Zeitschrift:	Helvetica Physica Acta
Band:	64 (1991)
Heft:	7
Artikel:	A remark on long-range Stark scattering
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DOI:	https://doi.org/10.5169/seals-116335

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A remark on long-range Stark scattering

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(26. IV. 1991)

Abstract. We propose a modified wave operator for long-range scattering in presence of an accelerating force. Its relevance in connection with classical scattering is addressed.

^{*}Work supported in part by the U.S. NSF grant DMS-9101715

Introduction

The wave operators for the pair (H, H_0) of Stark Hamiltonians with $H = H_0 + V$ and

$$H_0 = rac{p^2}{2} - g \cdot x \;, \qquad 0
eq g \in \mathbf{R}^{\mu}$$

on $\mathcal{H} = L^2(\mathbb{R}^{\nu})$ do not exist [6] if V = V(x) behaves as $|x|^{-\varepsilon}$, $\varepsilon \leq 1/2$ as $x \to \infty$. Recently, several proposals [3, 5, 8] for modified wave operators dealing with slowly decaying potentials were made. Here we add one more, based on a simple Dollard type argument: The classical free Stark trajectories $x(t) = gt^2/2 + p_0t + x_0$ suggest that $e^{-itH_0}e^{-i\Phi(t)}$ with

$$\Phi(t) := \int_0^t V(gs^2/2) \, ds$$

is a candidate for a comparison dynamics.

Theorem 1. Let $V \in C^1(\mathbb{R}^{\nu})$ be real-valued with

$$|\nabla V(x)| \le C(1+|x|)^{-(1+\varepsilon)} \tag{1}$$

for some $C, \varepsilon > 0$. Then the wave operator

$$\Omega := \mathbf{s} - \lim_{t \to \infty} \mathbf{e}^{\mathbf{i}tH} \mathbf{e}^{-\mathbf{i}tH_0} \mathbf{e}^{-\mathbf{i}\Phi(t)}$$
⁽²⁾

exists and is unitary.

The reason for stating this result comes from [6]. There a discrepancy between the quantum and the classical scattering problem was discovered, as for in the latter one the wave operators exist and are complete without modification basically as long as $V(x) = O(|x|^{-\varepsilon})$ for some $\varepsilon > 0$. We believe our result heals this discrepancy for the class of potentials we consider, since the comparison dynamics differs multiplicatively from the free Stark one only by a phase, which is physically irrelevant. Actually, no modification at all is needed if the asymptotic condition is formulated in the Heisenberg picture: The above result then implies that

$$\mu(A) := \mathbf{s} - \lim_{t \to \infty} \mathbf{e}^{\mathbf{i}tH} \mathbf{e}^{-\mathbf{i}tH_0} A \mathbf{e}^{\mathbf{i}tH_0} e^{-\mathbf{i}tH}$$

exists and is an automorphism on $\mathcal{L}(\mathcal{H})$. By contrast, this limit typically exists in longrange scattering only for $A \in \{H_0\}'$, the von Neumann algebra of operators commuting with bounded functions of H_0 [1]. In this respect (1) behaves as a short-range potential.

It follows from (1) that $\lim_{x\to\infty} V(x)$ exists. We may assume this limit to be zero, since (2) is not affected if the potential is shifted by a constant. Then $|V(x)| \leq C(1+|x|)^{-\varepsilon}$, which is assumed also by [3, 5, 8]. However there stronger oscillations of the potential are allowed, since (1) is replaced by a weaker decay.

Remark. The hypothesis (1) can be weakened logarithmically [6].

Proofs

Let us state some kinematical remarks beforehand. The following propagation estimate for the free dynamics without electric field is well-known (see e.g. [4], Lemma 6.3): Let $f \in C_0^{\infty}(\mathbb{R}^{\nu})$ with f(y) = 0 for $|y| \ge 1$, and let $\alpha > 1$. Then for $R > 0, t \ge 0$ and any N > 0

$$\left\|F(|x| > \alpha(R+t))e^{-itp^2/2}f(p)F(|x| < R)\right\| \le C_N(R+t)^{-N}$$

with C_N independent of R, t. States supported in momentum space in $\{|y| \leq v\}, v > 0$ are accounted for through scaling by v > 0: We apply the unitary dilation U satisfying $UpU^{-1} = p/v, UxU^{-1} = vx$ to the operator in the estimate above, replace t by v^2t and Rby vR, the result being

$$\left\|F(|x| > \alpha(R+vt))e^{-itp^{2}/2}f(p/v)F(|x| < R)\right\| \le C_{N}[v(R+vt)]^{-N}$$

An estimate for the free Stark dynamics is readily obtained by means of the Avron-Herbst formula

$$e^{-itH_0} = T(t)e^{-itp^2/2}$$
, $T(t) := e^{-ig^2t^3/6}e^{itg\cdot x}e^{-it^2g\cdot p/2}$

and of $T(t)x = (x - gt^2/2)T(t)$, namely

$$\left\|F(|x - gt^2/2| > \alpha(R + vt))e^{-itH_0}f(p/v)F(|x| < R)\right\| \le C_N[v(R + vt)]^{-N}.$$
 (3)

Proof of Theorem 1 (existence). By Cook's theorem we have to show that

$$\int_0^\infty \|U(x,t)\mathrm{e}^{-\mathrm{i}tH_0}\varphi\|\,dt<\infty$$

for φ in a dense set D, where $U(x,t) := V(x) - V(gt^2/2)$. Choosing

$$D := \left\{ f(p/v)F(|x| < R)\psi \, \middle| \, f \in C_0^\infty(|y| < 1), \, v, R > 0, \, \psi \in \mathcal{H} \right\} \,,$$

this follows from

$$\|U(x,t)e^{-itH_0}f(p/v)F(|x| < R)\psi\| \le \|U(x,t)F(|x - gt^2/2| \le \alpha(R+vt))\|\|\psi\| + C'_N[v(R+vt)]^{-N}\|\psi\| = O(t^{-1-2\varepsilon})\|\psi\| .$$
(4)

Here we estimated the first term on the second line as follows: If $|x - gt^2/2| \leq \alpha(R + vt)$, then $U(x,t) = \nabla V(\tilde{x}) \cdot (x - gt^2/2)$ for some \tilde{x} with $|\tilde{x} - gt^2/2| \leq \alpha(R + vt)$. Then $|\tilde{x}| > |g|t^2/4$ for t large enough, and hence $\nabla V(\tilde{x}) = O(t^{-2(1+\epsilon)})$, proving (4).

Immediate consequences of the existence of the wave operator are that it is an isometry and that

$$\Omega = \mathbf{s} - \lim_{s \to \infty} \mathbf{e}^{\mathbf{i}sH} \mathbf{e}^{-\mathbf{i}sH_0} \mathbf{e}^{-\mathbf{i}\Phi(s+t)}$$
(5)

for any $t \in \mathbf{R}$, since $\Phi(s+t) - \Phi(s) = \int_{s}^{s+t} V(g\tau^{2}/2) d\tau \to 0$ as $s \to \infty$. By comparing (5) with the definition (2) we get the intertwining relation $e^{itH}\Omega = \Omega e^{itH_{0}}$.

Another formulation of (3) is obtained using

$$e^{-itH_0}p = (p - gt)e^{-itH_0}$$
, $e^{-itH_0}x = (x - pt + gt^2/2)e^{-itH_0}$

which is

$$\left\|F(|x-gt^2/2| > \alpha(R+vt))f((p-gt)/v)F(|x-pt+gt^2/2| < R)\right\| \le C_N[v(R+vt)]^{-N} .$$
(6)

The same remark also leads to

Corollary 2.

$$\lim_{t \to \infty} \left\| (\Omega - e^{-i\Phi(t)}) f((p - gt)/v) F(|x - pt + gt^2/2| < R) \right\| = 0.$$
 (7)

Proof. We apply Cook's estimate to (5) yielding

$$\left\| (\Omega - \mathrm{e}^{-\mathrm{i}\Phi(t)}) \mathrm{e}^{-\mathrm{i}tH_0}\varphi \right\| \le \int_0^\infty \|U(x, s+t) \mathrm{e}^{-\mathrm{i}(s+t)H_0}\varphi\|\,ds = \int_t^\infty \|U(x, s) \mathrm{e}^{-\mathrm{i}sH_0}\varphi\|\,ds \,.$$

We then set $\varphi = f(p/v)F(|x| < R)e^{itH_0}\psi$ and use (4).

Let us now address the completeness question. We recall [2] that any $V \in L^{\infty}(\mathbb{R}^{\nu})$ with $\lim_{x\to\infty} V(x) = 0$ is relatively compact with respect to H_0 and to H. This follows by density if it holds for V with compact support. In this case all terms in

$$V(H_0 + i)^{-1} - V(p^2/2 + i)^{-1} = V(p^2/2 + i)^{-1}g \cdot x(H_0 + i)^{-1}$$

= $Vg \cdot x(p^2/2 + i)^{-1}(H_0 + i)^{-1}$
+ $iV(p^2/2 + i)^{-1}g \cdot p(p^2/2 + i)^{-1}(H_0 + i)^{-1}$

are compact. We will also use that $\mathcal{H}_{cont} = \mathcal{H}$ for the spectral decomposition with respect to H [2]. Without using this result, one could prove that $\operatorname{Ran} \Omega = \mathcal{H}_{cont}$ instead of unitarity of Ω by suitably modifying the proofs below.

Lemma 3. For $f \in C_{\infty}(\mathbb{R}^{\nu})$

$$s - \lim_{t \to \infty} e^{itH} f\left(\frac{p - gt}{t}\right) e^{-itH} = f(0) , \qquad (8)$$

$$s - \lim_{t \to \infty} e^{itH} f\left(\frac{x - pt + gt^2/2}{t^2}\right) e^{-itH} = f(0) .$$
 (9)

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Proof. We first remark that $D(p) \cap D(x)$ is invariant under e^{-itH} and that

$$e^{itH}(p-gt)e^{-itH}\psi = p\psi - \int_0^t e^{isH}\nabla V(x)e^{-isH}\psi \,ds , \qquad (10)$$
$$e^{itH}(x-pt+gt^2/2)e^{-itH}\psi = x\psi + \int_0^t se^{isH}\nabla V(x)e^{-isH}\psi \,ds$$

for $\psi \in D(p) \cap D(x)$.

It suffices to prove the lemma for f depending on a single coordinate of its argument, since linear combinations of products of such functions are dense in $C_{\infty}(\mathbb{R}^{\nu})$. By [7], Theorem VIII.20 (b) it is enough to show that (8, 9) hold for $f(y) = (y_i - z)^{-1}$, $z \notin \mathbb{R}$ and moreover strong and weak convergence are equivalent for resolvents. We thus set $p(t) := e^{itH} p e^{-itH}$, $x(t) := e^{itH} x e^{-itH}$ and have to show that for $\varphi \in \mathcal{H}$, $\psi \in D(p) \cap D(x)$

$$\left(\varphi, \left[\left(\frac{p_i(t) - g_i t}{t} - z\right)^{-1} - (-z)^{-1}\right]\psi\right) = z^{-1}\left(\left(\frac{p_i(t) - g_i t}{t} - \bar{z}\right)^{-1}\varphi, \frac{p_i(t) - g_i t}{t}\psi\right),$$

vanishes as $t \to \infty$. Indeed, by (10) this is bounded by a constant times

$$\frac{1}{t} \|p_i\psi\| + \frac{1}{t} \int_0^t \|\partial_i V(x) \mathrm{e}^{-\mathrm{i}sH}\psi\| \, ds \xrightarrow[t \to \infty]{} 0 \, .$$

The limit just stated holds by the RAGE theorem ([7], Theorem XI.115), because $\partial_i V$ is relatively compact with respect to H. The other observable is treated analogously.

Lemma 3 is still a rather weak result, since it roughly says that p(t) - gt = o(t), $x(t) - p(t)t + gt^2/2 = o(t^2)$, while these quantities are constants of motion for the free Stark problem. The idea to improve this is the following: Let

$$S_t := \left\{ (x, p) \, \big| \, |p - gt| < \delta t_0, \, |x - pt + gt^2/2| < \delta t_0^2 \right\}$$

for some $\delta, t_0 > 0$. Classically S_{t_0} at time t_0 is mapped into S_t at time $t \ge t_0$ under the free Stark evolution. Moreover

$$|x - gt^2/2| \le |x - pt + gt^2/2| + t|p - gt| < \delta t_0(t + t_0) \le 2\delta t^2 ,$$

$$|x| > (|g|/2 - 2\delta)t^2$$

i.e. the particle is far from the scatterer if $\delta < |g|/4$. Therefore S_{t_0} should be mapped approximately in S_t also under the full dynamics.

Let $f \in C_0^{\infty}(\mathbb{R}^{\nu})$ with f(y) = 0 for $|y| \ge \delta$ and set

$$f_1(t,t_0) := f\left(\frac{x - pt + gt^2/2}{t_0^2}\right), \quad f_2(t,t_0) := f\left(\frac{p - gt}{t_0}\right),$$

$$\Theta(t,t_0) := f_1(t,t_0)^* f_2(t,t_0)^* f_2(t,t_0) f_1(t,t_0).$$
(11)

Lemma 4. If $\delta < |g|/4$, then

$$\lim_{t_0 \to \infty} \sup_{t \ge t_0} \| e^{itH} \Theta(t, t_0) e^{-itH} - e^{it_0 H} \Theta(t_0, t_0) e^{-it_0 H} \| = 0.$$
(12)

Proof. The supremum in (12) is bounded by

$$\int_{t_0}^{\infty} \left\| D\Theta(t, t_0) \right\| dt , \qquad (13)$$

where $D \cdot = i[H, \cdot] + \partial \cdot / \partial t$, i.e.

$$D\Theta = (Df_1)^* f_2^* f_2 f_1 + f_1^* (Df_2)^* f_2 f_1 + f_1^* f_2^* (Df_2) f_1 + f_1^* f_2^* f_2 (Df_1) .$$

1

We start by computing

$$Df_1 = \int \hat{f}(s)i[V, e^{i(x-pt+gt^2/2)t_0^{-2} \cdot s}] \, ds = i \int \hat{f}(s)e^{i(x-pt+gt^2/2)t_0^{-2} \cdot s}(V(x+t_0^{-2}ts)-V(x)) \, ds$$

and split the integral into |s| > t and $|s| \le t$. The contribution to (13) of the first part is bounded by $O(t_0^{-N})$ for all N > 0, due to the decay of \hat{f} . For the other part, (6) implies that for $1 < \alpha < |g|(4\delta)^{-1}$

$$f_2^* f_2 f_1 = F(|x - gt^2/2| < \alpha \delta t_0(t + t_0)) f_2^* f_2 f_1 + O([t_0^2(t_0 + t)]^{-N}),$$

where the remainder contributes to (13) as little as $O(t_0^{-N})$. One is then left with estimating

$$\int_{|s| \le t} |\hat{f}(s)| \| (V(x + t_0^{-2} ts) - V(x)) F(|x - gt^2/2| < \alpha \delta t_0(t + t_0)) \| ds \le \text{const} \| s \hat{f} \|_1 t_0^{-2} t^{-(1 + 2\varepsilon)},$$

contributing $O(t_0^{-2(1+\epsilon)})$ to (13). Here we used that if $|x - gt^2/2| < \alpha \delta t_0(t+t_0) \leq 2\alpha \delta t^2$ then $|V(x + t_0^{-2}ts) - V(x)| \leq t_0^{-2}t|s||\nabla V(\tilde{x})|$ for some \tilde{x} with

$$|\tilde{x}| \ge |g|t^2/2 - |x - gt^2/2| - |x - \tilde{x}| \ge t^2(|g|/2 - 2\alpha\delta - t_0^{-2}),$$

where the bracket is positive for large t_0 . Terms arising from

$$Df_{2} = i \int \hat{f}(s) e^{i(p-gt)t_{0}^{-1} \cdot s} (V(x - t_{0}^{-1}s) - V(x)) \, ds$$

are dealt with similarly. The integral restricted to $|s| \leq t$ is estimated up to a constant by $\|s\hat{f}\|_1 t_0^{-1} t^{-2(1+\epsilon)}$, its contribution to (13) thus by $O(t_0^{-2(1+\epsilon)})$.

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Lemma 5. Let f be as in the previous lemma, and real-valued with $f \leq 1$, f(0) = 1 in addition. Then

$$\lim_{t_0 \to \infty} \sup_{t \ge t_0} \| (1 - f_2(t, t_0) f_1(t, t_0)) e^{-itH} \psi \| = 0$$
(14)

for any $\psi \in \mathcal{H}$.

Proof. Equations (8) and (9) imply $(e^{-it_0H}\psi, \Theta(t_0, t_0)e^{-it_0H}\psi) \to (\psi, \psi)$ as $t_0 \to \infty$. Then

$$\sup_{t \ge t_0} (\psi, \psi) - (\mathrm{e}^{-\mathrm{i}tH}\psi, \Theta(t, t_0) \mathrm{e}^{-\mathrm{i}tH}\psi) \xrightarrow[t_0 \to \infty]{} 0$$
(15)

follows from (12). Suppressing arguments (t, t_0) we have $1 - f_2 f_1 = (1 - f_1) + (1 - f_2)f_1$ and

$$\begin{aligned} \|(1-f_1)e^{-itH}\psi\|^2 + \|(1-f_2)f_1e^{-itH}\psi\|^2 \\ &\leq (\|\psi\|^2 - \|f_1e^{-itH}\psi\|^2) + (\|f_1e^{-itH}\psi\|^2 - \|f_2f_1e^{-itH}\psi\|^2) \\ &= (\psi,\psi) - (e^{-itH}\psi,\Theta(t,t_0)e^{-itH}\psi) , \end{aligned}$$

since $f_i^2 \leq f_i$, i = 1, 2. Hence (14) is immediate from (15).

Proof of Theorem 1 (completeness). Given $\psi \in \mathcal{H}$,

$$\begin{aligned} \|(\Omega - e^{-i\Phi(t)})e^{-itH}\psi\| \\ &\leq \|(\Omega - e^{-i\Phi(t)})f_2(t, t_0)f_1(t, t_0)e^{-itH}\psi\| + 2\|(1 - f_2(t, t_0)f_1(t, t_0))e^{-itH}\psi\| \end{aligned}$$

can be made arbitrarily small due to (7, 14) by first choosing t_0 and then t large enough. Thus

$$\psi = e^{itH} \Omega e^{i\Phi(t)} e^{-itH} \psi + o(1) = \Omega e^{itH_0} e^{i\Phi(t)} e^{-itH} \psi + o(1)$$

as $t \to \infty$, implying $\psi \in \overline{\operatorname{Ran} \Omega} = \operatorname{Ran} \Omega$.

Acknowledgements. We thank B. Simon for pointing out to us Ref. [1], and A. Jensen for useful discussions.

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