int d^3x\,\varepsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right). \tag{2.1}$$

We use the matrix notation, valid for all the fields φ encountered through the paper:

$$\varphi = \varphi^a \theta_a \ , \quad [\theta_a, \theta_b] = i f_{abc} \theta_c \ , \quad \mathrm{Tr}\,(\theta_a \theta_b) = \delta_{ab} \ , \tag{2.2}$$

where the θ_a's are the generators of the gauge group which is supposed to be simple.

Since we are going later to discuss the theory in presence of the plane boundary $x^1 = 0$, it is convenient to use the coordinates

$$u = x^1 \ , \quad z = \frac{1}{\sqrt{2}}(x^0 + x^2) \ , \quad \bar{z} = \frac{1}{\sqrt{2}}(x^0 - x^2) \ . \tag{2.3}$$

In these coordinates the action reads

$$\Sigma_{\mathrm{CS}}(A_\mu) = -\mathrm{Tr} \int du d^2z \, \left(A \bar{\partial} A_u + \bar{A} \partial_u A + A_u \partial \bar{A} + g A_u [A, \bar{A}] \right) \ , \tag{2.4}$$

with

$$A = A_z \ , \quad \bar{A} = A_{\bar{z}} \ , \qquad \partial = \partial_z \ , \quad \bar{\partial} = \partial_{\bar{z}} \ , \tag{2.5}$$

and where we have rescaled the fields with a factor $g = \sqrt{2\pi/k}$, which plays the role of the coupling constant.

As usual, one has to fix the gauge. A gauge adapted to our problem is of the axial type: "$A_u = 0$". The axial gauge is implemented by adding to the gauge invariant action (2.4) the following gauge fixing terms:

$$\Sigma_{\mathrm{gf}}(A_u, d, c, b) = \mathrm{Tr} \int du d^2z \, \left(d A_u + b(\partial_u c + g[A_u, c]) \right) \ . \tag{2.6}$$

Matrix notation (2.2) is used. d is a Lagrange multiplyer field, c and b are the Faddeev-Popov ghost and antighost fields.

The total action

$$\Sigma(A_\mu, d, c, b) = \Sigma_{\mathrm{CS}}(A_\mu) + \Sigma_{\mathrm{gf}}(A_u, d, c, b) \tag{2.7}$$

is invariant under the *BRS transformations*

$$\begin{aligned} sA_\mu &= -\partial_\mu c - g[A_\mu, c] \ , & sb &= d \ , \\ sc &= g c^2 \ , & sd &= 0 \ . \end{aligned} \tag{2.8}$$

It is also left invariant by the discrete *parity transformation* $z \leftrightarrow \bar{z}$, $u \to -u$, under which the fields transform as:

$$\begin{aligned} A &\leftrightarrow \bar{A} \ , & A_u &\to -A_u \ , & & \\ d &\to -d \ , & b &\to -c \ , & c &\to b \ . \end{aligned} \tag{2.9}$$

Three-dimensional Poincaré invariance is broken by the gauge fixing, but the action remains invariant under the *two-dimensional Poincaré transformations* of the plane $\{z, \bar{z}\}$.

	z	$\bar{z}$	u	A	$\bar{A}$	A_u	d	b	c
Dimension	-1	-1	-1	1	1	1	2	1	1
Helicity	-1	1	0	1	-1	0	0	0	0

Table 1: Dimensions and helicities.

Two-dimensional Lorentz invariance in particular is equivalent to *helicity conservation.* Helicity assignments are given in Table 1 together with the dimensions.

The theory is obviously *scale invariant*, since no dimensionful parameter has been introduced.

One finally notices that the gauge fixed action is still invariant under the *residual gauge invariance*

$$\begin{aligned} \delta A_\mu &= -\partial_\mu \omega - g[A_\mu, \omega] \,, \\ \delta\varphi &= -[\varphi, \omega] \,, \qquad \forall \varphi \neq A_\mu \,, \\ & \omega = \omega(z, \bar{z}) \,, \end{aligned} \tag{2.10}$$

where ω depends on z and $\bar{z}$, but not on u. One should of course also fix this residual gauge invariance [16] in order to define the theory. We shall see in the next subsection how this can be achieved, in the case where space-time has a boundary.

The Green functions derived from the action (2.7) obey, in the tree aproximation, equations of motion which can be written in a functional way as:

$$\begin{aligned} \partial_u \bar{A} - g[J_d, \bar{A}] + \bar{\partial} J_d + \bar{J} &= 0 \,, \\ -\partial_u A + g[J_d, A] - \partial J_d + J &= 0 \,, \\ d - \partial \bar{A} + \bar{\partial} A - g[A, \bar{A}] - g\{b, c\} + J_u &= 0 \,, \\ A_u + J_d &= 0 \,, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} \partial_u b - g[J_d, b] - J_c &= 0 \,, \\ \partial_u c - g[J_d, c] - J_b &= 0 \,. \end{aligned} \tag{2.12}$$

We have used the notation

$$\varphi = \frac{\delta Z_c}{\delta J_\varphi} \,, \qquad \varphi = A, \bar{A}, A_u, d, b, c \,, \tag{2.13}$$

where $Z_c(\bar{J}, J, J_u, J_d, J_b, J_c)$ is the generating functional of the connected Green functions

$$\langle \varphi_1(x_1) \cdots \varphi_N(x_N) \rangle \,. \tag{2.14}$$

The arguments of Z_c are the sources of the fields A, $\bar{A}$, A_u, d, b and c, respectively. In the tree approximation, Z_c is the Legendre transform of the classical action (2.7):

$$Z_c(J_\varphi) = \Sigma(\varphi) + \mathrm{Tr} \int d^3x \sum_\varphi J_\varphi \varphi \,. \tag{2.15}$$

Due to the linearity of the ghost equations (2.12), the Slavnov identity expressing BRS invariance takes the simple form of a local Ward identity[3]:

$$g\sum_{\varphi}[J_\varphi,\varphi]-\partial\bar{J}-\bar{\partial}J-\partial_u J_u-\partial_u d=0\ . \tag{2.16}$$

3 Theory with a boundary: the free propagators

Let us now introduce as boundary the plane $\mathcal{B}$ of equation $u=0$. We will define the Green functions in the entire space-time $\mathbb{R}^3$. So we shall determine the dynamics of the model by requiring the following two conditions, which specify its behaviour outside of the boundary:

Decoupling condition: The connected Green functions (2.14) vanish if the points x_n do not lie all in the same of the two half-spaces delimited by the plane $\mathcal{B}$.

Locality condition: For points lying all in the same half-space – and none on the border – we require the validity of the field equations (2.11) obeyed by the Green functions of the theory without boundary, *i.e.* those derived from the gauge fixed action (2.7) of the preceding section.

These requirements have to be understood in the distributional sense, Green functions being assumed to be tempered distributions. For the moment we do not specify more precisely the behaviour in the situation where some points are exactly on the boundary: this ambiguity will have to be fixed later on.

The prescriptions above are in the spirit of Symanzik's approach [12]: but there, boundary effects are taken into account by adding surface terms to the action, and the resulting Feynman rules give rise to singularities – or ambiguities [18] – which have to be renormalized.

It follows from the decoupling condition that the connected functional Z_c decomposes in two parts:

$$Z_c(J_\varphi)=Z_+(J_\varphi)+Z_-(J_\varphi)\ , \tag{3.1}$$

where Z_+, resp. Z_-, generates the connected Green functions in the right, resp. left half-space.

Let us see how the free propagators

$$\Delta_{\varphi_1\varphi_2}(x_1,x_2)=\langle\varphi_1(x_1)\,\varphi_2(x_2)\rangle \tag{3.2}$$

can be computed, beginning with the simplest case of the ghost propagator Δ_{bc}. According to (2.12), with $g=0$, its equation of motion reads, for both points in the same half-space but not on the boundary:

$$\partial_u\Delta_{bc}(x,x')=-\delta^3(x-x')=-\delta(u-u')\delta^2(z-z')\ . \tag{3.3}$$

[3]It is for this reason that the axial gauge may be considered as a "ghost free gauge" [17].

The general solution, invariant under the parity transformation (2.9) and fulfilling the decoupling condition, is

$$\Delta_{bc}(x,x') = -\left[\theta_+(\theta(u-u')+\rho)+\theta_-(\theta(u-u')+\rho)\right]\delta^2(z-z')\ , \tag{3.4}$$

where $\theta_\pm = \theta(\pm u)\theta(\pm u')$, and $\theta(u)$ is the Heavyside step function. Let us remark that the "integration constant" is a priori an arbitrary function of z and z'. Our choice $\rho\delta^2(z-z')$ is dictated by two-dimensional Poincaré invariance, scale invariance and regularity. Scale invariance fixes the dimension. Regularity – *i.e.* the fact that Green functions are tempered distributions – excludes ill-defined expressions like $1/\left((z-z')(\bar{z}-\bar{z}')\right)$.

The presence of θ-functions in the Ansatz (3.4) implies the occurrence of $\delta(u)$-terms in the right-hand-side of Eq. (3.3) or in the free ghost functional equations (2.12). With the assumption of scale invariance and taking into account the fact that the contributions of the boundary must be localized on $\mathcal{B}$, the most general form of the modified free ghost equations reads:

$$\begin{aligned} \partial_u b - J_c &= \delta(u)(\mu_+ b_+(z)+\mu_- b_-(z))\ , \\ \partial_u c - J_b &= -\delta(u)(\mu_- c_+(z)+\mu_+ c_-(z))\ . \end{aligned} \tag{3.5}$$

The second equation is deduced from the first one by parity invariance. We have used the functional notation (2.13). Moreover $\varphi_\pm(z)$ means the insertion of the field $\varphi(x)$ on the right, respectively on the left of the boundary $\mathcal{B}$:

$$\varphi_\pm(z) = \lim_{u\to\pm 0}\frac{\delta Z_c}{\delta J_\varphi(x)}\ . \tag{3.6}$$

The right-hand-side of (3.5) is thus the effect of the boundary. A first constraint on the parameters $\mu_\pm$ is provided by requiring the compatibility of both equations (3.5). In order to see this let us make use of the decomposition (3.1) of the connected functional, which allows to write these equations separately for Z_+ and Z_-:

$$\begin{aligned} M^b Z_\pm &\equiv \left\{\partial_u\frac{\delta}{\delta J_b} - \delta(u)\mu_\pm\frac{\delta}{\delta J_b}\Big|_{u=0}\right\} Z_\pm = J_c\ , \\ M^c Z_\pm &\equiv \left\{\partial_u\frac{\delta}{\delta J_c} + \delta(u)\mu_\mp\frac{\delta}{\delta J_c}\Big|_{u=0}\right\} Z_\pm = J_b\ . \end{aligned} \tag{3.7}$$

From the anticommutation relations

$$\left\{M^b(x), M^c(x')\right\} = 0\ , \tag{3.8}$$

applied to Z_+ or to Z_-, follows the condition

$$\mu_+ = \mu_- \equiv \mu\ . \tag{3.9}$$

The most general ghost propagator being of the form (3.4), it will obey the general ghost equation (3.5) provided the following constraints on the parameters ρ and μ hold:

$$\begin{aligned} \rho(\mu-1) &= 0\ , \\ (1+\rho)(\mu+1) &= 0\ . \end{aligned} \tag{3.10}$$

These constraints admit two distinct solutions:

$$\begin{aligned} &I: && \rho = -1\ , && \mu = 1\ , \\ &II: && \rho = 0\ , && \mu = -1\ . \end{aligned} \tag{3.11}$$

Since they are related by parity, it is sufficient to analyse the first one. The value -1 for ρ implies that

$$\begin{aligned} \langle b_-(z)\, c(x')\rangle &\equiv \lim_{u\to -0} \Delta_{bc}(x,x') = 0\ , \\ \langle b(x)\, c_+(z')\rangle &\equiv \lim_{u'\to +0} \Delta_{bc}(x,x') = 0\ . \end{aligned} \tag{3.12}$$

This result is interpreted as Dirichlet boundary conditions for the ghosts: b goes to zero as it tends to the boundary $\mathcal{B}$ from the left, whereas c goes to zero when it reaches $\mathcal{B}$ from the right.

The computation of the free propagators of the fields A, $\bar{A}$, A_u and d goes along the same lines. For points outside of the boundary one has to solve the system of equations (2.11) with $g = 0$. The general solution obeying the decoupling condition as well as all the dimension, symmetry and regularity constraints reads, in matrix form

$$\Delta(x,x') = \theta_+ \Delta_+(x,x') + \theta_- \Delta_-(x,x')\ , \tag{3.13}$$

with $\theta_\pm = \theta(\pm u)\theta(\pm u')$, and

$$\Delta_+(x,x') = \begin{pmatrix} \dfrac{\alpha}{2\pi i(z-z')^2} & -T_\gamma(x',x) & 0 & \partial T_{\gamma-\alpha}(x',x) \\ -T_\gamma(x,x') & \dfrac{\beta}{2\pi i(\bar{z}-\bar{z}')^2} & 0 & -\bar{\partial} T_{\gamma-\beta}(x,x') \\ 0 & 0 & 0 & -\delta^3(x-x') \\ -\partial T_{\gamma-\alpha}(x,x') & \bar{\partial} T_{\gamma-\beta}(x',x) & -\delta^3(x-x') & (-1-2\gamma+\alpha+\beta)\partial\bar{\partial}\delta^2 \end{pmatrix} \tag{3.14}$$

Δ_- is deduced from Δ_+ by parity. In (3.14) lines and columns are ordered according to the sequence $(A, \bar{A}, A_u, d)$. We have set

$$T_\xi(x,x') = (\theta(u-u') + \xi)\delta^2(z-z')\ . \tag{3.15}$$

This solution again is defined up to "integration constants", namely the functions of z and $\bar{z}$ parametrized by the constants α, β and γ. The expression $1/(z-z')^2$ is defined as a tempered distribution by:

$$\frac{1}{(z-z')^2} \equiv -\partial \frac{1}{z-z'} \equiv -\partial \lim_{\epsilon\to 0} \frac{1}{z-z'-i\epsilon(\bar{z}-\bar{z}')} \tag{3.16}$$

The two-dimensional "fermion propagator" $1/(z-z')$ obeys the equation

$$\bar{\partial}\frac{1}{z-z'} = 2\pi i \delta^2(z-z')\ . \tag{3.17}$$

The propagators (3.13) are the Green functions of free field equations of the type (2.11) (with $g = 0$), but now with surface terms. As in the case of the ghost propagator, let us write the most general form for the latter – with the usual requirements of dimension and symmetry:

$$\begin{aligned} \partial_u \bar{A} + \bar{\delta} J_d + \bar{J} &= -\delta(u)(\lambda_- \bar{A}_+ + \lambda_+ \bar{A}_-) \\ -\partial_u A - \partial J_d + J &= -\delta(u)(\lambda_+ A_+ + \lambda_- A_-) \\ d - \partial \bar{A} + \bar{\partial} A + J_u &= 0 \\ A_u + J_d &= 0 \end{aligned} \tag{3.18}$$

Like in the case of the ghost equation, compatibility of the first two equations imposes the restrictions

$$\lambda_+ = \lambda_- \equiv \lambda \,. \tag{3.19}$$

The general solution is given by (3.13), (3.14), with constraints analogous to the one encountered for the ghost propagator:

$$\begin{aligned} (1+\gamma)(\lambda - 1) &= 0 \,, & \alpha(\lambda - 1) &= 0 \,, \\ \gamma(\lambda + 1) &= 0 \,, & \beta(\lambda + 1) &= 0 \,. \end{aligned} \tag{3.20}$$

These constraints also admit two distinct solutions:

$$\begin{aligned} &I: && \gamma = 0 \,, && \alpha \text{ arbitrary} \,, && \beta = 0 \,, && \lambda = 1 \,, \\ &II: && \gamma = -1 \,, && \alpha = 0 \,, && \beta \text{ arbitrary} \,, && \lambda = -1 \,, \end{aligned} \tag{3.21}$$

related to each other by parity. The solution we shall retain for the rest of this work is solution I, for which the propagators (see (3.13)) read

$$\Delta_+(x, x') = \begin{pmatrix} \dfrac{\alpha}{2\pi i (z - z')^2} & -\theta(u' - u)\delta^2 & 0 & (\theta(u' - u) - \alpha)\partial\delta^2 \\ -\theta(u - u')\delta^2 & 0 & 0 & -\theta(u - u')\bar{\partial}\delta^2 \\ 0 & 0 & 0 & -\delta^3(x - x') \\ (\alpha - \theta(u - u'))\partial\delta^2 & \theta(u' - u)\bar{\partial}\delta^2 & -\delta^3(x - x') & (\alpha - 1)\partial\bar{\partial}\delta^2 \end{pmatrix} \tag{3.22}$$

where $\delta^2 = \delta^2(z - z')$. Notice that we are still left with the arbirary parameter α. One sees that this solution corresponds to the Dirichlet conditions:

$$\bar{A}_+ = \lim_{u \to +0} \bar{A} = 0 \,, \qquad A_- = \lim_{u \to -0} A = 0 \,. \tag{3.23}$$

4 The field equations and the Ward identity

Keeping the solution I from (3.21) we can write the field equations (3.18), now with interactions, as:

$$\begin{aligned} \partial_u \bar{A} - g[J_d, \bar{A}] + \bar{\delta} J_d + \bar{J} &= -\delta(u)\bar{A}_- \\ -\partial_u A + g[J_d, A] - \partial J_d + J &= -\delta(u) A_+ \\ d - \partial \bar{A} + \bar{\delta} A - g[A, \bar{A}] - g\{b, c\} + J_u &= 0 \\ A_u + J_d &= 0 \end{aligned} \tag{4.1}$$

We have not written the terms in $\bar{A}_+$ and A_- in the right-hand-sides of the first two equations. They are are indeed irrelevant due to the Dirichlet conditions (3.23).

The Slavnov identity (2.16) is expected to hold, but with a nonvanishing right-hand-side due to the boundary:

$$g \sum_{\varphi} [J_\varphi, \varphi] - \partial \bar{J} - \bar{\delta} J - \partial_u J_u - \partial_u d = \delta(u)(\bar{\delta} A_+ + \partial \bar{A}_-) \; . \tag{4.2}$$

Integrating on u and naively neglecting boudary terms at infinity would yield a Ward identity expressing the invariance of the theory under the residual gauge transformations (2.10):

$$\int_{-\infty}^{+\infty} du \left(g \sum_{\varphi} [J_\varphi, \varphi](x) - \partial \bar{J}(x) - \bar{\delta} J(x) \right) = \bar{\delta} A_+(z) + \partial \bar{A}_-(z) \; . \tag{4.3}$$

The identities (4.2) and (4.3) can be formally derived from the field equations (4.1). However, in the course of doing it, one meets three problems:

- One has to multiply field insertions, hence distributions, at the same point. This is the usual ultraviolet problem [18].
- The $\delta(u)$ terms of the field equations have to be multiplied by u-dependent field insertions, *i.e.* by u-dependent distributions. This effect of the boundary may give rise to new short distance singularities which would affect the right-hand-sides of (4.2) and (4.3).
- When integrating the local identity (4.2) in order to get (4.3), the neglect of terms at $u = \pm\infty$, generated by the u-integration of $\partial_u d$, cannot be done without special care because of the bad long distance behaviour in u of the Green functions, shown for instance in the free propagators (3.22). This difficulty is peculiar to the choice of an axial gauge.

The first short distance problem will be discussed in the next Section and will turn out to be absent, as one can expect from a theory which was shown [14, 15] to be ultraviolet finite[4]. The second and third problems are new and a priori more serious.

[4]The proof was given for the case of the Landau gauge.

We shall nevertheless postulate the validity of the Ward identity (4.3), with its right-hand-side as it stands. We shall see that this is possible, and that this completly fixes the parameters of the theory. As a preliminary exercise, one can apply it to the 2-point Green functions given by (3.13) and (3.22). The result is that it fixes the parameter α of the gauge field propagators:

$$\alpha = 1 \tag{4.4}$$

We shall see in the next section that the Ward identity (4.3) fixes as well all the arbitrary parameters appearing in the higher order Green functions when solving the equations of motions.

But let us for the time being interpret the Ward identity (4.3). Defining the functional operators

$$X^{\pm}_{a_1\cdots a_N}(z_1,\cdots,z_N) \equiv \prod_{k=1}^{N} \lim_{u_k\to\pm 0} \frac{\delta}{\delta \bar{J}^{a_k}(x_k)} \tag{4.5}$$

(where the a_k's are the group indices), applying *e.g.* X^+ to the Ward identity and then setting to zero all sources J_φ, we get the identities

$$\begin{aligned}
&\bar\delta\langle A_a(z)\, A_{a_1}(z_1)\cdots A_{a_N}(z_N)\rangle_+ = \\
&\quad ig\textstyle\sum_{k=1}^{N} f_{aa_kb}\delta^2(z-z_k)\Big\langle A_b(z)\, A_{a_1}(z_1)\cdots \widehat{A_{a_k}}(z_k)\cdots A_{a_N}(z_N)\Big\rangle_+ \\
&\quad -\delta_{1N}\delta_{aa_1}\partial\delta^2(z-z')
\end{aligned} \tag{4.6}$$

As usual the z's stand here for the z's and the $\bar z$'s. The hat on an argument of the Green function means omission of this argument. The suffix + means that the limit $u \to +0$ has being taken for all arguments.

Such Ward identities are known [19, 20, 11] to imply the conservation law

$$\bar\delta K_a(z) = 0\ , \tag{4.7}$$

and the chiral current algebra

$$[K_a(z), K_b(z')] = if_{abc}\delta(z-z')K_c(z) + \frac{k}{2\pi}\delta_{ab}\delta'(z-z')\ , \tag{4.8}$$

for the two-dimensional space-time operators

$$K_a(z) \equiv \sqrt{\frac{2\pi}{k}} \lim_{u\to+0} A(x)\ , \tag{4.9}$$

which live on the positive side of the boundary $\mathcal{B}$. One recognizes [19, 20, 11] a Kac-Moody algebra of level k. The parity conjugate operators

$$\bar K_a(z) = \sqrt{\frac{2\pi}{k}} \lim_{u\to-0} \bar A(x)\ , \tag{4.10}$$

living on the negative side of $\mathcal{B}$ of course obey the same algebra, z and $\bar z$ being interchanged.

5 General solution of the field equations and of the Ward identity

As announced we shall limit the discussion to the parametrization I of (3.21). Moreover, taking advantage of the decoupling condition and of the invariance under parity (2.9), we shall need only to consider in this section Green functions with arguments on the positive side $u \geq 0$ of the boundary $\mathcal{B}$.

Let us rewrite the field equations (4.1) in the following way (for $u \geq 0$; see (3.1) for the definition of Z_+):

$$M^{\bar{A}} Z_+ \equiv \left\{ \partial_u \frac{\delta}{\delta J} - g \left[J_d, \frac{\delta}{\delta J} \right] \right\} Z_+ = -\bar{\partial} J_d - \bar{J} \ , \tag{5.1}$$

$$M^{A} Z_+ \equiv \left\{ -\partial_u \frac{\delta}{\delta \bar{J}} + g \left[J_d, \frac{\delta}{\delta \bar{J}} \right] + \delta(u) \left(\frac{\delta}{\delta \bar{J}} \right)_+ \right\} Z_+ = \partial J_d - J \ , \tag{5.2}$$

$$M^{d} Z_+ \equiv \left\{ \frac{\delta}{\delta J_d} + \bar{\partial} \frac{\delta}{\delta \bar{J}} - \partial \frac{\delta}{\delta J} \right\} Z_+ - g \left[\frac{\delta Z_+}{\delta \bar{J}}, \frac{\delta Z_+}{\delta J} \right] = -J_u \ , \tag{5.3}$$

$$M^{A_u} Z_+ \equiv \frac{\delta}{\delta J_u} Z_+ = -J_d \ , \tag{5.4}$$

and similarly for the Ward identity (4.3):

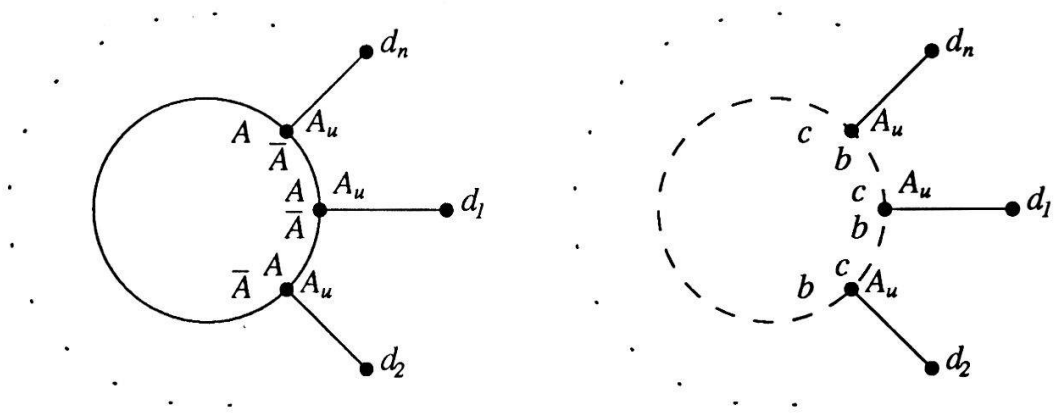
$$W(z) Z_+ \equiv \left\{ \sum_\varphi \int_{-\infty}^{+\infty} du \, g \left[J_\varphi, \frac{\delta}{\delta J_\varphi} \right] - \bar{\partial} \left(\frac{\delta}{\delta \bar{J}} \right)_+ \right\} Z_+ = \int_{-\infty}^{+\infty} du \, \left(\partial \bar{J} + \bar{\partial} J \right) \ . \tag{5.5}$$

The field equation (5.4) is trivial and we have already substituted it into the other field equations.

Eq. (5.3) contains the a-priori ill defined product of two field insertions A and $\bar{A}$ at the same space-time point. This field equation actually contains nonwritten terms of the form $\delta^2 Z_+ / \delta J \delta \bar{J}$ and $\delta^2 Z_+ / \delta J_b \delta J_c$, which generate loop graphs. But the only loop graphs of the theory are the one-loop graphs of Fig. 1 which contribute to the connected Green functions of the field d. All these graphs are ultraviolet divergent[5]. However each contribution factorizes into a regular u-dependent part and a singular product of z-plane Dirac distributions. The latter ultraviolet singularity can be subtracted in a usual way. The former factor is a product $\theta(u_1 - u_2)\theta(u_2 - u_3) \cdots \theta(u_n - u_1)$ which is equal to zero. Thus the contribution from each graph vanishes. This shows the absence of any loop correction and justifies the neglect of the nonwritten term in (5.3)[6].

[5] Irrespective of the number of external legs: this is a pathology of the axial gauge.

[6] This remark also makes obvious that one can practically forget the Faddeev-Popov ghosts.



(a) (b)

Figure 1: Contributions to the Green functions of the field d. (a) Loop of gauge field propagators. (b) Loop of ghost propagators.

On the other hand the two field equations (5.1) and (5.2) appear as linear equations. They can be solved, yielding connected generating functionals $\bar{A}$ and A, respectively – in the notation (2.13). These two functionals may then be inserted into the field equation (5.3), which in turn allows to compute d explicitly.

After having solved the field equations – possibly up to free parameters – the remaining task would be to check that the solutions found obey the Ward identity (5.5) at least for some value(s) of the free parameters. This program has been accomplished in the last section for the two-point functions: the general solution of the field equations has been given by (3.22), with one free parameter α; the Ward identity has fixed α to the value 1.

In order to solve the problem for the higher-point functions, we begin by looking at the field equation (5.1). In terms of Green functions this equation reads[7]

$$\begin{aligned} \partial_{u'} \left\langle \bar{A}_b(x')\, X \right\rangle &= i \textstyle\sum_{k=1}^{q} f_{bc_k e} \delta^3(x' - x_k'') \left\langle \bar{A}_e(x_k'')\, X \setminus d_{c_k}(x_k'') \right\rangle \\ &\quad - \delta_{X, d_c(x'')} \delta_{bc} \bar{\partial} \delta^3(x' - x'') - \delta_{X, A_a(x)} \delta_{ba} \delta^{(}x' - x) \,, \end{aligned} \tag{5.6}$$

where

$$X = A_{a_1}(x_1) \ldots A_{a_n}(x_n) \bar{A}_{b_1}(x_1') \ldots \bar{A}_{b_m}(x_m') d_{c_1}(x_1'') \ldots d_{c_q}(x_q'') \,, \tag{5.7}$$

and where $X \setminus \varphi$ means that φ is taken away from the string X, the result being zero if $\varphi \notin X$. Eq. (5.6) yields a recursion over the number of d's, whose starting point is given by the nonvanishing two-point functions $\left\langle \bar{A}\, A \right\rangle$ and $\left\langle \bar{A}\, d \right\rangle$.

Eq. (5.6) can be integrated explicitly. For $n + m + q \geq 2$ the result is:

$$\begin{aligned} \left\langle \bar{A}_b(x')\, X \right\rangle &= i \textstyle\sum_{k=1}^{q} f_{bc_k e} \delta^2(z' - z_k'') \theta(u' - u_k'') \theta(u') \left\langle \bar{A}_e(x_k'')\, X \setminus d_{c_k}(x_k'') \right\rangle \\ &= -i \textstyle\sum_{k=1}^{q} f_{bc_k e} \left\langle \bar{A}(x')\, A(x'') \right\rangle \left\langle \bar{A}_e(x_k'')\, X \setminus d_{c_k}(x_k'') \right\rangle \,. \end{aligned} \tag{5.8}$$

[7]Due to the decoupling condition, and as we consider the positive side of the boundary, a factor $\theta(u)$ for each u-variable is understood in front of every contribution to the Green functions.

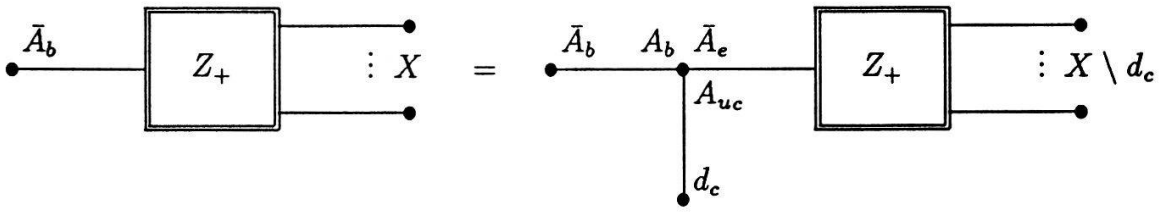


Figure 2: Diagrammatic representation of Eq. (5.8)

Indeed $\theta(u'-u'')\theta(u')$ is a primitive of $\delta(u'-u'')$, the positivity of the u-variables being taken into account, and this primitive is nothing else than minus the propagator $\langle \bar{A}\, A \rangle$ (see (3.22)). In principle (5.8) is defined up to an arbitrary u'-independent quantity. But such a term would violate the decoupling condition.

Taking into account the form of the propagator $\langle A_u\, d \rangle$ given in (3.22), we can represent (5.8) diagrammatically, as shown in Fig. 2. The vertex $A\bar{A}A_u$ is just the one given by the Chern-Simons action (2.4). We conclude that the Green functions containing at least one field $\bar{A}$ are made of tree graphs with, as Feynman rules, the propagators (3.22) and the Chern-Simons vertex. The nonvanishing ones have only one field $\bar{A}$ and at most one field A:

$$\left\langle \bar{A}\,(A)^n\,(\bar{A})^m\,(d)^q \right\rangle = 0 \quad \text{unless } n = 0 \text{ or } 1, \text{ and } m = 0\ . \tag{5.9}$$

The fulfillment of the Ward identity (5.5) by these Green functions follows simply from its fulfillment by the propagators.

We now turn to the Green functions without the field $\bar{A}$. We shall see that these are not simply given by the Feynman rules which one may derive from the action and the propagators (3.22). In particular arbitrary parameters appear when solving (5.2), which ought to be fixed by the Ward identity (4.3). But, instead of trying to solve the field equation (5.2) in full generality and then to show that one can fix the arbitrary parameters of the solution with the help of the Ward identity[8], we prefer to use first the Ward identity in order to define the field A on the border $\mathcal{B}$, and then to use the result as an initial condition for determining the solution of the field equation. The procedure hence begins by solving the Ward identity for A_+. The result reads, in functional form (recall the definitions (2.13) and (3.6)):

$$A_+(z) = \frac{g}{2\pi i z} * \int_{-\infty}^{+\infty} du \sum_{\varphi} [J_\varphi, \varphi](u,z) \equiv K Z_+\ , \tag{5.10}$$

where $1/z$ is the two-dimensional propagator defined in (3.16) and $*$ denotes the convolution product in the $(z,\bar{z})$-plane. Linear source terms are omitted here since we are interested in the Green functions of more than two fields.

[8]We have actually computed the three-point functions in this way: the solution of the field equation involves one free papameter, which is indeed fixed by the Ward identity. However the generalization to higher Green functions looks intricate.

The procedure then is iterative. Let us suppose that we know all n-point Green functions $\langle \varphi_1 \dots \varphi_n \rangle$ which solve the field equations and the Ward identity, for $n \leq N$. Eq. (5.10) gives the Green functions $\langle A(x)\varphi_1 \dots \varphi_N \rangle$ in the limit $u = +0$. The field equation (5.2) determines them for $u > 0$. The remaining Green functions $\langle d(x)\varphi_1 \dots \varphi_N \rangle$ are found with the field equation (5.3). But, since Green functions involving both A and d fields are clearly computed twice in the course of this procedure, there is a problem of compatibility. More precisely we have to show that the field equations (5.2) and (5.3), assumed to be valid for the N-point functions, are still valid after the insertion of A at $u = +0$ through (5.10). This follows indeed from the commutation relations

$$\begin{aligned} \left[K_a(z), M_b^A(x')\right] Z_+ &= igf_{abc}\delta^2(z-z')\frac{1}{2\pi i z'} * M_c^A(x')Z_+ \,, \\ \left(K_a(z)M_b^d(x') - M_b^{d(Z_+)}(x')K_a(z)\right) Z_+ &= igf_{abc}\delta^2(z-z')\frac{1}{2\pi i z'} * M_c^d(x')Z_+ \,, \end{aligned} \tag{5.11}$$

where we have introduced the Z_+-dependent linearized operator

$$M^{d(Z_+)}(x) \equiv \frac{\delta}{\delta J_d} + \bar{\partial}\frac{\delta}{\delta \bar{J}} - \partial\frac{\delta}{\delta J} - g\left[\frac{\delta Z_+}{\delta \bar{J}}, \frac{\delta}{\delta J}\right] - g\left[\frac{\delta Z_+}{\delta J}, \frac{\delta}{\delta \bar{J}}\right] . \tag{5.12}$$

6 Conclusions

We have thus shown the existence of a unique solution of the field equations (4.1) satisfying – by construction – the Ward identity (4.3) associated with the residual gauge invariance. More precisely, we have shown that this Ward identity is compatible with the field equations and that it characterizes uniquely their solution. It is this solution which leads to the chiral algebra (4.8) on the boundary $\mathcal{B}$. More precisely it leads to such an algebra on each side of the boundary. The current operators involved on one side have their helicity opposite to that of the operators on the other side.

The Green functions of the theory turn out to be free of any radiative corrections, and hence are ultraviolet finite as expected. On the other hand, the infrared problem linked to the axial gauge and which manifests itself by the occurence of long-distance ambiguities – see *e.g.* the parameters α, β and γ in the propagators (3.14) – has been resolved by two requirements. The first one is the choice of the parametrization, I or II, in (3.21). This choice, I in our case, is implied by the explicit form (4.1) we have choosen for the field equations. The second requirement is the residual gauge Ward identity.

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References

[1] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Topological field theory*, preprint CERN-TH. 6045/91, to be published in *Phys. Reports*;

[2] A.S. Schwarz, Baku International Topological Conference, Abstracts, vol. 2, p.345 (1987);

[3] E. Witten, *Comm. Math. Phys.* 121 (1989) 351;

[4] G. Moore and N. Seiberg, *Phys. Lett.* B220 (1989) 422;

[5] V.V. Fock and Y.I. Kogan, *Mod. Phys. Lett.* A5 (1990) 1365;

[6] R. Floreanini, R. Percacci and R. Rajaraman, *Phys. Lett.* B231 (1989) 119;

[7] G.V. Dunne and C.A. Trugenberger, *Phys. Lett.* B248 (1990) 311;

[8] C.M. Hull and B. Spence, *Mod. Phys. Lett.* A6 (1991) 969;

[9] V.G. Kac, *Funct. Anal. App. 1 (1967) 328)*;
R.V. Moody, *Bull. Amer. Math. Soc. 73 (1967) 217*;

[10] E. Witten, *Comm. Math. Phys.* 92 (1984) 455;

[11] A. Blasi and R. Collina, *Chern-Simons models in the Landau gauge and its connection to the Kac-Moody Algebra*, Proceedings of the 4th Annecy Meeting on Theoretical Physics, 1990;
The Chern-Simons model with boundary: A cohomological approach, Preprint GEF-Th-27/1990 (Genua, 1990);

[12] K. Symanzik, *Nucl. Phys.* B190 (1981) 1;

[13] C.P. Martin, *Phys. Lett.* B263 (1991) 69;

[14] A. Blasi and R. Collina, *Nucl. Phys.* B345 (1990) 472;

[15] F. Delduc, C. Lucchesi, O. Piguet and S.P. Sorella, *Nucl. Phys.* B346 (1990) 313;

[16] N. Nakanishi, *Progr. Theor. Phys.* 67 (1982) 965;

[17] W. Kummer, *Acta Phys. Austriaca, Suppl. XV* (1976) 423 (Schladming Lectures 1976);

[18] H. Epstein and V. Glaser, *Ann. Inst. Henri Poincaré* 19 (1973) 211;

[19] G. Mack, *Introduction to conformal invariant quantum field theory in two and more dimensions*, Cargèse lectures July 1987, ed. by G. 't Hooft et al., Nonperturbative Quantum Field Theory, Plenum Press, N Y 1988;

[20] C. Becchi and O. Piguet, *Nucl. Phys.* B315(1989)153.