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## A remark on long-range Stark scattering

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**Abstract.** We propose a modified wave operator for long-range scattering in presence of an accelerating force. Its relevance in connection with classical scattering is addressed.

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## Introduction

The wave operators for the pair  $(H, H_0)$  of Stark Hamiltonians with  $H = H_0 + V$  and

$$H_0 = \frac{p^2}{2} - g \cdot x, \quad 0 \neq g \in \mathbb{R}^\nu$$

on  $\mathcal{H} = L^2(\mathbb{R}^\nu)$  do not exist [6] if  $V = V(x)$  behaves as  $|x|^{-\varepsilon}$ ,  $\varepsilon \leq 1/2$  as  $x \rightarrow \infty$ . Recently, several proposals [3, 5, 8] for modified wave operators dealing with slowly decaying potentials were made. Here we add one more, based on a simple Dollard type argument: The classical free Stark trajectories  $x(t) = gt^2/2 + p_0t + x_0$  suggest that  $e^{-itH_0}e^{-i\Phi(t)}$  with

$$\Phi(t) := \int_0^t V(gs^2/2) ds$$

is a candidate for a comparison dynamics.

**Theorem 1.** *Let  $V \in C^1(\mathbb{R}^\nu)$  be real-valued with*

$$|\nabla V(x)| \leq C(1 + |x|)^{-(1+\varepsilon)} \quad (1)$$

*for some  $C, \varepsilon > 0$ . Then the wave operator*

$$\Omega := s\text{-}\lim_{t \rightarrow \infty} e^{itH} e^{-itH_0} e^{-i\Phi(t)} \quad (2)$$

*exists and is unitary.*

The reason for stating this result comes from [6]. There a discrepancy between the quantum and the classical scattering problem was discovered, as for in the latter one the wave operators exist and are complete without modification basically as long as  $V(x) = O(|x|^{-\varepsilon})$  for some  $\varepsilon > 0$ . We believe our result heals this discrepancy for the class of potentials we consider, since the comparison dynamics differs multiplicatively from the free Stark one only by a phase, which is physically irrelevant. Actually, no modification at all is needed if the asymptotic condition is formulated in the Heisenberg picture: The above result then implies that

$$\mu(A) := s\text{-}\lim_{t \rightarrow \infty} e^{itH} e^{-itH_0} A e^{itH_0} e^{-itH}$$

exists and is an automorphism on  $\mathcal{L}(\mathcal{H})$ . By contrast, this limit typically exists in long-range scattering only for  $A \in \{H_0\}'$ , the von Neumann algebra of operators commuting with bounded functions of  $H_0$  [1]. In this respect (1) behaves as a short-range potential.

It follows from (1) that  $\lim_{x \rightarrow \infty} V(x)$  exists. We may assume this limit to be zero, since (2) is not affected if the potential is shifted by a constant. Then  $|V(x)| \leq C(1 + |x|)^{-\varepsilon}$ , which is assumed also by [3, 5, 8]. However there stronger oscillations of the potential are allowed, since (1) is replaced by a weaker decay.

**Remark.** The hypothesis (1) can be weakened logarithmically [6].

## Proofs

Let us state some kinematical remarks beforehand. The following propagation estimate for the free dynamics without electric field is well-known (see e.g. [4], Lemma 6.3): Let  $f \in C_0^\infty(\mathbf{R}^\nu)$  with  $f(y) = 0$  for  $|y| \geq 1$ , and let  $\alpha > 1$ . Then for  $R > 0, t \geq 0$  and any  $N > 0$

$$\|F(|x| > \alpha(R+t))e^{-itp^2/2}f(p)F(|x| < R)\| \leq C_N(R+t)^{-N},$$

with  $C_N$  independent of  $R, t$ . States supported in momentum space in  $\{|y| \leq v\}$ ,  $v > 0$  are accounted for through scaling by  $v > 0$ : We apply the unitary dilation  $U$  satisfying  $UpU^{-1} = p/v$ ,  $UxU^{-1} = vx$  to the operator in the estimate above, replace  $t$  by  $v^2t$  and  $R$  by  $vR$ , the result being

$$\|F(|x| > \alpha(R+vt))e^{-itp^2/2}f(p/v)F(|x| < R)\| \leq C_N[v(R+vt)]^{-N}.$$

An estimate for the free Stark dynamics is readily obtained by means of the Avron-Herbst formula

$$e^{-itH_0} = T(t)e^{-itp^2/2}, \quad T(t) := e^{-ig^2t^3/6}e^{itg \cdot x}e^{-it^2g \cdot p/2}$$

and of  $T(t)x = (x - gt^2/2)T(t)$ , namely

$$\|F(|x - gt^2/2| > \alpha(R+vt))e^{-itH_0}f(p/v)F(|x| < R)\| \leq C_N[v(R+vt)]^{-N}. \quad (3)$$

**Proof of Theorem 1** (existence). By Cook's theorem we have to show that

$$\int_0^\infty \|U(x,t)e^{-itH_0}\varphi\| dt < \infty$$

for  $\varphi$  in a dense set  $D$ , where  $U(x,t) := V(x) - V(gt^2/2)$ . Choosing

$$D := \{f(p/v)F(|x| < R)\psi \mid f \in C_0^\infty(|y| < 1), v, R > 0, \psi \in \mathcal{H}\},$$

this follows from

$$\begin{aligned} & \|U(x,t)e^{-itH_0}f(p/v)F(|x| < R)\psi\| \\ & \leq \|U(x,t)F(|x - gt^2/2| \leq \alpha(R+vt))\|\|\psi\| + C'_N[v(R+vt)]^{-N}\|\psi\| = O(t^{-1-2\epsilon})\|\psi\|. \end{aligned} \quad (4)$$

Here we estimated the first term on the second line as follows: If  $|x - gt^2/2| \leq \alpha(R+vt)$ , then  $U(x,t) = \nabla V(\tilde{x}) \cdot (x - gt^2/2)$  for some  $\tilde{x}$  with  $|\tilde{x} - gt^2/2| \leq \alpha(R+vt)$ . Then  $|\tilde{x}| > |g|t^2/4$  for  $t$  large enough, and hence  $\nabla V(\tilde{x}) = O(t^{-2(1+\epsilon)})$ , proving (4).  $\square$

Immediate consequences of the existence of the wave operator are that it is an isometry and that

$$\Omega = s\text{-}\lim_{s \rightarrow \infty} e^{isH}e^{-isH_0}e^{-i\Phi(s+t)} \quad (5)$$

for any  $t \in \mathbf{R}$ , since  $\Phi(s+t) - \Phi(s) = \int_s^{s+t} V(g\tau^2/2) d\tau \rightarrow 0$  as  $s \rightarrow \infty$ . By comparing (5) with the definition (2) we get the intertwining relation  $e^{itH}\Omega = \Omega e^{itH_0}$ .

Another formulation of (3) is obtained using

$$e^{-itH_0}p = (p - gt)e^{-itH_0}, \quad e^{-itH_0}x = (x - pt + gt^2/2)e^{-itH_0},$$

which is

$$\|F(|x - gt^2/2| > \alpha(R+vt))f((p-gt)/v)F(|x - pt + gt^2/2| < R)\| \leq C_N[v(R+vt)]^{-N}. \quad (6)$$

The same remark also leads to

**Corollary 2.**

$$\lim_{t \rightarrow \infty} \|(\Omega - e^{-i\Phi(t)})f((p-gt)/v)F(|x - pt + gt^2/2| < R)\| = 0. \quad (7)$$

**Proof.** We apply Cook's estimate to (5) yielding

$$\|(\Omega - e^{-i\Phi(t)})e^{-itH_0}\varphi\| \leq \int_0^\infty \|U(x, s+t)e^{-i(s+t)H_0}\varphi\| ds = \int_t^\infty \|U(x, s)e^{-isH_0}\varphi\| ds.$$

We then set  $\varphi = f(p/v)F(|x| < R)e^{itH_0}\psi$  and use (4). □

Let us now address the completeness question. We recall [2] that any  $V \in L^\infty(\mathbf{R}^\nu)$  with  $\lim_{x \rightarrow \infty} V(x) = 0$  is relatively compact with respect to  $H_0$  and to  $H$ . This follows by density if it holds for  $V$  with compact support. In this case all terms in

$$\begin{aligned} V(H_0 + i)^{-1} - V(p^2/2 + i)^{-1} &= V(p^2/2 + i)^{-1}g \cdot x(H_0 + i)^{-1} \\ &= Vg \cdot x(p^2/2 + i)^{-1}(H_0 + i)^{-1} \\ &\quad + iV(p^2/2 + i)^{-1}g \cdot p(p^2/2 + i)^{-1}(H_0 + i)^{-1} \end{aligned}$$

are compact. We will also use that  $\mathcal{H}_{cont} = \mathcal{H}$  for the spectral decomposition with respect to  $H$  [2]. Without using this result, one could prove that  $\text{Ran } \Omega = \mathcal{H}_{cont}$  instead of unitarity of  $\Omega$  by suitably modifying the proofs below.

**Lemma 3.** For  $f \in C_\infty(\mathbf{R}^\nu)$

$$s\text{-}\lim_{t \rightarrow \infty} e^{itH}f\left(\frac{p-gt}{t}\right)e^{-itH} = f(0), \quad (8)$$

$$s\text{-}\lim_{t \rightarrow \infty} e^{itH}f\left(\frac{x-pt+gt^2/2}{t^2}\right)e^{-itH} = f(0). \quad (9)$$

**Proof.** We first remark that  $D(p) \cap D(x)$  is invariant under  $e^{-itH}$  and that

$$\begin{aligned} e^{itH}(p - gt)e^{-itH}\psi &= p\psi - \int_0^t e^{isH} \nabla V(x) e^{-isH} \psi \, ds, \\ e^{itH}(x - pt + gt^2/2)e^{-itH}\psi &= x\psi + \int_0^t s e^{isH} \nabla V(x) e^{-isH} \psi \, ds \end{aligned} \quad (10)$$

for  $\psi \in D(p) \cap D(x)$ .

It suffices to prove the lemma for  $f$  depending on a single coordinate of its argument, since linear combinations of products of such functions are dense in  $C_\infty(\mathbf{R}^\nu)$ . By [7], Theorem VIII.20 (b) it is enough to show that (8, 9) hold for  $f(y) = (y_i - z)^{-1}$ ,  $z \notin \mathbf{R}$  and moreover strong and weak convergence are equivalent for resolvents. We thus set  $p(t) := e^{itH} p e^{-itH}$ ,  $x(t) := e^{itH} x e^{-itH}$  and have to show that for  $\varphi \in \mathcal{H}$ ,  $\psi \in D(p) \cap D(x)$

$$(\varphi, [(\frac{p_i(t) - g_i t}{t} - z)^{-1} - (-z)^{-1}] \psi) = z^{-1} ((\frac{p_i(t) - g_i t}{t} - \bar{z})^{-1} \varphi, \frac{p_i(t) - g_i t}{t} \psi),$$

vanishes as  $t \rightarrow \infty$ . Indeed, by (10) this is bounded by a constant times

$$\frac{1}{t} \|p_i \psi\| + \frac{1}{t} \int_0^t \|\partial_i V(x) e^{-isH} \psi\| \, ds \xrightarrow[t \rightarrow \infty]{} 0.$$

The limit just stated holds by the RAGE theorem ([7], Theorem XI.115), because  $\partial_i V$  is relatively compact with respect to  $H$ . The other observable is treated analogously.  $\square$

Lemma 3 is still a rather weak result, since it roughly says that  $p(t) - gt = o(t)$ ,  $x(t) - p(t)t + gt^2/2 = o(t^2)$ , while these quantities are constants of motion for the free Stark problem. The idea to improve this is the following: Let

$$S_t := \{(x, p) \mid |p - gt| < \delta t_0, |x - pt + gt^2/2| < \delta t_0^2\}$$

for some  $\delta, t_0 > 0$ . Classically  $S_{t_0}$  at time  $t_0$  is mapped into  $S_t$  at time  $t \geq t_0$  under the free Stark evolution. Moreover

$$\begin{aligned} |x - gt^2/2| &\leq |x - pt + gt^2/2| + t|p - gt| < \delta t_0(t + t_0) \leq 2\delta t^2, \\ |x| &> (|g|/2 - 2\delta)t^2 \end{aligned}$$

i.e. the particle is far from the scatterer if  $\delta < |g|/4$ . Therefore  $S_{t_0}$  should be mapped approximately in  $S_t$  also under the full dynamics.

Let  $f \in C_0^\infty(\mathbf{R}^\nu)$  with  $f(y) = 0$  for  $|y| \geq \delta$  and set

$$\begin{aligned} f_1(t, t_0) &:= f\left(\frac{x - pt + gt^2/2}{t_0^2}\right), \quad f_2(t, t_0) := f\left(\frac{p - gt}{t_0}\right), \\ \Theta(t, t_0) &:= f_1(t, t_0)^* f_2(t, t_0)^* f_2(t, t_0) f_1(t, t_0). \end{aligned} \quad (11)$$

**Lemma 4.** *If  $\delta < |g|/4$ , then*

$$\lim_{t_0 \rightarrow \infty} \sup_{t \geq t_0} \|e^{itH} \Theta(t, t_0) e^{-itH} - e^{it_0 H} \Theta(t_0, t_0) e^{-it_0 H}\| = 0. \quad (12)$$

**Proof.** The supremum in (12) is bounded by

$$\int_{t_0}^{\infty} \|D\Theta(t, t_0)\| dt, \quad (13)$$

where  $D \cdot = i[H, \cdot] + \partial \cdot / \partial t$ , i.e.

$$D\Theta = (Df_1)^* f_2^* f_2 f_1 + f_1^* (Df_2)^* f_2 f_1 + f_1^* f_2^* (Df_2) f_1 + f_1^* f_2^* f_2 (Df_1).$$

We start by computing

$$Df_1 = \int \hat{f}(s) i[V, e^{i(x-pt+gt^2/2)t_0^{-2} \cdot s}] ds = i \int \hat{f}(s) e^{i(x-pt+gt^2/2)t_0^{-2} \cdot s} (V(x+t_0^{-2}ts) - V(x)) ds$$

and split the integral into  $|s| > t$  and  $|s| \leq t$ . The contribution to (13) of the first part is bounded by  $O(t_0^{-N})$  for all  $N > 0$ , due to the decay of  $\hat{f}$ . For the other part, (6) implies that for  $1 < \alpha < |g|(4\delta)^{-1}$

$$f_2^* f_2 f_1 = F(|x - gt^2/2| < \alpha\delta t_0(t + t_0)) f_2^* f_2 f_1 + O([t_0^2(t_0 + t)]^{-N}),$$

where the remainder contributes to (13) as little as  $O(t_0^{-N})$ . One is then left with estimating

$$\int_{|s| \leq t} |\hat{f}(s)| \| (V(x+t_0^{-2}ts) - V(x)) F(|x - gt^2/2| < \alpha\delta t_0(t + t_0)) \| ds \leq \text{const} \|s\hat{f}\|_1 t_0^{-2} t^{-(1+2\epsilon)},$$

contributing  $O(t_0^{-2(1+\epsilon)})$  to (13). Here we used that if  $|x - gt^2/2| < \alpha\delta t_0(t + t_0) \leq 2\alpha\delta t^2$  then  $|V(x + t_0^{-2}ts) - V(x)| \leq t_0^{-2}t|s| |\nabla V(\tilde{x})|$  for some  $\tilde{x}$  with

$$|\tilde{x}| \geq |g|t^2/2 - |x - gt^2/2| - |x - \tilde{x}| \geq t^2(|g|/2 - 2\alpha\delta - t_0^{-2}),$$

where the bracket is positive for large  $t_0$ . Terms arising from

$$Df_2 = i \int \hat{f}(s) e^{i(p-gt)t_0^{-1} \cdot s} (V(x - t_0^{-1}s) - V(x)) ds$$

are dealt with similarly. The integral restricted to  $|s| \leq t$  is estimated up to a constant by  $\|s\hat{f}\|_1 t_0^{-1} t^{-2(1+\epsilon)}$ , its contribution to (13) thus by  $O(t_0^{-2(1+\epsilon)})$ .  $\square$

**Lemma 5.** *Let  $f$  be as in the previous lemma, and real-valued with  $f \leq 1$ ,  $f(0) = 1$  in addition. Then*

$$\lim_{t_0 \rightarrow \infty} \sup_{t \geq t_0} \|(1 - f_2(t, t_0)f_1(t, t_0))e^{-itH}\psi\| = 0 \quad (14)$$

for any  $\psi \in \mathcal{H}$ .

**Proof.** Equations (8) and (9) imply  $(e^{-it_0H}\psi, \Theta(t_0, t_0)e^{-it_0H}\psi) \rightarrow (\psi, \psi)$  as  $t_0 \rightarrow \infty$ . Then

$$\sup_{t \geq t_0} (\psi, \psi) - (e^{-itH}\psi, \Theta(t, t_0)e^{-itH}\psi) \xrightarrow{t_0 \rightarrow \infty} 0 \quad (15)$$

follows from (12). Suppressing arguments  $(t, t_0)$  we have  $1 - f_2f_1 = (1 - f_1) + (1 - f_2)f_1$  and

$$\begin{aligned} \|(1 - f_1)e^{-itH}\psi\|^2 + \|(1 - f_2)f_1e^{-itH}\psi\|^2 \\ \leq (\|\psi\|^2 - \|f_1e^{-itH}\psi\|^2) + (\|f_1e^{-itH}\psi\|^2 - \|f_2f_1e^{-itH}\psi\|^2) \\ = (\psi, \psi) - (e^{-itH}\psi, \Theta(t, t_0)e^{-itH}\psi), \end{aligned}$$

since  $f_i^2 \leq f_i$ ,  $i = 1, 2$ . Hence (14) is immediate from (15).  $\square$

**Proof of Theorem 1** (completeness). Given  $\psi \in \mathcal{H}$ ,

$$\begin{aligned} \|(\Omega - e^{-i\Phi(t)})e^{-itH}\psi\| \\ \leq \|(\Omega - e^{-i\Phi(t)})f_2(t, t_0)f_1(t, t_0)e^{-itH}\psi\| + 2\|(1 - f_2(t, t_0)f_1(t, t_0))e^{-itH}\psi\| \end{aligned}$$

can be made arbitrarily small due to (7, 14) by first choosing  $t_0$  and then  $t$  large enough. Thus

$$\psi = e^{itH}\Omega e^{i\Phi(t)}e^{-itH}\psi + o(1) = \Omega e^{itH_0}e^{i\Phi(t)}e^{-itH}\psi + o(1)$$

as  $t \rightarrow \infty$ , implying  $\psi \in \overline{\text{Ran } \Omega} = \text{Ran } \Omega$ .  $\square$

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