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Condensation in Some Perturbed Meanfield Models of a Bose Gas

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Abstract: We examine the existence of Bose-Einstein condensation in a perturbed meanfield model of a Bose gas in which the interaction is given by a Gaussian kernel. We find that: for negative values of the chemical potential μ there is never condensation; there is a $\mu_0 > 0$ such that for $\mu \in (0, \mu_0)$ there is condensation for all temperatures below a critical temperature $T_C(\mu)$; there is a $\tilde{\mu}_0 > 2\mu_0$ such that for $\mu \in (\mu_0, \tilde{\mu}_0)$ at sufficiently low temperatures, there is no condensation.

§.1 Introduction

The Hamiltonian for a system of bosons interacting through a pair potential $\phi(x - x')$ can be written as

$$H = T + U$$

where T is the kinetic energy operator and U is the potential energy operator,

$$U = \frac{1}{2} \int \int \phi(x - x')\psi^*(x)\psi^*(x')\psi(x)\psi(x')dxdx'$$

where $\psi(x)$ and $\psi^*(x)$ satisfy the canonical commutation relations. For particles in a cube Λ of volume V with periodic boundary conditions, the Hamiltonian can be written in terms of momentum space operators using

$$\psi(x) = \frac{1}{V} \sum_k a_k e^{ikx} \qquad \text{and} \qquad v(k) = \int_\Lambda \phi(x)e^{-ikx} dx \ :$$

$$T = \sum_k \epsilon(k) n_k$$

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and

$$
\begin{aligned}
U &= \frac{1}{2V}\sum_q\sum_k\sum_{k'} v(q)a^*_{k+q}a^*_{k'-q}a_{k'}a_k \\
&= \frac{v(0)}{2V}(N^2-N) + \frac{1}{2V}\sum_k\sum_{k'\neq k} v(k-k')n_k n_{k'} \\
&+ \frac{1}{2V}\sum_k\sum_{k'}\sum_{\substack{q\neq 0\\ q\neq k-k'}} v(q)a^*_{k+q}a^*_{k'-q}a_{k'}a_k
\end{aligned}
$$

where $n_k = a^*_k a_k$ and $N = \sum_k n_k$. The last term in the righthand side is generally believed to be of less importance at low densities. The remaining terms are diagonal in the occupation numbers $\{n_k\}$ and models utilizing these terms have been studied by many authors ([1, 2] for example).

It is convenient to distinguish four models:

$$
\begin{aligned}
U^{MF} &= \frac{a}{2V}N^2, \quad a>0; \\
U^{HYL} &= U^{MF} + \frac{a}{2V}\{N^2 - \sum_k n_k^2\}; \\
U^{PMF} &= \frac{1}{2V}\sum_k\sum_{k'} v(k-k')n_k n_{k'}; \\
U^{FD} &= U^{PMF} + \frac{a}{2V}\{N^2 - \sum_k n_k^2\}.
\end{aligned}
$$

This paper is one in a series in which we study these models using the techniques of Varadhan's Large Deviation Theory. The first of these models, the meanfield model, has been studied exhaustively; the first rigorous treatment was given by Davies [3]. It was studied in [13] for a more general class of kinetic energy operators and in the present framework in [4, 5, 6]. The first rigorous treatment of the second model, the Huang-Yang-Luttinger model [2], was given in [6, 7] as part of the present programme. The fourth model, the full diagonal model, is the subject of a subsequent paper [8]. The third model, the perturbed meanfield model, was studied in [4]; in this paper we expressed the pressure as the infimum of a functional on the space of measures. Our aim in the present paper is to study the variational problem in some examples and to relate it to the existence of Bose-Einstein condensation.

The main object of our study is the variational problem for the Gaussian kernel

$$v(k,k') = v_0 e^{-\delta\|k-k'\|^2}, \tag{1.1}$$

where v_0 and δ are positive constants. This comes, of course, from a Gaussian pair-interaction in configuration space. We proceed by comparision with simpler kernels; v can be written in the form

$$v(k,k') = v_0 e^{-\delta(\|k\|^2+\|k'\|^2)} e^{2\delta k\cdot k'}. \tag{1.2}$$

First we consider the kernel in which $e^{2\delta k\cdot k'}$ is replaced by 1:

$$v_1(k,k') = v_0 e^{-\delta(\|k\|^2+\|k'\|^2)}. \tag{1.3}$$

This has the great advantage that it is separable. Next we consider finite sums of separable kernels:

$$v_2(k,k') = v_0 e^{-\delta(\|k\|^2+\|k'\|^2)}(1+\sigma_1 k\cdot k'), \tag{1.4}$$

$$v_3(k,k') = v_0 e^{-\delta(\|k\|^2+\|k'\|^2)}(1+\sigma_1 k\cdot k' + \sigma_2(k\cdot k')^2). \tag{1.5}$$

Our final result for the Gaussian kernel (Theorem 8) can be summarized as follows: for negative values of the chemical potential μ there is never condensation; there is a $\mu_0(\delta) > 0$ such that for $\mu \in (0, \mu_0(\delta))$ there is condensation for all temperatures below a critical temperature $T_C(\mu)$; there is a $\tilde{\mu}_0(\delta) > 2\mu_0(\delta)$ such that for $\mu \in (\mu_0(\delta), \tilde{\mu}_0(\delta))$, at sufficiently low temperatures, there is no condensation.

The paper is organized as follows:

in §2 we derive the variational expression for the pressure along the lines of [5] but making modifications necessary to accomodate kernels which are functions on momentum space;

in §3 we study the simplifications that arise when the model has spherical symmetry proving that in this case there exists a unique minimizing measure m^* for the variational problem, m^* is absolutely continuous with respect to Lebesgue measure apart from a possible atom at zero momentum and the amount of condensate is equal to the weight of the atom;

in §4 we study the model with the Gaussian kernel (1.1) and make some remarks about other kernels.

§2. Large Deviation results and a variational expression for the pressure.

As in [5] we consider the occupation numbers as random variables rather than as operators. The probability space on which we define our random variables is the countable set Ω of terminating sequences of non-negative integers; an element ω of Ω is a sequence $\{\omega(j) \in \mathbf{N} : j = 1, 2, \ldots\}$ satisfying $\sum_{j\geq 1} \omega(j) < \infty$. The basic random variables are the occupation numbers $\{\sigma_j : j = 1, 2, \ldots\}$; they are the evaluation maps $\sigma_j : \Omega \to \mathbf{N}$ defined by $\sigma_j(\omega) = \omega(j)$ for each ω in Ω. The total number of particles in the configuration ω is defined by

$$N(\omega) = \sum_{j\geq 1} \sigma_j(\omega)\ . \tag{2.1}$$

Let $\Lambda_1, \Lambda_2, \ldots$ be a sequence of regions in $\mathbf{R}^d$ and denote the volume of Λ_l by V_l; we assume that $V_l \to \infty$ as $l \to \infty$. We associate with the region Λ_l the free-gas Hamiltonian H_l given by

$$H_l(\omega) = \sum_{j\geq 1} \epsilon(k_l(j))\sigma_j(\omega)\ , \tag{2.2}$$

where $\epsilon : \mathbf{R}^d \to \mathbf{R}$ is a continuous positive map having bounded level sets and satisfying the condition $\inf_{k\in\mathbf{R}^d} \epsilon(k) = 0$, and $k_l(1), k_l(2)\ldots$ is a sequence in $\mathbf{R}^d$.

The Hamiltonian, $\tilde{H}_l$, of the perturbed mean-field model considered in this paper given by

$$\tilde{H}_l(\omega) = H_l(\omega) + \frac{1}{2V_l}\sum_{j,j'\geq 1} v(k_l(j), k_l(j'))\sigma_j(\omega)\sigma_{j'}(\omega). \tag{2.3}$$

The free-gas pressure, $p_l(\mu)$, is defined for $\mu < 0$ by

$$e^{\beta V_l p_l(\mu)} = \sum_{\omega\in\Omega} e^{\beta\{\mu N(\omega)-H(\omega)\}}\ ; \tag{2.4}$$

it is given in terms of $k_l(j)$ by

$$p_l(\mu) = \int_{\mathbf{R}^d} p(\mu|k)\nu_l(dk), \tag{2.5}$$

where ν_l is the measure on $\mathbf{R}^d$ defined by

$$\nu_l(A) = (V_l)^{-1}\sharp\{j : k_l(j) \in A\} \tag{2.6}$$

and $p(\mu|k)$ is the partial pressure given by

$$p(\mu|k) = \beta^{-1}\ln\left(1 - e^{\beta(\mu-\epsilon(k))}\right)^{-1}. \tag{2.7}$$

The pressure $\tilde{p}_l(\mu)$ in the perturbed mean-field model is given by

$$\tilde{p}_l(\mu) = \frac{1}{\beta V_l}\ln\sum_{\omega\in\Omega} e^{\beta\{\mu N(\omega)-\tilde{H}_l(\omega)\}}. \tag{2.8}$$

Proceeding as in [5] we shall rewrite it as an integral over E, the space of bounded positive measures on $\mathbf{R}^d$ equipped with the narrow topology. First we need some definitions:

the free-gas canonical measure is defined for $\alpha < 0$ by

$$\mathbf{P}_l^\alpha[\omega] = e^{\beta\{\alpha N(\omega)-H_l(\omega)-V_l p_l(\alpha)\}}; \tag{2.9}$$

the occupation measure L_l is defined for each Borel subset A of $\mathbf{R}^d$ and ω in Ω, by

$$L_l[\omega; A] = \frac{1}{V_l}\sum_{j\geq 1}\sigma_j(\omega)\delta_{k_l(j)}[A]; \tag{2.10}$$

for each ω in Ω, $L_l[\omega;\cdot]$ is an element of E. For each $m \in E$ define

$$\langle m, Vm\rangle = \int\int_{\mathbf{R}^d\times\mathbf{R}^d} v(k,k')m(dk)m(dk') \tag{2.11}$$

and put

$$G^\mu[m] = \mu\|m\| - \frac{1}{2}\langle m, Vm\rangle, \tag{2.12}$$

where

$$\|m\| = \int_{\mathbf{R}^d} m(dk). \tag{2.13}$$

We let $\mathbf{K}_l^\alpha$ be the probability measure induced on E by L_l :

$$\mathbf{K}_l^\alpha = \mathbf{P}_l^\alpha \circ L_l^{-1}; \tag{2.14}$$

and rewrite (2.8) as

$$\tilde{p}_l(\mu) = p_l(\alpha) + \frac{1}{\beta V_l} \ln \int_E e^{\beta V_l G^{\mu-\alpha}[m]} \mathbf{K}_l^\alpha[dm]. \tag{2.15}$$

We impose conditions on $\{k_l(j)\}$ to ensure the existence of the limit $p(\alpha) = \lim_{l\to\infty} p_l(\alpha)$ and that the large deviation principle holds for the induced measures.
(T1) *There exists a measure ν on $\mathbf{R}^d$ such that, for $\beta > 0$,*

$$\int_{\mathbf{R}^d} e^{-\beta\epsilon(k)} \nu(dk) < \infty \tag{2.16}$$

and the sequence $\{e^{-\beta\epsilon(k)}\nu_l(dk)\}$ converges to $e^{-\beta\epsilon(k)}\nu(dk)$ in the narrow topology.
(T2) *ν is absolutely continuous with respect to Lebesgue measure with a density which is strictly positive almost everywhere.*

The condition (T1) implies that $p(\alpha) = \lim p_l(\alpha)$ exists for $\alpha < 0$ and is given by

$$p(\alpha) = \int_{\mathbf{R}^d} p(\alpha|k)\nu(dk). \tag{2.17}$$

In the case in which $\epsilon(k_l(j))$, $j = 1,2,\dots$ are the eigenvalues of the Laplacian with periodic boundary conditions on the cube of side $V_l^{1/d}$ condition (T1) is easily checked: here $\epsilon(k) = \|k\|^2$, $k_l(j) = \frac{2\pi}{V_l^{1/d}} n(j)$, $n(j) \in \mathbf{Z}^d$, and $\frac{1}{V_l}\sum_{j\geq 1} e^{-\beta\epsilon(k_l(j))} \to \int_{\mathbf{R}^d} e^{-\beta\epsilon(k)} \frac{dk}{(2\pi)^d}$; it follows that for each bounded continuous function f on $\mathbf{R}^d$ we have

$$\left| \int_{\mathbf{R}^d} f(k)e^{-\beta\epsilon(k)}\nu_l(dk) - \int_{\mathbf{R}^d} f(k)e^{-\beta\epsilon(k)}\nu(dk)\right|$$

$$\leq \left| \frac{1}{V_l} \sum_{\{j:\ \|k_l(j)\|<R\}} f(k_l(j))e^{-\beta\epsilon(k_l(j))} - \int_{\{k\in\mathbf{R}^d:\|k\|<R\}} f(k)e^{-\beta\epsilon(k)} \frac{dk}{(2\pi)^d} \right|$$

$$+\|f\|_\infty e^{-\frac{\beta R^2}{2}} \left(\frac{1}{V_l}\sum_{j\geq 1} e^{-\frac{1}{2}\beta\epsilon(k_l(j))} + \int_{\mathbf{R}^d} e^{-\frac{1}{2}\beta\epsilon(k)} \frac{dk}{(2\pi)^d} \right).$$

Fix R such that the second term in the righthand side is less than $\frac{1}{2}\epsilon$ for all l and then the first term can be made less than $\frac{1}{2}\epsilon$ by choosing l sufficiently large since on a compact set the Riemann sum converges to the integral.

Theorem 1. *Suppose that (T1) and (T2) hold; then, for each $\alpha < 0$ the sequence of probability measures $\{\mathbf{K}_l^\alpha\}$ satisfies the large deviation principle with constants $\{\beta V_l\}$ and rate function $I^\alpha : E \to [0,\infty]$ given by*

$$I^\alpha[m] = f[m] + p(\alpha) - \alpha\|m\| \tag{2.18}$$

where

$$f[m] = \int_{\mathbf{R}^d} \epsilon(k)m(dk) - \beta^{-1}\int_{\mathbf{R}^d} s\left(\frac{dm}{d\nu}(k)\right)\nu(dk) \tag{2.19}$$

and

$$s(x) = (1+x)\ln(1+x) - x\ln x \quad (x \geq 0). \tag{2.20}$$

The proof of this theorem is exactly parallel with that of Theorem 3 of [5] and we shall omit it.

Next we use Varadhan's theorem to establish a variational expression for the pressure as in [5]. To include the special potentials mentioned in the introduction we make the following assumptions on v:

(P) $v : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ *is a bounded, continuous, positive definite function; there exists a continuous, strictly positive, symmetric function* $v_o : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ *such that for all* $m \in E$,

$$\langle m, Vm\rangle \geq \langle m, V_o m\rangle$$

where $\langle m, V_o m\rangle = \int\int_{\mathbf{R}^d \times \mathbf{R}^d} v_o(k,k')m(dk)m(dk')$.

The form of Varadhan's theorem given in [5] cannot be applied here since clearly $G^{\mu-\alpha}$ is not bounded above; we use the following version which is also used in [9]:

Varadhan's Theorem [10]. *Let* $\{\mathbf{K}_l\}$ *be a sequence of Radon probability measures on a regular Hausdorff space* E *satisfying the large deviation principle with rate function* I *and constants* $\{a_l\}$. *Suppose* $G : E \to \mathbf{R}$ *is continuous and satisfies*

$$\lim_{A\to\infty} \limsup_{l\to\infty} \frac{1}{a_l} \ln \int_{\{x\in E: G(x)\geq A\}} e^{a_l G(x)} \mathbf{K}_l[dx] = -\infty\ . \tag{2.21}$$

Then

$$\lim_{l\to\infty} \frac{1}{a_l} \ln \int_E e^{a_l G(x)} \mathbf{K}_l[dx] = \sup_{x\in E}\{G(x) - I(x)\}\ . \tag{2.22}$$

To verify that the function $G^{\mu-\alpha}$ satisfies the hypothesis of Varadhan's theorem, we note that the continuity of G follows as in Lemma 4.1 in [5] by replacing the Laplace transform by the Fourier transform; to check (2.21) is more troublesome and we do it in the following two lemmas.

Lemma 2.1. *Suppose* C *is a non-empty compact subset of* $\mathbf{R}^d$; *then there is a constant* $b(C) > 0$ *such that for all* $m \in E$, *we have*

$$G^\mu[m] \leq \frac{\mu^2}{2b(C)} + \mu m(C^c).$$

Proof : Let $b(C) = \inf\{v_o(k,k') : (k,k') \in C \times C\}$; we have for $m \in \mathcal{M}_+^b(C)$, the space of bounded positive measures on C, that

$$\langle m, V_o m\rangle \geq b(C)\|m\|^2$$

and since $C \times C$ is compact $b(C) > 0$. We now split up the measures $m \in E$ into two parts: $m = m' + m''$ where $m' = m|_C$ and $m'' = m|_{C^c}$. Since $\langle m, Vm\rangle \geq \langle m, V_o m\rangle$ for all $m \in E$ we have

$$\begin{aligned}
G^\mu[m] &\leq \mu\|m\| - \frac{1}{2}\langle m, V_o m\rangle \\
&= \mu\|m'\| + \mu\|m''\| - \frac{1}{2}\langle m', V_o m'\rangle - \frac{1}{2}\langle m'', V_o m''\rangle - \langle m', V_o m''\rangle \\
&\leq \mu\|m'\| + \mu\|m''\| - \frac{1}{2}\langle m', V_o m'\rangle \\
&\leq \mu\|m'\| - \frac{1}{2}b(C)\|m'\|^2 + \mu\|m''\| \leq \frac{\mu^2}{2b(C)} + \mu\|m''\|.
\end{aligned}$$

□

Lemma 2.2. *The functional $G^{\mu-\alpha}[\,\cdot\,]$ satisfies (2.21) with respect to the measures* $\mathbf{K}_l^\alpha$.

Proof: To prove (2.21) it is sufficient to prove that for $\zeta > 1$,

$$\limsup_{l\to\infty} \frac{1}{a_l} \ln \int_E e^{\zeta a_l G(x)} \mathbf{K}_l(dx) < \infty \,. \tag{2.23}$$

We see this as follows: If $\zeta > 1$

$$\begin{aligned}\frac{1}{a_l} \ln \int_{\{x\in E: G(x)\ge A\}} e^{a_l G(x)} \mathbf{K}_l(dx) &\le \frac{1}{a_l} \ln \int_E e^{\{\zeta a_l G(x) - (\zeta-1)a_l A\}} \mathbf{K}_l(dx) \\ &= -(\zeta-1)A + \frac{1}{a_l}\ln \int_E e^{\zeta a_l G(x)} \mathbf{K}_l(dx).\end{aligned}$$

If (2.23) is satisfied then the last term is bounded and thus (2.21) holds.

Let $\gamma > \max\,(2\mu-\alpha, 0)$ and let $C = \{k : k \in \mathbf{R}^d,\ \epsilon(k) \le \gamma\}$. C is compact and therefore by Lemma 2.1

$$G^{\mu-\alpha}[m] \le \frac{(\mu-\alpha)^2}{2b(C)} + (\mu-\alpha)m(C^c)\,.$$

Thus

$$\int_E e^{2\beta V_l G^{\mu-\alpha}[m]} \mathbf{K}_l^\alpha[dm] \le e^{\beta \frac{(\mu-\alpha)^2}{b(C)} V_l} \int_E e^{2\beta V_l (\mu-\alpha) m(C^c)} \mathbf{K}_l^\alpha[dm]. \tag{2.24}$$

We can compute explicitly the integral in the righthand side of (2.24):

$$\begin{aligned}\int_E e^{2\beta V_l(\mu-\alpha)m(C^c)} \mathbf{K}_l^\alpha[dm] &= \sum_{\omega\in\Omega} \exp\left\{ 2\beta(\mu-\alpha) \sum_{\{j:\epsilon(k_l(j))>\gamma\}} \sigma_j(\omega)\right\} \mathbf{P}_l^\alpha[\omega] \\ &= \exp\left\{\beta V_l \int_{C^c} (p_l(2\mu-\alpha|k) - p_l(\alpha|k))\nu_l(dk)\right\}.\end{aligned}$$

Therefore

$$\limsup_{l\to\infty} \frac{1}{\beta V_l} \ln \int_E e^{2\beta V_l G^{\mu-\alpha}[m]} \mathbf{K}_l^\alpha[dm] \le \frac{(\mu-\alpha)^2}{b(C)} + \int_{C^c} p(2\mu-\alpha|k)\nu(dk) \ < \infty$$

since $\epsilon(k) > 2\mu - \alpha$ on the complement of C. Thus we have proved 2.23 with $\zeta = 2$.

□

We are now ready to give a variational formula for the perturbed mean-field model on which the rest of this work is based. This follows by applying Varadhan's theorem to $G^{\mu-\alpha}[m]$, using the preceding lemma.

Theorem 2. *Suppose that (T1) and (T2) hold and that the potential v satisfies (P); then the pressure $\tilde{p}(\mu) = \lim_{l\to\infty} \tilde{p}_l(\mu)$ exists for the perturbed mean-field model determined by (2.3), and is given by*

$$\tilde{p}(\mu) = -\inf_E \mathcal{E}^\mu[m]$$

where

$$\begin{aligned}\mathcal{E}^\mu[m] &= I^\alpha[m] - G^{\mu-\alpha}[m] - p(\alpha)\\ &= f[m] + \frac{1}{2}\langle m, Vm\rangle - \mu\|m\|,\end{aligned} \tag{2.25}$$

and $f[m]$ is the free energy functional for free bosons given by (2.19).

Remark The results so far hold also if instead of assuming that (P) holds we suppose that v satisfies:
(P′) $v : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ *is a continuous, positive, strictly positive definite function.*

The proof of Lemma 2.1 is modified as follows: Since C is compact the unit sphere in $\mathcal{M}_+^b(C)$, the space of bounded positive measures on C, is compact in the narrow topology. Because $m \mapsto \langle m, Vm\rangle$ is continuous and $\langle m, Vm\rangle > 0$ for $m \neq 0$,

$$\inf\{\langle m, Vm\rangle \ : \ m \in \mathcal{M}_+^b(C),\ \|m\| = 1\} > 0.$$

Thus there is a constant $b(C) > 0$ such that

$$\langle m, Vm\rangle \geq b(C)\|m\|^2 \text{ for all } m \in \mathcal{M}_+^b(C).$$

§3. Existence, uniqueness and spherical symmetry of the minimizer.

We shall begin by proving that the minimizer m^* of $\mathcal{E}^\mu[m]$ defined in (2.25) exists and satisfies the Euler-Lagrange equations. Next we assume that the model has rotational symmetry; in that case we can reformulate the problem in terms of measures on $[0,\infty)$ and obtain a formula for the pressure similar to that derived in [5]. We prove also the uniqueness of the minimizing measure.

The proof of existence is more complex than in [5] because we do not have a bound of the form $a\|m\|^2$ for $G^\mu[m]$; however we can use the weaker bound of Lemma 2.1.

Lemma 3.1. *Let $e = \inf_{m\in E} \mathcal{E}^\mu[m]$; then there exists m^* in E such that $\mathcal{E}^\mu[m^*] = e$.*

Proof: Since $\mathcal{E}^\mu[0] = 0$ we have $e \leq 0$ and it follows that there exists a sequence $\{m_n\}$ in E such that $e \leq \mathcal{E}^\mu[m_n] \leq 0$ and $\lim_{n\to\infty} \mathcal{E}^\mu[m_n] = e$. Recall that $\mathcal{E}^\mu[m]$ can be expressed in the form (see Theorem 2):

$$\mathcal{E}^\mu[m] = I^\alpha[m] - G^{\mu-\alpha}[m] - p(\alpha).$$

Since I^α is lower semi-continuous and $G^{\mu-\alpha}$ is continuous, $\mathcal{E}^\mu$ is lower semi-continuous; therefore it is sufficient to prove that $\{m_n\}$ has a convergent subsequence. Because $\mathcal{E}^\mu[m_n] \leq 0$, we have

$$I^\alpha[m_n] \leq G^{\mu-\alpha}[m_n] + p(\alpha)\ . \tag{3.1}$$

As in Lemma 2.2 let $\gamma > \max(2\mu - \alpha, 0)$ and let $C = \{k : k \in \mathbf{R}^d, \epsilon(k) \leq \gamma\}$; then by Lemma 2.1 and the inequality (3.1) it follows that

$$I^\alpha[m_n] \leq \frac{(\mu-\alpha)^2}{2b(C)} + (\mu - \alpha)m_n(C^c) + p(\alpha) \tag{3.2}$$

Let $I^\alpha_{\frac{1}{2}\beta}[m]$ be the same as $I^\alpha[m]$ with β replaced by $\frac{1}{2}\beta$, that is

$$I^\alpha_{\frac{1}{2}\beta}[m] = p_{\frac{1}{2}\beta}(\alpha) + \int_{\mathbf{R}^d} (\epsilon(k) - \alpha)m(dk) - \frac{2}{\beta}\int_{\mathbf{R}^d} s\left(\frac{dm}{d\nu}(k)\right)\nu(dk), \tag{3.3}$$

where

$$p_{\frac{1}{2}\beta}(\alpha) = -\frac{2}{\beta}\int_{\mathbf{R}^d} \ln(1 - e^{-\frac{1}{2}\beta(\epsilon(k)-\alpha)})\nu(dk).$$

It is easy to check that $\inf_{m\in E} I^\alpha_{\frac{1}{2}\beta}[m] = 0$.

Now

$$\begin{aligned} I^\alpha[m] &= p(\alpha) - \frac{1}{2}p_{\frac{1}{2}\beta}(\alpha) + \frac{1}{2}\int_{\mathbf{R}^d}(\epsilon(k)-\alpha)m(dk) + \frac{1}{2}I^\alpha_{\frac{1}{2}\beta}[m] \\ &\geq p(\alpha) - \frac{1}{2}p_{\frac{1}{2}\beta}(\alpha) + \frac{1}{2}(\gamma - \alpha)m(C^c). \end{aligned}$$

Combining this inequality with (3.2) we get

$$\frac{1}{2}(\gamma - 2\mu + \alpha)m_n(C^c) \leq \frac{(\mu-\alpha)^2}{2b(C)} + \frac{1}{2}p_{\frac{1}{2}\beta}(\alpha)\ .$$

This means that the sequence $\{m_n(C^c)\}$ is bounded and therefore $\{I^\alpha[m_n]\}$ is bounded by (3.2); thus the sequence $\{m_n\}$ lies inside a level set of I^α which, since I^α is a rate function, is compact. Hence $\{m_n\}$ contains a convergent subsequence.

□

The Euler-Lagrange equations for the variational formula given in (2.25) are as follows:

$$L^\mu(m;k) = 0 \qquad m_s - a.e. \tag{3.4a}$$

$$L^\mu(m;k) = \beta^{-1}s'(\rho(k)) \quad \nu - a.e. \tag{3.4b}$$

where

$$m(dk) = m_s(dk) + \rho(k)\nu(dk)$$

is the Lebesgue decomposition of m with respect to ν and $L^\mu(m;k)$ is defined by

$$L^\mu(m;k) = \epsilon(k) + (Vm)(k) - \mu \tag{3.5}$$

with

$$(Vm)(k) = \int_{\mathbf{R}^d} v(k,k')m(dk')\ .$$

The following results can be proved exactly as in [5]; we therefore omit the proof.

Theorem 3.
a. *Let m be a minimizer of $\mathcal{E}^\mu$; then $\rho(k) > 0$ a.e. with respect to ν.*
b. *A measure m in E is a minimizer of $\mathcal{E}^\mu$ if and only if it satisfies the Euler-Lagrange equations (3.4a, b).*
c. *If m_1 and m_2 are minimizers of $\mathcal{E}^\mu$ then their absolutely continuous parts coincide.*
We now study the problem when the model has rotational symmetry; from now on we shall assume that ϵ, v and ν have the following properties:
For each $R \in O(d)$ the group of rotations in $\mathbf{R}^d$,
(R1) $\epsilon \circ R = \epsilon$,
(R2) $v(Rk, Rk') = v(k,k')$ *for all* $k, k' \in \mathbf{R}^d$,
(R3) $\nu \circ R^{-1} = \nu$.

Lemma 3.2. *Suppose (R1), (R2) and (R3) are satisfied and let $m \in E$ be a minimizer of $\mathcal{E}^\mu$; then the absolutely continuous part of m with respect to ν is rotation invariant, that is $\rho \circ R = \rho$ for all $R \in O(d)$.*

Proof: From (R1), (R2) and (R3) it easily follows that $\mathcal{E}^\mu[m \circ R^{-1}] = \mathcal{E}^\mu[m]$ for all $m \in E$ and $R \in O(d)$. Thus if m is a minimizer so is $m \circ R^{-1}$. But we know from Theorem 3 that the absolutely continuous parts of $m \circ R^{-1}$ and m must coincide.

□

Let $\tilde{E} = \mathcal{M}_+^b[0,\infty)$ the space of positive bounded measures on $[0,\infty)$ and for $k \in \mathbf{R}^d$ let $p(k) = \|k\|$. If $m \in E$ is rotationally invariant we can express $\mathcal{E}^\mu[m]$ in terms of $m \circ p^{-1} \in \tilde{E}$. Let $\hat{e}$ be an arbitrary fixed unit vector in $\mathbf{R}^d$ and define $\tilde{\epsilon} : \mathbf{R}_+ \to \mathbf{R}_+$ by $\tilde{\epsilon}(r) = \epsilon(r\hat{e})$. If we now assume that $\tilde{\epsilon}$ is invertible then we can write $\mathcal{E}[m]$ in terms of $\tilde{m} = m \circ p^{-1} \circ \tilde{\epsilon}^{-1} = m \circ \epsilon^{-1} \in \tilde{E}$:

$$\mathcal{E}^\mu[m] = \tilde{\mathcal{E}}^\mu[\tilde{m}],$$

where

$$\tilde{\mathcal{E}}^\mu[\tilde{m}] = \int_{[0,\infty)} \lambda \tilde{m}(d\lambda) + \frac{1}{2}\langle \tilde{m}, U\tilde{m}\rangle - \beta^{-1}\int_{[0,\infty)} s(\tilde{\rho}(\lambda))dF(\lambda) - \mu\|\tilde{m}\|, \tag{3.6}$$

$$\begin{aligned} \langle \tilde{m}, U\tilde{m}\rangle &= \int\int_{[0,\infty)\times[0,\infty)} u(\lambda,\lambda')\tilde{m}(d\lambda)\tilde{m}(d\lambda) \ , \\ u(\lambda,\lambda') &= \int_{O(d)} v(\tilde{\epsilon}^{-1}(\lambda)\hat{e}, \tilde{\epsilon}^{-1}(\lambda')R\hat{e})\Omega(dR) \ , \\ dF(\lambda) &= (\nu \circ \epsilon)^{-1}(d\lambda) \ , \\ \tilde{\rho}(\lambda) &= \frac{d\tilde{m}}{dF}(\lambda) = \rho(\tilde{\epsilon}^{-1}(\lambda)\hat{e}) \ . \end{aligned}$$

In the remainder of this section we shall assume that $\tilde{\epsilon}$ is invertible. Since $\tilde{\epsilon}$ is invertible and ϵ has compact level sets $\tilde{\epsilon}$ must be strictly increasing; therefore $\epsilon(0) = \inf \epsilon(k) = 0$ and $\epsilon(k) = 0$ if and only if $k = 0$.
Let R_ν denote the subset of $\mathbf{R}^d$ on which the function $k' \mapsto (\epsilon(k) - \epsilon(k'))^{-1}$ is locally ν-integrable:

$$\begin{aligned} R_\nu &= \left\{ k \in \mathbf{R}^d \ : \ k' \mapsto (\epsilon(k) - \epsilon(k'))^{-1} \in \mathcal{L}^1_{loc}(\mathbf{R}^d, \nu) \right\} \\ &= \left\{ k \in \mathbf{R}^d \ : \ \lambda \mapsto (\epsilon(k) - \lambda)^{-1} \in \mathcal{L}^1_{loc}(\mathbf{R}_+, dF) \right\} . \end{aligned}$$

Lemma 3.3. *Let v be such that for any $\tilde{m} \in \tilde{E}$, $\lambda \mapsto (U\tilde{m})(\lambda)$ is continuously differentiable; then the singular part m_s of a minimizing measure m is concentrated on the set R_ν: $m_s(R_\nu^c) = 0$.*

Proof: For $m \in E$ let m_a denote the component in the Lebesgue decomposition of m which is absolutely continuous with respect to ν. Let Ω be the normalized invariant measure on $O(d)$ and let

$$\overline{m} = \int_{O(d)} m \circ R^{-1}\Omega(dR).$$

Since v is positive definite $\langle(\overline{m} - m \circ R^{-1}), V(\overline{m} - m \circ R^{-1})\rangle \geq 0$; integrating this inequality with respect to the measure Ω we get $\langle \overline{m}, V\overline{m}\rangle \leq \langle m, Vm\rangle$ and therefore

$$\int_{\mathbf{R}^d} (\epsilon(k) - \mu)\overline{m}(dk) + \frac{1}{2}\langle \overline{m}, V\overline{m}\rangle - \frac{1}{\beta}\int_{\mathbf{R}^d} s(\rho(k))\nu(dk) \leq \mathcal{E}^\mu[m].$$

Now if m is a minimizer of $\mathcal{E}^\mu$ by Lemma 3.2 we have that $\overline{m}_a = (\overline{m_s})_a + m_a$; therefore if $\overline{\rho}(k) = \frac{d\overline{m}}{d\nu}(k)$ then $\overline{\rho}(k) \geq \rho(k)$ and since $x \mapsto s(x)$ is increasing $-\frac{1}{\beta}\int_{\mathbf{R}^d} s(\overline{\rho}(k))\nu(dk) \leq -\frac{1}{\beta}\int_{\mathbf{R}^d} s(\rho(k))\nu(dk)$. This combined with the first inequality gives

$$\mathcal{E}^\mu[\overline{m}] \leq \mathcal{E}^\mu[m]$$

and so $\overline{m}$ is also a minimizer of $\mathcal{E}^\mu$. Thus from Lemma 3.2 we have $\overline{m}_a = m_a = \overline{m_a}$. But then $\overline{m} = \overline{m_s} + \overline{m_a} = \overline{m_s} + m_a$ and $\overline{m} = \overline{m}_s + \overline{m}_a = \overline{m}_s + m_a$ and consequently $\overline{m}_s = \overline{m_s}$. Thus since R_ν is rotation invariant $m_s(R_\nu^c) = \overline{m}_s(R_\nu^c)$ and therefore it is sufficient to prove that $\overline{m}_s(R_\nu^c) = 0$. But since $\overline{m}$ minimizes $\mathcal{E}^\mu$, $\tilde{\overline{m}} = \overline{m} \circ \epsilon^{-1} \in \tilde{E}$ minimizes $\tilde{\mathcal{E}}$ given in (3.6). Therefore as in Lemma 5.4 of [5], $\tilde{\overline{m}}_s$ is concentrated on $R_F = \{\lambda : \lambda' \mapsto (\lambda - \lambda')^{-1} \in \mathcal{L}^1_{loc}(\mathbf{R}_+, dF)\}$ and so $\overline{m}_s$ is concentrated on $\epsilon^{-1}R_F = R_\nu$.

□

Theorem 4. *Suppose that v satisfies the smoothness condition of Lemma 3.3 and that for $\lambda > 0$, $F'(\lambda)$ is continuous and $F'(\lambda) > 0$; then a minimizer m of $\mathcal{E}^\mu$ has the following properties:*

1. *if $\lambda \mapsto \frac{1}{\lambda}$ is not locally dF-integrable at 0 then $m_s = 0$,*
2. if $m_s \neq 0$ then m_s is concentrated at $k = 0$,
3. *m is the unique minimizer of $\mathcal{E}^\mu$.*

Proof: If $F'(\lambda)$ is continuous and $F'(\lambda) > 0$ for $\lambda > 0$ then $R_F \subset \{0\}$ and thus $R_\nu \subset \{k : \epsilon(k) = 0\} = \{0\}$ so that (2) holds.

If $\lambda \to \frac{1}{\lambda}$ is not locally dF-integrable at 0 then $R_\nu = \emptyset$ so that (1) holds. Since $x \mapsto x^2 v(0,0)$ is strictly convex, m_s is unique; but by Theorem 3 we know that ρ is unique, therefore m is unique.

□

If the conditions of Theorem 4 are satisfied the minimizer of $\mathcal{E}^\mu$ is rotationally symmetric since ρ is symmetric and m_s is concentrated at $k = 0$. We can therefore reduce the problem to one over $\tilde{E}$:

Theorem 5. *Suppose that v and F satisfy the conditions of Theorem 4 then the pressure $\tilde{p}(\mu)$ is given by*

$$\tilde{p}(\mu) = -\inf_{\tilde{m} \in \tilde{E}} \tilde{\mathcal{E}}^\mu[\tilde{m}]$$

where $\tilde{\mathcal{E}}^\mu$ is given by (3.6).

The minimizer $\tilde{m}^* \in \tilde{E}$ of $\tilde{\mathcal{E}}^\mu$ is unique and obeys the Euler-Lagrange equations:

$$\tilde{L}^\mu[\tilde{m},\lambda] = 0 \qquad \tilde{m}_s - a.e. \tag{3.7a}$$

$$\tilde{L}^\mu[\tilde{m},\lambda] = \beta^{-1}s'(\tilde{\rho}(\lambda)) \quad dF - a.e. \tag{3.7b}$$

where

$$\tilde{L}^\mu[\tilde{m},\lambda] = \lambda - \mu + (U\tilde{m})(\lambda). \tag{3.8}$$

Conversely if $\tilde{m}^*$ satisfies (3.7a) and (3.7b) then $\tilde{m}^*$ is the unique minimizer of $\tilde{\mathcal{E}}^\mu$. If $m^* \in E$ is the unique minimizer of $\mathcal{E}^\mu$ then $\tilde{m}^* = m^* \circ \epsilon^{-1}$.

We finally relate the atom in m^* to Bose-Einstein condensation. Following [11] we define the generalized condensate $\Delta(\mu)$ by

$$\Delta(\mu) = \lim_{\delta\downarrow 0} \Delta(\mu;\delta) \tag{3.9}$$

where

$$\Delta(\mu;\delta) = \lim_{l\to\infty} \tilde{\mathsf{E}}^\mu_l[X^\delta_l] \tag{3.10}$$

and X^δ_l is the random variable

$$X^\delta_l(\omega) = \frac{1}{V_l} \sum_{\{j:\epsilon(k_l(j))\le\delta\}} \sigma_j(\omega) \tag{3.11}$$

and the expectation $\tilde{\mathsf{E}}^\mu_l$ is with respect to the grand-canonical probability measure on Ω given by

$$\tilde{\mathsf{P}}^\mu_l[\omega] = \exp\left\{\beta[\mu N(\omega) - \tilde{H}_l(\omega) - V_l\tilde{p}_l(\mu)]\right\}. \tag{3.12}$$

Let $\tilde{\mathsf{K}}^\mu_l$ be the probability measure induced on E by L_l:

$$\tilde{\mathsf{K}}^\mu_l = \tilde{\mathsf{P}}^\mu_l \circ L_l^{-1}; \tag{3.13}$$

or more explicitly

$$\tilde{\mathsf{K}}^\mu_l[dm] = e^{\beta V_l\{G^{\mu-\alpha}[m] - \tilde{p}_l(\mu) - p_l(\alpha)\}}\mathsf{K}^\alpha_l[dm]. \tag{3.14}$$

The sequence of probability measures $\{\tilde{\mathsf{K}}^\mu_l\}$ satisfies the large deviation principle with constants $\{\beta V_l\}$ and rate function

$$\tilde{I}^\mu[m] = \mathcal{E}^\mu[m] - \tilde{p}(\mu). \tag{3.15}$$

Under the assumptions of Theorem 4 $\tilde{I}^\mu$ has a unique minimizer m^*. If F is a closed subset of E not containing m^* and and $\inf\{\mathcal{E}^\mu[m] : m \in E\} = \inf\{\mathcal{E}^\mu[m] : m \in F\}$, then by the argument in Lemma 3.1 the set F must contain a minimizer of $\mathcal{E}^\mu$, contradicting the uniqueness of m^*; thus $\inf\{\mathcal{E}^\mu[m] : m \in E\} < \inf\{\mathcal{E}^\mu[m] : m \in F\}$. Therefore by Theorem 3.6 of [10] if $g : E \to \mathbf{R}$ is continuous

$$\int_E g[m]\tilde{\mathsf{K}}^\mu_l[dm] \to g(m^*) \tag{3.16}$$

as $l \to \infty$.

In terms of $\{\tilde{\mathsf{K}}_l^\mu\}$, $\Delta(\mu)$ is given by

$$\Delta(\mu;\delta) = \lim_{l\to\infty} \int_E \langle m, 1_\delta \circ \epsilon\rangle \tilde{\mathsf{K}}_l^\mu[dm]. \tag{3.17}$$

where 1_δ is the indicator function of the interval $[0,\delta]$. $m \mapsto \langle m, 1_\delta \circ \epsilon\rangle$ is not continuous in the narrow topology; however by using (3.16) we get the following bounds for $\Delta(\mu;\delta)$.

$$\sup_{\substack{t\in C^b(\mathbf{R}^d)\\ t\le 1_\delta\circ\epsilon}} \langle m^*, t\rangle \le \Delta(\mu;\delta) \le \inf_{\substack{t\in C^b(\mathbf{R}^d)\\ t\ge 1_\delta\circ\epsilon}} \langle m^*, t\rangle\ . \tag{3.18}$$

This gives

$$m^*\{k : \epsilon(k) < \delta\} \le \Delta(\mu;\delta) \le m^*\{k : \epsilon(k) \le \delta\}$$

or equivalently

$$\tilde{m}^*[0,\delta) \le \Delta(\mu;\delta) \le \tilde{m}^*[0,\delta].$$

Since $\tilde{m}^*$ is absolutely continuous except at $\lambda = 0$, the two bounds are equal. Therefore

$$\Delta(\mu) = \lim_{\delta\downarrow 0} \Delta(\mu;\delta) = \tilde{m}^*\{0\} = m^*\{0\}. \tag{3.19}$$

§4. The models

In this section we study the variational problem for the pressure for some special kernels $v(\cdot,\cdot)$. Throughout §4 we take $\epsilon(k) = a\|k\|^2$ with $a > 0$, we assume that ν is rotationally invariant in the sense of (R3) and that for $\lambda > 0, F'(\lambda)$ is continuous and strictly positive. We see from Theorem 4 that if $\lambda \mapsto \frac{1}{\lambda}$ is not dF - integrable at 0 then there is no condensation; we therefore concentrate on the cases where there is a possibility that the model exhibits Bose-Einstein condensation and assume that $\lambda \mapsto \frac{1}{\lambda}$ is dF - integrable at 0. Moreover in the cases that we shall consider the spherically averaged kernel $u(\cdot,\cdot)$ given in (3.6) is strictly positive. Condensation requires by (3.7) that $(Um)(0) = \mu$ which is impossible if $\mu \le 0$; we shall therefore take $\mu > 0$.

Our main objective is to study the variational problem for the Gaussian kernel

$$v(k,k') = v_0 e^{-\delta\|k-k'\|^2}, \tag{4.1}$$

where v_0 and δ are positive constants. v clearly satisfies (P) and (R2) (in fact v is also strictly positive definite). As discussed in §1 we proceed by comparison with simpler kernels; we write v in the form

$$v(k,k') = v_0 e^{-\delta(\|k\|^2+\|k'\|^2)} e^{2\delta k\cdot k'} \tag{4.2}$$

and replace $e^{2\delta k\cdot k'}$ by by 1, to get

$$v(k,k') = v_0 e^{-\delta(\|k\|^2+\|k'\|^2)}. \tag{4.3}$$

As this kernel is separable, the corresponding operator is of rank one. In this case v is not strictly positive definite; however it is positive definite and strictly positive so that it satisfies (P). The spherically averaged kernel corresponding to (4.3) is

$$u(\lambda,\lambda') = u_0 e^{-\alpha(\lambda+\lambda')}, \tag{4.4}$$

where u_0 and α are positive constants. Next we consider the approximation

$$v(k,k') = v_0 e^{-\delta(\|k\|^2+\|k'\|^2)}(1+2\delta k\cdot k'). \tag{4.5}$$

This is positive definite but not strictly so and also not positive; but it still satisfies (P) since $\langle m, Vm\rangle \geq \langle m, V_1 m\rangle$ where V_1 is given by the kernel in (4.3). The spherically averaged kernel in this case is the same as for (4.3), that is, it is given by (4.4) and so the variational problem is the same.

We then consider

$$v(k,k') = v_0 e^{-\delta(\|k\|^2+\|k'\|^2)}(1+\sigma_1 k\cdot k' + \sigma_2 (k\cdot k')^2). \tag{4.6}$$

This satisfies (P) by the same argument as above and the corresponding u is given by

$$u(\lambda,\lambda') = u_0 e^{-\alpha(\lambda+\lambda')}(1+\gamma\lambda\lambda'), \tag{4.7}$$

where u_0, α and γ are positive constants.

Before embarking on the detailed study of these models we make the following two remarks. If $m \in \tilde{E}$ is a minimizer of $\tilde{\mathcal{E}}^\mu$ then from equation (3.7b) we see, since $F'(\lambda) > 0$ and F' is continuous for $\lambda > 0$, that

$$\lambda - \mu + (Um)(\lambda) \geq 0 \;\text{ a.e for }\; \lambda > 0; \tag{4.8}$$

but since $\lambda \mapsto \lambda - \mu + (Um)(\lambda)$ is continuous we have

$$\lambda - \mu + (Um)(\lambda) \geq 0 \;\text{ for all }\; \lambda \in [0,\infty), \tag{4.9}$$

and in particular

$$(Um)(0) \geq \mu\ . \tag{4.10}$$

If the model exhibits Bose-Einstein condensation that is, $m(\{0\}) \neq 0$ then from (3.7a) we get

$$(Um)(0) = \mu. \tag{4.11}$$

We now proceed with the study of the models mentioned above. We start with those given by (4.3) and (4.5).

Let $g_0(\lambda) = \lambda - \mu + \mu e^{-\alpha\lambda}$; if $\mu \in (0,\alpha^{-1}]$, $g_0(\lambda) > 0$ for $\lambda > 0$. $g_0(\lambda) \sim \lambda(1-\alpha\mu)$ for small λ so that if $\mu \in (0,\alpha^{-1})$

$$\int_0^\infty \frac{e^{-\alpha\lambda}}{e^{\beta g_0(\lambda)}-1}\,dF(\lambda) < \infty;$$

the integral is strictly decreasing in β, tends to ∞ as $\beta \to 0$ and to 0 as $\beta \to \infty$. For $\mu \in (0,\alpha^{-1})$ let $\beta_c(\mu)$ be the unique solution of

$$\mu = u_0 \int_0^\infty \frac{e^{-\alpha\lambda}}{e^{\beta g_0(\lambda)}-1}\,dF(\lambda)\ . \tag{4.12}$$

If $\mu = \alpha^{-1}$, $g_0(\lambda) \sim \mu\frac{\alpha^2\lambda^2}{2}$ for small λ; if $\lambda \mapsto \frac{1}{\lambda^2}$ is dF-integrable at 0, then we define $\beta_c(\alpha^{-1})$ as above, otherwise we put $\beta_c(\alpha^{-1}) = \infty$.

Theorem 6. *The perturbed meanfield model with interaction given by the kernel in (4.3) or in (4.5) has the following behaviour:*
(a) *If $\mu \in (0, \alpha^{-1}]$, the model exhibits Bose-Einstein condensation for $\beta > \beta_c(\mu)$ and no condensation for $\beta \leq \beta_c(\mu)$.* (b) *If $\mu > \frac{1}{\alpha}$, there is no condensation.*

Proof: Let $m \in \tilde{E}$ be the minimizer of $\tilde{\mathcal{E}}^\mu$.
(a) Suppose that $\mu \in (0, \alpha^{-1}]$ and $\beta > \beta_c(\mu)$. Using (4.10) we get

$$(Um)(\lambda) = e^{-\alpha\lambda}(Um)(0) \geq \mu e^{-\alpha\lambda}, \tag{4.13}$$

and thus

$$\rho(\lambda) \leq \frac{1}{e^{\beta g_0(\lambda)} - 1}. \tag{4.14}$$

Therefore if there is no condensation

$$(Um)(0) = \int_0^\infty u_0 e^{-\alpha\lambda}\rho(\lambda)dF(\lambda) \leq u_0 \int_0^\infty \frac{e^{-\alpha\lambda}}{e^{\beta g_0(\lambda)} - 1}dF(\lambda) < \mu, \tag{4.15}$$

which contradicts (4.10), and so there must be condensation.

Now let $\beta \geq \beta_c(\mu)$ and suppose there is condensation so that $(Um)(0) = \mu$. Then

$$\rho(\lambda) = \frac{1}{e^{\beta g_o(\lambda)} - 1} \tag{4.16}$$

and consequently

$$\mu = (Um)(0) > u_0 \int_0^\infty \frac{e^{-\alpha\lambda}}{e^{\beta g_0(\lambda)} - 1}dF(\lambda) \geq \mu; \tag{4.17}$$

contradiction. Hence there is no condensation for $\beta \geq \beta_c(\mu)$.
(b) Finally let $\mu > \alpha^{-1}$ and again suppose there is condensation so that (4.16) holds. This contradicts $\rho(\lambda) > 0$ since $g_0(\lambda) < 0$ for some values of λ if $\mu > \alpha^{-1}$.

□

We now turn to the model with the interaction given by (4.7). In this case we are not able to give the full behaviour of the model but we can describe what happens for low temperatures. We break up the proof into several lemmas and combine the results in Theorem 7.

Lemma 4.1 *For the model with interaction given by (4.6) with $\mu \in (0, 2\alpha^{-1})$, $\mu \neq \alpha^{-1}$, there exists $\beta(\mu) > 0$ such that for all $\beta > \beta(\mu)$ there is condensation for $\mu \in (0, \alpha^{-1})$ and no condensation for $\mu \in (\alpha^{-1}, 2\alpha^{-1})$.*

Proof: Suppose $\mu \in (0, \alpha^{-1})$ and let $m \in \tilde{E}$ be the minimizer of $\tilde{\mathcal{E}}^\mu$. Since $(Um)(\lambda) \geq e^{-\alpha\lambda}(Um)(0)$ the argument of Theorem 6 applies and we have condensation for $\beta > \beta(\mu) = \beta_c(\mu)$.

Let $\mu \in (\alpha^{-1}, 2\alpha^{-1})$ and let

$$g(\lambda) = \beta^{-1} s'(\rho(\lambda)) \ ,$$
$$x = \int_{[0,\infty)} u_0 e^{-\alpha\lambda} m(d\lambda) \ ,$$
$$y = \int_{[0,\infty)} \gamma u_0 \lambda e^{-\alpha\lambda} m(d\lambda)$$

and

$$\rho_0 = u_0 m(\{0\}) \ .$$

We then have

$$x = u_0 \int_0^\infty \frac{e^{-\alpha\lambda}}{e^{\beta g(\lambda)} - 1} dF(\lambda) + \rho_0 \ , \tag{4.18}$$

$$y = \gamma u_0 \int_0^\infty \frac{\lambda e^{-\alpha\lambda}}{e^{\beta g(\lambda)} - 1} dF(\lambda) \ , \tag{4.19}$$

and from the Euler-Lagrange equation (3.7b),

$$g(\lambda) = \lambda - \mu + x e^{-\alpha\lambda} + y\lambda e^{-\alpha\lambda} \ . \tag{4.20}$$

If there is condensation, then $x = (Um)(0) = \mu$; but then $g(0) = 0$ and, since $g(\lambda)$ cannot be negative, $g'(0) = y - \alpha\mu + 1 \geq 0$ or $y \geq \alpha\mu - 1$. Let

$$g_1(\lambda) = \lambda - \mu + \mu e^{-\alpha\lambda} + (\alpha\mu - 1)\lambda e^{-\alpha\lambda},$$

$g_1(0) = 0, \quad g_1'(0) = 0$ and

$$g_1''(\lambda) = \alpha(2 - \alpha\mu)e^{-\alpha\lambda} + \alpha^2(\alpha\mu - 1)\lambda e^{-\alpha\lambda} \ .$$

Thus g_1 is convex, increasing and $g(\lambda) > 0$ for $\lambda > 0$. We know that $g(\lambda) \geq g_1(\lambda)$ and so

$$y \leq \gamma u_0 \int_0^\infty \frac{\lambda e^{-\alpha\lambda}}{e^{\beta g_1(\lambda)} - 1} dF(\lambda). \tag{4.21}$$

Now $g_1(\lambda) \sim \frac{\alpha^2\lambda^2}{2}(2\alpha^{-1} - \mu)$ for small λ; thus

$$\int_0^\infty \frac{\lambda e^{-\alpha\lambda}}{e^{\beta g_1(\lambda)} - 1} dF(\lambda) < \infty$$

and is strictly decreasing in β. Therefore if $\beta > \beta(\mu)$ where $\beta(\mu)$ is the solution of

$$u_0\gamma \int_0^\infty \frac{\lambda e^{-\alpha\lambda}}{e^{\beta g_0(\lambda)} - 1} = \alpha\mu - 1,$$

we get a contradition: $y < \alpha\mu - 1$. Hence there is no condensation for $\mu \in (\alpha^{-1}, 2\alpha^{-1})$ and $\beta > \beta(\mu)$.

□

Remark If $\lambda \mapsto 1/\lambda^2$ is dF - integrable at 0, then for $\mu = \alpha^{-1}$ we have condensation and for $\mu = 2\alpha^{-1}$ we have no condensation for β sufficiently large. These can be deduced from the proof of lemma 4.1.

In the presence of condensation we must have $x = \mu$; let

$$g_1(\lambda; y) = \lambda - \mu + \mu e^{-\alpha\lambda} + y\lambda e^{-\alpha\lambda}. \tag{4.22}$$

For $\mu \in (\alpha^{-1}, 2\alpha^{-1}]$ we saw that the only constraint on y for this to be positive is $y \geq \alpha\mu - 1$. For $\mu > 2\alpha^{-1}$ let $\lambda(\mu)$ be the unique positive solution of

$$e^{-\alpha\lambda} = \frac{\alpha}{\mu}\lambda^2 - \alpha\lambda + 1 \tag{4.23}$$

and let

$$y(\mu) = e^{\alpha\lambda(\mu)}(\alpha\mu - \alpha\lambda(\mu) - 1). \tag{4.24}$$

We have the following bounds for $\lambda(\mu)$ and $y(\mu)$:

Lemma 4.2
(a) $\mu - \frac{2}{\alpha} < \lambda(\mu) < \frac{\alpha\mu-2}{\alpha\mu-1}\mu$;
(b) $y(\mu) > \alpha\mu - 1$.

Proof:
(a) Let $r(\lambda) = e^{-\alpha\lambda} - (\frac{\alpha}{\mu}\lambda^2 - \alpha\lambda + 1)$; then $r(\lambda) > 0$ if and only if $0 < \lambda < \lambda(\mu)$ and $r(\lambda) < 0$ if and only if $\lambda > \lambda(\mu)$. Now for $y > 0$ we have $\sqrt{1 + \frac{y^2}{4}} < 1 + \frac{y^2}{6}$ and therefore $1 + \frac{y^2}{2} + y\sqrt{1 + \frac{y^2}{4}} < 1 + y + \frac{y^2}{2} + \frac{y^3}{6} < e^y$ or

$$e^{-y} - \frac{1}{1 + \frac{y^2}{2} + y\sqrt{1 + \frac{y^2}{4}}} < 0.$$

Putting $y = \alpha\lambda_+(\mu)$ where $\lambda_+(\mu) = \frac{\alpha\mu-2}{\alpha\mu-1}\mu$ we get $r(\lambda_+(\mu)) < 0$ so that $\lambda_+(\mu) > \lambda(\mu)$. To obtain the lower bound for $\lambda(\mu)$ we use the inequality

$$e^{-y} - \frac{2-y}{2+y} > 0$$

for $y > 0$. Letting $y = \alpha\lambda_-(\mu)$ where $\lambda_-(\mu) = \mu - \frac{2}{\alpha}$ we get $r(\lambda_-(\mu)) > 0$ and therefore $\lambda_-(\mu) < \lambda(\mu)$.
(b) We can rewrite the upper bound in (a) in the form:

$$\frac{\lambda(\mu)}{\mu} - 1 < -\frac{1}{\alpha\mu - 1}.$$

Therefore

$$\frac{\alpha}{\mu}\lambda(\mu)^2 - \alpha\lambda(\mu) + 1 < 1 - \frac{\alpha\lambda(\mu)}{\alpha\mu - 1}$$

or

$$e^{-\alpha\lambda(\mu)} < \frac{\alpha\mu - 1 - \alpha\lambda(\mu)}{\alpha\mu - 1};$$

Thus

$$e^{\alpha\lambda(\mu)}(\alpha\mu - 1 - \alpha\lambda(\mu)) > \alpha\mu - 1 ,$$

or equivalently

$$y(\mu) > \alpha\mu - 1 .$$

□

We now obtain the range of y consistent with $x = \mu$ and $\rho(\lambda) > 0$.

Lemma 4.3 *Let $g_1(\lambda; y)$ be as in (4.22); then $g_1(\lambda; y) > 0$ for all $\lambda > 0$ if and only if $y > y(\mu)$.*

Proof: Let $g_2(\lambda) = g_1(\lambda; y(\mu))$; $\lambda(\mu)$ is defined in such a way that $g_2(\lambda(\mu)) = 0$ and $g_2'(\lambda(\mu)) = 0$. Since $g_2(0) = 0$, $g_2'(0) = 1 - \alpha\mu + y(\mu) > 0$ by Lemma 4.2 and $g_2(\lambda) \to \infty$ as $\lambda \to \infty$, $g_2'(\lambda) = 0$ must have two solutions λ_1, λ_2 with $0 < \lambda_1 < \lambda_2$ say; λ_1 must be a local maximum and λ_2 a local minimum. But then $g_2(\lambda_1) > 0$ and therefore $\lambda(\mu) = \lambda_2$ so that $g_2(\lambda) \geq \min(0, g_2(\lambda(\mu)) = 0$. We thus have that if $y > y(\mu)$,

$$g_1(\lambda; y) > g_2(\lambda) \geq 0 \text{ for } \lambda > 0.$$

Conversely if $y \leq y(\mu)$, $g_1(\lambda(\mu); y) \leq g_1(\lambda(\mu)) = 0$

□

For $y > y(\mu)$,

$$\gamma u_0 \int_0^\infty \frac{\lambda e^{-\alpha\lambda}}{e^{\beta g_1(\lambda;y)} - 1} dF(\lambda)$$

is strictly decreasing as a function of y; it tends to ∞ as $y \to y(\mu)$ since $g_2(\lambda(\mu)) = 0$ and tends to 0 as $y \to \infty$. Therefore the equation

$$y = \gamma u_0 \int_0^\infty \frac{\lambda e^{-\alpha\lambda}}{e^{\beta g_1(\lambda;y)} - 1} dF(\lambda) \tag{4.25}$$

has a unique solution $y_0(\mu) > y(\mu)$.

To introduce the next lemma we make some heuristic remarks. As $\beta \to \infty$ the integral in (4.25) tends to 0 if $y > y(\mu)$; thus as $\beta \to \infty, y_0(\mu) \to y(\mu)$. As $\beta \to \infty$ then the integrand in (4.25) with $y = y_0(\mu)$ must peak around $\lambda = \lambda(\mu)$ and must behave like $\dfrac{y(\mu)}{\gamma u_0}\delta(\lambda - \lambda(\mu))$. But then $u_0 \displaystyle\int_0^\infty \frac{e^{-\alpha\lambda}}{e^{\beta g_1(\lambda;y_0(\mu))} - 1} dF(\lambda) \sim \frac{y(\mu)}{\gamma\lambda(\mu)}$. If the latter quantity exceeds μ then there is no condensation since this contradicts (4.18); if it is less than μ, then $x = \mu$, $y = y_0(\mu)$ is a solution of (4.18) and (4.19) with $\rho_0 > 0$ and therefore there is condensation for large β.

Lemma 4.4 *As $\beta \to \infty$, $y_0(\mu) \to y(\mu)$ and*

$$u_0 \int_0^\infty \frac{e^{-\alpha\lambda}}{e^{\beta g_1(\lambda;y_0(\mu))} - 1} dF(\lambda) \to \frac{y(\mu)}{\gamma\lambda(\mu)}.$$

Proof: Given ϵ choose β_0 such that

$$u_0\gamma \int_0^\infty \frac{\lambda e^{-\alpha\lambda}}{e^{\beta_0 g_1(\lambda;y(\mu)+\epsilon)} - 1} dF(\lambda) < y(\mu) + \epsilon ,$$

and suppose $y_0(\mu) > y(\mu) + \epsilon$. Then for $\beta > \beta_0$ we have

$$\begin{aligned} y_0(\mu) &= u_0\gamma \int_0^\infty \frac{\lambda e^{-\alpha\lambda}}{e^{\beta g_1(\lambda; y_0(\mu))} - 1} dF(\lambda) \\ &< u_0\gamma \int_0^\infty \frac{\lambda e^{-\alpha\lambda}}{e^{\beta_0 g_1(\lambda; y(\mu)+\epsilon)} - 1} dF(\lambda) \\ &< y(\mu) + \epsilon < y_0(\mu). \end{aligned}$$

Since this is a contradiction we must have $y_0(\mu) \leq y(\mu) + \epsilon$ for all $\beta > \beta_0$.

It is clear that for any $\delta > 0$

$$\lim_{\beta\to\infty} \int_{(\lambda(\mu)-\delta, \lambda(\mu)+\delta)^c} \frac{e^{-\alpha\lambda}}{e^{\beta g_1(\lambda; y_0(\mu))} - 1} dF(\lambda) = 0$$

and

$$\lim_{\beta\to\infty} \int_{(\lambda(\mu)-\delta, \lambda(\mu)+\delta)^c} \frac{\lambda e^{-\alpha\lambda}}{e^{\beta g_1(\lambda; y_0(\mu))} - 1} dF(\lambda) = 0;$$

thus we have

$$\begin{aligned} \liminf_{\beta\to\infty} u_0 \int_0^\infty \frac{e^{-\alpha\lambda}}{e^{\beta g_1(\lambda; y_0(\mu))} - 1} dF(\lambda) &\geq \liminf_{\beta\to\infty} u_0 \int_{\lambda(\mu)-\delta}^{\lambda(\mu)+\delta} \frac{e^{-\alpha\lambda} dF(\lambda)}{e^{\beta g_1(\lambda; y_0(\mu))} - 1} \\ &\geq \lim_{\beta\to\infty} \frac{y_0(\mu)}{\gamma(\lambda(\mu)+\delta)} = \frac{y(\mu)}{\gamma(\lambda(\mu)+\delta)} \end{aligned}$$

and

$$\limsup_{\beta\to\infty} u_0 \int_0^\infty \frac{e^{-\alpha\lambda}}{e^{\beta g_1(\lambda; y_0(\mu))} - 1} dF(\lambda) \leq \frac{y(\mu)}{\gamma(\lambda(\mu)-\delta)}.$$

Since δ is arbitrary, we have proved the lemma.

$\square$

Lemma 4.5 *Let $f(\mu) = \frac{y(\mu)}{\lambda(\mu)}$. For $\mu > 2\alpha^{-1}$ there is $\beta(\mu) > 0$ such that for $\beta > \beta(\mu)$ there is condensation for $f(\mu) < \gamma\mu$ and there is no condensation for $f(\mu) > \gamma\mu$.*

Proof: Suppose first $f(\mu) > \gamma\mu$ and suppose there is a sequence β_n increasing to ∞ such that for $\beta = \beta_n$ there is condensation; for n sufficiently large, by Lemma 4.4,

$$u_0 \int_0^\infty \frac{e^{-\alpha\lambda}}{e^{\beta_n g_1(\lambda; y_0(\mu))} - 1} dF(\lambda) > \mu.$$

This contradicts (4.18) and therefore there is no condensation for β large enough.

On the otherhand, if $f(\mu) < \gamma\mu$, for β sufficiently large

$$u_0 \int_0^\infty \frac{e^{-\alpha\lambda}}{e^{\beta g_1(\lambda; y_0(\mu))} - 1} dF(\lambda) < \mu$$

and therefore

$$x = \mu$$
$$y = y_0(\mu)$$
$$\rho_0 = \mu - u_0 \int_0^\infty \frac{e^{-\alpha\lambda}}{e^{\beta g_1(\lambda; y_0(\mu))} - 1} dF(\lambda)$$

is a solution of (4.18) and (4.19).

□

In the preceding lemma we saw that to determine whether there is condensation we have to compare $f(\mu)$ with $\gamma\mu$; this is done in the following lemma:

Lemma 4.6 *There is a critical value γ_c such that, for $\gamma < \gamma_c$, $f(\mu) > \gamma\mu$ for all $\mu > 2\alpha^{-1}$; for $\gamma > \gamma_c$ there are two values of μ, μ_1 and μ_2, $2\alpha^{-1} < \mu_1 < \mu_2$, such that $f(\mu) > \gamma\mu$ if $\mu \in (2\alpha^{-1}, \mu_1) \cup (\mu_2, \infty)$ and $f(\mu) < \gamma\mu$ if $\mu \in (\mu_1, \mu_2)$.*

Proof: We first check that $f(\mu)$ is strictly convex by showing that $f''(\mu) > 0$ for $\mu > 2\alpha^{-1}$. Using the relation

$$\lambda'(\mu) = \frac{\lambda(\mu)}{\mu(2 - \alpha\mu + \alpha\lambda(\mu))}$$

we find that

$$f'(\mu) = \frac{e^{\alpha\lambda(\mu)}}{\lambda(\mu)} \frac{\alpha\mu - \alpha\lambda + 1 - \alpha^2(\mu - \lambda(\mu))^2}{\mu(2 - \alpha\mu + \alpha\lambda(\mu))}$$

$$f''(\lambda) = \frac{e^{\alpha\lambda(\mu)}}{\mu^2\lambda(\mu)}\{\alpha\mu[2 - \alpha\mu + \alpha\lambda(\mu)]\}^{-3}\{\alpha^2\mu^2[2\alpha\lambda(\mu) - 1]$$
$$+\alpha\mu[5 - 4\alpha\lambda(\mu) - 4\alpha^2\lambda(\mu)^2] + 2\alpha^3\lambda(\mu)^3 + 5\alpha^2\lambda(\mu)^2 - 6\}. \tag{4.26}$$

Since by Lemma 4.2 (a) the denominator in (4.26) is positive, it is sufficient to prove that the numerator in (4.26) is strictly positive; using (4.23) we eliminate μ from the numerator to write the latter as:

$$e^{-\alpha\lambda(\mu)}\{e^{-\alpha\lambda(\mu)} + \alpha\lambda(\mu) - 1\}^{-2} g(\alpha\lambda(\mu))$$

where $g : (0, \infty) \to \mathbf{R}$ is defined by

$$g(x) = e^{2x}(x^3 - 6x^2 + 12x - 6) + e^x(2x^3 - 5x^2 - 12x + 12) + (2x^3 + 5x^2 - 6).$$

We can expand g in powers of x:

$$g(x) = \sum_{n=5}^{\infty} c_n x^n,$$

where

$$c_n = (n!)^{-1}[2^{n-3}(n^3 - 15n^2 + 62n - 48) + (2n^3 - 11n^2 - 3n + 12)].$$

It is easy to check that the cubic polynomial $z^3 - 15z^2 + 62z - 48$ is increasing for $z > 7.08$ and takes the value zero at $z = 8$, on the other hand the cubic polynomial $2z^3 - 11z^2 - 3z + 12$ is increasing for $z > 3.8$ and takes the value 30 at $z = 6$. Therfore both polynomials are positive for $z \geq 8$; thus $c_n > 0$ for $n \geq 8$. We also have $c_5 = \frac{1}{6}$, $c_6 = \frac{1}{24}$, $c_7 = \frac{1}{120}$. Therefore $c_n > 0$ for $n \geq 5$ and so $g(x) > 0$ for all $x > 0$ and consequently $f''(\mu) > 0$ for $\mu > 2\alpha^{-1}$. Now as $\mu \to 2\alpha^{-1}$, $\lambda(\mu) \to 0$ and $y(\mu) \to 1$ so that $f(\mu) \to \infty$. On the other hand if $\alpha\mu > 3$

$$\alpha\mu - \alpha\lambda(\mu) - 1 > \alpha\mu\frac{2\alpha\mu - 3}{\alpha\mu - 1} - 1 > \frac{(\alpha\mu)^2 - \alpha\mu}{\alpha\mu} > \frac{2}{3}\alpha\mu.$$

Therefore $f(\mu) > \frac{2e^{\alpha\lambda(\mu)}}{\lambda(\mu)}\alpha\mu$ and thus $f(\mu) \to \infty$ exponentially as $\mu \to \infty$. The lemma now follows immediately.

□

We now collect the results of the last six lemmas in a theorem giving the low temperature behaviour of the model. With γ_c, μ_1, μ_2 as in the last lemma we have:

Theorem 7. *The perturbed meanfield model with interaction given by the kernel in (4.6) has the following low temperature behaviour:*
(a) *If $\gamma < \gamma_c$, then for each $\mu > 0$, $\mu \neq \alpha^{-1}$, $\mu \neq 2\alpha^{-1}$ there is a $\beta(\mu) > 0$ such that for $\beta > \beta(\mu)$ the model exhibits condensation for $\mu < \alpha^{-1}$ and no condensation for $\mu > \alpha^{-1}$, $\mu \neq 2\alpha^{-1}$.*
(b) *If $\gamma > \gamma_c$, then for each $\mu > 0$, $\mu \neq \alpha^{-1}$, $\mu \neq 2\alpha^{-1}$, $\mu \neq \mu_1$, $\mu \neq \mu_2$ there is a $\beta(\mu) > 0$ such that if $\beta > \beta(\mu)$ there is condensation for $\mu \in (0, \alpha^{-1}) \cup (\mu_1, \mu_2)$ and no condensation for $\mu \in (\alpha^{-1}, \mu_1) \cup (\mu_2, \infty)$, $\mu \neq 2\alpha^{-1}$.*

Proof: For $\mu \in (0, 2\alpha^{-1})$ the result follows from Lemma 4.1. For $\mu > 2\alpha^{-1}$ we obtain the result by combining Lemmas 4.5 and 4.6.

□

We finally come to the Gaussian kernel (4.1). The spherically averaged kernel corresponding to (4.1) is

$$u(\lambda, \lambda') = u_0 e^{-\alpha(\lambda+\lambda')}\frac{\sinh 2\alpha\sqrt{\lambda\lambda'}}{2\alpha\sqrt{\lambda\lambda'}}. \tag{4.27}$$

We are not able to give the full low temperature behaviour in this case; we can deal only with the range of chemical potential $\mu < \tilde{\mu}_0$ where $\tilde{\mu}_0 \geq (1 + \frac{1}{2}\sqrt{10})\alpha^{-1}$ is determined in the following manner: For $\lambda \geq 0$ let

$$\tilde{g}(\lambda) = \lambda - \mu + e^{-\alpha\lambda}\left[\mu + (\alpha\mu - 1)\lambda + \frac{3}{10\mu}(\alpha\mu - 1)^2\lambda^2\right].$$

$\tilde{g}(0) = 0$, $\tilde{g}'(0) = 0$ and

$$\tilde{g}''(\lambda) = \frac{e^{-\alpha\lambda}}{10\mu}\{2(3 + 4\alpha\mu - 2\alpha^2\mu^2) + 2\alpha(\alpha\mu - 1)(6 - \alpha\mu)\lambda + 3\alpha^2(\alpha\mu - 1)^2\lambda^2\}.$$

Thus for $1 < \alpha\mu < 1 + \frac{1}{2}\sqrt{10}$, $\tilde{g}$ is convex, increasing and $\tilde{g}(\lambda) > 0$ if $\lambda > 0$. Therfore there is a maximal interval $(\alpha^{-1}, \tilde{\mu}_0)$ with $\tilde{\mu}_0 \geq (1 + \frac{1}{2}\sqrt{10})\alpha^{-1}$ such that for $\mu \in (\alpha^{-1}, \tilde{\mu}_0)$, $\tilde{g}(\lambda) > 0$ if $\lambda > 0$.

The methods used in Theorem 8 are those of Theorem 6 and Lemma 4.1.

Theorem 8. *For the perturbed meanfield model with interaction kernel given by (4.1) the following holds: If $\mu \in (0, \tilde{\mu}_0)$, $\mu \neq \alpha^{-1}$, then there exists $\beta(\mu) > 0$ such that if $\beta > \beta(\mu)$ the model exhibits Bose-Einstein condensation for $\mu < \alpha^{-1}$ and no condensation for $\mu > \alpha^{-1}$.*

Proof: Let $m \in \tilde{E}$ be the minimizer of $\tilde{\mathcal{E}}^{\mu}$. Since $u(\lambda, \lambda') \geq u_0 e^{-\alpha(\lambda+\lambda')}$ we have $(Um)(\lambda) \geq e^{-\alpha\lambda}(Um)(0)$ and therefore for $\mu \in (0, \alpha^{-1})$ we can argue as in Theorem 6 to show that there is no condensation for $\beta > \beta(\mu) = \beta_c(\mu)$.

For $\mu > \alpha^{-1}$, we use the argument in Lemma 4.1 with a slight improvement. Let

$$g(\lambda) = \beta^{-1} s'(\rho(\lambda)) = \lambda - \mu + (Um)(\lambda);$$

if there is condensation $(Um)(0) = \mu$ and so $g(0) = 0$. Now $g'(0) = 1 - \alpha\mu + y$ where

$$y = \frac{2}{3}\alpha^2 u_0 \int_{[0,\infty)} \lambda e^{-\alpha\lambda} m(d\lambda);$$

thus $y \geq \alpha\mu - 1$. We also have the inequality

$$u(\lambda, \lambda') \geq u_0 e^{-\alpha(\lambda+\lambda')} \left[1 + \frac{2\alpha^2}{3}\lambda\lambda' + \frac{2}{15}\alpha^4\lambda^2\lambda'^2\right]$$

so that

$$g(\lambda) \geq \lambda - \mu + e^{-\alpha\lambda}[\mu + y\lambda + z\lambda^2]$$

where

$$z = \frac{2\alpha^4}{15} u_0 \int_{[0,\infty)} \lambda^2 e^{-\alpha\lambda} m(d\lambda).$$

Using the Schwarz inequality we have

$$\begin{aligned} y^2 &= \frac{4}{9}\alpha^4 u_0^2 \left(\int_{[0,\infty)} \lambda e^{-\alpha\lambda} m(d\lambda)\right)^2 \\ &\leq \frac{4}{9}\alpha^4 u_0^2 \left(\int_{[0,\infty)} \lambda^2 e^{-\alpha\lambda} m(d\lambda)\right)\left(\int_{[0,\infty)} e^{-\alpha\lambda} m(d\lambda)\right) \\ &= \frac{10}{3} z\mu; \end{aligned}$$

therefore $z > \frac{3y^2}{10\mu}$ and thus

$$g(\lambda) \geq \lambda - \mu + e^{-\alpha\lambda}\left[\mu + y\lambda + \frac{3}{10}\frac{y^2}{\mu}\lambda^2\right] \geq \tilde{g}(\lambda).$$

Since $\tilde{g}(0) = 0$, $\tilde{g}'(0) = 0$ and for $\mu \in (\alpha^{-1}, \tilde{\mu}_0)$, $\tilde{g}(\lambda) > 0$ if $\lambda > 0$, the rest of the proof is the same as in Lemma 4.1.

□

The behaviour of the models discussed above is in marked contrast with the meanfield model of a Bose-gas; in the latter v is a constant equal to v_0 say. Let $\rho_c(\beta)$ be the free Bose-gas critical density at inverse temperature β, that is

$$\rho_c(\beta) = \int_0^\infty \frac{1}{e^{\beta\lambda} - 1} dF(\lambda), \tag{4.28}$$

and for $\mu > 0$ let $\beta_c(\mu)$ be the unique value of β such that $\mu = v_0\rho_c(\beta)$. We have proved [4, 5] that for $\beta > \beta_c(\mu)$ the meanfield model exhibits Bose-Einstein condensation while for $\beta \leq \beta_c(\mu)$ it does not. We find that a similar behaviour occurs also for the perturbed meanfield models with v given by

$$v(k, k') = v_0 e^{-\delta|\|k\|^2 - \|k'\|^2|} \tag{4.29}$$

and

$$v(k, k') = v_0 e^{-\delta(\|k\|^2 - \|k'\|^2)^2}. \tag{4.30}$$

In the case of (4.29) the variational problem can be solved completely by exploiting the fact that $u(\lambda, \lambda') = u_0 e^{-\alpha|\lambda - \lambda'|}$ is the kernel of the inverse of $\left(-\frac{d^2}{d\lambda^2} - \alpha^2\right)$. This is done in [12] . The other model with interaction given by (4.30) can be dealt with very simply by extending the idea of the proof of Theorem 4.1; we end with a theorem which gives the low temperature behaviour of this model.

Theorem 9. *If in the perturbed meanfield model the interaction is given by (4.30), then for all $\mu > 0$ there exists $\beta(\mu)$ such that for $\beta > \beta(\mu)$ there is Bose-Einstein condensation.*

Proof: The spherically averaged kernel is

$$u(\lambda, \lambda') = u_0 e^{-\alpha(\lambda - \lambda')^2}; \tag{4.31}$$

clearly for $\lambda, \lambda' \geq 0$, $u(\lambda, \lambda') \geq u_0 e^{-\alpha(\lambda^2 + \lambda'^2)}$ and therefore if $m \in \tilde{E}$ is the minimizer of $\tilde{\mathcal{E}}^\mu$ then $(Um)(\lambda) \geq e^{-\alpha\lambda^2}(Um)(0) \geq e^{-\alpha\lambda^2}\mu$. If as before we let

$$g(\lambda) = \beta^{-1}s'(\rho(\lambda)) = \lambda - \mu + (Um)(\lambda)$$

we have $g(\lambda) \geq \lambda - \mu + \mu e^{-\alpha\lambda^2}$. Now there exists $\mu_0 > 0$ such that if $\mu < \mu_0$ then $\lambda - \mu + \mu e^{-\alpha\lambda^2} > 0$ for $\lambda > 0$. Then for $\mu < \mu_0$, by the usual argument, we must have condensation for β sufficiently large.

If $\mu \geq \mu_0$ let λ_0 be the smallest value of λ greater than 0 for which

$$\lambda - \mu + \mu e^{-\alpha\lambda^2} = 0 ;$$

then $\lambda - \mu + \mu e^{-\alpha\lambda^2} > 0$ for $0 < \lambda < \lambda_0$ and therefore

$$\lim_{\beta\to\infty} u_0 \int_0^{\lambda_0/2} \frac{e^{-\alpha\lambda^2}}{e^{\beta g(\lambda)} - 1} dF(\lambda) = 0.$$

Suppose there is a sequence β_n increasing to infinity such that for $\beta = \beta_n$ there is no condensation. Choose n_0 such that for $n > n_0$

$$u_0 \int_0^{\lambda_0/2} \frac{e^{-\alpha\lambda^2}}{e^{\beta_n g(\lambda)} - 1} dF(\lambda) < \frac{\lambda_0}{4};$$

then

$$u_0 \int_{\lambda_0/2}^{\infty} \frac{e^{-\alpha\lambda^2}}{e^{\beta_n g(\lambda)} - 1} dF(\lambda) > \mu - \frac{\lambda_0}{4}.$$

It follows that

$$(Um)(\lambda) > u_0 \int_{\lambda_0/2}^{\infty} e^{2\alpha\lambda\lambda'} \frac{e^{-\alpha(\lambda^2+\lambda'^2)}}{e^{\beta_n g(\lambda')}-1} dF(\lambda')$$
$$> u_0 e^{\alpha\lambda\lambda_0} e^{-\alpha\lambda^2} \int_{\lambda_0/2}^{\infty} \frac{e^{-\alpha\lambda'^2}}{e^{\beta_n g(\lambda')}-1} dF(\lambda')$$
$$> e^{-\alpha(\lambda-\frac{1}{2}\lambda_0)^2} e^{\frac{1}{4}\alpha\lambda_0^2} \left(\mu - \frac{\lambda_0}{4}\right).$$

Thus for $\lambda \in (\frac{1}{2}\lambda_0, \lambda_0)$,

$$g(\lambda) > \lambda - \mu + e^{-\alpha(\lambda-\frac{1}{2}\lambda_0)^2} e^{\frac{1}{4}\alpha\lambda_0^2}(\mu - \frac{\lambda_0}{4})$$
$$> \frac{1}{2}\lambda_0 - \mu + (\mu - \frac{\lambda_0}{4}) = \frac{\lambda_0}{4} > 0;$$

consequently

$$\lim_{n\to\infty} u_0 \int_0^{\lambda_0} \frac{e^{-\alpha\lambda^2}}{e^{\beta_n g(\lambda)}-1} dF(\lambda) = 0.$$

We repeat the argument and choose n_1 such that for $n > n_1$

$$u_0 \int_0^{\lambda_0} \frac{e^{-\alpha\lambda^2}}{e^{\beta_n g(\lambda)}-1} dF(\lambda) < \frac{\lambda_0}{4} = 0$$

which implies that

$$u_0 \int_{\lambda_0}^{\infty} \frac{e^{-\alpha\lambda^2}}{e^{\beta_n g(\lambda)}-1} dF(\lambda) > \mu - \frac{\lambda_0}{4} = 0$$

and, as above,

$$(Um)(\lambda) > e^{-(\lambda-\lambda_0)} e^{\alpha\lambda_0^2}(\mu - \frac{\lambda_0}{4}).$$

For $\lambda \in (\lambda_0, 2\lambda_0)$

$$g(\lambda) > \lambda - \mu + e^{-\alpha(\lambda-\lambda_0)^2} e^{\alpha\lambda_0^2}(\mu - \frac{\lambda_0}{4})$$
$$> \lambda_0 - \mu + (\mu - \frac{\lambda_0}{4}) = \frac{3\lambda_0}{4} > 0$$

and so

$$\lim_{n\to\infty} u_0 \int_0^{2\lambda_0} \frac{e^{-\alpha\lambda^2}}{e^{\beta_n g(\lambda)}-1} dF(\lambda) = 0.$$

Choose N such that $2^N\lambda_0 > 2\mu$; then by repeating the argument N times we have

$$\lim_{n\to\infty} u_0 \int_0^{2\mu} \frac{e^{-\alpha\lambda^2}}{e^{\beta_n g(\lambda)}-1} dF(\lambda) = 0.$$

But

$$\lim_{n\to\infty} u_0 \int_{2\mu}^{\infty} \frac{e^{-\alpha\lambda^2}}{e^{\beta_n g(\lambda)}-1} dF(\lambda) < \lim_{n\to\infty} \frac{u_0}{e^{\beta_n\mu}-1} \int_{2\mu}^{\infty} e^{-\alpha\lambda^2} dF(\lambda) = 0,$$

and therefore $\lim_{n\to\infty}(Um)(0) = 0$ contradicting $(Um)(0) \geq \mu$. It follows that there exists $\beta(\mu)$ such that, for $\beta > \beta(\mu)$, there is condensation.

□

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