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A remark on long-range Stark scattering

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Abstract. We propose a modified wave operator for long-range scattering in presence of an accelerating force. Its relevance in connection with classical scattering is addressed.

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Introduction

The wave operators for the pair (H, H_0) of Stark Hamiltonians with $H = H_0 + V$ and

$$H_0 = \frac{p^2}{2} - g \cdot x \, , \qquad 0 \neq g \in \mathbf{R}^\nu$$

on $\mathcal{H} = L^2(\mathbf{R}^\nu)$ do not exist [6] if $V = V(x)$ behaves as $|x|^{-\varepsilon}$, $\varepsilon \leq 1/2$ as $x \to \infty$. Recently, several proposals [3, 5, 8] for modified wave operators dealing with slowly decaying potentials were made. Here we add one more, based on a simple Dollard type argument: The classical free Stark trajectories $x(t) = gt^2/2 + p_0 t + x_0$ suggest that $\mathrm{e}^{-\mathrm{i}tH_0}\mathrm{e}^{-\mathrm{i}\Phi(t)}$ with

$$\Phi(t) := \int_0^t V(gs^2/2)\, ds$$

is a candidate for a comparison dynamics.

Theorem 1. *Let $V \in C^1(\mathbf{R}^\nu)$ be real-valued with*

$$|\nabla V(x)| \leq C(1 + |x|)^{-(1+\varepsilon)} \tag{1}$$

for some $C, \varepsilon > 0$. Then the wave operator

$$\Omega := \mathrm{s-}\lim_{t \to \infty} \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_0} \mathrm{e}^{-\mathrm{i}\Phi(t)} \tag{2}$$

exists and is unitary.

The reason for stating this result comes from [6]. There a discrepancy between the quantum and the classical scattering problem was discovered, as for in the latter one the wave operators exist and are complete without modification basically as long as $V(x) = \mathrm{O}(|x|^{-\varepsilon})$ for some $\varepsilon > 0$. We believe our result heals this discrepancy for the class of potentials we consider, since the comparison dynamics differs multiplicatively from the free Stark one only by a phase, which is physically irrelevant. Actually, no modification at all is needed if the asymptotic condition is formulated in the Heisenberg picture: The above result then implies that

$$\mu(A) := \mathrm{s-}\lim_{t \to \infty} \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_0} A \mathrm{e}^{\mathrm{i}tH_0} e^{-\mathrm{i}tH}$$

exists and is an automorphism on $\mathcal{L}(\mathcal{H})$. By contrast, this limit typically exists in long-range scattering only for $A \in \{H_0\}'$, the von Neumann algebra of operators commuting with bounded functions of H_0 [1]. In this respect (1) behaves as a short-range potential.

It follows from (1) that $\lim_{x \to \infty} V(x)$ exists. We may assume this limit to be zero, since (2) is not affected if the potential is shifted by a constant. Then $|V(x)| \leq C(1+|x|)^{-\varepsilon}$, which is assumed also by [3, 5, 8]. However there stronger oscillations of the potential are allowed, since (1) is replaced by a weaker decay.

Remark. The hypothesis (1) can be weakened logarithmically [6].

Proofs

Let us state some kinematical remarks beforehand. The following propagation estimate for the free dynamics without electric field is well-known (see e.g. [4], Lemma 6.3): Let $f \in C_0^\infty(\mathbf{R}^\nu)$ with $f(y) = 0$ for $|y| \geq 1$, and let $\alpha > 1$. Then for $R > 0, t \geq 0$ and any $N > 0$

$$\left\| F(|x| > \alpha(R+t))\mathrm{e}^{-itp^2/2} f(p) F(|x| < R) \right\| \leq C_N (R+t)^{-N} ,$$

with C_N indepedent of R, t. States supported in momentum space in $\{|y| \leq v\}$, $v > 0$ are accounted for through scaling by $v > 0$: We apply the unitary dilation U satisfying $UpU^{-1} = p/v$, $UxU^{-1} = vx$ to the operator in the estimate above, replace t by $v^2 t$ and R by vR, the result being

$$\left\| F(|x| > \alpha(R+vt))\mathrm{e}^{-itp^2/2} f(p/v) F(|x| < R) \right\| \leq C_N [v(R+vt)]^{-N} .$$

An estimate for the free Stark dynamics is readily obtained by means of the Avron-Herbst formula

$$\mathrm{e}^{-itH_0} = T(t)\mathrm{e}^{-itp^2/2} , \quad T(t) := \mathrm{e}^{-ig^2t^3/6}\mathrm{e}^{itg\cdot x}\mathrm{e}^{-it^2 g\cdot p/2}$$

and of $T(t)x = (x - gt^2/2)T(t)$, namely

$$\left\| F(|x - gt^2/2| > \alpha(R+vt))\mathrm{e}^{-itH_0} f(p/v) F(|x| < R) \right\| \leq C_N [v(R+vt)]^{-N} . \tag{3}$$

Proof of Theorem 1 (existence). By Cook's theorem we have to show that

$$\int_0^\infty \| U(x,t)\mathrm{e}^{-itH_0}\varphi \| \, dt < \infty$$

for φ in a dense set D, where $U(x,t) := V(x) - V(gt^2/2)$. Choosing

$$D := \left\{ f(p/v)F(|x| < R)\psi \,\middle|\, f \in C_0^\infty(|y| < 1),\ v, R > 0,\ \psi \in \mathcal{H} \right\} ,$$

this follows from

$$\begin{aligned} &\| U(x,t)\mathrm{e}^{-itH_0} f(p/v) F(|x| < R)\psi \| \\ &\leq \| U(x,t) F(|x - gt^2/2| \leq \alpha(R+vt)) \| \|\psi\| + C_N' [v(R+vt)]^{-N} \|\psi\| = \mathrm{O}(t^{-1-2\varepsilon}) \|\psi\| . \end{aligned} \tag{4}$$

Here we estimated the first term on the second line as follows: If $|x - gt^2/2| \leq \alpha(R+vt)$, then $U(x,t) = \nabla V(\tilde{x}) \cdot (x - gt^2/2)$ for some $\tilde{x}$ with $|\tilde{x} - gt^2/2| \leq \alpha(R+vt)$. Then $|\tilde{x}| > |g|t^2/4$ for t large enough, and hence $\nabla V(\tilde{x}) = \mathrm{O}(t^{-2(1+\varepsilon)})$, proving (4). □

Immediate consequences of the existence of the wave operator are that it is an isometry and that

$$\Omega = \mathrm{s-}\lim_{s \to \infty} \mathrm{e}^{isH}\mathrm{e}^{-isH_0}\mathrm{e}^{-i\Phi(s+t)} \tag{5}$$

for any $t \in \mathbf{R}$, since $\Phi(s+t) - \Phi(s) = \int_s^{s+t} V(g\tau^2/2)\, d\tau \to 0$ as $s \to \infty$. By comparing (5) with the definition (2) we get the intertwining relation $\mathrm{e}^{\mathrm{i}tH}\Omega = \Omega \mathrm{e}^{\mathrm{i}tH_0}$.

Another formulation of (3) is obtained using

$$\mathrm{e}^{-\mathrm{i}tH_0} p = (p - gt)\mathrm{e}^{-\mathrm{i}tH_0} \,, \quad \mathrm{e}^{-\mathrm{i}tH_0} x = (x - pt + gt^2/2)\mathrm{e}^{-\mathrm{i}tH_0} \,,$$

which is

$$\|F(|x - gt^2/2| > \alpha(R + vt)) f((p - gt)/v) F(|x - pt + gt^2/2| < R)\| \le C_N [v(R + vt)]^{-N} \,. \tag{6}$$

The same remark also leads to

Corollary 2.

$$\lim_{t\to\infty} \|(\Omega - \mathrm{e}^{-\mathrm{i}\Phi(t)}) f((p - gt)/v) F(|x - pt + gt^2/2| < R)\| = 0 \,. \tag{7}$$

Proof. We apply Cook's estimate to (5) yielding

$$\|(\Omega - \mathrm{e}^{-\mathrm{i}\Phi(t)})\mathrm{e}^{-\mathrm{i}tH_0}\varphi\| \le \int_0^\infty \|U(x, s + t)\mathrm{e}^{-\mathrm{i}(s+t)H_0}\varphi\|\, ds = \int_t^\infty \|U(x, s)\mathrm{e}^{-\mathrm{i}sH_0}\varphi\|\, ds \,.$$

We then set $\varphi = f(p/v)F(|x| < R)\mathrm{e}^{\mathrm{i}tH_0}\psi$ and use (4). □

Let us now address the completeness question. We recall [2] that any $V \in L^\infty(\mathbf{R}^\nu)$ with $\lim_{x\to\infty} V(x) = 0$ is relatively compact with respect to H_0 and to H. This follows by density if it holds for V with compact support. In this case all terms in

$$\begin{aligned} V(H_0 + \mathrm{i})^{-1} - V(p^2/2 + \mathrm{i})^{-1} &= V(p^2/2 + \mathrm{i})^{-1} g \cdot x (H_0 + \mathrm{i})^{-1} \\ &= V g \cdot x (p^2/2 + \mathrm{i})^{-1}(H_0 + \mathrm{i})^{-1} \\ &\quad + \mathrm{i}V(p^2/2 + \mathrm{i})^{-1} g \cdot p (p^2/2 + \mathrm{i})^{-1}(H_0 + \mathrm{i})^{-1} \end{aligned}$$

are compact. We will also use that $\mathcal{H}_{cont} = \mathcal{H}$ for the spectral decomposition with respect to H [2]. Without using this result, one could prove that $\operatorname{Ran}\Omega = \mathcal{H}_{cont}$ instead of unitarity of Ω by suitably modifying the proofs below.

Lemma 3. *For $f \in C_\infty(\mathbf{R}^\nu)$*

$$\text{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} f\Big(\frac{p - gt}{t}\Big)\mathrm{e}^{-\mathrm{i}tH} = f(0) \,, \tag{8}$$

$$\text{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} f\Big(\frac{x - pt + gt^2/2}{t^2}\Big)\mathrm{e}^{-\mathrm{i}tH} = f(0) \,. \tag{9}$$

Proof. We first remark that $D(p) \cap D(x)$ is invariant under $\mathrm{e}^{-\mathrm{i}tH}$ and that

$$\mathrm{e}^{\mathrm{i}tH}(p - gt)\mathrm{e}^{-\mathrm{i}tH}\psi = p\psi - \int_0^t \mathrm{e}^{\mathrm{i}sH}\nabla V(x)\mathrm{e}^{-\mathrm{i}sH}\psi \, ds \,, \tag{10}$$

$$\mathrm{e}^{\mathrm{i}tH}(x - pt + gt^2/2)\mathrm{e}^{-\mathrm{i}tH}\psi = x\psi + \int_0^t s\mathrm{e}^{\mathrm{i}sH}\nabla V(x)\mathrm{e}^{-\mathrm{i}sH}\psi \, ds$$

for $\psi \in D(p) \cap D(x)$.

It suffices to prove the lemma for f depending on a single coordinate of its argument, since linear combinations of products of such functions are dense in $C_\infty(\mathbf{R}^\nu)$. By [7], Theorem VIII.20 (b) it is enough to show that (8, 9) hold for $f(y) = (y_i - z)^{-1}$, $z \notin \mathbf{R}$ and moreover strong and weak convergence are equivalent for resolvents. We thus set $p(t) := \mathrm{e}^{\mathrm{i}tH}p\mathrm{e}^{-\mathrm{i}tH}$, $x(t) := \mathrm{e}^{\mathrm{i}tH}x\mathrm{e}^{-\mathrm{i}tH}$ and have to show that for $\varphi \in \mathcal{H}$, $\psi \in D(p) \cap D(x)$

$$(\varphi, [(\frac{p_i(t) - g_i t}{t} - z)^{-1} - (-z)^{-1}]\psi) = z^{-1}((\frac{p_i(t) - g_i t}{t} - \bar{z})^{-1}\varphi, \frac{p_i(t) - g_i t}{t}\psi) \,,$$

vanishes as $t \to \infty$. Indeed, by (10) this is bounded by a constant times

$$\frac{1}{t}\|p_i\psi\| + \frac{1}{t}\int_0^t \|\partial_i V(x)\mathrm{e}^{-\mathrm{i}sH}\psi\| \, ds \xrightarrow[t\to\infty]{} 0 \,.$$

The limit just stated holds by the RAGE theorem ([7], Theorem XI.115), because $\partial_i V$ is relatively compact with respect to H. The other observable is treated analogously. $\square$

Lemma 3 is still a rather weak result, since it roughly says that $p(t) - gt = \mathrm{o}(t)$, $x(t) - p(t)t + gt^2/2 = \mathrm{o}(t^2)$, while these quantities are constants of motion for the free Stark problem. The idea to improve this is the following: Let

$$S_t := \{(x,p) \,|\, |p - gt| < \delta t_0, \, |x - pt + gt^2/2| < \delta t_0^2\}$$

for some $\delta, t_0 > 0$. Classically S_{t_0} at time t_0 is mapped into S_t at time $t \geq t_0$ under the free Stark evolution. Moreover

$$|x - gt^2/2| \leq |x - pt + gt^2/2| + t|p - gt| < \delta t_0(t + t_0) \leq 2\delta t^2 \,,$$
$$|x| > (|g|/2 - 2\delta)t^2$$

i.e. the particle is far from the scatterer if $\delta < |g|/4$. Therefore S_{t_0} should be mapped approximately in S_t also under the full dynamics.

Let $f \in C_0^\infty(\mathbf{R}^\nu)$ with $f(y) = 0$ for $|y| \geq \delta$ and set

$$f_1(t, t_0) := f\Big(\frac{x - pt + gt^2/2}{t_0^2}\Big) \,, \quad f_2(t, t_0) := f\Big(\frac{p - gt}{t_0}\Big) \,,$$

$$\Theta(t, t_0) := f_1(t, t_0)^* f_2(t, t_0)^* f_2(t, t_0) f_1(t, t_0) \,. \tag{11}$$

Lemma 4. *If $\delta < |g|/4$, then*

$$\lim_{t_0 \to \infty} \sup_{t \geq t_0} \| e^{itH} \Theta(t, t_0) e^{-itH} - e^{it_0 H} \Theta(t_0, t_0) e^{-it_0 H} \| = 0 . \tag{12}$$

Proof. The supremum in (12) is bounded by

$$\int_{t_0}^{\infty} \| D\Theta(t, t_0) \| \, dt , \tag{13}$$

where $D \cdot = \mathrm{i}[H, \cdot] + \partial \cdot / \partial t$, i.e.

$$D\Theta = (Df_1)^* f_2^* f_2 f_1 + f_1^* (Df_2)^* f_2 f_1 + f_1^* f_2^* (Df_2) f_1 + f_1^* f_2^* f_2 (Df_1) .$$

We start by computing

$$Df_1 = \int \hat{f}(s) \mathrm{i}[V, e^{\mathrm{i}(x - pt + gt^2/2) t_0^{-2} \cdot s}] \, ds = \mathrm{i} \int \hat{f}(s) e^{\mathrm{i}(x - pt + gt^2/2) t_0^{-2} \cdot s} (V(x + t_0^{-2} ts) - V(x)) \, ds$$

and split the integral into $|s| > t$ and $|s| \leq t$. The contribution to (13) of the first part is bounded by $O(t_0^{-N})$ for all $N > 0$, due to the decay of $\hat{f}$. For the other part, (6) implies that for $1 < \alpha < |g|(4\delta)^{-1}$

$$f_2^* f_2 f_1 = F(|x - gt^2/2| < \alpha \delta t_0 (t + t_0)) f_2^* f_2 f_1 + O([t_0^2 (t_0 + t)]^{-N}) ,$$

where the remainder contributes to (13) as little as $O(t_0^{-N})$. One is then left with estimating

$$\int_{|s| \leq t} |\hat{f}(s)| \| (V(x + t_0^{-2} ts) - V(x)) F(|x - gt^2/2| < \alpha \delta t_0 (t + t_0)) \| \, ds \leq \text{const} \, \| s \hat{f} \|_1 t_0^{-2} t^{-(1+2\varepsilon)} ,$$

contributing $O(t_0^{-2(1+\varepsilon)})$ to (13). Here we used that if $|x - gt^2/2| < \alpha \delta t_0 (t + t_0) \leq 2\alpha \delta t^2$ then $|V(x + t_0^{-2} ts) - V(x)| \leq t_0^{-2} t |s| |\nabla V(\tilde{x})|$ for some $\tilde{x}$ with

$$|\tilde{x}| \geq |g| t^2/2 - |x - gt^2/2| - |x - \tilde{x}| \geq t^2 (|g|/2 - 2\alpha\delta - t_0^{-2}) ,$$

where the bracket is positive for large t_0. Terms arising from

$$Df_2 = \mathrm{i} \int \hat{f}(s) e^{\mathrm{i}(p - gt) t_0^{-1} \cdot s} (V(x - t_0^{-1} s) - V(x)) \, ds$$

are dealt with similarly. The integral restricted to $|s| \leq t$ is estimated up to a constant by $\| s \hat{f} \|_1 t_0^{-1} t^{-2(1+\varepsilon)}$, its contribution to (13) thus by $O(t_0^{-2(1+\varepsilon)})$. □

Lemma 5. *Let f be as in the previous lemma, and real-valued with $f \le 1$, $f(0) = 1$ in addition. Then*

$$\lim_{t_0\to\infty} \sup_{t\ge t_0} \|(1 - f_2(t,t_0)f_1(t,t_0))\mathrm{e}^{-\mathrm{i}tH}\psi\| = 0 \tag{14}$$

for any $\psi \in \mathcal{H}$.

Proof. Equations (8) and (9) imply $(\mathrm{e}^{-\mathrm{i}t_0H}\psi, \Theta(t_0,t_0)\mathrm{e}^{-\mathrm{i}t_0H}\psi) \to (\psi,\psi)$ as $t_0 \to \infty$. Then

$$\sup_{t\ge t_0} (\psi,\psi) - (\mathrm{e}^{-\mathrm{i}tH}\psi, \Theta(t,t_0)\mathrm{e}^{-\mathrm{i}tH}\psi) \xrightarrow[t_0\to\infty]{} 0 \tag{15}$$

follows from (12). Suppressing arguments (t,t_0) we have $1 - f_2f_1 = (1-f_1) + (1-f_2)f_1$ and

$$\begin{aligned}\|(1-f_1)\mathrm{e}^{-\mathrm{i}tH}\psi\|^2 &+ \|(1-f_2)f_1\mathrm{e}^{-\mathrm{i}tH}\psi\|^2 \\ &\le (\|\psi\|^2 - \|f_1\mathrm{e}^{-\mathrm{i}tH}\psi\|^2) + (\|f_1\mathrm{e}^{-\mathrm{i}tH}\psi\|^2 - \|f_2f_1\mathrm{e}^{-\mathrm{i}tH}\psi\|^2) \\ &= (\psi,\psi) - (\mathrm{e}^{-\mathrm{i}tH}\psi, \Theta(t,t_0)\mathrm{e}^{-\mathrm{i}tH}\psi)\,,\end{aligned}$$

since $f_i^2 \le f_i$, $i = 1,2$. Hence (14) is immediate from (15). □

Proof of Theorem 1 (completeness). Given $\psi \in \mathcal{H}$,

$$\begin{aligned}&\|(\Omega - \mathrm{e}^{-\mathrm{i}\Phi(t)})\mathrm{e}^{-\mathrm{i}tH}\psi\| \\ &\quad\le \|(\Omega - \mathrm{e}^{-\mathrm{i}\Phi(t)})f_2(t,t_0)f_1(t,t_0)\mathrm{e}^{-\mathrm{i}tH}\psi\| + 2\|(1 - f_2(t,t_0)f_1(t,t_0))\mathrm{e}^{-\mathrm{i}tH}\psi\|\end{aligned}$$

can be made arbitrarily small due to (7, 14) by first choosing t_0 and then t large enough. Thus

$$\psi = \mathrm{e}^{\mathrm{i}tH}\Omega\mathrm{e}^{\mathrm{i}\Phi(t)}\mathrm{e}^{-\mathrm{i}tH}\psi + \mathrm{o}(1) = \Omega\mathrm{e}^{\mathrm{i}tH_0}\mathrm{e}^{\mathrm{i}\Phi(t)}\mathrm{e}^{-\mathrm{i}tH}\psi + \mathrm{o}(1)$$

as $t \to \infty$, implying $\psi \in \overline{\operatorname{Ran}\Omega} = \operatorname{Ran}\Omega$. □

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