Zeitschrift: Helvetica Physica Acta
Band: 64 (1991)
Heft: 4

Artikel: Random surfaces, statistical mechanics and string theory

Autor: Alvarez-Gaumé, L.

DOI: https://doi.org/10.5169/seals-116310

Helvetica Physica Acta
Vol. 64 (1991)

0018-0238/91/040359-168$1.50+0.20/0

Random Surfaces, Statistical Mechanics and String Theory

L. Alvarez-Gaumé
Theory Division, CERN
CH-1211 Genève 23

(4. IV. 1991)

TABLE OF CONTENTS

Lecture 1. General Introduction

1.1. Introduction

During the past year there has been a great deal of activity in the newly found subject of sub-critical strings. Many results have already accumulated, giving some clues on the non-perturbative properties of string theories in realistic dimensions. These new developments are a consequence of the investigations over the last two decades on subjects such as conformal field theory, lattice and continuum integrable models, quantum field theory in the large N limit, string theory, critical systems on random surfaces, etc.

These notes are based on an eight-lecture Cours de Troisième Cycle de Physique de la Suisse Romande, held at Lausanne in the Fall of 1990. The aim of the course was to provide a reasonably self-contained introduction to the subject of sub-critical strings and matrix models, and their connection with some problems in statistical mechanics. Since the audience contained both students and faculty members with a wide range of research interests, we assumed very little in terms of prerequisites, and we made an effort to assume as little previous knowledge as possible. The various preliminary subjects required to understand the double scaling limit construction of two-dimensional conformally invariant systems coupled to gravity are reviewed in some detail. Consequently, the lectures do not contain an exhaustive treatment of the subject, but should be considered instead as an invitation to study the new subject of sub-critical strings, and as an introduction to such a fast-growing field. What constitutes a pedagogical presentation of a subject is almost always a matter of personal taste. Accordingly, the author alone should be held responsible for the choice of preliminary subjects intended to present the concepts and techniques of this new subject as transparently as possible.

In preparing these notes, we have followed faithfully the content of the lectures, without adding any extra material. The outline of the course is as follows. Lecture 1 provides a general introduction to string theory and to its relations with the theory of random surfaces and conformally invariant theories. Lectures 2 and 3 contain a mini–course on conformally invariant quantum field theories, the minimal conformal models and the Coulomb gas representation of their correlation functions. In lecture 4 we derive some of the main results due to Polyakov [1] and Knizhnik, Polyakov and Zamolodchikov [2], using the presentation of David [3] and Distler and Kawai [4]. This provides us with the gravitational dressing of conformal primary fields and with their gravitationally dressed dimensions. These results are derived in the continuum and they constitute good reference points for the subsequent lectures, concentrating mostly on the discrete definition of two–dimensional quantum gravity and its coupling to matter. The next three lectures (5, 6 and 7) present the use of large N field theory techniques to simulate the sum over two–dimensional geometries, and the properties and definitions of critical exponents for statistical mechanical models coupled to fluctuating surfaces. We obtain the Kazakov multi-critical points [5], one–matrix models, orthogonal polynomials, and a first look at the double scaling limit [6, 7, 8]. Finally, in lecture 8 we survey the relation between sub-critical strings, the KdV hierarchy, the Douglas equations [9], the double scaling limit for the Kazakov multi-critical points, the $c = 1$ string theory, and the two–dimensional Ising model in the presence of gravity and an external magnetic field. We briefly comment on the loop equations, the Virasoro constraints, and the non–perturbative properties of pure two–dimensional gravity. All these subjects are now being investigated vigorously. Due to the lack of time, we have not covered these subjects in more detail. This is the reason why we only present details throughout these lectures for the cases of pure gravity, Kazakov's multi-critical points, and Ising and $c = 1$ theories in the presence of gravity. Many other models are studied in the literature: the relevant references are quoted in the text. The approach to many of these subjects using topological field theories has also been omitted, again due to lack of space–time. The interested reader is referred to [10, 11, 12, 13, 14, 15, 16] and references therein.

Acknowledgements

I would like to thank Prof. H. Ruegg and the Troisième Cycle de Physique de la Suisse Romande for the kind invitation to give this course. My warmest thanks go to Martí Ruiz–Altaba for his invaluable and intelligent assistance in the preparation of these notes, and for many discussions which helped to improve the course. Discussions with and comments by Philippe Zaugg on many parts of these lectures are also gratefully acknowledged. I would like to thank the participants in the course for their interest and also for their patience with the style and speed of the lecturer.

I am grateful to C. Bachas, F. David, C. Gómez, D. Gross, V. Kazakov, I. Kostov, J. Lacki, G. Moore, S. Mukhi, G. Sierra, E. Verlinde and H. Verlinde for sharing with me their insights on the subject of these lectures.

1.2. String Theory as a Theory of Random Surfaces

Problems involving the statistical mechanical properties of random surfaces appear in many different subjects of theoretical physics. Among them, let us mention two–dimensional gravity and Liouville field theory, biological membranes, interphase boundaries, wetting, string theory, three–dimensional critical phenomena, etc.

In ordinary particle mechanics, the time evolution of a particle generates a world line. Its quantum properties can be obtained by summing over all possible trajectories. For a free particle, the natural action functional is proportional to the length of the world–line. Thus for a relativistic particle

$$S[x(\tau)] = -m \int_{\tau_i}^{\tau_f} d\tau \tag{1.1}$$

where τ is the proper time along the particle trajectory and m is the particle's mass. Similarly, the most natural quantity to describe the action of a string (a closed one–dimensional object) moving in some flat space–time is the area of the world–surface swept out by the string. Thorough presentations of string theory with references can be found in [17, 18, 19]. Let σ and τ be the space and time co-ordinates of the world-sheet, respectively, and let $X^\mu(\sigma,\tau)$ be the function describing its embedding in space-time. For a d-dimensional Minkowski space, the action is then given by the area of the world-sheet with respect to the reduced metric:

$$\begin{aligned} ds^2 &= \eta_{\mu\nu} dX^\mu dX^\nu \\ &= \eta_{\mu\nu}\partial_\tau X^\mu \partial_\tau X^\nu d\tau^2 + \eta_{\mu\nu}\partial_\sigma X^\mu \partial_\sigma X^\nu d\sigma^2 + 2\eta_{\mu\nu}\partial_\tau X^\mu \partial_\sigma X^\nu d\sigma d\tau \end{aligned} \tag{1.2}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,\ldots,-1)$ is the metric of the embedding space M^d. Letting $\xi^0 = \tau$, $\xi^1 = \sigma$, the above expression can be written as

$$ds^2 = g_{ij}(X) d\xi^i d\xi^j \tag{1.3}$$

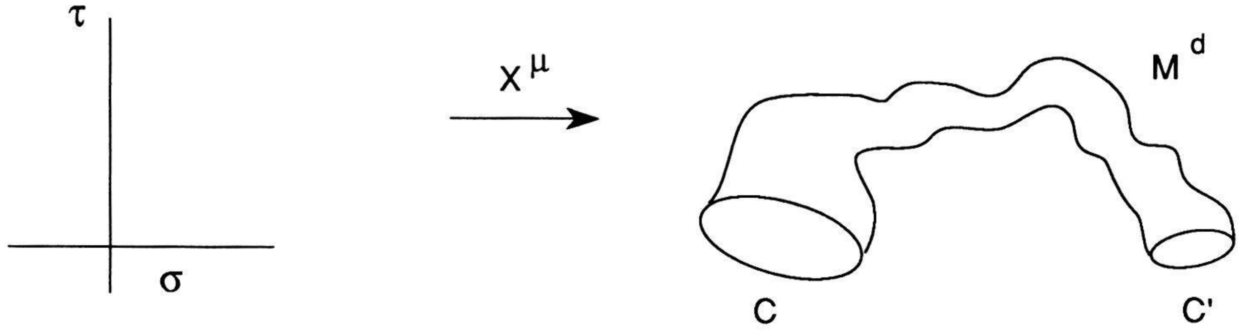


Figure 1.1. A typical configuration of the embedding of a string into space-time M^d.

The action of the string history $X^\mu(\xi^0,\xi^1)$ is proportional to the area $\int \sqrt{\det g}$:

$$S[X(\xi^i)] = T\int d\sigma d\tau \sqrt{\dot{X}^2 X'^2 - (\dot{X}\cdot X')^2} \tag{1.4}$$

with

$$\begin{aligned} \dot{X}^\mu &= \partial_\tau X^\mu \\ X'^\mu &= \partial_\sigma X^\mu \\ A\cdot B &= \eta_{\mu\nu} A^\mu B^\nu \end{aligned} \tag{1.5}$$

The string tension T is the analogue of the particle mass. It has units of mass over length or $(\text{mass})^2$, since we always set $\hbar = c = 1$.

Depending on the mapping X^μ, the embedding of the string world–sheet can be arbitrarily complex and crumpled. Clearly, (1.4) is a rather complicated action and if one wants to quantize it, a simplification would be most helpful. Such a simplification is indeed available at the expense of introducing an auxiliary metric g_{ij} on the two–dimensional world–sheet.

Instead of (1.4), consider the action (see [17, 18, 19] for details)

$$S[X,g] = -\frac{T}{2}\int d^2\xi\sqrt{g}g^{ij}\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} \tag{1.6}$$

which is quadratic in X. The metric g_{ij} on the surface swept by the string is a new field. Since no derivatives of g_{ij} appear in (1.6), its equations of motion will implement some constraints on the dynamical field $X^\mu(\sigma,\tau)$. Recalling from general relativity that the variation of the action with respect to the metric is equal to the energy–momentum tensor T_{ij}, we learn that the classical theory defined by (1.6) satisfies the equations of motion

$$T_{ij} = \partial_i X^\mu \partial_j X_\mu - \frac{1}{2}g_{ij}g^{k\ell}\partial_k X^\mu \partial_\ell X_\mu = 0 \tag{1.7}$$

$$\frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j X^\mu\right) = \Delta X^\mu = 0 \tag{1.8}$$

Therefore, in the presence of the metric g_{ij} the field $X^\mu(\sigma,\tau)$ is a free scalar field which does not carry two–dimensional energy or momentum. We can use (1.7) to solve for the metric tensor g_{ij} in terms of X^μ. Writing (1.7) as $\partial_i X\cdot\partial_j X = \frac{1}{2}g_{ij}g^{k\ell}\partial_k X\cdot\partial_\ell X$ and taking determinants on both sides, we find that (1.6) is equivalent to the Nambu–Goto action (1.4). Thus (1.6) provides the simplification we sought.

Several things should be remarked about the equivalence between (1.4) and (1.6):

i) The free propagation of a string in M^d is described by a free two–dimensional field theory.

ii) The actions (1.4) and (1.6) are both invariant under arbitrary reparametrizations $\xi^i \to f^i(\xi)$ of the surface, under which

$$g_{ij} \to \partial_i f^k \partial_j f^\ell g_{k\ell} \tag{1.9}$$

The form (1.6) of the action gives the minimal coupling of d two-dimensional scalar fields X^μ ($\mu = 0, \ldots, d-1$) with two-dimensional gravity.

iii) The energy momentum tensor T_{ij} in (1.7) is traceless with respect to g_{ij}: $g^{ij}T_{ij} = 0$. This tracelessness reflects the local Weyl invariance of the action (1.6). If we rescale the metric as

$$g_{ij} \to e^{\phi(\sigma,\tau)} g_{ij} \tag{1.10}$$

we find that $\sqrt{g} g^{ij}$ remains unchanged. This is true only in two dimensions.

Thus, classically, the action has three types of symmetries: diffeomorphism or reparametrization invariance, local Weyl or conformal symmetry, and the space-time symmetries depending on the isometries of the metric $\eta_{\mu\nu}$. For instance, if M^d is the standard Minkowski space, the action enjoys d-dimensional Poincaré symmetry: $X^\mu \to \Lambda^\mu{}_\nu X^\nu + b^\mu$ where Λ^μ_ν is a Lorentz transformation.

We have just met conformal invariance for the first time: it will play an important role in subsequent lectures. We have learned that string theory is described by the coupling of a conformally invariant field theory (CFT) to two-dimensional gravity.

The simplest topology of the world-sheet is a cylinder. In this case, the space-time picture of string propagation is shown in figure 1.2 and the sum over all embeddings with this topology describes the propagation of a free string. For this simple topology, we can use the diffeomorphism invariance to fix $\sqrt{g} g_{ij} = \eta_{ij}$, the Minkowski metric on the cylinder: $\eta_{\tau\tau} = 1$, $\eta_{\sigma\sigma} = -1$, $\eta_{\tau\sigma} = 0$. In this gauge (the conformal gauge), the action (1.6) looks particularly simple:

$$S = -\frac{T}{2} \int d\tau d\sigma \left(\dot{X}^2 - X'^2 \right) \tag{1.10}$$

and the equations of motion in this gauge are those of a free scalar field on the cylinder.

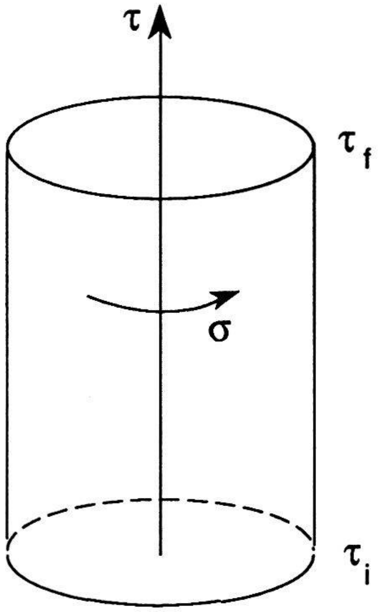


Figure 1.2. Propagation of a free closed string.

For future reference, it is convenient to consider the dynamics of the theory on the punctured complex plane $\mathbf{C}^* = \mathbf{C} - \{0\}$, in order to exploit the power of complex analysis. We do this by first Wick–rotating the time τ to euclidean time, and then using light–cone co-ordinates:

$$\begin{aligned} \sigma^+ &= \tau + \sigma \rightarrow -i(\tau + i\sigma) = -iw \\ \sigma^- &= \tau - \sigma \rightarrow -i(\tau - i\sigma) = -i\overline{w} \end{aligned} \tag{1.11}$$

The metric element is thus $ds^2 = d\tau^2 - d\sigma^2 = d\sigma^+ d\sigma^- = -dw d\overline{w}$, and the null geodesics are the straight lines $\sigma^\pm$ =constant, for which $ds^2 = 0$. The causal structure is thus preserved by any transformations of the form $\sigma^+ \rightarrow f(\sigma^+)$, $\sigma^- \rightarrow g(\sigma^-)$. Since both f

and g are arbitrary, the conformal group is twice infinite–dimensional.

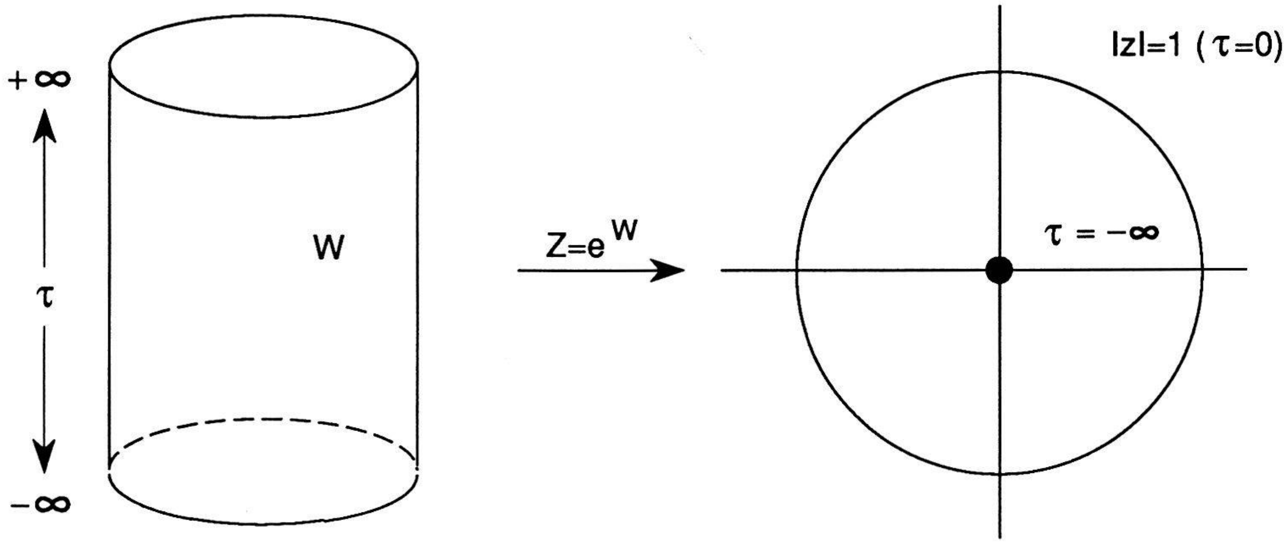


Figure 1.3. Conformal mapping of the cylinder to the punctured complex plane.

It follows from (1.11) that Fourier expansions in $\tau \pm \sigma$ become expansions in e^w $(e^{\overline{w}})$. We may now implement the conformal transformation

$$z = e^w \tag{1.12}$$

The lower end of the cylinder $\tau = -\infty$ is mapped to the origin of $\mathbf{C}$, and $\tau = \infty$ is mapped to the point at infinity (of the Riemann sphere). Hence this conformal transformation maps the cylinder to the sphere with its standard conformal structure and with both the North and South poles ($z = 0, \infty$) removed. Fourier expansions in the original variables $\sigma^{\pm}$ become Laurent expansions in the variables z and $\overline{z}$. This description in terms of (anti)holomorphic co-ordinates is useful because the equations of motion following from (1.10) imply

$$\partial_+ \partial_- X^\mu = 0 \tag{1.13}$$

or

$$\partial_z \partial_{\overline{z}} X^\mu = 0 \tag{1.14}$$

whose most general single–valued solution is

$$X^\mu = q^\mu - ip^\mu \log|z|^2 + i\sum_{n\neq 0}\frac{a_n}{n}z^{-n} + i\sum_{n\neq 0}\frac{\tilde{a}_n}{n}\overline{z}^{-n} \tag{1.15}$$

In this expression, q^μ describes the location of the string's center of mass and p^μ its total momentum, whereas the oscillator modes a_n (respectively $\tilde{a}_n$) describe the left–moving (respectively the right–moving) string excitations.

In holomorphic co-ordinates, the energy–momentum tensor has components T_{zz}, $T_{\overline{z}\,\overline{z}}$ and $T_{z\overline{z}} = T_{\overline{z}\,z}$. The tracelessness of T_{ij} implies

$$T_{z\,\overline{z}} = 0 \tag{1.16}$$

and the conservation of energy and momentum can be written as

$$\begin{aligned}\partial_{\overline{z}}T_{zz} + \partial_z T_{\overline{z}\,z} &= 0\\ \partial_z T_{\overline{z}\,\overline{z}} + \partial_{\overline{z}}T_{z\overline{z}} &= 0\end{aligned} \tag{1.17}$$

which, combined with (1.16) imply

$$\begin{aligned}\partial_{\overline{z}}T_{zz} &= 0\\ \partial_z T_{\overline{z}\,\overline{z}} &= 0\end{aligned} \tag{1.18}$$

In words, this means that $T = T_{zz}$ (respectively $\overline{T} = T_{\overline{z}\,\overline{z}}$) is a holomorphic (respectively anti-holomorphic) function in $\mathbf{C}^*$. Hence, we may Laurent expand T:

$$T(z) = \sum_{n\in\mathbf{Z}} L_n z^{-n-2} \tag{1.19}$$

and similarly† for $\overline{T}$. The factor z^{-2} in (1.19) has a simple explanation in terms of the conformal mapping (1.12). In the original cylinder co-ordinates, T is a quadratic differential *i.e.*, $T_{ww}dw^2$ is a scalar. After implementing (1.12), $dw = dz/z$ and therefore $dw^2 = dz^2/z^2$ giving rise to the overall factor of z^{-2} in (1.19). Similarly, a general tensor

† From now on, we shall write explicit formulas only for the holomorphic part of the theory; the anti-holomorphic equations are completely analogous.

$T_{w\cdots w\overline{w}\cdots\overline{w}}$ of type $(j,\overline{\jmath})$, with j holomorphic and $\overline{\jmath}$ anti-holomorphic indices, will contain an overall factor of $z^{-j}\overline{z}^{-\overline{\jmath}}$.

After quantization, the coefficients L_n satisfy the celebrated Virasoro algebra. Geometrically, these operators generate (through Poisson brackets in the classical theory or commutators in the quantum theory) the algebra of infinitesimal conformal transformations

$$z \to z + \varepsilon z^{n+1} = z + \varepsilon v_n^z(z) \tag{1.20}$$

The vector field $v^z(z)$ may have poles or zeroes only at $z = 0$ or ∞. It is clear that, if $\partial_{\overline{z}} T(z) = 0$, then also $\partial_{\overline{z}} z^n T(z) = 0$, and hence $z^n T(z)$ is the local density of the Noether charge implementing (1.20) on the space of states. In field theory, we usually compute this charge by integrating on the $\tau = 0$ surface. In holomorphic co-ordinates, $\tau = 0$ corresponds to the circle $|z| = 1$. Therefore, the charge is given by the contour integral

$$L_n = \oint_{|z|=1} z^{n+1} T(z) \tag{1.21}$$

Here and throughout, $\oint$ stands for $\oint \frac{dz}{2\pi i}$. Since $T(z)$ is analytic in $\mathbf{C}^*$, L_n is shown to be conserved by a simple contour deformation argument (see figure 1.4).

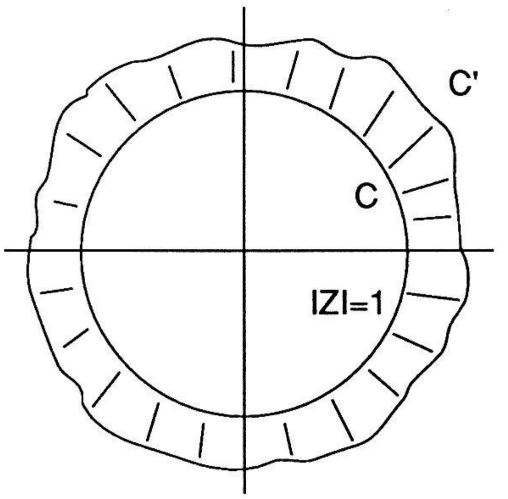


Figure 1.4. Two different integration contours for L_n.

On the z–plane, time evolution is equivalent to radial evolution. Constant τ "surfaces" correspond to circles centered about the origin $|z| = e^{\tau}$. Consider two homologous (*i.e.*, smoothly deformable into each other) contours C and C' and the annulus A bounded by them. As long as no sources of energy or momentum are present in A, the analyticity of T implies that

$$L_n = \oint_C z^{n+1}T(z) = \oint_{C'} z^{n+1}T(z) \tag{1.22}$$

For the special case of C' a circle concentric with $C = \{|z| = 1\}$, this is the statement that the charge is time–independent. Notice, however, that we have a much larger symmetry, since L_n will not change as long as C' is homologous to C. This large symmetry is a reflection of the conformal invariance of the theory.

The transformations associated with $L_{\pm 1}$ and L_0 are especially significant. They generate the so–called Möbius transformations

$$z \to \frac{az+b}{cz+d} \qquad a,b,c,d \in \mathbf{C}, \quad ad-bc=1 \tag{1.23}$$

which are the conformal automorphisms of the sphere onto itself. The infinitesimal generators for these transformations are

$$\begin{aligned} L_{-1}: &\quad z \to z+\varepsilon_{-1} \\ L_0: &\quad z \to (1+\varepsilon_0)z \\ L_1: &\quad z \to z+\varepsilon_1 z^2 \end{aligned} \tag{1.24}$$

which are indeed infinitesimal translations, dilatations and special conformal transformations respectively. From these relations and the analogous anti-holomorphic ones, we learn that $L_0 + \overline{L}_0$ generates time translations $\tau \to \tau + \varepsilon$, and $L_0 - \overline{L}_0$ generates rotations $\sigma \to \sigma + \varepsilon'$.

The algebra satisfied by the infinitesimal generators of conformal transformations

$$v_n = -z^{n+1}\frac{d}{dz} \tag{1.25}$$

is given by

$$[v_n, v_m] = (n-m)v_{n+m} \tag{1.26}$$

where [,] is the commutator of the vectors (1.25), viewed as differential operators. After quantization, we may ask whether the representation†

$$v \to T(v) = \oint_C v(z)T(z) \tag{1.27}$$

still satisfies (1.26) This is not necessary, and in fact it is not the case.

Notice, first of all, that the states in quantum mechanics are represented by rays in a Hilbert space. Therefore, in general we will have a projective representation of (1.26):

$$[T(v), T(w)] = T([v, w]) + c - \text{number} \tag{1.28}$$

From a mathematical point of view, we can determine the general form of the central term as follows. Since v_n is represented by L_n, we can write (1.28) in full generality as

$$[L_n, L_m] = (n-m)L_{n+m} + c_{n,m} \tag{1.29}$$

The antisymmetry of the commutator and the Jacobi identity imply

$$\begin{aligned} c_{n,m} &= -c_{m,n} \\ (n-m)c_{n+m,k} &+ (m-k)c_{m+k,n} + (k-n)c_{k+n,m} = 0 \end{aligned} \tag{1.30}$$

The general solution to these equations is

$$c_{n,m} = (an^3 + bn)\delta_{n+m,0} \tag{1.31}$$

Furthermore, it is possible to redefine the generators in such a way as to make $b = -a$. We want to keep explicitly the SL_2 symmetry of the theory: since $L_{\pm 1}$, L_0 generate the conformal automorphisms of the sphere, we effect the redefinition so that $c_{m,n} = 0$, $m, n \in \{-1, 0, 1\}$. Mathematically, $c_{m,n}$ is a cocycle, and (1.30) are the cocycle conditions. The solution $c_{m,n} = bn\delta_{m+n,0}$ is a co-boundary. The geometrical meaning of this co-cycle can be found in [20].

† The tensor is T_{zz}, and the vector is v^z, so $v^z T_{zz}$ is a one–differential whose contour integral is well defined.

The other reason why one expects a central extension has to do with the positivity of the Hilbert space. It is conventional to write (1.31) as

$$c_{n,m} = \frac{c}{12}(n^3 - n)\delta_{n+m,0} \tag{1.32}$$

so that the algebra (1.28), known as the Virasoro algebra, takes the form

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{m+n,0} \tag{1.33}$$

The translationally invariant ground state satisfies the so–called Virasoro conditions

$$L_n\,|0\rangle = 0 \qquad n \geq -1 \tag{1.34}$$

Using (1.33), we can compute the two–point function of $T(z)$ on the SL_2 invariant state $|0\rangle$. A simple computation yields

$$\langle T(z)T(\zeta)\rangle = \frac{c/2}{(z-\zeta)^4} \tag{1.35}$$

We can use $T(z)$ to construct some physical states in the Hilbert space, and (1.35) can be interpreted as the scalar product for these states. If c was either zero or negative, then the theory would not satisfy the positivity requirement on the Hilbert space. We will see some examples of conformal field theories in lecture 2. We remark only that every free massless scalar field contributes a unit of c. Hence, in our previous example we have a conformally invariant theory in two dimensions described by the fields $X^\mu(\sigma,\tau)$. If there are d dimensions, the total contribution to the central charge c is d. This allows us to define in string theory a generalized notion of dimension. We can identify the "dimension" of a unitary conformal field theory as the value of its central extension of the Virasoro algebra.†

So far, we have only considered the propagation of a free string because we have concentrated only on the cylindrical topology for the world–sheet. We are following, by analogy,

† For theories which are not unitary, the generalized dimension is $d = c - 24h_{\text{min}}$ where h_{min} is the minimal conformal weight of all the primaries in the theory. In the unitary case, $h_{\text{min}} = 0$.

the first quantized formulation in quantum field theory. Using Feynman's path integral formulation of quantum mechanics, we can give a general (perturbative) prescription for the computation of an arbitrary n–point function. For simplicity, we analyze the case of a scalar field theory with a $\lambda\phi^3$ interaction. We can define a diagrammatic algorithm to evaluate the Green's functions. Suppose, for instance, that we want to compute the four–point function. We draw all possible networks joining the out-going and in-coming lines in such a way that we allow a total of only three lines at each node of the network, with strength $i\lambda$. Furthermore, apart from the initial and final points, we integrate over the position of the nodes in the network. Each straight segment between nodes contributes also a factor equal to the free propagator $K_0(x,t|x',t')$. The sum over all such networks yields the probability amplitude between the initial and final states. From a field–theoretic point of view, the previous rules are nothing but the Feynman rules in configuration space. Figure 1.5 represents schematically the free scalar propagator, whereas in figure 1.6 some contributions to the four–point function in a $\lambda\phi^3$ theory are shown.

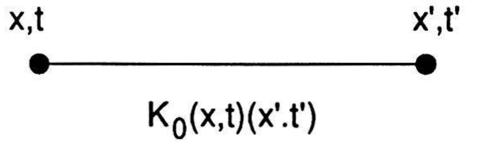


Figure 1.5. The free propagator is computed by summing over all possible paths joining (x,t) and (x',t'), each weighted with the free classical action for the path.

From the path integral point of view, the graph in figure 1.5 means that we sum over all paths between (x,t) and (x',t') without any branching. In string theory, the line between (x,t) and (x',t') is replaced by a cylinder between two configurations C and C' of the strings, as in figure 1.1. Similarly, the lines in figure 1.6 are thickened to become tubes.

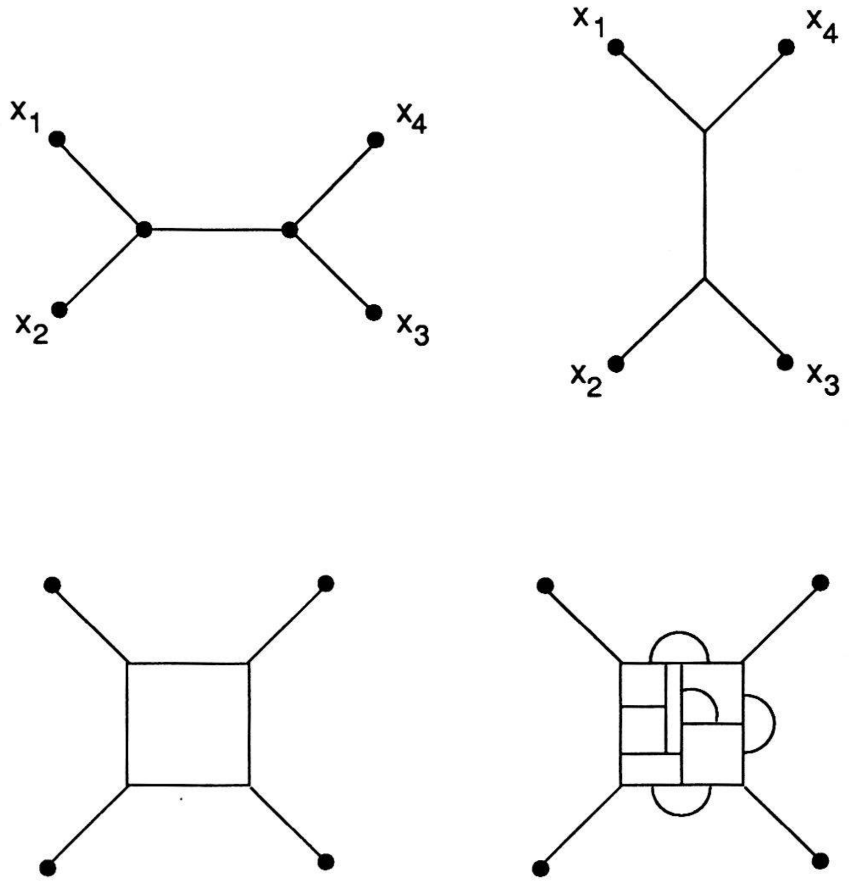


Figure 1.6. Some contributions to the four–particle amplitude in a $\lambda\phi^3$ theory.

At this elementary level, we can already see some non–trivial differences between particle theory and string theory. Quite generally, a local and relativistically invariant field theory can have interaction vertices (the nodes in the network) with arbitrary complexity. In principle, any number of lines can be in-coming or out-going. This would describe a ϕ^n vertex if the total number of lines at the vertex were n, because the branching of the network is seen at the same space–time point by all observers. This is not the case in string theory, however. First of all, notice that any space–time string diagram can always be split into components made up of the free string propagator and a three–pronged vertex. Some illustrative examples are shown in figure 1.7a and b.

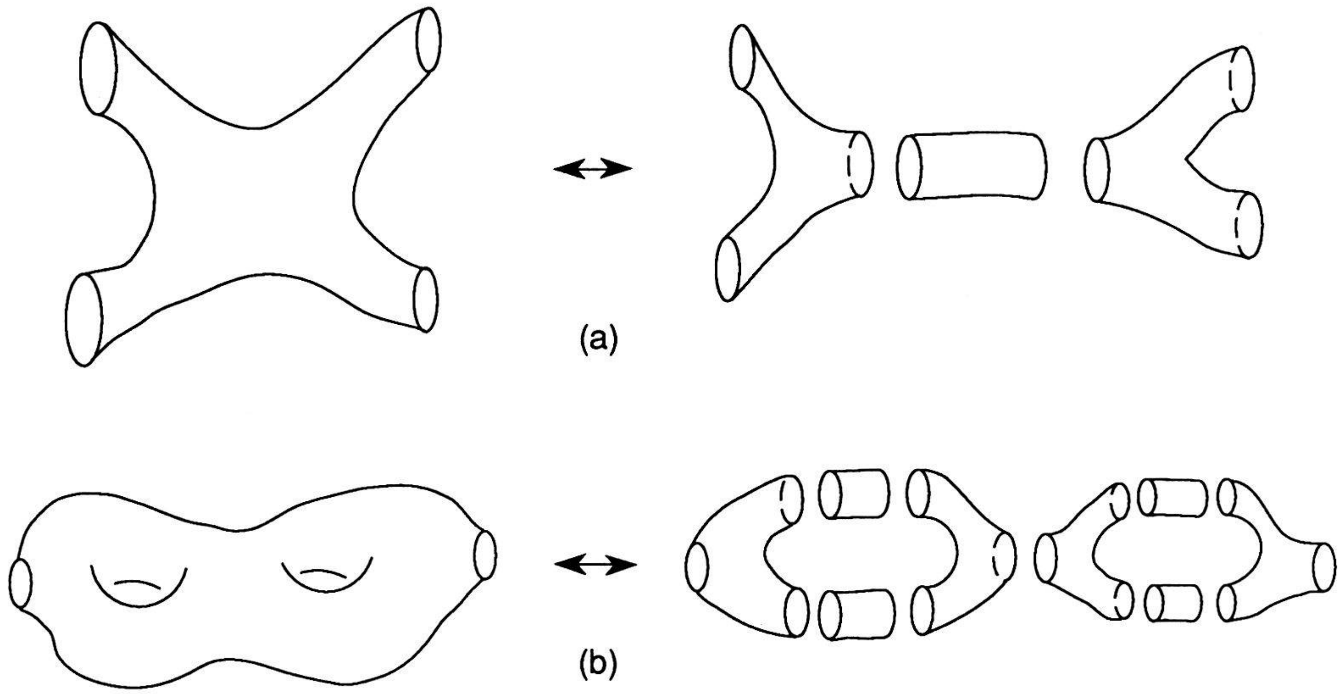


Figure 1.7. Decomposition of some string diagrams into propagators and three–point vertices. (a) Tree–level contribution to the four–string amplitude. (b) Two–loop contribution to the string two–point function.

We can describe the three–string vertex in terms of a time history, as in figure 1.8. The breaking of one string into two occurs locally at the point P. Even though the strings are extended objects, they interact only locally; and yet, the breaking point P is different for different observers. The reader can easily convince herself/himself of this by drawing a few space–time diagrams of the vertex for different observers. This feature is also responsible for the high degree of uniqueness of the possible interaction vertices in the theory of relativistic strings.

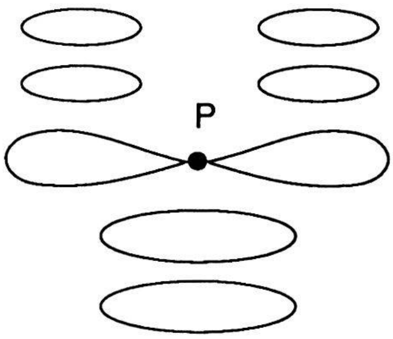


Figure 1.8. Snapshots of the string vertex in FIgure 1.7(b).

We can associate a genus to the string diagrams of figure 1.7a if we think of the graph as a topological surface: the expansion in genus (number of handles) is equivalent to the expansion in powers of $\hbar$ in quantum field theory.

To summarize, we have learned that in the first quantized approach to string theory we must consider a conformal field theory with a given value of c, and then sum over all possible surfaces interpolating between some given initial and final states.

We now outline Polyakov's approach [21] to the quantization of string theory (see also [22] and the references at the beginning of this section). In our previous arguments we have dealt mostly with the classical theory of strings. When we quantize them, we find a number of complications which allow us to understand the difference between critical and sub-critical strings. The Nambu–Goto action (1.4) was written in a simpler form in (1.6) by introducing the two-dimensional metric g_{ij} as an auxiliary field. Both (1.4) and (1.6) are invariant under arbitrary reparametrizations of the two-dimensional world-sheet. We also found the classical action (1.6) to be invariant under Weyl rescalings (1.9). When we quantize using path integrals, we have to sum not only over the embedding variables $X^\mu(\sigma,\tau)$ (or any other variables describing the two-dimensional conformal field theory living on the string world-sheet) but also over the metric g_{ij}. To avoid over-counting, we have to "divide" the integral by the volume of the group of diffeomorphisms of the two-dimensional surface. Classically, one would think that the volume of the group of Weyl transformations should also be divided out. This is a dynamical issue. We can think of the Liouville field $\phi(\sigma,\tau)$ in (1.9) as another field in the theory. When we quantize the theory, two things may happen:

i) The Liouville field decouples completely. Then it is legitimate to divide out by the Weyl group because (1.9) is a true symmetry of the full quantum theory. When this happens, we say that the string theory is critical. For bosonic strings, this requires $c = 26$.

ii) The Liouville field does not decouple at the quantum level. The Weyl transformations cannot be divided out and the string theory contains the coupling of the conformal field theory to the Liouville field. In this case, we say that the string theory is non–critical. We will see further ahead that, in order to solve the string theory in this case, one needs to know how to solve the quantum Liouville theory. Unfortunately, it is not yet known how to do this in general.

We can write formally the vacuum–to–vacuum amplitude (the cosmological constant) in string theory as follows:

$$Z = \sum_h \lambda^h \int \frac{(\mathcal{D}g_{ij})'}{Vol(Diff)} \int (\mathcal{D}X^\mu) e^{iS[X,g]} \tag{1.36}$$

First in (1.36), we sum over the different topologies of the world–sheet: sphere, torus, double torus, etc. In two dimensions, a closed oriented surface is completely characterized topologically by the genus or number of handles h. The parameter λ is a loop–counting parameter. For each topology, we sum over all possible embeddings (the $\mathcal{D}X^\mu$ integral) and over all possible metrics in that fixed topology, modulo diffeomorphisms. This explains the factor $Vol(Diff)$.

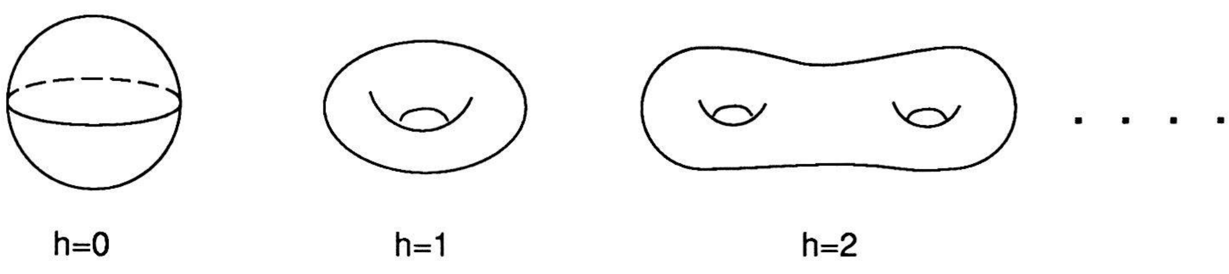


Figure 1.9. The first three topologies contributing to the sum in (1.36).

The prime in the integration over g_{ij} in (1.36) distinguishes between critical and sub-critical strings. In the critical case, we still have to divide by the volume of the Weyl group, and the integration over geometries simplifies drastically: it reduces to a finite–dimensional integral. A two–dimensional metric has three independent components g_{11}, g_{12} and g_{22}. A general diffeomorphism depends on two arbitrary functions. The Weyl rescaling provides another arbitrary function. It can be shown that any metric g_{ij} on a Riemann surface Σ can be written in the form

$$g = e^{\phi} f^{*}(\hat{g}(t)) \tag{1.37}$$

where f^* represents the action of a diffeomorphism, and e^{ϕ} is a Weyl rescaling. The parameters t_i are known as the moduli of the surface. The sphere has no moduli, the torus has one complex parameter and, for genus $h > 1$, the number of complex moduli equals $3h - 3$. In (1.37), $\hat{g}(t)$ represents some slice in the space of metrics cutting once each orbit of the group of diffeomorphisms and Weyl transformations. Hence, for critical strings, the integration over geometries is reduced to integration over the moduli space of Riemann surfaces. This is a difficult mathematical problem, but certainly simpler than solving the quantum Liouville theory necessary for sub-critical strings.

Once (1.36) has been reached, it is clear why string theory can be thought of as a theory of random surfaces. We would like to be able to evaluate (1.36) and also to obtain non–perturbative information about the theory. In order to do this, we have to be able to define the different pieces in (1.36) in a non–perturbative way. One possibility (perhaps the only one known) is to discretize the problem. We can replace the conformal theory by a statistical mechanical system at criticality, and the sum over metrics can be replaced by a sum over random triangulated surfaces of arbitrary topology. In the continuum limit of this approximation, one should recover a model for (1.36).

In general, one would imagine that solving a spin system on a random lattice should be much more difficult than studying it in the continuum. The big surprise at the end of 1989 was that for some sub-critical strings (with $c \leq 1$) it is possible to obtain exact expressions for (1.36). More precisely, the partition functions were found to satisfy some non–linear ordinary differential equations intimately related to the KdV hierarchy. It is still mysterious why complicated objects such as (1.36) are related with the theory of integrable models.

In these lectures, we will attempt to explain how (1.36) can be computed exactly starting with random triangulations and using the double scaling limit, simultaneously discovered last year by three different groups: E. Brézin and V. Kazakov in Paris, M. Douglas and S. Shenker at Rutgers, and D. Gross and A.A. Migdal at Princeton [6, 7, 8].

This will require us to review first some aspects of conformal field theories, of large N matrix models as an efficient method to generate random lattices, of the theory of integrable systems, and of two-dimensional gravity. It is also quite surprising that on random lattices one can often obtain more results than on an ordinary square lattice. In 1988, Polyakov [1] and Knizhnik, Polyakov and Zamolodchikov (KPZ) [2] were able to obtain the exact critical exponents of the minimal conformal theories coupled to two-dimensional gravity. In other words, they were able to obtain exact information about the behaviour of some conformal field theory coupled to the Liouville field. Since this work sheds a lot of light on the issues of string theory, we shall present the results of KPZ in this course.

Before leaving this quick introduction to string theory, we would like to mention in passing that the scattering processes involving string excitations of critical strings can be represented in terms of expectation values of vertex operators. A pictorial argument can be given as follows. Consider the four-point scattering amplitude in figure 1.10, where the incoming (respectively outgoing) lines come from the past (respectively future) asymptotic region. By a conformal transformation we can map figure 1.10 into a sphere with four distinguished points (figure 1.11).

Hence, it is plausible that the string excitations can be represented by local operators on the world-sheet. In the case of the bosonic string, these operators take the form

$$V(k,z) = P(\partial X)e^{ik_\mu X^\mu} \tag{1.38}$$

where $P(\partial X)$ is a polynomial in the derivatives of X^{μ}.

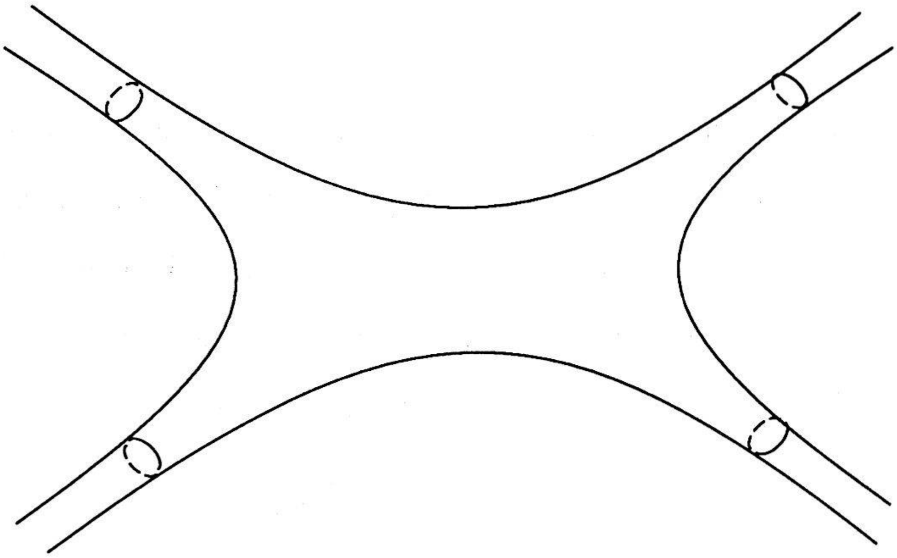


Figure 1.10. Tree–level contribution to the scattering of two strings.

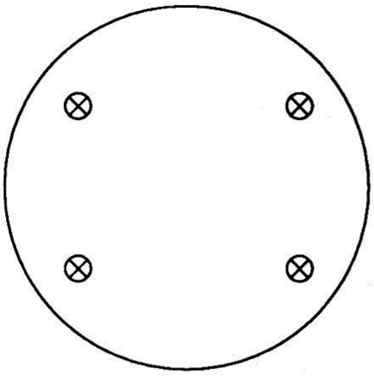


Figure 1.11. A conformally equivalent description of figure 1.10.

1.3. General Remarks

We would like to close this lecture with some comments about other aspects of the statistics of random surfaces. The string action we considered above contained only the area of the embedded surface. In the language of the renormalization group, this term is strongly relevant and should be the leading term in the infra-red description of the theory. Nevertheless, other terms should also be important in the study of phenomena involving random surfaces in the research domains of interfaces, biophysics, molecular membranes, etc. In many statistical models in three dimensions one can easily see that the understanding of fluctuating surfaces should play an important role in the classification of universality classes.

As a typical example, consider the three–dimensional Ising model. By a duality transformation, it can be transformed into a Z_2 gauge theory [23]. Indeed, consider a three–dimensional square lattice and associate with each link ℓ a variable σ_ℓ taking values ± 1. If we want to make the theory invariant under Z_2 gauge transformations, we write the analogue of the Wilson action and obtain a partition function

$$Z = 2^{-3N} \sum_{\{\sigma_\ell\}} \exp \beta \sum_p \sigma(\partial p) \tag{1.39}$$

were $\sigma(\partial p)$ stands for the product of the four link variables around the plaquette p, and $\sum_p$ is the sum over all plaquettes of the lattice. Expanding the exponent and using the fact that $\sigma^2(\partial p) = 1$, we obtain:

$$Z = 2^{-3N} \sum_{\{\sigma_\ell\}} \prod_p \cosh \beta (1 + \sigma(\partial p) \tanh \beta) \tag{1.40}$$

The summation over σ_ℓ gives either 0 or 1 according to the parity of the power of σ_ℓ in each term in the expansion of the product. Each contributing term can be represented geometrically as a closed surface which may self–intersect. However, each plaquette belonging to the surface may appear only once, and at most four plaquettes may share the same link. Hence

$$Z = (\cosh \beta)^{3N} \sum_{\substack{\text{closed} \\ \text{surfaces}}} (\tanh \beta)^n \tag{1.41}$$

Here, n is the number of plaquettes making the closed surface.

A closed surface is also characterized by the different volumes it separates. By duality in three dimensions, a cube in the direct lattice L is associated to a site in the dual lattice L^*; this site can be visualized at the center of the original cube. Similarly, a plaquette is dual to a link. To write n in (1.41) more conveniently, we define a new variable s_i associated with the sites of L^*. The value $s_i = +1$ corresponds to the cube being exterior to the closed surface, and $s_i = -1$ for a cube belonging to the inner volume. Each configuration of $\{s_i\}$, defined up to a global sign, is in one–to–one correspondence with each closed surface. Since a plaquette belongs to the closed surface when the two s_i's joined by its dual link have opposite signs, we can write

$$n = \frac{1}{2} \sum_{<i,j>} (1 - s_i s_j) \tag{1.42}$$

where the notation $< i, j >$ stands for pairs of nearest neighbour sites. Replacing (1.42) in (1.41), we obtain

$$Z = \frac{1}{2}(\cosh\beta)^{3N}(\tanh\beta)^{3N/2} \sum_{\{s_i\}} e^{-\frac{1}{2}\log\tanh\beta \sum_{<i,j>} s_i s_j} \tag{1.43}$$

The overall factor of $\frac{1}{2}$ is due to the overall sign ambiguity in relating closed surfaces to dual spins s_i.

Thus for the free energies

$$F(\beta) = F_{\text{Ising}}(\beta^*) + \frac{3}{2}\log\sinh 2\beta - \frac{1}{2}\log 2 \tag{1.44}$$

with

$$\beta^* = -\frac{1}{2}\log\tanh\beta \tag{1.45}$$

or, equivalently,

$$\sinh 2\beta \sinh 2\beta^* = 1 \tag{1.46}$$

Therefore, in the high–temperature phase of the three–dimensional Ising model, the partition function can be written as an expansion in random surfaces of some special type.

From a more phenomenological point of view, one could try to derive the generic action describing a theory of fluctuating surfaces by taking into account not only their area but also their bending in ambient space (for details and references, see the contribution by

F. David in [24]). As before, let X^μ describe the embedding of the surface in some D–dimensional space. So far, we have only used the area in the induced metric,

$$g_{ij} = \partial_i X \cdot \partial_j X \tag{1.47}$$

Some additional possibilities are offered by the extrinsic curvature tensor

$$K^\mu_{ij} = D_i D_j X^\mu \tag{1.48}$$

and the intrinsic Ricci scalar

$$R = K^i_i \cdot K^j_j - K^i_j \cdot K^j_i \tag{1.49}$$

With these three ingredients, we can write the most general action

$$S = \Lambda \int d^2\sigma\sqrt{g} + \frac{\kappa}{2}\int d^2\sigma\sqrt{g}K^i_i \cdot K^j_j + \frac{\overline{\kappa}}{2}\int d^2\sigma\sqrt{g}R \tag{1.50}$$

By dimensional analysis, Λ has dimension of (mass)^2 while κ and $\overline{\kappa}$ have dimension zero and are marginal parameters. In the study of the large distance properties of the surface, any other terms are irrelevant.

Depending on the embedding dimension, other terms may be added. For example, in $D = 4$ a two–dimensional surface will self–intersect only at points. We may add to the action a term proportional to the self–intersection number of the surface. This topological term would be analogous to the θ term in four–dimensional non–abelian gauge theories.

The action (1.50) was introduced by A.M. Polyakov to study the "fine structure" of strings and to make contact with the confined phase of four-dimensional gauge theories. In the context of statistical mechanics, (1.50) had been considered before in the study of membranes and vesicles. The coefficient κ is known, in this context, as the bending rigidity modulus, and $\overline{\kappa}$ as the Gaussian rigidity modulus. The coefficient Λ is the two–dimensional cosmological constant, and when the area of the surface is not fixed, it can be thought of as the chemical potential of surface elements.

The correlation length ξ is known as the persistence length. It is a non–trivial problem to determine whether the surface will be crumpled (finite ξ) or flat (infinite ξ). At long distances (*i.e.*, bigger than the persistence length) there are general arguments which suggest that the effective theory is controlled by the cosmological term in (1.50) plus the action (1.6). In this case, an important critical exponent is the string susceptibility obtained by counting the number of configurations $n(A)$ of a surface with fixed area A. For $D < 1$ it was shown by KPZ [2] that

$$n(A) \propto A^{\gamma-3} e^{\Lambda_o A} \tag{1.51}$$

with

$$\gamma = \frac{1}{12}\left[D - 1 - \sqrt{(D-1)(D-25)}\right] \tag{1.52}$$

for a surface of spherical topology. This and many other impressive results obtained by KPZ generated a good deal of activity which led to the construction of exact solutions to string theory for $D \leq 1$. In (1.51), Λ_o is the critical value of the cosmological constant.

Another example where random surfaces play an important role is gauge theory in four dimensions. In these lectures we concentrate on the theory of sub-critical strings, and we shall not dwell on other subjects where fluctuating surfaces show up. It should be clear, nevertheless, that any advance in the study of $D > 1$ string theory is very likely to provide useful information on three–dimensional critical phenomena and on four–dimensional gauge theories. As we shall see, at $D = 1$ the Liouville theory enters a strong coupling regime which has not yet been characterized in any detail. It is actually quite possible that the surfaces stop behaving as two–dimensional objects at large distances.

2. Conformally Invariant Theories

2.1. Preliminary Technicalities

The reason why scale–invariant field theories in $D = 2$ are very special is because the conformal group in two dimensions is infinite–dimensional. A useful preliminay result in the study of conformal field theories is to prove that in any relativistically (or euclidean) invariant field theory, scale invariance implies automatically conformal invariance. The basic reference in conformal field theories is [25]. More thorough reviews than the one presented in this and the following lecture are for example [26, 27, 28, 29].

Let $\Theta_{\mu\nu}$ be the energy–momentum tensor of the theory. It can be obtained via Noether's theorem or else by placing the theory in an external gravitational field, computing the functional variation of the action with respect to the metric, and evaluating the result at the trivial metric:

$$\Theta_{\mu\nu} = \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}\bigg|_{g=\delta} \tag{2.1}$$

The energy–momentum tensor generates co-ordinate transformations in D dimensions. Given a vector field V^{μ}, we can construct a current

$$j_{\mu}(V) = V^{\nu}\Theta_{\nu\mu} \tag{2.2}$$

If the energy–momentum is conserved (we assume so), the current j_{μ} will be conserved provided

$$\partial_{\mu} j^{\mu} = (\partial_{\mu} V_{\nu})\Theta^{\mu\nu} + V_{\nu}\partial_{\mu}\Theta^{mu\nu} = \frac{1}{2}(\partial_{\mu}V_{\nu} + \partial_{\nu}V_{\mu})\Theta^{\mu\nu} = 0 \tag{2.3}$$

Therefore if

$$\partial_{\mu}V_{\nu} + \partial_{\nu}V_{\mu} = 0 \tag{2.4}$$

then the current $j_{\mu}(V)$ is conserved.

Particular examples of this method are given by infinitesimal translations and Lorentz transformations. For dilatations or scale transformations, the vector field is $V^{\mu} = x^{\mu}$ and the dilatation current is

$$D_\mu = x^\nu \Theta_{\mu\nu} \tag{2.5}$$

It follows that

$$\partial_\mu D^\mu = \Theta_\mu^{\ \mu} \tag{2.6}$$

and dilatations are a symmetry of the theory as long as the energy–momentum tensor is traceless.

If the theory is scale–invariant *i.e.*, if $\Theta_\mu^{\ \mu} = 0$ and $\partial_\mu \Theta^{\mu\nu} = 0$, we now prove that the theory is invariant under all the conformal transformations. For a generic metric, a conformal transformation is a change of variables characterized by the equation

$$g_{\mu\nu}(x')dx'^\mu dx'^\nu = \Omega^2(x) g_{\mu\nu} dx^\mu dx^\nu \tag{2.7}$$

In other words, the angles are preserved. Infinitesimally, $\Omega = 1+\lambda(x)$ and $x'^\mu = x^\mu + V^\mu(x)$. A simple computation then yields

$$\partial_\mu V_\nu + \partial_\nu V_\mu - \frac{2}{D}\eta_{\mu\nu}\partial_\alpha V^\alpha = 0 \tag{2.8}$$

where we have restricted the computation to the flat space–time metric.

The solution to (2.8) when $D > 2$ is

$$V_\mu = b_\mu + \omega_{\mu\nu}x^\nu + \lambda x_\mu + (c\cdot x)x_\mu - \frac{1}{2}c_\mu x^2 \tag{2.9}$$

with $\omega_{\mu\nu} = -\omega_{\nu\mu}$. If V_μ satisfies (2.8) and $\Theta_\mu^{\ \mu} = 0$, then

$$\partial_\mu(V_\nu \Theta^{\nu\mu}) = \partial_\mu V_\nu \Theta^{\nu\mu} = \frac{1}{2}\left(\partial_\mu V_\nu + \partial_\nu V_\mu - \frac{2}{D}\eta_{\mu\nu}\partial_\alpha V^\alpha\right)\Theta^{\mu\nu} = 0 \tag{2.10}$$

For a Minkowski space of signature (p, q), it is left as an exercise to verify that the generators (29) satisfy the commutation relations of the Lie algebra $SO(p+1, q+1)$. Hence for $D > 2$, when scale invariance is promoted to conformal invariance, D more symmetry generators appear.

For $D = 2$, using the light–cone co-ordinates (1.11), the equation (2.8) reduces to

$$\partial_+ V_+ = \partial_- V_- = 0 \tag{2.11}$$

yielding an infinity of solutions

$$V_+ = V_+(\sigma^-) \quad , \quad V_- = V_-(\sigma^+) \tag{2.12}$$

This can be understood easily if we write the two–dimensional Minkowski metric $ds^2 = d\tau^2 - d\sigma^2$ in terms of $\sigma^\pm$:

$$ds^2 = d\sigma^+ d\sigma^- \tag{2.13}$$

If we make a transformation

$$\sigma^+ \to f(\sigma^+) \quad , \quad \sigma^- \to g(\sigma^-) \tag{2.14}$$

it is clear that

$$ds^2 \to f'(\sigma^+) g'(\sigma^-) ds^2 \tag{2.15}$$

and therefore the angles and the light–cones are left invariant. After Wick–rotating $\sigma^\pm$ as in lecture 1, we find arbitrary analytic redefinitions of w, $\overline{w}$. Since the cylinder is conformally equivalent to the sphere minus two points ($z = 0$ and ∞, the South and North poles), using the z, $\overline{z}$ variables (1.12) the generators of conformal transformations are vector fields

$$v_n(z) = z^{n+1} \frac{d}{dz} \tag{2.16}$$

with poles only at $z = 0$ or $z = \infty$ depending on the value of n.

When $D > 2$, the conformal group is bigger than the euclidean group (or the Poincaré group, depending on the signature) but it is nevertheless a finite–dimensional group, actually not very useful in the computation of correlation functions and anomalous dimensions either in critical theories or in field theory. In $D = 2$, we shall see that conformal invariance is extremely useful and, in some cases, strong enough to allow us to compute all correlation functions.

2.2. Scale Invariance and the Renormalization Group

One of the major advances in our understanding of critical systems has been the renormalization group. Two of the basic results are the scaling laws by which all critical exponents of a given system can be expressed in terms of a smaller number, and the principle of universality according to which large classes of microscopic hamiltonians share the same critical behaviour. There are many good books and reviews on the renormalization group in the literature, see for instance [30, 31]. We will only stress some of its most salient features.

The problem of co-operative phenomena in condensed matter physics is a difficult one. It is rather surprising to find that, out of microscopic hamiltonians with short-range interactions, one can obtain phenomena with infinite correlation length. This is what happens at criticality. For some special models, an exact derivation of their critical behaviour has been obtained: the two-dimensional Ising model, the six-vertex model, the eight-vertex model, the hard hexagons, and some generalizations thereof [32]. All these models describe two-dimensional systems. Their partition functions, as well as many of their critical exponents, are known. In the exactly solved models, one can test some of the hypotheses of the renormalization group and also obtain further insights into the hidden symmetries which are responsible for the exact solvability: Yang–Baxter equations [32], quantum groups, etc.

From a more realistic point of view, the existence of exactly solved models (in two dimensions) is not very helpful in understanding the many features of co-operative phenomena. The renormalization group provides a symmetry which allows us to obtain useful information about the macroscopic behaviour of critical systems without the need to know how to solve them exactly. This is a common case in physics. The hamiltonian of the strong interactions describing nuclei and low–mass baryons and mesons is neither completely known nor solvable. Nevertheless, since strong interactions are approximately

$SU(3)$ flavor symmetric and the electromagnetic interaction is proportional to the third component of isospin, we can understand many features of the hadron spectrum without having to solve the theory completely. A similar argument can be made in atomic physics beyond the hydrogen atom. This "symmetry" approach to critical systems is provided by the renormalization group, which consists essentially of two transformations:

i) Kadanoff's block spin transformation;

ii) Rescaling to the original lattice size.

Imagine we start with a D–dimensional lattice with L^D sites (L very large) and lattice spacing a. For simplicity, let $\phi(x)$ be a continuous set of variables describing the lattice degrees of freedom whose interactions are described by the hamiltonian $H[\phi]$. (The assumption that $\phi(x)$ is a continuous variable is irrelevant, the same arguments go through for discrete variables.) We may Fourier–transform the field $\phi(x)$:

$$\phi(x) = L^{-D/2} \sum_{|k|<\Lambda} \phi(k) e^{ik\cdot x} \tag{2.17}$$

where Λ is a cut–off, given for a square lattice by the boundary of the Brillouin zone π/a. Regardless of whether the hamiltonian $H[\phi]$ is fundamental (say the exact hamiltonian describing the interactions between the atoms in a solid) or it is already a simplified model, the restriction $|k| < \Lambda$ means that we are not interested in describing details of the system at lengths shorter than $\sim \Lambda^{-1}$. The Kadanoff transformation allows us to thin out systematically the fast (or microscopic) degrees of freedom by integrating out the modes with momentum in the shell $\Lambda/s \leq k \leq \Lambda$. In terms of the Fourier components, this leads us to an effective hamiltonian for the degrees of freedom with $k < \Lambda/s$:

$$e^{-H_{\Lambda/s}[\phi]/T} = \int \prod_{\Lambda/s\leq k\leq\Lambda} d\phi(k) e^{-H[\phi]/T} \tag{2.18}$$

In principle, no information is lost in this transformation, as long as we are interested in very low wave–numbers (long distances). We can define these transformations as

$$H_{\Lambda/s}[\phi] \equiv R_s H[\phi] \tag{2.19}$$

If we work in real space, we can divide the lattice in blocks of size sa, each containing s^D variables. Then we can define an average variable in each block:

$$\phi(x) = s^{-1} \sum_{y \text{ in block } x} \phi(y) \tag{2.20}$$

For the coarse-grained variables,

$$e^{-H_s[\phi]/T} = \int [d\phi] \prod_x \delta \left(\phi(x) - s^{-D} \sum_{y \text{ in block } x} \phi(y) \right) e^{-H[\phi]/T} \tag{2.21}$$

the new theory is defined on a lattice with $(L/s)^D$ sites and latttice spacing sa. We can define R_s, $R_{s'}$ for different s and s', and we can easily check that

$$R_s R_{s'} = R_{ss'} \tag{2.22}$$

whereby renormalization operations do form a semi-group.

Notice that after a few steps like (2.20), even discrete original variables would become effectively continuous. If, for instance, $\phi(y) = \pm 1$, then the block variable can take s^{D+1} values between -1 and $+1$. When s increases, ϕ becomes for all practical purposes a continuous variable.

The second step in the renormalization group is to relabel the variables and rescale back to the original size. This means to change $\phi(x) \to \lambda_s \phi(x/s)$, which is a scale transformation. Steps i) and ii) can be implemented simultaneously if the δ-function in (2.21) is replaced by

$$\delta \left(\lambda_s \phi(\frac{x}{s}) - s^{-D} \sum_{y \text{ in block } x} \phi(y) \right) \tag{2.23}$$

The factors λ_s satisfy a functional equation as a consequence of (2.22):

$$\lambda_s \lambda_{s'} = \lambda_{ss'} \tag{2.24}$$

with solution

$$\lambda_s = h^s \tag{2.25}$$

Although the lattice shrinks in size, we shall assume that L is large enough that we do not have to worry about finite–size or edge effects. When we study bulk properties of the material, we shall take $L \to \infty$ *i.e.,* the thermodynamic limit.

In our simplified discussion, only one field variable was considered. More generally, the renormalization group tranformation will transform the original fields $\phi_i(x)$ into a new set with a linear relation of the form

$$\phi_i(x) \overset{R_s}{\rightarrow} \sum_j M_{ij}(s)\phi_j(x/s) \tag{2.26}$$

The fields $\phi_j(x)$ which diagonalize the matrix $M(s)$ are called scaling fields:

$$\phi_j \overset{R_s}{\rightarrow} s^{h_j}\phi_j(x/s) \tag{2.27}$$

The exponent h_j is the scaling dimension of the field ϕ_j. These dimensions are intimately connected with the critical exponents.

Similarly, if we could also define rotations on the ϕ_j fields, we could also associate a spin to them:

$$\phi_j(x) \rightarrow e^{i\ell_j\theta}\phi_j(R_\theta^{-1}x) \tag{2.28}$$

where R_θ stands for a rotation through an angle θ about some axis.

It should be clear from steps i) and ii) that, if on the original lattice the correlation length is ξ (in lattice units), after applying R_s it will reduce to ξ/s. The correlation length ξ is determined by looking at the long distance behaviour of two–point functions

$$\langle\phi(x)\phi(0)\rangle \sim_{|x|\to\infty} e^{-|x|/\xi} \tag{2.29}$$

It should also be mentioned at this point that the scaling fields represent the order parameters. As we approach the critical temperature $|T - T_c| \to 0$, the critical exponents follow from h_j. At the critical point the correlation length goes to infinity $\xi \to \infty$. Now we can understand qualitatively why the system is scale–invariant at the critical point. In second-order phase transitions, the correlation length of the fluctuations diverges. Mathematically, the scale ξ is taken to infinity. In the renormalization group, however, we have been thinning out the degrees of freedom and it is plausible that at criticality the memory of microscopic lengths is lost, hence the physical phenomena in this region have no scale and become scale-invariant.

There is a nice way to illustrate the renormalization group. Imagine that we have a very powerful microscope which we use to look at the most minute details of a system. In the microscope picture we see the arrays of atoms, etc. If we gradually lose magnification power, and the sample is large enough, it will appear as if it was receding from us. Eventually, we will lose all knowledge of the microscopic details and the macroscopic structures will appear (domains in ferromagnets, droplets in liquid-gas transitions, etc.)

From the previous arguments, it should be clear that one needs to study the fixed points of the renormalization group transformation in order to obtain information about the critical behaviour. In other words, we look for hamiltonians satisfying

$$R_s H^* = H^* \tag{2.30}$$

Since in the renormalization group transformations we always rescale to the original lattice, we can picture the iteration of R_s as a trajectory in the space of possible hamiltonians. This is perhaps not very precise mathematically, but quite clear physically. Starting with some initial H_0, we obtain a sequence H_0, $R_s H_0$, $R_s^2 H_0, \ldots$ which we can approximate as a flow in the space of coupling constants. If $\lambda = (\lambda_1, \lambda_2, \ldots)$ stands for the coupling constants of the problem, then the action of the renormalization group generates a trajectory $\lambda(s)$. Suppose we can find a fixed point of R_s:

$$R_s \lambda^* = \lambda^* \tag{2.31}$$

true for any value of s. The fixed points may be discrete or form subvarieties in coupling constant space. One defines the critical surface of the fixed point λ^* as the subspace of

λ–space having the property that it contracts under R_s to λ^* as $s \to \infty$:

$$\lim_{s\to\infty} R_s\lambda = \lambda^* \tag{2.32}$$

All points of the critical surface are eventually driven to a fixed point. In the neighbourhood of λ^*, we can linearize the renormalization group transformation. If λ is close to λ^*, we can write formally $\lambda = \lambda^* + \delta\lambda$ and for the transformed point $\lambda' = \lambda^* + \delta\lambda'$. Hence, $\lambda' = R_s\lambda$ becomes

$$\delta\lambda' = \tilde{R}_s\delta\lambda \tag{2.33}$$

with the linear operator

$$(\tilde{R}_s)_{ij} = \frac{\partial\lambda'_i}{\partial\lambda_j}\bigg|_{\lambda=\lambda^*} \tag{2.34}$$

The eigenvalues and eigenvectors of this matrix will determine the scaling behaviour near λ^*. The eigenvalues are of the form s^{h_j}, and we can expand $\delta\lambda$ in the basis of eigenvectors e_j, $\delta\lambda = \sum_j t_j e_j$. Under a renormalization group tranformation,,

$$\delta\lambda' = \sum_j t_j s^{h_j} e_j \quad ; \qquad t'_j = t_j s^{h_j} \tag{2.35}$$

The parameters in the theory can be divided as follows:

Relevant when they grow as s increases; or

Irrelevant when they decrease as s increases; or

Marginal when they do not change as s increases.

The fundamental hypothesis linking the renormalization group with critical phenomena is that at zero external field, $\lambda(T_c, 0, \ldots)$ is a point on the critical surface of a fixed point λ^*. For example, in a ferromagnet the two important parameters are the temperature T and the external magnetic field h. Then $\lambda(T_c, 0)$ is at the critical surface, and it is driven to λ^* as $s \to \infty$, whereas $\lambda(T, h)$ is not on the critical surface if $T \neq T_c$ or $h \neq 0$. In terms of hamiltonians, we can rewrite (2.35) as

$$H(t) = H^* + \sum t_j H_j \tag{2.36}$$

Similarly, the operators H_j can be classified as relevant, irrelevant or marginal. It is now clear that, since $\lambda(T_c)$ is on the critical surface, if $T - T_c$ is very small, then $R_s\lambda(T)$ will move closer to λ^* as $s \to \infty$. The critical dependence of $\lambda(T)$ as a function of $(T - T_c)$ is then determined by the spectrum of anomalous dimensions h_j. Finding the h_j is one of the basic problems in the renormalization group approach to co-operative phenomena.

Remark 1) We saw in section 2.1 that any local field theory, invariant under the euclidean roation group and scale transformations, is actually invariant under the whole conformal group. In an important paper that appeared in 1984 [25], A. Belavin, A. Polyakov and A. Zamolodchikov (BPZ) made the hypothesis that scale invariance in two dimensions should be extended to full conformal symmetry. Using this hypothesis, they were able to compute exactly the critical exponents of many critical systems. In standard critical phenomena, one has a finite number of order parameters. From the previous arguments and using the BPZ hypothesis, one can ask if it is possible to classify all two-dimensional conformal field theories with a finite number of primary fields. The question was answered in the positive by BPZ; we will explain some of their results in lecture 3.

Remark 2) Partly motivated by the renormalization group, there has been a large amount of research on dynamical systems and the theory of iterated maps. This permits a rigorous study of many of the ideas of the renormalization group: universality classes, types of fixed points, period doubling, chaotic behaviour, etc. In analogy with the renormalization group, we can think of iterated maps as a discrete dynamical system. For instance, one of the basic examples is provided by the map $f(x) = -\mu x(1-x)$. The series obtained by iteration x, $f(x)$, $f(f(x)) = f^2(x)$, $f^3(x)$, $\ldots$ displays a very rich and complex behaviour depending on the value of the coupling constant μ. This simple example exhibits many of the interesting features of one-dimensional dynamical systems.

Remark 3) There are many situations where the ideas of the renormalization group can be applied succesfully, such as brownian motions, self-avoiding random walks, polymers, and many other systems which can be described by "geometric scaling". Some of these systems can also be described in the infra-red by (non–unitary) conformal field theories.

2.3. Simple Consequences of Conformal Invariance

In the last part of this lecture we shall consider some simple properties of conformal field theories. We will follow [25] and rely mostly on the properties of the Virasoro algebra (1.33). The largest set of commuting generators is $\{L_0, c\}$. Since the central term is fixed by the theory, we can label the states by their L_0 (and $\overline{L}_0$) eigenvalues. Recall that $(L_0 + \overline{L}_0)$ is the energy operator. To define the ground state, we look for the largest set of compatible conditions we can impose on it. An examination of the commutation relations (1.33) indicates that we can require L_n $(n \geq -1)$ to annihilate the vacuum:

$$L_n \left|0\right\rangle = 0 \qquad n \geq -1 \tag{2.37}$$

Since $[L_0, L_n] = -nL_n$, it is reasonable to require that $|0\rangle$ be the lowest energy eigenstate: this is precisely what (2.37) indicates. The simplest scaling fields are those transforming as tensors under conformal transformations. If $\Phi_{h,\overline{h}}$ has scaling eigenvalues h, $\overline{h}$, and we consider two overlapping patches U_α, U_β (with $U_\alpha \cap U_\beta \neq \emptyset$), the quantity $\Phi_{h,\overline{h}} dz^h d\overline{z}^{\overline{h}}$ is invariantly defined:

$$\Phi^{(\alpha)}_{h,\overline{h}} dz_\alpha^h d\overline{z}\alpha^{\overline{h}} = \Phi^{(\beta)}_{h,\overline{h}} dz_\beta^h d\overline{z}_\beta^{\overline{h}} \tag{2.38}$$

which can be rewritten as

$$\Phi^{(\beta)}_{h,\overline{h}}(z_\beta) = \Phi^{(\alpha)}_{h,\overline{h}}(z_\alpha) \left(\frac{dz_\alpha}{dz_\beta}\right)^h \left(\frac{d\overline{z}_\alpha}{d\overline{z}_\beta}\right)^{\overline{h}} \tag{2.39}$$

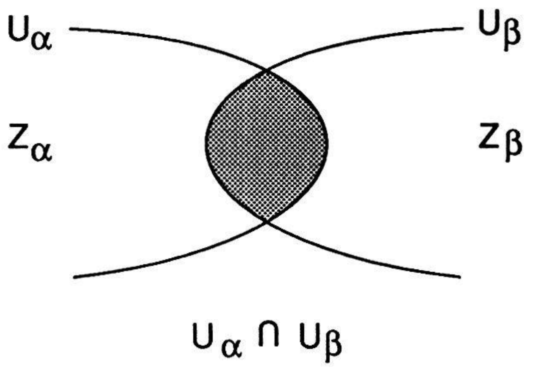


Figure 2.1. Two overlapping patches on a surface.

Under an infinitesimal transformation $z \to z + \varepsilon(z)$,

$$\partial_\varepsilon \Phi_{h,\bar{h}} = \left(\varepsilon \frac{\partial}{\partial z} + h \frac{\partial \varepsilon}{\partial z} \right) \Phi_{h,\bar{h}} \tag{2.40}$$

The anti-holomorphic transformations work similarly; from now on we shall not write them explicitly unless needed. Since the energy–momentum generates conformal transformations, we can write

$$\partial_\varepsilon \Phi_{h,\bar{h}} = \left[T(\varepsilon), \Phi_{h,\bar{h}} \right] = \varepsilon \frac{\partial \Phi_{h,\bar{h}}}{\partial z} + h \frac{\partial \varepsilon}{\partial z} \Phi_{h,\bar{h}} \tag{2.41}$$

If we particularize to $\varepsilon_n = z^{n+1}$, then $T(\varepsilon) = L_n$ and

$$\delta_n \Phi = [L_n, \Phi] = \left(z^{n+1} \frac{d}{dz} + h(n+1) z^n \right) \Phi \tag{2.42}$$

Since $\tau = -\infty$ on the cylinder corresponds to the origin of the z–plane, we can associate a state to the field Φ_h by acting with it on $|0\rangle$ at $z = 0$:

$$|h\rangle = \Phi_h(0)\, |0\rangle \tag{2.43}$$

The state $|0\rangle$ is invariant under SL_2 transformations because $L_{\pm 1}$ and L_0 annihilate it. This is not the case for $|h\rangle$; however with the help of (2.37) and (2.42) for $n \geq 0$ we obtain

$$\begin{aligned} L_n\, |h\rangle &= 0 \quad n > 0 \\ L_0\, |h\rangle &= h\, |h\rangle \end{aligned} \tag{2.44}$$

The states satisfying (2.44) are called highest weight states, and the representation generated from $|h\rangle$ by the L_{-n}, $n = 1, 2, \ldots$ is denoted by $V(h, c)$.

The Hilbert space of any conformal field theory can be split into representations of the Virasoro algebra Vir and the anti-Virasoro algebra $\overline{Vir}$:

$$H = \bigoplus_{h,\overline{h}} V(h, c) \otimes V(\overline{h}, c) \tag{2.45}$$

The states in $V(h, c)$ other than $|h\rangle$ are known as descendant states.

A way of counting the number of states in $V(h, c)$ is to introduce the character of Vir:

$$\mathcal{X}_h(q) = \mathrm{tr}_{\,V(h,c)} q^{L_0 - c/24} \tag{2.46}$$

with $q = e^{2\pi i\tau}$, and τ a complex number (with $\mathrm{Im}(\tau) > 0$). The constant term $-c/24$ will be explained further ahead. For the time being, consider it simply as an unusual normalization factor. The states in $V(h, c)$ are

$$\begin{array}{ll} |h\rangle & L_0 = h \\ L_{-1}\,|h\rangle & L_0 = h + 1 \\ L_{-1}^2\,|h\rangle\,, L_{-2}\,|h\rangle & L_0 = h + 2 \\ \cdots & \cdots \\ L_{-1}^{n_1} \cdots L_{-p}^{n_p}\,|h\rangle & L_0 = h + n_1 + 2n_2 + \cdots + pn_p \end{array} \tag{2.47}$$

Assuming that $V(h, c)$ has a single highest weight vector (no null vectors at any level), one easily obtains

$$\mathcal{X}_h(q) = \frac{q^{h-c/24}}{\prod_{n=1}^{\infty}(1 - q^n)} \tag{2.48}$$

because we can act on $|h\rangle$ with any power of L_{-p} independently of the powers of any other $L_{-p'}$. It follows that the character (2.46) is formally equivalent to the partition function of an infinite number of oscillators with energies $E_n = n$. The total number of states at level N (with $L_0 = h + N$) is just the number of partitions $\pi(N)$ which count the number of ways we can write N as a sum of positive integers. The denominator in (2.48) is the generating function for $\pi(N)$:

$$\prod_{n=1}^{\infty}(1-q^n)^{-1} = \sum_{N=0}^{\infty} \pi(N)q^N \tag{2.49}$$

To the states in (2.47) we can associate fields, as follows. To the highest weight $|h\rangle$ we associate the primary field $\Phi(z)$, while to the descendants we associate products of $T(w)$ with $\Phi(z)$. More precisely, the state $L_{-n_1}\cdots L_{-n_p}\,|h\rangle$ is created by the field

$$\oint_{C_1}(w_1-z)^{-n_1+1}T(w_1)\oint_{C_2}(w_2-z)^{-n_2+1}T(w_2)\cdots\oint_{C_p}(w_p-z)^{-n_p+1}T(w_p)\Phi(z) \tag{2.50}$$

acting at the origin $z=0$, with the contours $C_1 \supset C_2 \supset \ldots \supset C_p$. The conformal symmetry of the theory gives a one–to–one correspondence between fields and states.

A very useful way of relating equal–time commutators to operator product expansions (OPE) is obtained as follows: consider two analytic fields $A(z)$, $B(w)$ and for arbitrary functions $f(z)$, $g(w)$ construct

$$\begin{aligned} A(f) &= \oint_0 f(z)A(z) \\ B(g) &= \oint_0 g(w)B(w) \end{aligned} \tag{2.51}$$

where the contours are circles around the origin, $|z|=1$ and $|w|=1$. To construct the equal–time commutator $[A(f),B(g)]_{\mathrm{E.T.}}$, write it as

$$[A(f),B(g)]_{\mathrm{E.T.}} = \oint_{C_1} f(z)A(z)\oint_{C_2} g(w)B(w) - \oint_{C_2} g(w)B(w)\oint_{C_1} f(z)A(z) \tag{2.52}$$

This can be visualized as in figure 2.2. Next, in the second integral, fix a value of w and deform C_1 to make it go through w as in figure 2.3.

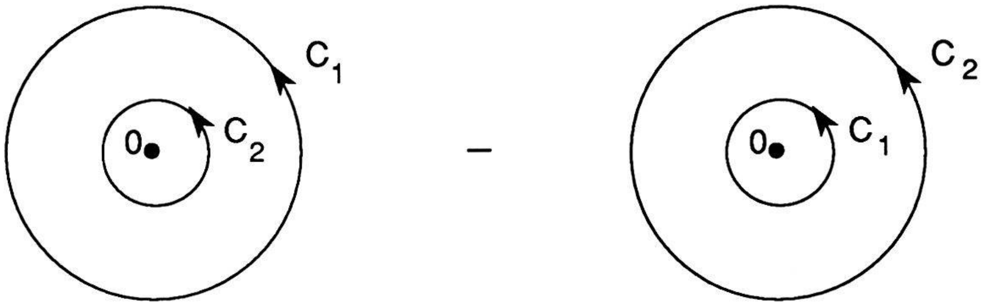


Figure 2.2. Integration contours in (2.52).

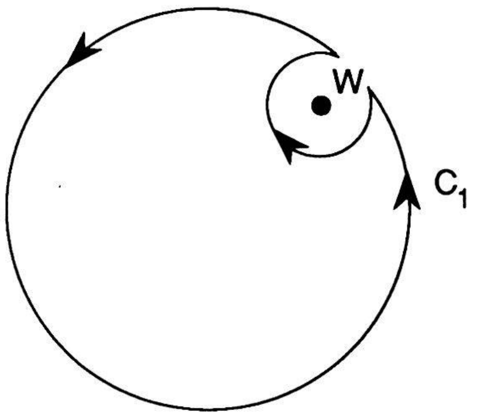


Figure 2.3. Contour manipulation for the evaluation of (2.52).

The final result is thus

$$[A(f), B(g)] = \oint_0 g(w) \oint_w A(z)B(w)f(z) \tag{2.53}$$

and the contour around w is as small as we wish. Thus, $\oint_w$ is given by the singularities in the operator product expansion of $A(z)B(w)$. This information, combined with (2.41), immediately results in

$$T(z)\Phi(w) = \frac{h}{(z-w)^2}\Phi(w) + \frac{1}{z-w}\partial\Phi(w) + \text{analytic} \tag{2.54}$$

Similarly, we can represent the Virasoro algebra (1.33) in terms of an operator product expansion:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \text{analytic} \tag{2.55}$$

The reader is encouraged to substitute (2.55) in (2.53) and verify (1.33).

Assuming the theory is unitary, we find $L_n^\dagger = L_{-n}$. Therefore, the out vacuum satisfies

$$\langle 0| L_{-n} = 0 \qquad n \geq -1 \tag{2.56}$$

and we easily obtain

$$\langle 0| T(z)T(w) |0\rangle = \frac{c/2}{(z-w)^4} \tag{2.57}$$

Note that this result was used in (1.35).

The operator product expansions (2.54,2.55) can be used to derive the correlation functions of descendant fields once the primary field correlators are known. As a simple example, consider

$$\langle T(z)\Phi_1(z_1)\cdots\Phi_N(z_N)\rangle \tag{2.58}$$

This correlation function can be thought of as a quadratic differential in z which vanishes at $z=\infty$ (if all $z_i \neq \infty$), and whose singularities are prescribed by (2.54). Then

$$\langle T(z)\Phi_1(z_1)\cdots\Phi_N(z_N)\rangle = \sum_{i=1}^{N}\left(\frac{h_i}{(z-z_i)^2} + \frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right)\langle\Phi_1(z_1)\cdots\Phi_N(z_N)\rangle \tag{2.59}$$

The reader is invited to work out more complicated examples and to consult the original work of BPZ.

In the next lecture, we will study conformal field theory in more detail, and delve into the Coulomb gas representation of correlation functions [33].

3.Minimal Conformal Field Theories. Coulomb Gas Representation

3.1. Consequences of SL_2 Invariance. Schwartzian Derivatives

The vacuum states $|0\rangle$, $\langle 0|$ are annihilated by L_0, $L_{\pm 1}$. Using (2.42) we obtain for the correlation of primary fields

$$
\begin{aligned}
\sum_{i=1}^{N} \frac{\partial}{\partial z_i} \langle \Phi_1(z_1) \cdots \Phi_N(z_N) \rangle = 0 \qquad (L_{-1}) \\
\sum_{i=1}^{N} \left(z_i \frac{\partial}{\partial z_i} + h_i \right) \langle \Phi_1(z_1) \cdots \Phi_N(z_N) \rangle = 0 \qquad (L_0) \\
\sum_{i=1}^{N} \left(z_i^2 \frac{\partial}{\partial z_i} + 2h_i z_i \right) \langle \Phi_1(z_1) \cdots \Phi_N(z_N) \rangle = 0 \qquad (L_1)
\end{aligned}
\tag{3.1}
$$

and similarly for $\overline{L}_{\pm 1}$, $\overline{L}_0$. The behaviour of correlation functions under finite SL_2 transformations follows from (3.1):

$$
\begin{aligned}
g(z) &= \frac{az+b}{cz+d} \\
\frac{dg}{dz} &= \frac{1}{(cz+d)^2} \\
z_i - z_j &\to g(z_i) - g(z_j) = \frac{z_i - z_j}{(cz_i+d)(cz_j+d)}
\end{aligned}
\tag{3.2}
$$

Since $\phi(z)dz^k$ is invariant under $z \to g(z)$, we obtain

$$
\langle \Phi_1(z_1) \cdots \Phi_N(z_N) \rangle = \left(\prod_{i=1}^{N} (cz_i + d)^{2h_i} \right) \langle \Phi_1(z_1) \cdots \Phi_N(z_N) \rangle \tag{3.3}
$$

Given four points w_1, w_2, w_3, w_4, we can construct an SL_2 invariant in terms of the harmonic ratio

$$
\eta = \frac{w_{12} w_{34}}{w_{13} w_{24}} \qquad w_{ij} = w_i - w_j \tag{3.4}
$$

Hence, we can solve (3.1) by expressing $\langle \Phi_1 \cdots \Phi_N \rangle$ in terms of $N-3$ harmonic ratios η_1, $\ldots$, η_{N-3}. We write

$$
\langle \Phi_1(z_1) \cdots \Phi_N(z_N) \rangle = \left(\prod_{i<j} z_{ij}^{-\gamma_{ij}} \right) f(\eta_a) \tag{3.5}
$$

To determine the exponents, we impose (3.3) on the prefactor in (3.5). This yields

$$\gamma_{ij} = \gamma_{ji}$$
$$\sum_{j\neq i} \gamma_{ij} = 2h_i \tag{3.6}$$

This determines completely the two– and three–point functions:

$$\left\langle \Phi_{h,\overline{h}}(z,\overline{z})\Phi_{h',\overline{h}'}(w,\overline{w})\right\rangle = \delta_{h,h'}\delta_{\overline{h},\overline{h}'} \tag{3.7}$$

$$\langle \Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\rangle = z_{12}^{-\gamma_{12}} z_{13}^{-\gamma_{13}} z_{23}^{-\gamma_{23}} \overline{z}_{12}^{-\overline{\gamma}_{12}} \overline{z}_{13}^{-\overline{\gamma}_{13}} \overline{z}_{23}^{-\overline{\gamma}_{23}}\ C_{123} \tag{3.8}$$

$$\gamma_{12} = h_1 + h_2 - h_3 \quad , \quad \overline{\gamma}_{12} = \overline{h}_1 + \overline{h}_2 - \overline{h}_3 \quad \text{etc.} \tag{3.9}$$

We have normalized the fields such that the coefficient of the two–point function is one. The two–point function can be used, in fact, to define a scalar product in the Hlbert space of states. The dynamical information contained in (3.8) are the structure constants C_{ijk} of the operator algebra. The dimensions h_i, $\overline{h}_i$ and the structure constants C_{ijk} determine the algebra completely.

When one studies the group of fractional linear transformations (3.2), it is known form elementary complex analysis that they are the only conformal maps for which the schwartzian derivatives vanish:

$$\{w,z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2 \tag{3.10}$$

To lighten the notation, we have used $w' = dw/dz$, etc. The solution to $\{w,z\} = 0$ is precisely (3.2). The schwartzian derivative determines the transformation rules of $T(z)$ under conformal transformations. From the operator product expansion (2.55), we obtain

$$\delta_\varepsilon T(z) = [T(\varepsilon), T(z)] = \left(\varepsilon\frac{d}{dz} + 2\frac{d\varepsilon}{dz}\right)T(z) + \frac{c}{12}\varepsilon'''(z) \tag{3.11}$$

For finite transformations,

$$T'(z) = \left(\frac{dw}{dz}\right)^2 T(w) + \frac{c}{12}\{w,z\} \tag{3.12}$$

Therefore, the energy–momentum operator $T(z)$ behaves like a tensor only under Möbius transformations.

The property (3.12) explains the factor of $q^{-c/24}$ in (2.46). Recall that in lecture 1 we started to study a conformal field theory on the cylinder, with complex variables w, $\overline{w}$. Then we mapped the cylinder to the punctured plane using $z = e^w$. From (3.10) we obtain

$$\{e^w, w\} = -\frac{1}{2} \tag{3.13}$$

and therefore

$$\begin{aligned} T_{\rm cyl}(w) &= z^2 T(z) - \frac{c}{24} \\ T_{\rm cyl}(w) &= \sum_{n \notin Z} \left(L_n - \frac{c}{24}\delta_{n,0} \right) e^{-nw} \end{aligned} \tag{3.14}$$

and

$$(L_0)_{\rm cyl} = (L_0)_{\rm plane} - \frac{c}{24} \tag{3.15}$$

The contribution $-c/24$ can be thought of as a Casimir energy. The schwartzian derivative gives a complete account of the contribution $q^{-c/24}$ to the Virasoro characters.

3.2. Massless Scalar Fields in Two Dimensions

The simplest example of a conformal field theory is provided by a single–valued two–dimensional massless scalar field ϕ. Its operator equation of motion is $\partial\overline{\partial}\phi = 0$, so

$$\partial\phi(z) = \sum_{n \in Z} \alpha_n z^{-n-1} \tag{3.16}$$

Under canonical quantization,

$$\begin{aligned} \{\alpha_n, \alpha_m\} &= n\delta_{n+m,0} \\ \{\alpha_n, \overline{\alpha}_m\} &= 0 \\ \{\overline{\alpha}_n, \overline{\alpha}_m\} &= n\delta_{n+m,0} \end{aligned} \tag{3.17}$$

The basic two–point function follows from the normal–ordering formula

$$\partial\phi(z)\partial\phi(w) = \frac{-1}{(z-w)^2} + :\partial\phi(z)\partial\phi(w): \tag{3.18}$$

In : ... : the annihilation operators are pushed to the right and the creation operators to the left. The energy–momentum tensor is given by

$$T(z) = -\frac{1}{2} : \partial\phi(z)\partial\phi(z) : \tag{3.19}$$

and c is computed from the leading singularity in the operator product expansion

$$T(z)T(w) = \frac{1}{4} : \partial\phi(z)\partial\phi(z) : : \partial\phi(w)\partial\phi(w) := \frac{1}{2}\frac{1}{(z-w)^4} + \text{less singular terms} \tag{3.20}$$

Therefore, $c = 1$ for a free two–dimensional massless scalar field.

We can also construct primary fields with ϕ: $\partial\phi$, for instance, behaves as a (1,0) primary field:

$$T(z)\partial\phi(w) = \frac{1}{(z-w)^2}\partial\phi(w) + \frac{1}{z-w}\partial^2\phi(w) + \cdots \tag{3.21}$$

We also have the vertex operators $V_k(z,\overline{z})$,

$$V_k =: \exp[ik\phi(z,\overline{z})] : \tag{3.22}$$

with conformal dimensions

$$h(V_k) = \frac{k^2}{2} \qquad \overline{h}(V_k) = \frac{k^2}{2} \tag{3.23}$$

We can verify (3.23) in some detail:

$$\begin{aligned} T(z)V_k(w) &= -\frac{1}{2} : \partial\phi(z)\partial\phi(z) : \sum_{n=0}^{\infty} \frac{1}{n!}(ik)^n : \phi^n : \\ &= -\frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{n!}(ik)^n n(n-1)\frac{1}{(z-w)^2} : \phi^{n-2} : + \sum_{n=0}^{\infty} \frac{1}{n!}(ik)^n n \frac{1}{z-w} : \partial\phi(z)\phi^{n-1} : \\ &= \frac{k^2/2}{(z-w)^2}V_k(w) + \frac{1}{z-w}\partial_w V_k(w) + \text{analytic} \end{aligned} \tag{3.24}$$

To establish this result, we have used the fact that the two–point function of the ϕ–field is

$$\langle \phi(z,\overline{z})\phi(w,\overline{w})\rangle = -\log|z-w|^2 \tag{3.25}$$

The scalar field ϕ is not a primary field itself, because its correlation function behaves logarithmically. Clearly, though, with vertex operators we can in principle obtain any conformal dimension:

$$\langle V_k(z,\overline{z})V_{-k}(w,\overline{w})\rangle = |z-w|^{-k^2} \tag{3.26}$$

This led Kadanoff to conjecture that the correlation functions of any conformal field theory can be represented in terms of free massless scalar fields. In the Coulomb gas representation of minimal conformal field theories we will see a realization of these ideas.

3.3. Conformal Blocks and Duality

General properties of a conformal field theory can be obtained by requiring associativity of the operator product expansion. This gives a set of crossing or duality relations for the structure constants ${C_{ij}}^k$ of the operator algebra. This is one of the pillars of the BPZ analysis of conformal field theories.

Let us begin by defining the out state created by a primary field as

$$\langle h,\overline{h}| = \lim_{z,\overline{z}\to\infty} \langle 0|\, \Phi_{h,\overline{h}}(z,\overline{z}) z^{2L_0}\overline{z}^{2\overline{L}_0} \tag{3.27}$$

With this definition, the three-point function (3.8) becomes

$$\langle n|\, \Phi_m(z,\overline{z})\,|\ell\rangle = C^{\ell}_{\ nm} z^{h_n-h_m-h_\ell}\overline{z}^{\overline{h}_n-\overline{h}_m-\overline{h}_\ell} \tag{3.28}$$

From (3.5), we know that we lose no information if for the four-point function we choose three of the points at 0, 1 and ∞. We shall study in some detail the four-point function

$$\langle k|\, \Phi_\ell(1,1)\Phi_n(x,\overline{x})\,|m\rangle \tag{3.29}$$

Consider the operator product expansion of two primary fields

$$\Phi_n(z,\overline{z})\Phi_m(0,0) = \sum_p \sum_{\{k,\overline{k}\}} C_{nm}^{p\{k,\overline{k}\}} z^{(h_p-h_n-h_m+\sum k}\overline{z}^{\overline{h}_p-\overline{h}_n-\overline{h}_m+\sum \overline{k})}\ \Phi_p^{\{k,\overline{k}\}}(0,0) \tag{3.30}$$

The index p runs over the primary fields in the theory (representations of Virasoro). For a given family p, the notation $\{k, \overline{k}\}$ labels the descendant states:

$$\Phi_p^{\{k,\overline{k}\}}(0) = L_{-k_1} \cdots L_{-k_N} \overline{L}_{-\overline{k}_1} \cdots \overline{L}_{-\overline{k}_M} \Phi_p(0,0) \tag{3.31}$$

In analogy with the Wigner–Eckart theorem, the structure constants factorize:

$$C_{nm}^{p\{k,\overline{k}\}} = C_{nm}^p \beta_{nm}^{p\{k\}} \overline{\beta}_{nm}^{p\{\overline{k}\}} \tag{3.32}$$

The β, $\overline{\beta}$ coefficients are analogous to the $3j$–symbols and they follow from the conformal symmetry (see BPZ for details). Hence

$$\begin{aligned}&\langle k|\, \Phi_\ell(1,1)\Phi_n(x,\overline{x})\, |m\rangle \\ &= \langle k|\, \Phi_\ell(1,1) \sum_p C_{nm}^p x^{h_p - h_n - h_m} \overline{x}^{\overline{h}_p - \overline{h}_n - \overline{h}_m} \sum_{\{k,\overline{k}\}} \beta_{nm}^{p\{k\}} \overline{\beta}_{nm}^{p\{\overline{k}\}} x^{(\sum k} \overline{x}^{\sum \overline{k})}\, \Phi_p^{\{k,\overline{k}\}}(0,0)\, |0\rangle\end{aligned} \tag{3.33}$$

Define

$$\mathcal{F}_{nm}^{\ell k}(p|x) = x^{h_p - h_n - h_m} \sum_{\{k\}} \beta_{nm}^{p\ \{k\}} \frac{\langle k|\, \Phi_\ell(1,1) L_{-k_1} \cdots L_{-k_N}\, |p\rangle}{\langle k|\, \Phi_\ell(1,1)\, |p\rangle}\, x^{\sum k} \tag{3.34}$$

and similarly for the anti-holomorphic part. Then the four–point function becomes

$$G_{nm}^{\ell k}(x, \overline{x}) = \sum C^p{}_{nm} C_{k\ell p} \mathcal{F}_{nm}^{\ell k}(p|x) \overline{\mathcal{F}_{nm}^{\ell k}(p|x)} \tag{3.35}$$

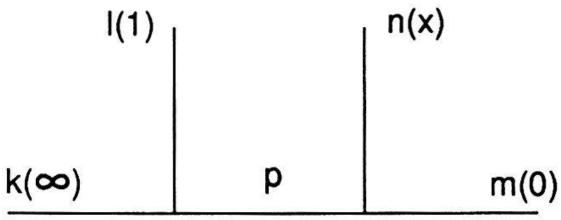


Figure 3.1. A four–point conformal block.

Graphically, each function $\mathcal{F}$ is represented as a skeleton graph indicating the order in which we perform the operator product expansion (figure 3.1). The functions $\mathcal{F}(p|x)$ are generally multi-valued and they are known as conformal blocks. The blocks are normalized so that

$$\mathcal{F}^{\ell k}_{nm}(p|x) \underset{x \to 0}{\simeq} x^{h_p - h_n - h_m}\,(1 + \ldots) \tag{3.36}$$

This is the convention chosen by BPZ.

We can consider more general n–point functions. If we fix the external legs to belong to the families $[\Phi_{i_1}]$, ..., $[\Phi_{i_n}]$ and fix also a particular order in which to perform the operator product expansion (as in figure 3.2), we obtain a basis in the space of conformal blocks. Each element of this basis is labelled by a collection of indices $(p_1, \ldots p_{n-3})$ indicating the conformal families appearing in the internal legs. The correlation function can be written as a "metric" in the space of blocks:

$$G_{i_1 \cdots i_n}(z_{i_1}, \bar{z}_{i_1}, \ldots z_{i_n} \bar{z}_{i_n}) = \sum_{p,p'} \mathcal{F}_p(Z) \overline{\mathcal{F}_{p'}(Z)}\, h_{pp'} \tag{3.37}$$

and the blocks $\mathcal{F}_p(Z)$ (with $Z = (z_1, \ldots, z_n)$ for convenience) can be thought of geometrically as sections of a flat holomorphic vector bundle over the moduli space of the sphere with n marked points. We shall not pursue this geometrical picture in the following.

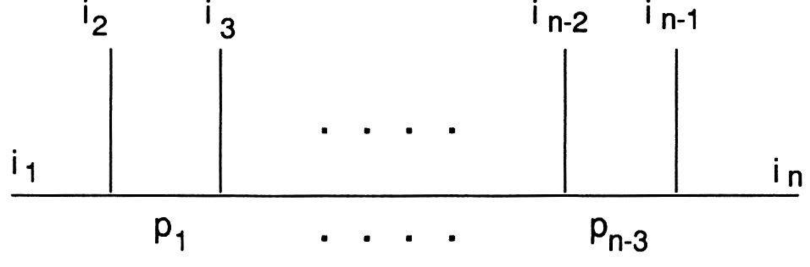


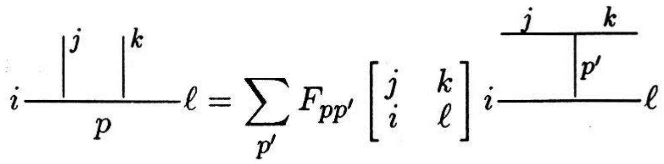
Figure 3.2. A particular basis for the n–point block.

Requiring now associativity of the operator product expansion gives us the desired equations for the structure constants: the bootstrap, duality or crossing relations. In the graph of figure 3.1, we perform first the operator product expansion between Φ_n and Φ_m, and then we compute the operator product expansion of the result with Φ_ℓ. Associativity implies that the physical amplitude should not depend on whether we do first the operator product expansion between n and m, or between ℓ and n.

Using first the SL_2 transformation $x \to 1-x$ to obtain $\langle k|\,\Phi_m(1,1)\Phi_n(1-x,1-\overline{x})\,|\ell\rangle$, and following next the steps leading to (3.35), we find

$$\sum_p C^p{}_{nm} C_{k\ell p} \mathcal{F}^{\ell k}_{nm}(p|x)\overline{\mathcal{F}^{\ell k}_{nm}(p|x)} = \sum_q C^q{}_{n\ell} C_{kmq} \mathcal{F}^{mk}_{n\ell}(q|1-x)\overline{\mathcal{F}^{mk}_{n\ell}(q|1-x)} \tag{3.38}$$

Since $\mathcal{F}$ and $\overline{\mathcal{F}}$ are determined by the conformal dimensions and the conformal Ward identities, we can interpret (3.38) as a set of rather non–trivial conditions characterizing the structure constants C^k_{ij} which appear in a conformal field theory. From the point of view of conformal blocks, the duality equation (3.38) tells us that the basis defined by the graph $\perp\!\perp$ and the basis represented by the decomposition $\top\!\!\!\perp$ are linearly equivalent. This can be represented graphically as follows:



(3.39)

The fusion coefficients $F_{pp'}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix}$ play a central role in the polynomial equations for a conformal field theory written down by G. Moore and N. Seiberg [34]. The fusion coefficients are very closely related to the braiding coefficients determining the monodromy properties of the conformal blocks.

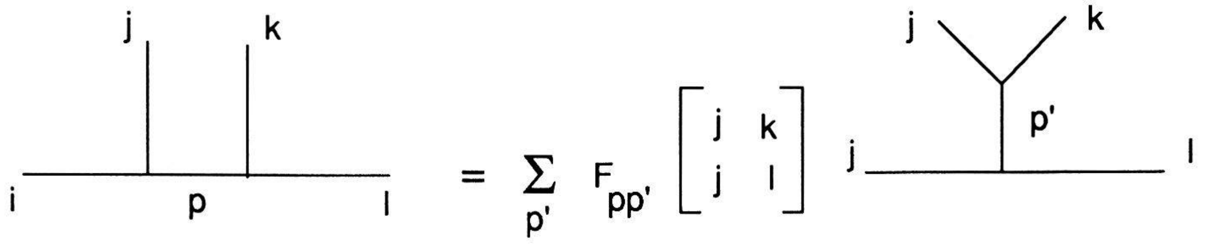
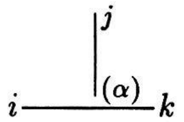


Figure 3.3. Graphical representation of the bootstrap equations.

The crudest pieces of information contained in the structure constants $C_{ij}{}^k$ are the fusion rules. A detailed and deep analysis of them is carried out in [35]. They simply count in how many different ways can we couple the i, j and k families. Let us write this number as a positive or zero integer $N_{ij}{}^{k}$. For $N_{ij}{}^{k} > 1$, we label the basic three–point vertex as



$$\alpha = 1, \ldots N_{ij}{}^{k} \tag{3.40}$$

The coefficients $N_{ij}{}^{k}$ satisfy $N_{ij}{}^{k} = N_{ji}{}^{k}$ and the associativity of the operator product expansion implies

$$\sum_p N_{ik}{}^{p} N_{jp}{}^{\ell} = \sum_p N_{jk}{}^{p} N_{ip}{}^{\ell} \tag{3.41}$$

Introducing the matrices N_i,

$$(N_i)_j{}^{k} = N_{ij}{}^{k}, \tag{3.42}$$

the expression (3.41) can be rewritten as

$$N_i N_j = N_j N_i \tag{3.43}$$

Furthermore, since the operator product expansion with the identity or any of its descendants does not change the conformal family,

$$N_{i0}{}^{0} = \delta_i^j \tag{3.44}$$

and we can define a "charge conjugation" metric

$$C_{ij} = N_{ij}{}^{0} \tag{3.45}$$

which keeps track of what pairs of families have the identity in their operator product expansion. We can use C_{ij} to lower the upper index in $N_{ij}{}^{k}$:

$$N_{ijk} = N_{ij}{}^{\ell} C_{\ell k} \tag{3.46}$$

We can also think of C as a mapping $C : \Phi_i \to \Phi_{\bar{i}}$ such that $C^2 = 1$. The coefficients N_{ijk} are totally symmetric. Since $N_{ijk} = N_{jik}$, all we need to verify is $N_{ijk} = N_{ikj}$. For this, it suffices to set $\ell = 0$ in (3.41).

We can define abstractly a fusion algebra which is both commutative and associative. If, for simplicity, we assume that the number of primary fields is finite, we may introduce a generator x_i of the algebra for each family $i = 0, 1, \ldots, n-1$, with the understanding that x_0 represents the identity. The defining relations for the algebra are

$$x_i x_j = N_{ij}{}^k x_k \tag{3.47}$$

Its regular representation is $x_i \to N_i$, and there are n one–dimensional representations given by the eigenvalues of the matrices N_i. If $\lambda_i^{(\ell)}$ is the ℓ–th eigenvalue of N_i, then

$$\lambda_i^{(\ell)} \lambda_j^{(\ell)} = N_{ij}{}^k \lambda_k^{(\ell)} \tag{3.48}$$

This algebra has very surprising properties, the most remarkable being that the matrix S which diagonalizes all N_i's is the matrix which implements the modular transformation $\tau \to -1/\tau$ acting on the characters of the representations $V(c, h_i)$ of the Virasoro algebra [35]. We will have the occasion to investigate some interesting fusion rules further ahead.

3.4. Degenerate Conformal Families

We present now some general properties of conformal field theories with null vectors [25]. Many of the facts that will be stated are proved in the next section in terms of the Coulomb gas representation of the conformal blocks [33].

In the theory of group representations, often we study only the details for simple or semi-simple groups: S_n, $SU(n)$, $SO(2n)$ $n \geq 2$, $Sp(2n)$, $SU(n) \times Sp(2m)$, etc. A common feature of their finite–dimensional representations is their full reducibility: every reducible representation can be written uniquely (up to similarity transformations) as a direct sum of irreducible representations. Representations that are not fully reducible can be characterized by the presence of null vectors: there is more than one highest–weight vector in the representation. In the case of Vir, we say that a representation $V(c, h)$ has a null vector $|\mathcal{X}\rangle$ at level N if it satisfies

$$\begin{aligned} L_0 |\mathcal{X}\rangle &= (h+N) |\mathcal{X}\rangle \\ L_n |\mathcal{X}\rangle &= 0 \quad n>0 \end{aligned} \tag{3.49}$$

We should mention in passing that, for unitary representations, h is a positive number. This is because $||L_{-1}|h\rangle||^2 = \langle h| L_1 L_{-1} |h\rangle = 2\langle h| L_0 |h\rangle = 2h$. If $L_n^\dagger = L_{-n}$, the state $|\mathcal{X}\rangle$ is orthogonal to all states in $V(c,h)$, because they can be written as $|\psi\rangle = \prod L_{-n_i}|h\rangle$. But since $L_n|\mathcal{X}\rangle = 0$ $(n>0)$, then we have $\langle\psi|\mathcal{X}\rangle = 0$. The conclusion holds for any $|\psi\rangle$. Hence, we can set $|\mathcal{X}\rangle = 0$, which is equivalent to taking the quotient by the null vectors and the subrepresentations of Vir they generate. If $\Phi_h(z)$ is the field generating $V(c,h)$, we say that the family is degenerate and that it has a null vector at level N.

As a simple example, let us look for the condition guaranteeing the existence of a null vector at level two. It must be of the general form

$$|\mathcal{X}\rangle = \left(aL_{-2} + bL_{-1}^2\right)|h\rangle \tag{3.50}$$

For all L_n $(n>0)$ to annihilate $|\mathcal{X}\rangle$, it suffices to require $L_1|\mathcal{X}\rangle = L_2|\mathcal{X}\rangle = 0$ because all other L_n's are generated by the commutators of L_1 and L_2. Imposing these two constraints, we obtain

$$\begin{aligned} |\mathcal{X}\rangle &= \left(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2\right)|h\rangle \\ h &= \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right) \end{aligned} \tag{3.51}$$

where we have normalized, for convenience, with $a=1$. If we now examine the decoupling of $|\mathcal{X}\rangle$ from any correlation function, we obain a second order differential equation for correlators involving $\Phi_h(z)$:

$$\begin{aligned} 0 &= \langle \mathcal{X}_h(z)\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle \\ &= \left(\frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2} - \sum_{i=1}^n \frac{h_i}{(z-z_i)^2} - \sum_{i=1}^n \frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right)\langle\Phi_h(z)\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle \end{aligned} \tag{3.52}$$

This equation can be derived by writing $(L_{-2}\Phi)(z) - (3/2(2h_1))(L_{-1}^2\Phi)(z)$ as a small contour integral about Φ and then deforming the contour to the "other side of the surface."

This, together with the operator product expansion between $T(z)$ and $\Phi_i(z_i)$, yield (3.52). If in (3.52) we take $n = 4$ and set $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$, we obtain the hypergeometric equation.

The table of Virasoro representations with null vectors was given by V. Kac [36], and his proof completed by Feigin and Fuks [37]. To obtain this set, one has to consider for a given $V(c,h)$ and level N, the determinant of the scalar products among all the states at level N. To write the result in a simple way, we introduce three number $\alpha_\circ$, α_+, α_-:

$$\begin{aligned} c &= 1 - 24\alpha_\circ^2 \\ \alpha_+ + \alpha_- &= 2\alpha_\circ \\ \alpha_+\alpha_- &= -1 \end{aligned} \tag{3.53}$$

The representations with null vectors are labelled by two integers $m, n \geq 1$ with dimension

$$h(m,n) = -\frac{1}{4}\alpha_\circ^2 + \frac{1}{4}(m\alpha_+ + n\alpha_-)^2 \tag{3.54}$$

This result will be derived in the next subsection with the help of the Coulomb gas representation. The representation $V(h_{m,n})$ has a null vector at level $m \cdot n$. Using the null vectors, one can obtain a great deal of information about the theory. In particular, we can derive useful information concerning the fusion rules.

Consider, for example, $h = h(1,2)$ or $h(2,1)$. The null vector appears at level two and the decoupling equation is just (3.52). If at z_1 we have the field $\Phi_1(z_1)$ with dimension h_1 and we let $z \approx z_1$, we have the operator product expansion

$$\Phi_h(z)\Phi_{h_1}(z_1) = const. \times (z - z_1)^k \left[\Phi_{h'}(z_1) + \cdots\right] \tag{3.55}$$

This expansion solves (3.52) provided

$$\begin{aligned} h' &= -\frac{1}{4}\alpha_\circ^2 + \frac{1}{4}\alpha'^2 \\ h_1 &= -\frac{1}{4}\alpha_\circ^2 + \frac{1}{4}\alpha^2 \end{aligned} \tag{3.56}$$

with

$$h = h(1,2) \qquad \alpha' = \alpha \pm \alpha_- \\ h = h(2,1) \qquad \alpha' = \alpha \pm \alpha_+ \tag{3.57}$$

giving the fusion rules

$$\Phi_{(1,2)}\Phi_{(\alpha)} = [\Phi_{(\alpha-\alpha_-)}] + [\Phi_{(\alpha+\alpha_-)}] \\ \Phi_{(2,1)}\Phi_{(\alpha)} = [\Phi_{(\alpha-\alpha_+)}] + [\Phi_{(\alpha+\alpha_+)}] \tag{3.58}$$

The normalization remains to be determined. With this argument, we only know that the conformal families $[\phi_{(\alpha-\alpha_\pm)}]$ can in principle appear in the operator product expansion between $\Phi_{(1,2)}$ (or $\Phi_{(2,1)}$) and $\Phi_{(\alpha)}$. Examples are given by taking $[\Phi_{(\alpha)}]$ a degenerate family itself:

$$\Phi_{(1,2)}\Phi_{(1,2)} = [\Phi_{(1,1)}] + [\Phi_{(1,3)}] \\ \Phi_{(2,1)}\Phi_{(2,1)} = [\Phi_{(1,1)}] + [\Phi_{(3,1)}] \tag{3.59}$$

The field $\Phi_{(1,1)}$ has zero conformal dimension and it is identified with the identity operator. The null vector at level one is clearly $L_{-1}\Phi_{(1,1)} = 0$, meaning that the identity operator is translationally invariant.

A naive application of the rules (3.59) might seem to generate fields with m and/or n zero or negative. This is not the case, because the operator product expansion truncates from below. Consider, for instance, $\Phi_{(1,2)}\Phi_{(2,1)}$. Using the first relation in (3.58), we find

$$\Phi_{(1,2)}\Phi_{(2,1)} = c_1[\Phi_{(2,0)}] + c_2[\Phi_{(2,2)}] \tag{3.60}$$

But using the second relation in (3.58), we find instead

$$\Phi_{(1,2)}\Phi_{(2,1)} = c_1'[\Phi_{(0,2)}] + c_2'[\Phi_{(2,2)}] \tag{3.61}$$

Consistency requires then $c_1 = c_1' = 0$, $c_2 = c_2'$. Therefore,

$$\Phi_{(1,2)}\Phi_{(2,1)} = [\Phi_{(2,2)}] \tag{3.61}$$

Using $\Phi_{(2,1)}$ and $\Phi_{(1,2)}$, we can in principle reach any family $\Phi_{(m,n)}$. Although we shall not derive it until next section, we quote here the general fusion rules for degenerate families:

$$\Phi_{(m_1,n_1)}\Phi_{(m_2,n_2)} = \sum_{k=|m_1-m_2|+1}^{m_1+m_2-1} \sum_{\ell=|n_1-n_2|+1}^{n_1+n_2-1} [\Phi_{(k,\ell)}] \tag{3.62}$$

Note that the sums are restricted: if $|m_1 - m_2|$ is odd (even), then k runs only over even (odd) integers. The same applies to n_1, n_2 and ℓ.

One of the main results of Belavin, Polyakov and Zamolodchikov was to find conformal field theories where all fields are degenerate, and containing a finite number of primary fields. They showed that this happens when α_-/α_+ is a rational number. In this case, each Verma module $V(h_{m,n})$ contains an infinite number of null vectors. If p, p' are relatively prime positive integers, the minimal models of BPZ satisfy

$$\begin{aligned}
&\frac{\alpha_-}{\alpha_+} = \frac{-p}{p'} \\
&c = 1 - 6\frac{(p-p')^2}{pp'} \\
&h(m,n) = \frac{1}{4pp'}\left[(mp' - np)^2 - (p-p')^2\right]
\end{aligned} \tag{3.63}$$

If one imposes unitarity, Friedan, Qiu and Shenker [38] showed that the only allowed values are $p' = p + 1$, $p \geq 2$. Notice also the reflection symmetry

$$h(m,n) = h(p-m, p'-n) \tag{3.64}$$

The simplest non–trivial example is the Ising model, with $p = 3$ and $p' = 4$. Now $c = 1/2$, and the conformal dimensions of the primary fields are

$$\begin{aligned}
h(1,1) = h(2,3) &= 0 \qquad (1) \\
h(1,2) = h(2,2) &= \frac{1}{16} \qquad (\sigma) \\
h(2,1) = h(1,3) &= \frac{1}{2} \qquad (\epsilon)
\end{aligned} \tag{3.65}$$

The fields $\Phi_{(1,1)}$, $\Phi_{(1,2)}$ and $\Phi_{(2,1)}$ are identified with the identity, the spin density and the energy density, respectively. These fields can be drawn on a grid, as shown in figure 3.4.

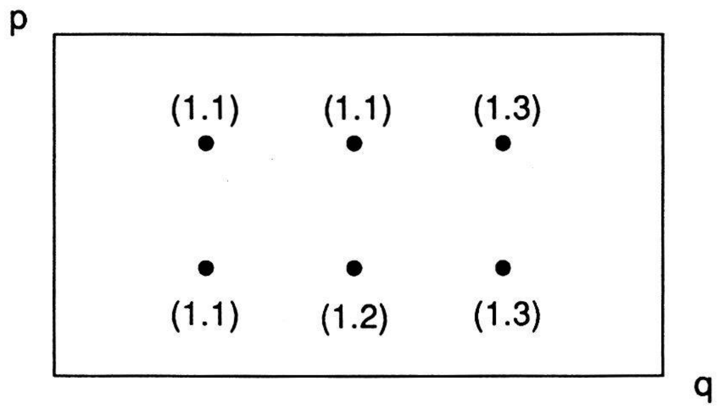


Figure 3.4. Grid of (m,n) fields in the Ising model *i.e.,* the minimal model with $(p,p') = (3,4)$.

We shall see below that the fusion rules of the fields in (3.65) are

$$\begin{aligned} \epsilon \cdot \epsilon &= [1] \\ \epsilon \cdot \sigma &= [\sigma] \\ \sigma \cdot \sigma &= [1] + [\epsilon] \end{aligned} \tag{3.66}$$

This concludes the generalities on degenerate conformal families. Next, we find an explicit representation for their correlation functions. We will also derive some of the results of this section.

3.5. Coulomb Gas Representation

The properties of all minimal (p,p') models can be obtained in terms of a single scalar field ϕ. In section 3.3, we learned that any scaling dimension can be obtained using vertex operators. It is not so clear how to obtain a central charge $c \neq 1$ for a single scalar field. To overcome this hurdle, imagine coupling the scalar field ϕ to an arbitrary metric on the sphere. To the standard kinetic term in the lagrangian $\frac{1}{2}g^{ij}\partial_i\phi\partial_j\phi$ we can envisage adding a term of the form $R\phi$, where R is the trace of the Ricci tensor of g. Now the equations of motion take the form $\Delta\phi \propto R$, implying that the curent $j_\mu = \partial_\mu\phi$ is ot conserved. This is equivalent to having a background charge on the world–sheet. Varying the action with respect to g_{ij} about the standard metric, leads to the following energy–momentum tensor:

$$T(z) = -\frac{1}{4}\partial\phi(z)\partial\phi(z) + i\alpha_o\partial^2\phi \tag{3.67}$$

We have rescaled the field ϕ to agree with the notation of Dotsenko and Fateev [33], so that

$$\langle\phi(z)\phi(w)\rangle = -2\log(z-w) \tag{3.68}$$

The lagrangian from which this energy–momentum tensor is derived is

$$2\pi\sqrt{g}\mathcal{L} = \frac{1}{4}g^{ij}\partial_i\phi\partial_j\phi + \frac{i\alpha_o}{4}R\phi \tag{3.69}$$

With respect to $T(z)$, the operator $\partial\phi$ has an anomaly. Indeed, the operator product expansion between T and $\partial\phi$ is

$$T(z)\partial\phi(w) = \frac{1}{(z-w)^2}\partial\phi(w) + \frac{1}{z-w}\partial^2\phi(w) + \frac{4i\alpha_o}{(z-w)^3} \tag{3.70}$$

With the aid of the Gauss–Bonnet formula, we can compute the violation of the total charge. The vertex operators are still primary fields. If we consider the theory using the functional integral, the correlation function of several vertex operators takes the form

$$\left\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i,\overline{z}_i)\right\rangle = \int(\mathcal{D}\phi)V_{\alpha_1}(z_1,\overline{z}_1)\cdots V_{\alpha_n}(z_n,\overline{z}_n)e^{-S_0-i(\alpha_o/8\pi)\int\sqrt{g}R\phi} \tag{3.71}$$

Under the shift by a constant, $\phi \to \phi + c$, the vertex operators contribute a factor $\exp(ic\sum\alpha_i)$, whereas the curvature term in the action contributes $\exp(-i\alpha_o c\int\sqrt{g}R/8\pi)$. The Gauss–Bonnet theorem states that $\int\sqrt{g}R = 8\pi\mathcal{X}$, where $\mathcal{X}$ is the Euler number of the surface. If the surface is triangulated, then $\mathcal{X}$ is equal to the number of faces plus the number of vertices minus the number of edges. For the sphere, $\mathcal{X} = 2$, and therefore the correlator (3.71) vanishes unless

$$\sum\alpha_i = 2\alpha_o \tag{3.72}$$

Equivalently, we can view the background as contributing to the total charge, by an amount $-2\alpha_o$.

The seemingly innocent change in $T(z)$ has far-reaching consequences. First, let us compute the new central charge. Contributions to the fourth-order pole come from

$$\frac{1}{16} : \partial\phi\partial\phi(z) : : \partial\phi\partial\phi(w) := \frac{1/2}{(z-w)^4} + \cdots \tag{3.73}$$

and from

$$-\alpha_o^2 \partial^2\phi(z)\partial^2\phi(w) = -\frac{12\alpha_o^2}{(z-w)^4} + \cdots \tag{3.74}$$

Therefore,

$$c = 1 - 24\alpha_o^2 \tag{3.75}$$

The conformal dimension of vertex operators changes as well. A simple computation shows that

$$h(: e^{i\alpha\phi} :) = \alpha(\alpha - 2\alpha_o) \tag{3.76}$$

From this formula, we learn of the existence of two fields of dimension one. The equation $\alpha(\alpha - 2\alpha_o) = 1$ has two roots α_+, α_- satisfying

$$\begin{aligned} \alpha_+ + \alpha_- &= 2\alpha_o \\ \alpha_+\alpha_- &= -1 \end{aligned} \tag{3.77}$$

as in (3.53).

Another important piece of information we learn from (3.76) is that vertex operators of charge α and $2\alpha_o - \alpha$ have the same dimension, so from a representation-theoretic point of view they represent the same object. From now on, we shall use

$$\overline{\alpha} = 2\alpha_o - \alpha \tag{3.78}$$

In particular, the identity can be written either as 1 or as $: \exp(2i\alpha_o\phi) :$.

Finally, since there are two currents of dimension one

$$J_\pm =: e^{i\alpha_\pm\phi} : \tag{3.79}$$

we can introduce two charges

$$Q_{\pm} = \oint J_{\pm}(z)dz \tag{3.80}$$

The conformal properties of correlation functions are not changed by the insertion of $Q_{\pm}$, although these insertions do affect the charge balance.

To represent the correlation functions of minimal models, we write any of the primary fields as a vertex operator $V_{\alpha}(z)$ or $V_{\overline{\alpha}}(z)$. We can write the four-point conformal block in the form $\langle V_\alpha V_\alpha V_\alpha V_{\overline{\alpha}}\rangle$. The sum of the charges is $2\alpha + (\alpha + \overline{\alpha}) = 2\alpha + 2\alpha_{\circ}$. The four-point block will vanish unless

$$2\alpha = -m\alpha_{+} - n\alpha_{-} \tag{3.81}$$

in which case the charge can be screened to zero by the introduction of m Q_+'s and n Q_-'s. The spectrum of vertex operators with non-vanishing four-point functions is thus

$$\alpha_{m,n} = \frac{1-m}{2}\alpha_{+} + \frac{1-n}{2}\alpha_{-} \tag{3.82}$$

The dimensions of these fields are

$$\begin{aligned} h_{m,n} &= \alpha_{m,n}(\alpha_{m,n} - 2\alpha_{\circ}) \\ &= -\alpha_{\circ}^2 + \frac{1}{4}(m\alpha_{+} + n\alpha_{-})^2 \end{aligned} \tag{3.83}$$

which agree with the Kac table.

A four-point block will take the form

$$\begin{gathered} \oint_{C_1} dt_1 \cdots \oint_{C_m} dt_m \oint_{C_1'} dt_1' \cdots \oint_{C_n'} dt_n' \\ \langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)V_{\alpha_4}(z_4)J_{+}(t_1)\cdots J_{+}(t_m)J_{-}(t_1')\cdots J_{-}(t_n')\rangle \end{gathered} \tag{3.84}$$

where the correlation function is understood to be computed in the presence of a background charge of $-2\alpha_{\circ}$.

The simplest application of this construction is the derivation of the fusion rules (3.62) [29]. We want to find the families $\Phi_{(k,\ell)}$ appearing in the operator product expansion of

$\Phi_{(m,n)}$ and $\Phi_{(r,s)}$. We only need the three–point functions for this. Using SL_2 invariance, there are three equivalent representations of the three point function:

$$\left\langle V_{\overline{(k,\ell)}}(\infty)V_{(m,n)}(1)V_{(r,s)}(0)Q_+^{\cdots}Q_-^{\cdots}\right\rangle \tag{3.85a}$$

$$\left\langle V_{(k,\ell)}(\infty)V_{\overline{(m,n)}}(1)V_{(r,s)}(0)Q_+^{\cdots}Q_-^{\cdots}\right\rangle \tag{3.85b}$$

$$\left\langle V_{(k,\ell)}(\infty)V_{(m,n)}(1)V_{\overline{(r,s)}}(0)Q_+^{\cdots}Q_-^{\cdots}\right\rangle \tag{3.85c}$$

The powers of $Q_\pm$ are determined by balancing the charge. In (3.85a), the charge of the first three fields is

$$2\alpha_\circ - \alpha_{k,\ell} + \alpha_{m,n} + \alpha_{r,s} = 2\alpha_\circ + \frac{k-m-r+1}{2}\alpha_+ + \frac{\ell-n-s+1}{2}\alpha_- \tag{3.86}$$

and the charge can be screened if

$$\begin{array}{ll} k \le m+r-1 & k+m+r \text{ odd} \\ \ell \le n+s-1 & \ell+n+s \text{ odd} \end{array} \tag{3.87}$$

Doing the same computations for (3.85b ,3.85c), we obtain the conditions

$$\begin{array}{ll} k+m+r \text{ odd} & \ell+n+s \text{ odd} \\ k \le m+r-1 & \ell \le n+s-1 \\ m \le r+k-1 & n \le s+\ell-1 \\ r \le k+m-1 & s \le \ell+n-1 \end{array} \tag{3.88}$$

implying the fusion rules

$$\Phi_{(m,n)}\Phi_{(r,s)} = \sum_{\substack{k=|m-r|+1 \\ k+m+r \text{ odd}}}^{m+r-1} \ \sum_{\substack{\ell=|n-s|+1 \\ \ell+n+s \text{ odd}}}^{n+s-1} [\Phi_{(k,\ell)}] \tag{3.89}$$

Although a truncation below is already implemented (we never find a family with $k < 1$ or $\ell < 1$), unless we impose some extra conditions the operator algebra will not truncate from above. A truncation to a finite operator algebra appears when we restrict $\alpha_\pm^2$ to be rational numbers. Since $\alpha_+\alpha_- = -1$, we take

$$\alpha_+ = \sqrt{\frac{p'}{p}} \qquad \alpha_- = -\sqrt{\frac{p}{p'}} \tag{3.90}$$

and then

$$\alpha_o = \frac{p' - p}{2\sqrt{pp'}}$$
$$c = 1 - 6\frac{(p-p')^2}{pp'} \tag{3.91}$$

and the conformal dimensions of the degenerate families are

$$h_{m,n} == \alpha_o^2 + \frac{1}{4}(m\alpha_+ + n\alpha_-)^2 = \frac{1}{4pp'}\left[(mp' - np)^2 - (p-p')^2\right] \tag{3.92}$$

A very useful property of (3.92) is the symmetry

$$h_{m,n} = h_{p-m,p'-n} \tag{3.93}$$

which suggests the truncation from above. From the point of view of representation theory, we can identify the (m, n) and $(p - m, p' - n)$ families, and we can compute (3.89) in a different way:

$$\Phi_{(p-m,p'-n)}\Phi_{(p-r,p'-s)} = \sum_{\substack{k=|m-r|+1 \\ k+m+r \text{ odd}}}^{2p-m-r-1} \sum_{\substack{\ell=|n-s|+1 \\ \ell+n+s \text{ odd}}}^{2p'-n-s-1} [\Phi_{(k,\ell)}] \tag{3.94}$$

Now (3.89) and (3.94) are compatible provided

$$\Phi_{(m,n)}\Phi_{(r,s)} = \sum_{\substack{k=|m-r|+1 \\ k+m+r \text{ odd}}}^{\min(m+r-1,2p-m-r-1)} \sum_{\substack{\ell=|n-s|+1 \\ \ell+n+s \text{ odd}}}^{\min(n+s-1,2p'-n-s-1)} [\Phi_{(k,\ell)}] \tag{3.95}$$

which is the fusion algebra of minimal (p, p') models.

A good mnemonic for (3.89) is to introduce ordinary $SU(2)$ spins. Let $m = 2j_1 + 1$, $r = 2j_2 + 1$, $n = 2j_1' + 1$, $s = 2j_2' + 1$. The composition of angular momentum $[j_1] \times [j_2] = [|j_1 - j_2|] + \ldots + [j_1 + j_2]$ is exactly the rule (3.89), which can be rewritten as

$$\Phi^{(j_1,j_2)} \times \Phi^{(j_1',j_2')} = \sum_{|j_1-j_1'|}^{j_1+j_1'} \sum_{|j_2-j_2'|}^{j_2+j_2'} \Phi^{(j,j')} \tag{3.96}$$

where we have used the notation

$$\Phi^{(j_1,j_2)} = \Phi_{(m,n)} \qquad m = 2j_1 + 1 \ , \ n = 2j_2 + 1 \tag{3.97}$$

Finally, defining $p = k+2$, $p' = p'+2$, equation (3.95) becomes

$$\Phi^{(j_1,j_2)} \times \Phi^{(j'_1,j'_2)} = \sum_{|j_1-j'_1|}^{\min(j_1+j'_1,k-j_1-j'_1)} \sum_{|j_2-j'_2|}^{\min(j_2+j'_2,k'-j_2-j'_2)} \Phi^{(j,j')} \tag{3.98}$$

which are the fusion rules for the primary fields of the Kac–Moody algebra $SU(2)_k \times SU(2)_{k'}$ [39].

Remark 1) A simple way to understand why (3.93) is a consequence of (3.90) is to write

$$\alpha_{p-m,p'-n} = \frac{1-p+m}{2}\alpha_+ + \frac{1-p'+n}{2}\alpha_- = \alpha_\circ + \frac{m\alpha_+ + n\alpha_-}{2} - (p\alpha_+ + p'\alpha_-) \tag{3.99}$$

and to use $p\alpha_+ + p'\alpha_- = 0$ for $\alpha_\pm$ as in (3.90). Moreover,

$$\alpha_{p-m,p'-n} = 2\alpha_\circ - \alpha_{m,n} \tag{3.100}$$

and thus the reflection symmetry (3.93) is simply $\alpha \to \overline{\alpha}$.

Remark 2) The range of (m,n) in $\Phi_{(m,n)}$ for (p,p') models can be derived from (3.95) and (3.98):

$$0 < m < p \qquad 0 < n < p' \tag{3.101}$$

The total number of families in the (p,p') model is $\frac{1}{2}(p-1)(p'-1)$.

Next, we illustrate some sample computations of four–point functions. The simplest ones are

$$\left\langle V_{(\overline{m,n})} V_{(1,2)} V_{(1,2)} V_{(m,n)} \right\rangle \tag{3.102}$$

and

$$\left\langle V_{(\overline{m,n})} V_{(2,1)} V_{(2,1)} V_{(m,n)} \right\rangle \tag{3.103}$$

In the first one, the total charge is $2\alpha_\circ + 2\alpha_{1,2} = 2\alpha_\circ - \alpha_-$, whereas in the second one it is $2\alpha_\circ + 2\alpha_{2,1} = 2\alpha_\circ - \alpha_+$. In both cases, it suffices to introduce one single screening operator. We are then faced with the evaluation of integrals of the form

$$\oint dt\,\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)V_{\alpha_4}(z_4)J_\pm(t)\rangle = \left[\prod_{i<j} z_{ij}^{2\alpha_i\alpha_j}\right] I(\{z_i\})$$
$$\text{with} \qquad I(\{z_i\}) = \oint dt \prod_i (z_i - t)^{2\alpha_i\alpha_\pm} \tag{3.104}$$

We can write (3.104) more conveniently using the SL_2 transformations:

$$\left\langle \prod_i V_i(z_i)J_\pm(t)\right\rangle = \left[\prod_{i=1}^{4}(cz_i+d)^{-2h_i}(ct+d)^{-2}\right]\left\langle \prod_i V_i\left(\frac{az_i+b}{cz_i+d}\right) J_\pm\left(\frac{at+b}{ct+d}\right)\right\rangle \tag{3.105}$$

Since the currents $J_\pm$ are one–forms and t is integrated over, we can forget about the t transformation. Now choose a, b, c, d such that $z_1 \to \infty$, $z_2 \to 1$, $z_3 \to \eta$, $z_4 \to 0$, where

$$\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}} \tag{3.106}$$

This is achieved by

$$z \longrightarrow \frac{(z-z_4)(z_1-z_2)}{(z_1-z)(z_2-z_4)} \tag{3.107}$$

which leads to

$$\oint dt \left\langle \prod_i V_i(z_i)J_\pm(t)\right\rangle = \left(\frac{z_{12}z_{14}}{z_{24}}\right)^{h_2+h_3+h_4-h_1}$$
$$\frac{(1-\eta)^{2\alpha_2\alpha_3}\eta^{2\alpha_3\alpha_4}}{z_{12}^{2h_2}z_{13}^{2h_3}z_{14}^{2h_4}} \oint_C dt(1-t)^{2\alpha_2\alpha_\pm}(\eta-t)^{2\alpha_3\alpha_\pm}t^{2\alpha_4\alpha_\pm} \tag{3.108}$$

This can be expressed in terms of hypergeometric functions

$$F(a,b,c,;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 dt\, t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} \tag{3.109}$$

In (3.108), we have two possible independent contours, shown in figure 3.5.

The different contours are related to the internal states of the conformal block. We have written the integration contours as "open" contours in figure 3.4, assuming that all the points 0, 1, η and ∞ are non–trivial branch points. Instead of integrating along a contour, we can integrate along the cut. The difference between the two procedures is only a normalization constant which is fixed by the condition (3.36). The results may differ when one of the points is not a branch point. In this case, one of the contour integrals will vanish, and this gives constraints on the fusion rules.

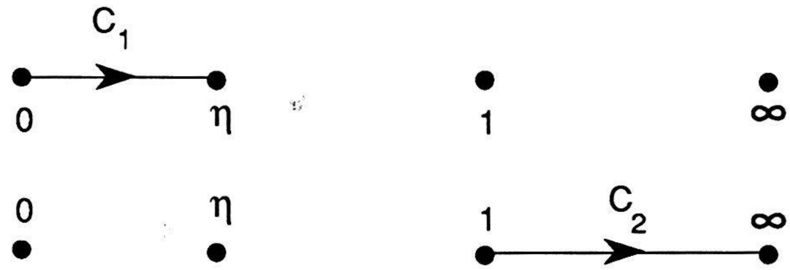


Figure 3.5. The two possible integration contours in (3.108).

As an example, consider the Ising model (3.65) with $p = 3$, $p' = 4$, $c = 1/2$. Consider the four–point function $\langle\epsilon\epsilon\epsilon\epsilon\rangle$, $\epsilon \sim V_{(2,1)}$, $\alpha_{2,1} = -\alpha_+/2$. The second contour does not contribute because $t = 0$ is not a branch point. The first contour yields

$$\langle\epsilon\epsilon\epsilon\epsilon\rangle = \frac{1}{z_{13}z_{24}}\left[\eta(1-\eta)\right]^{2/3}\left[\eta(1-\eta)\right]^{-5/3} F(-2,-1/3,-2/3;\eta) \tag{3.110}$$

and using the properties of hypergeometric functions we find

$$\langle\epsilon\epsilon\epsilon\epsilon\rangle = \frac{1}{z_{13}z_{24}}\frac{1-\eta+\eta^2}{\eta(1-\eta)} = \mathcal{F}_{\epsilon\epsilon\epsilon\epsilon}(0|\eta) \tag{3.111}$$

which satifies (3.36). From (3.111) we can read the fusion rule $\epsilon \cdot \epsilon = 1$. Many other examples can be found in the literature. It is a good exercise to compute all the four–point blocks in the Ising model. The integrals needed are

$$\int_0^\eta dt(1-t)^\alpha(\eta-t)^\beta t^\gamma = \frac{\Gamma(1+\gamma)\Gamma(1+\beta)}{\Gamma(2+\beta+\gamma)}\eta^{1+\beta+\gamma}F(-\alpha,1+\gamma,2+\beta+\gamma;\eta)$$
$$\int_1^\infty dt(1-t)^\alpha(\eta-t)^\beta t^\gamma = \frac{\Gamma(1+\alpha)\Gamma(-\alpha-\beta-\gamma-1)}{\Gamma(-\beta-\gamma)}F(-\beta,-\alpha-\beta-\gamma-1,-\beta-\gamma;\eta) \tag{3.112}$$

This concludes our brief survey of conformal field theories and their Coulomb gas representation. Using this representation of the minimal theories it is possible to derive the

Virasoro characters for their representations [40]. One can also give an explicit presentation of the null vectors in terms of contour integrals (see the paper by Fateev and Zamolodchikov [41]).

Final Remarks. The minimal (p,p') models provide an example of a rational conformal field theory (RCFT). The notion of rationality depends on the chiral algebra of the theory. For the minimal theories, this is the Virasoro algebra. In general, the chiral algebra will be $\mathcal{A} = \mathcal{A}_L \times \mathcal{A}_R$ with $Vir \subset \mathcal{A}_L, \mathcal{A}_R$. The theory is a rational conformal field theory if the Hilbert space decomposes into a finite number of irreducible representations of $\mathcal{A}$:

$$\mathcal{H} = \bigoplus_{a,\overline{a}} V_a \otimes V_{\overline{a}} \tag{3.113}$$

with a and $\overline{a}$ running over a finite range of labels. When the number of primary fields is not finite, we say that the conformal field theory is irrational. If the number of primary fields is countable, the theory is said to be compact, and non–compact otherwise. Rational conformal field theories are the simplest to study, and all their duality and modular properties can be summarized in terms of a set of polynomial equations. The largest class of solutions to these equations is provided by the representation theory of quantum groups when the deformation parameter is a root of unity ([29, 42] and references therein).

4. Coupling to Two–Dimensional Gravity

4.1. Path Integrals and the Liouville Field

Before analyzing two–dimensional gravity and its coupling to conformal field theory in terms of random triangulations and with large N methods, it is useful to work with the theory in the continuum to get the flavour of the kind of results one should obtain on the lattice. The original derivations of the results in this lecture are due to Polyakov [1] and to Knizhnik, Polyakov and Zamolodchikov (KPZ) [2]. They quantized two-dimensional gravity in the light–cone gauge, where they found a residual $SL_2(R)$ current algebra which played a crucial role in the determination of anomalous dimensions. Here, we will follow the approach of David [3] and Distler and Kawai [4], who used instead the conformal gauge. This has some advantages from the pedagogical point of view, and it also allows for a straightforward generalization of the KPZ results to surfaces of higher genus. To avoid unnecessary distractions with technical points, we have collected some well–known facts about the conformal properties of determinants of laplacians on Riemann surfaces in an Appendix.

In this lecture, we shall derive the change in the dimensions $h_{(m,n)}$ of primary fields in a (p,p') minimal model as a consequence of its coupling to gravity. They have been checked explicitly in various cases, where the statistical mechanical model on a random surface have been solved. The details used below in setting up the integral can be found in [21, 22, 43].

The partition function for the bosonic string (or any other conformal system with central charge $c = d$) is given by

$$Z = \int \frac{\mathcal{D}g\mathcal{D}X}{Vol(Diff)} e^{-S_M(X;g) - \frac{\mu_0}{2\pi}\int \sqrt{g} d^2\xi} \tag{4.1}$$

where μ_0 is the bare cosmological constant, and S_M is the conformally invariant action representing the matter fields X. For a free bosonic string,

$$S_M = \frac{1}{8\pi} \int d^2\xi \sqrt{g} g^{ab} \partial_a \vec{X} \partial_b \vec{X} \tag{4.2}$$

The integration measures in (4.1) must be described more explicitly. For $\mathcal{D}X$, we construct the measure by normalizing the functional integral of the gaussian of a (quantum field) fluctuation

$$\begin{aligned} \int \mathcal{D}_g \delta X e^{-||\delta X||_g^2} &= 1 \\ ||\delta X||_g^2 &= \int d^2\xi \sqrt{g} \delta X \cdot \delta X \end{aligned} \tag{4.3}$$

For the metrics, $\mathcal{D}g$ is more difficult to define. However, given a particular point g_{ij} in the space of metrics on a genus h surface, we can define the measure over a fluctuation δg from

$$\begin{aligned} \int \mathcal{D}\delta g e^{-\frac{1}{2}||\delta g||_g^2} &= 1 \\ ||\delta g||_g^2 &= \int d^2\xi \left(g^{ac} g^{bd} + 2 g^{ab} g^{cd} \right) \delta g_{ab} \delta g_{cd} \end{aligned} \tag{4.4}$$

It is clear from (4.3) and (4.4) that the measures are invariant under the diffeomorphism group. The invariance under conformal transformations is not assured, however. Since $||\delta X||_g^2$ depends on g, we will have $\mathcal{D}_{e^\sigma g} X \neq \mathcal{D}_g X$. In fact (see the Appendix),

$$\mathcal{D}_{e^\sigma g} X = e^{(d/48\pi) S_L(\sigma)} \mathcal{D}_g X \tag{4.5}$$

with the Liouville action

$$S_L(\sigma) = \int d^2\xi \sqrt{g} \left(\frac{1}{2} g^{ab} \partial_a \sigma \partial_b \sigma + R\sigma + \mu e^\sigma \right) \tag{4.6}$$

One way to obtain this result is to decompose the measure over a scalar field ϕ in terms of the orthonormal set of eigenfunctions of the laplacian $\Delta = -(\sqrt{g})^{-1} \partial_i (g^{ij} \sqrt{g} \partial_j)$:

$$\Delta \phi_n = \lambda_n \phi_n \qquad (\phi_n, \phi_m) = \int d^2\xi \sqrt{g} \phi_n \phi_m = \delta_{n,m} \tag{4.7}$$

The laplacian always has a zero mode $\phi_0 = (\int d^2\xi \sqrt{g})^{-1/2}$. Write next $\phi(\xi) = \sum a_n \phi_n(\xi)$, and since the basis is orthonormal, $\mathcal{D}\phi = \prod_n da_n$. The zero mode of X^μ is to be interpreted as the string's center of mass co-ordinate, and hence we give up the integration da_0 in favour of d(C.M.). For the non-zero modes, we integrate over da_n. The action S_M is quadratic in the a_n's, and therefore we obtain

$$\int d(\text{C.M.}) \left(\frac{\det' \Delta}{\int \sqrt{g}}\right)^{-d/2} \tag{4.8}$$

The integral in this expression gives the total volume of space–time – we shall ignore this factor from now on. The notation det$'$ means that the zero mode is removed.

A useful method to compute determinants is provided by heat kernels and ζ–functions. Using the identity

$$\log \frac{a}{b} = -\int_0^\infty \frac{dt}{t}\left(e^{-at} - e^{-bt}\right) \tag{4.9}$$

we can define

$$\log \det{}' \Delta = -\int_\varepsilon^\infty \frac{dt}{t}\, \mathrm{Tr}' e^{-t\Delta} \tag{4.10}$$

which is very difficult to compute in general, although some of its variations can be computed in closed form. This is done for Weyl transformations in the Appendix, and the result is (4.6).

For the metric measure $\mathcal{D}g$, we mentioned in lecture 1 that the space of metrics on a compact topological surface modulo diffeomorphisms and Weyl transformations is a finite–dimensional space $\mathcal{M}_g$. Suppose we choose some representative metric $\hat{g}_{ij}(\tau)$ for every point $\tau \in \mathcal{M}_g$. Then the orbits generated by *Diff* and *Weyl* acting on $\hat{g}_{ij}(\tau)$ generate the space of metrics on Σ_g. This is shown schematically in figure 4.1.

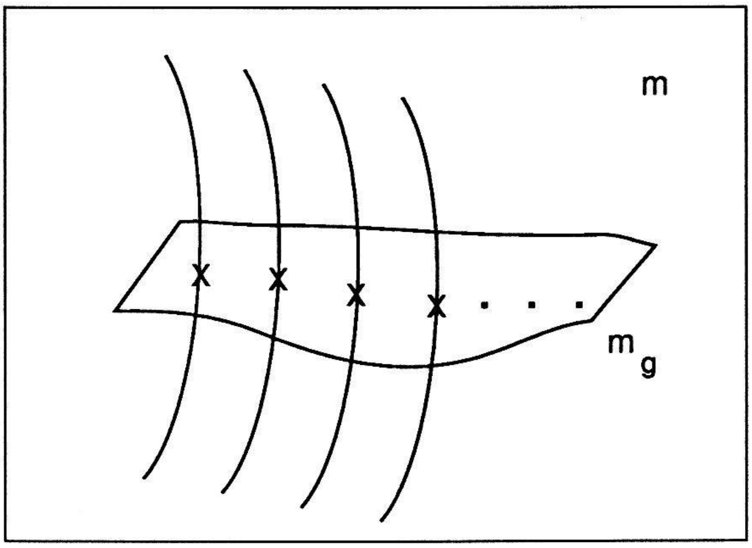


Figure 4.1. Slice $\mathcal{M}_g$ in the space of metrics $\mathcal{M}$, and the orbits of the diffeomorphism and Weyl groups.

Given the slice $\hat{g}(\tau)$, we can represent any metric in the form

$$f^* g = e^{\phi_0} \hat{g}(\tau) \tag{4.11}$$

wherer f^* represents the action of a diffeomorphism $f : \Sigma \to \Sigma$. Since the metric (4.3) is *Diff*-invariant, we have to divide by the "orbit" of *Diff*. This is done using Faddeev–Popov ghosts, as in gauge theories. If we denote an infinitesimal co-ordinate change of the metric in complex co-ordinates as

$$\begin{aligned} \delta g_{zz} &= \nabla_z \xi_z \\ \delta g_{\overline{z}\overline{z}} &= \nabla_{\overline{z}} \xi_{\overline{z}} \end{aligned} \tag{4.12}$$

then the measure $\mathcal{D}_g$ at $\hat{g}(\tau)$ will split into three pieces. One is an integration over moduli, $\mathcal{D}\tau$. The second one is an integration over the conformal factor $\mathcal{D}\phi_0$. The third integration is over diffeomorphisms $\mathcal{D}\xi\mathcal{D}\overline{\xi}$. going from $\mathcal{D}\delta g_{zz}\mathcal{D}\delta g_{\overline{z}\overline{z}}$ to $\mathcal{D}\xi\mathcal{D}\overline{\xi}$, we pick up the Jacobian "$\det \nabla_z \det \nabla_{\overline{z}}$", which can be exponentiated by introducing anti-commuting ghost variables b_{zz}, c^z, $b_{\overline{z}\overline{z}}$, $c^{\overline{z}}$. The ghost b_{zz} is a holomorphic quadratic differential, whereas c^z is a holomorphic vector. The final result for this Jacobian is

$$\mathcal{D}_g b \mathcal{D}_g c \mathcal{D}_g \overline{b} \mathcal{D}_g \overline{c} e^{-S_{\mathrm{gh}}(b,c,g) - S_{\mathrm{gh}}(\overline{b},\overline{c},g)} \tag{4.13}$$

where

$$S_{\rm gh}(b,c,g)=\int d^2\xi\ b_{zz}\nabla_{\bar z}c^z$$
$$S_{\rm gh}(b,c,g)=\int d^2\xi\ b_{\bar z\bar z}\nabla_z c^{\bar z} \tag{4.14}$$

The ghosts will not play an important role in our argument, except for their contribution to the conformal anomaly. Once again, the measure is not invariant under $g\to e^\sigma g$, and the result of a conformal transformation is [21, 22, 43]

$$\mathcal{D}_{e^\sigma g}\mathcal{D}_{e^\sigma g}c=\mathcal{D}_g b\mathcal{D}_g c e^{(-26/48\pi)S_L(\sigma,g)} \tag{4.15}$$

The path integral we want to study is

$$Z=\int [d\tau]\mathcal{D}_g\phi_o\mathcal{D}_g b\mathcal{D}_g c\mathcal{D}_g X e^{-\left(S_M+S_{\rm gh}+(\mu_o/2\pi)\int d^2\xi\sqrt{g}\right)} \tag{4.16}$$

The first difficulty we find is with the integration over ϕ_o: the measure depends implicitly on ϕ_o:

$$||\delta\phi_o||_g^2=\int\sqrt{g}(\delta\phi_o)^2=\int d^2\xi\sqrt{\hat g}e^{\phi_o}(\delta\phi_o)^2 \tag{4.17}$$

We would like to transform this metric into a free field metric,

$$\int d^2\xi\sqrt{\hat g}(\delta\phi_o)^2 \tag{4.18}$$

After choosing the slice $\hat g$, the measure changes:

$$\mathcal{D}_{e^{\phi_o}\hat g}\phi_o\mathcal{D}_{e^{\phi_o}\hat g}b\mathcal{D}_{e^{\phi_o}\hat g}c\mathcal{D}_{e^{\phi_o}\hat g}X=\mathcal{D}_{\hat g}\phi_o\mathcal{D}_{\hat g}b\mathcal{D}_{\hat g}c\mathcal{D}_{\hat g}X\,J(\phi_o\hat g) \tag{4.19}$$

The Jacobian was easy to compute for the system containing matter and ghost fields, but some extra information is needed to evaluate the contribution of the ϕ_o field. David, and Distler and Kawai (DDK) made the plausible assumption that $J(\phi_o,\hat g)$ is the exponential of a local action similar to the Liouville action. Some justification for this hypothesis is provided in [44, 45]. For some applications of the DDK prescription, see also [46]. Since this assumption has very important consequences, we shall present DDK's arguments in detail.

In terms of the slice $\hat{g}$ and the measures defined in (4.2), (4.4) and (4.18), the partition function is of the form

$$Z = \int [d\tau] \mathcal{D}_{\hat{g}}\phi_0 \mathcal{D}_{\hat{g}} b \mathcal{D}_{\hat{g}} c \mathcal{D}_{\hat{g}} X \, e^{-S_M(X,\hat{g}) - S_{\text{gh}}(b,c,\hat{g})}$$
$$\exp\left[-\int d^2\xi \sqrt{\hat{g}} \left(a\hat{g}^{ab}\partial_a\phi\partial_b\phi + b\hat{R}\sqrt{\hat{g}}\phi + \mu e^{c\phi} \right) \right] \tag{4.20}$$

Notice that a rescaling of ϕ is implied in the term $e^{c\phi}$. At the end of the argument, therefore, we will find that the physical metric has been changed to $e^{\alpha\phi}\hat{g}$. But recall that we defined the path integral in a reparametrization–invariant way, and Z is only a function of $e^{\phi}\hat{g} = g$ (up to diffeomorphisms). Hence, (4.20) must be invariant under the change

$$\delta\hat{g} = \varepsilon(\xi)\hat{g}$$
$$\delta\phi = -\varepsilon(\xi) \tag{4.21}$$

The answers to physical questions should not depend on how the slice $\hat{g}$ is chosen. We can use the computed conformal anomaly for ϕ, X, b, c together with the explicit variation of the last exponential in (4.20) to determine the values of a , b, c. We first determine the coefficients a and b.

The transformations (4.21) produce two terms, one proportional to $\varepsilon\Delta\phi$ and another one proportional to εR (see the Appendix). The contributions of the form $\varepsilon\Delta\phi$ come only from the variation of ϕ in the term of (4.20) proportional to a, and from the metric variation in $b\hat{R}\sqrt{\hat{g}}\phi$. They yield

$$(2a - b) \int d^2\xi \sqrt{\hat{g}} \varepsilon \Delta\phi \tag{4.22}$$

The term $\varepsilon\sqrt{g}R$ receives contributions from all the fields. The combined conformal anomalies of the matter and ghost sectors yield $(d-26)/48\pi$. From the measure $\mathcal{D}_{\hat{g}}\phi$, we obtain an extra $1/48\pi$, and finally from the variation of ϕ in the term proportional to b in (4.20), we get a contribution of precisely b. The total is

$$\left(\frac{d - 26 + 1}{48\pi} + b \right) \int d^2\xi \sqrt{\hat{g}} \hat{R} \tag{4.23}$$

Invariance under (4.21) implies then

$$b = \frac{25-d}{48\pi} \qquad a = \frac{b}{2} \tag{4.24}$$

Since the coefficient of the total Liouville action is proportional to $(25-d)$, the effective coupling constant behaves as $(25-d)^{-1}$, and therefore the classical limit is obtained as $d \to -\infty$. This will prove useful later on. Furthermore, we can rescale $\phi \to \sqrt{12/(25-d)}\phi$ to have a canonical kinetic term of the form $(8\pi)^{-1}\int(\partial\phi)^2$ so that, on the sphere, $\phi(z)\phi(w) = -\log(z-w) + \cdots$:

$$\begin{aligned}\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\left(\frac{25-d}{12}\hat{g}^{ab}\partial_a\phi\partial_b\phi + \frac{25-d}{6}\hat{R}\phi\right) \to \\ \to \frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\left(\hat{g}^{ab}\partial_a\phi\partial_b\phi + Q\hat{R}\phi\right)\end{aligned} \tag{4.25}$$

with

$$Q = \sqrt{\frac{25-d}{3}} \tag{4.26}$$

The contribution to the energy–momentum of ϕ coming from (4.25) is obtained by computing $\delta/\delta g^{ab}$. All we need is $\delta/\delta g^{zz}$. The result follows if we take two identities into account. The first one simply states that the two–dimensional Einstein equations are satisfied identically. In two dimensions, the curvature tensor for a metric g_{ab} takes the form

$$R_{abcd} = \frac{R}{2}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) \tag{4.27}$$

with R the scalar curvature. Then $R_{ac} = Rg_{ac}/2$. We also need the general metric variation of the Ricci tensor:

$$\begin{aligned}\delta R_{ab} = \frac{1}{2}\Big(& -\nabla^c\nabla_c\delta g_{ab} - \nabla_a\nabla_b g^{cd}\delta g_{cd} \\ & + \nabla^c\nabla_a\delta g_{cb} + \nabla^c\nabla_b\delta g_{ca}\Big)\end{aligned} \tag{4.28}$$

Therefore,

$$\delta\int\sqrt{g}R\phi = \int\sqrt{g}\left(\frac{1}{2}Rg^{ab} - R^{ab}\right)\delta g_{ab}\phi + \int\sqrt{g}g^{ab}\delta R_{ab}\phi \tag{4.29}$$

The term proportional to the Einstein equations vanishes and, after partially integrating by parts, we are left with

$$\int \sqrt{g} \left(-g^{ab} \delta g_{ab} \nabla^c \nabla_c \phi + \delta g_{ab} \nabla^a \nabla^b \phi \right) \tag{4.30}$$

Since $\delta g^{ab} = -g^{ac} g^{bd} \delta g_{bd}$, the variations of the form δg^{zz}, $\delta g^{\bar{z}z}$ about a metric $ds^2 = g_{z\bar{z}} dz d\bar{z}$ are such that only the second term in (4.30) contributes. The energy–momentum tensor is then

$$T = -\frac{1}{2} \partial_z \phi \partial_z \phi + \frac{Q}{2} \partial_z^2 \phi \tag{4.31}$$

with Virasoro central charge

$$c_{\text{Liouville}} = 1 + 3Q^2 \tag{4.32}$$

Adding now $d - 26$ from the matter and ghost sectors, and requiring that the total central charge vanish leads again to (4.26) for Q, thus providing a consistency check on the previous arguments.

Next, we determine the coefficient c in (4.20). Since we have rescaled ϕ, we can write this term as $e^{\gamma\phi}$. Geometrically, $\int e^{\gamma\phi} \sqrt{\hat{g}}$ represents the area of the surface for the metric g. The coefficient γ is determined by requiring $e^{\gamma\phi}$ to behave as a (1,1) conformal field in order to implement the symmetry (4.21), or rather its renormalized forma $\delta\hat{g} = \varepsilon\hat{g}$, $\delta\phi = -\varepsilon/\gamma$. Adapting the Coulomb gas derivations in the previous lecture to the present case leads to

$$h(e^{\gamma\phi}) = -\frac{1}{2}\gamma(\gamma - Q) = 1 \tag{4.33}$$

or equivalently,

$$Q = \frac{2}{\gamma} + \gamma \tag{4.34}$$

The classical limit $d \to -\infty$ can be thought of as $\gamma \to 0$. This classical limit is not the mean field theory limit, which corresponds to $d \to +\infty$. It should also be pointed out that we have tuned the bare comological constant to cancel the exponential term $\mu^2 e^{\gamma\phi}$ in the action. This is not strictly necessary, and in fact the Liouville energy–momentum tensor is still given by (4.31) at the classical level if one uses the equations of motion for the field ϕ.

In (4.33), we may instead solve for γ in terms of Q:

$$\gamma = \frac{Q \pm \sqrt{Q^2 - 8}}{2} = \frac{\sqrt{25-d} \pm \sqrt{1-d}}{\sqrt{12}} \tag{4.35}$$

To recover the classical limit $Q \to \infty$, $\gamma \to 0$ we must choose the minus sign:

$$\gamma = \frac{Q - \sqrt{Q^2 - 8}}{2} = \frac{\sqrt{25-d} - \sqrt{1-d}}{\sqrt{12}} \tag{4.36}$$

There are three regimes to consider:

a) $d \leq 1$: γ is real and the theory is well defined. With the choice (4.36), Q is real and $\gamma \leq Q/2$.

b) $d \geq 25$; γ and Q are purely imaginary. In order to have a real metric $e^{\gamma\phi}\hat{g}$, we have to Wick rotate $\phi \to -i\phi$ and this changes the sign of the kinetic term. The quanta of the ϕ field in perturbative quantization have negative metric. Exactly at $d = 25$ one can interpret the continued field $X^0 = -i\phi$ as a time co-ordinate in space–time. More precisely, if we start with a free field theory describing the embedding of the string histories in flat 25–dimensional euclidean space–time, then the Liouville field becomes effectively a time co-ordinate and the full theory is equivalent to the 26–dimensional string in Minkowski space–time [47]. This is very intriguing, and one is immediately led to speculate whether the signature of space–time with its causal structure could arise as a result from quantum string theory.

c) $1 < d < 25$; γ is complex and Q is purely imaginary. very little is known about this region. What kind of phase transition describes the passage from $c < 1$ to $c > 1$ is anybody's guess. We mentioned in lecture 1 that the three–dimensional Ising model and the confined phase of four–dimensional gauge theories are closely related to theories of fluctuating surfaces or, equivalently, sub-critical string theory. Although it is a very hard problem, the reward for solving string theory is indeed a hefty one. Much physical knowledge is likely to be gained once this range of d is understood in Liouville theory. When $d = 7$, 13 or 19, there seems to be a consistent description (a unitary truncation) of the quantum Liouville theory which can be used to construct strings in those dimensions [48].

There is one more puzzling aspect of sub-critical strings worth mentioning before closing this section. One of the basic and universal features of critical strings is the existence of massless spin–two excitations. In an interacting theory, this is equivalent to having a theory of gravity. Friedan proved in his doctoral thesis that the Einstein equations follow from critical string theory. When the string is sub-critical, however, no trace has been found yet of the graviton. We lack some basic understanding in this regard.

After Polyakov's papers [21], Curtright, Thorn, Braaten and Ghandour studied [49, 50, 51] the quantization of the Liouville theory preserving conformal invariance. They were the first to derive the results (4.34,4.35,4.36) and they also found an explicit operator solution to the Liouville equations of motion by expressing the Liouville field in terms of a free field through a quantum version of the classical Bäcklund transformation which solves the classical equations of motion. Their results also shed light on the properties of the spectrum of the theory. They were not able, however, to obtain a prescription for evaluating general correlation functions in the theory. The study of the quantum Liouville theory was also carried out by J.L. Gervais and A. Neveu [52] with open and closed string boundary conditions. Whenever the results of [49, 50, 51, 52] can be compared, they agree. There are also attempts to quantize Liouville theory using quantum groups [53, 48, 54, 55] and, although the preliminary results are encouraging, much remains to be done. A good and incisive discussion of Liouville theory can be found in the review by Seiberg [56].

Unless otherwise stated, we shall restrict our discussion in this lecture to the simplest case $d < 1$.

4.2. Critical Exponents

We impose a fixed area constraint on the partition function:

$$Z(A) = \int \mathcal{D}\phi\mathcal{D}X e^{-S}\, \delta\left(\int e^{\gamma\phi}\sqrt{\hat{g}}d^2\xi - A\right) \tag{4.37}$$

We have dumped both the ghost determinant and the integration over moduli into $\mathcal{D}X$, for notational simplicity. The string susceptibility Γ is defined by†

$$Z(A) \sim KA^{\Gamma-3} \quad \text{when } A \to \infty \tag{4.38}$$

We can determine Γ using a very simple scaling argument. Shifting $\phi \to \phi + \rho/\gamma$, ρ a constant, the measure does not change. In the Liouville action, only the term proportional to $\hat{R}$ contributes:

$$\frac{Q}{8\pi}\int d^2\xi\sqrt{\hat{g}}\hat{R}\phi \to \frac{Q}{8\pi}\int d^2\xi\sqrt{\hat{g}}\hat{R}\phi + \frac{Q}{8\pi}\frac{\rho}{\gamma}\int d^2\xi\sqrt{\hat{g}}\hat{R} \tag{4.39}$$

If the surface Σ has G handles, the Gauss–Bonnet theorem implies

$$\frac{1}{8\pi}\int_\Sigma d^2\xi\sqrt{\hat{g}}\hat{R} = 1 - G \tag{4.40}$$

and therefore

$$Z(A) = e^{-\frac{Q\rho}{\gamma}(1-G)-\rho}Z(e^{-\rho}A) \tag{4.41}$$

where we have used $\delta(\lambda x) = |\lambda|^{-1}\delta(x)$. Choosing $e^\rho = A$, we obtain

$$Z(A) = A^{-1-\frac{Q}{\gamma}(1-G)}Z(1) \tag{4.42}$$

and hence

$$\Gamma_G = 2 - \frac{Q}{\gamma}(1-G) \tag{4.43}$$

At genus zero (for spherical topology),

$$\Gamma_0 \equiv \Gamma = 2 - \frac{Q}{\gamma} = \frac{d - 1 - \sqrt{(25-d)(1-d)}}{12} \tag{4.44}$$

† The definition of critical exponents of statistical systems is analyzed in some detail in lecture 7 below.

When $d = 0$ (pure gravity case), $\Gamma_0 = -1/2$. If we take d in the minimal unitary series, $d = 1 - 6/p(p+1)$, then $\Gamma_0 = -1/p$. For $d = 1$, $\Gamma_0 = 0$ and we should expect some logarithmic dependence of $Z(A)$ on the area A.

Next, we compute the dimensions of fields. Let Φ_o be a spinless primary field of the conformal theory whose conformal dimension in flat space is $h_o = h(\Phi_o) = \overline{h}(\Phi_o)$. The net effect of the gravitational interactions of the field Φ_o is to dress it, modifying its dimensions (h_o, h_o) so that the total dressed field Φ is a (1,1) field (a measure). In this way, the dressed field can be integrated over Σ preserving the symmetry (4.21). We write the dressed field as

$$\Phi = \Phi_o e^{\beta\phi} \tag{4.45}$$

with ϕ the Liouville field. The dressing factor $e^{\beta\phi}$ is the "wave–function renormalization" which allows the field to couple consistently to gravity. The value of β is determined by the requirement that the dimension of Φ be indeed one:

$$h_o - \frac{1}{2}\beta(\beta - Q) = 1 \tag{4.46}$$

We can associate a critical exponent with the dressed field Φ by considering its one–point function at fixed area A in the limit $A \to \infty$:

$$F_\Phi(A) = \frac{1}{Z(A)} \int \mathcal{D}\phi \mathcal{D}X e^{-S} \delta\left(\int e^{\gamma\phi}\sqrt{\hat{g}}d^2\xi - a\right) \int \Phi_o e^{\beta\phi}\sqrt{\hat{g}}d^2\xi \tag{4.47}$$

The gravitational scaling dimension h is defined by

$$F_\Phi(A) \underset{A \to \infty}{\longrightarrow} \tilde{K} A^{1-h} \tag{4.48}$$

Quite generally, exponents (such as Γ or h) are physically meaningful quantities even if the (local) operator product expansions are of dubious validity because they come from integrated expressions, smeared over the whole surface.

The same scaling argument leading to (4.42) yields

$$h = 1 - \frac{\beta}{\gamma} \tag{4.49}$$

Writing this as $\beta = \gamma(1-h)$, substituting β into (4.46), and using (4.33), we easily obtain

$$h - h_\circ = \frac{\gamma^2}{2} h(1-h) \tag{4.50}$$

which is the famous KPZ result. Choosing in (4.46) the branch $\beta \leq Q/2$ yields

$$\beta = \frac{\sqrt{25-d} - \sqrt{1-d+24h_\circ}}{\sqrt{12}} = \frac{Q}{2} - \sqrt{\frac{Q^2}{4} - 2 + 2h_\circ} \tag{4.51}$$

and

$$h = \frac{\sqrt{1-d+24h_\circ} - \sqrt{1-d}}{\sqrt{25-d} - \sqrt{1-d}} \tag{4.52}$$

4.3. Gravitational Dressing of the Minimal Series

In this section, we compute explicitly the values $h_{m,n}$ and $\beta_{m,n}$ for the minimal (p, p') series (recall p and p' are co-prime, and $p' > p$). Since $d = 1 - 6(p'-p)^2/pp'$,

$$Q = \frac{2(p+p')}{\sqrt{2pp'}} \tag{4.53}$$

and

$$\gamma = \sqrt{\frac{2p}{p'}} \qquad Q = \frac{2}{\gamma} + \gamma \tag{4.54}$$

In the minimal series,

$$h^\circ_{m,n} = -\alpha_\circ^2 + \frac{1}{4}(m\alpha_+ + n\alpha_-)^2 = \frac{(mp'-np)^2 - (p'-p)^2}{4pp'} \tag{4.55}$$

From (4.51), we obtain

$$\beta_{m,n} = \frac{Q}{2} - \frac{|mp'-np|}{\sqrt{2pp'}} \tag{4.56}$$

and finally

$$h_{m,n} = 1 - \frac{\beta_{m,n}}{\gamma} \tag{4.57}$$

As for the undressed case, we have the reflection symmetry

$$h_{m,n} = h_{p-m,p'-n} \tag{4.58}$$

In the Ising model, for example, $p = 3$ and $p' = 4$, so $h_{1,1} = h_{2,3} = 0$, $h_{1,2} = h_{2,2} = 1/6$ and $h_{1,3} = h_{2,1} = 2/3$. The exponents (4.56) are in agreement with all the known exact results of statistical models on fluctuating lattices. So far, there are exact results for self–avoiding walks, $O(n)$ models, the $Q = 1$ Potts model, bond percolation, Ising model, and a few others. They all agree with (4.57).

It is interesting to re-interpret the values of Q, γ and $\beta_{m,n}$ in terms of the Kac table and the Coulomb gas representation of minimal models:

$$c_L = 1 + 3Q^2 = 1 - 24(-\frac{Q^2}{8}) = 13 + \left(\gamma^2 + \frac{4}{\gamma^2}\right) \tag{4.59}$$

Naturally, we have two fields of dimension one:

$$\begin{aligned} J_+ &= e^{\gamma\phi} \\ J_- &= e^{2\phi/\gamma} \end{aligned} \tag{4.60}$$

This should be expected because $-\frac{1}{2}\gamma(\gamma - Q)$ is symmetric under $\gamma \leftrightarrow Q - \gamma$, and $e^{\gamma\phi}$ has dimension 1. In striking contrast, the coefficient γ in the exponent of J_+ is $\leq Q/2$, whereas for J_-, $2/\gamma \geq Q/2$†. Define

$$\begin{aligned} \beta_o &= -i\frac{Q}{2\sqrt{2}} \\ \beta_+ &= -i\frac{\gamma}{\sqrt{2}} \\ \beta_- &= -i\frac{\sqrt{2}}{\gamma} \\ \beta_+ + \beta_- &= 2\beta_o = -i\frac{Q}{\sqrt{2}} \end{aligned} \tag{4.61}$$

Then

$$\beta_{m,n} = \sqrt{2}\left(\frac{1+m}{2}\beta_+ + \frac{1-n}{2}\beta_-\right) \tag{4.62}$$

or, more elegantly,

$$i\beta_{m,n} = \frac{Q}{2} + \frac{m\gamma}{2} - \frac{n}{\gamma} \tag{4.63}$$

† In the operator solution [49, 50] to the quantum Liouville theory, it is possible to define the fields $e^{\alpha\phi}$ using free field normal ordering when $\alpha \leq Q/2$. Whether these fields can be defined when $\alpha > Q/2$ by any other means is not known.

which agrees with (4.56). Hence, the dressed field is, in the Coulomb gas representation,

$$\Phi_{m,n} = e^{i\alpha_{m,n}\eta} e^{i\beta_{m,n}\phi} \tag{4.64}$$

Here, η is the free scalar field we used for the Coulomb gas representation of minimal models in lecture 3, and ϕ is the Liouville field, which we shall manipulate as if it really was a free field as well.

We already have four dimension–one currents at our disposal:

$$e^{i\alpha_{\pm}\eta} \quad , \quad e^{\gamma\phi} \quad , \quad e^{2\phi/\gamma} \tag{4.65}$$

Formally, however, we can construct an infinite number of possible screening charges [4] *i.e.*, infinitely many unitarily inequivalent representations of the canonical commutation relations. This is very reminiscent of the picture–changing operators in superstring theory. The equation determining the possible dimension one fields is, in the Coulomb gas language,

$$1 = h\left(e^{i\alpha\eta}e^{\beta\phi}\right) = -\frac{1}{2}\beta(\beta - Q) + \frac{1}{2}\alpha(\alpha - 2\alpha_0) \tag{4.66}$$

consistent with an energy–momentum tensor

$$T = -\frac{1}{2}\partial\phi\partial\phi + \frac{Q}{2}\partial^2\phi - \frac{1}{2}\partial\eta\partial\eta + i\alpha_0\partial^2\eta \tag{4.67}$$

Using the identities

$$\begin{aligned} \alpha_{\pm} &= \alpha_0 + \frac{Q}{2} \\ \alpha_0 &= \frac{1}{\gamma} - \frac{\gamma}{2} \end{aligned} \tag{4.68}$$

we can find many solutions to (4.66). For example, we can generalize the minimal model screening currents:

$$\begin{aligned} J_+^{(b)} &= e^{i(b\alpha_0+\alpha_+)\eta}e^{-b\alpha_0\phi} \\ J_-^{(b)} &= e^{i(b\alpha_0+\alpha_-)\eta}e^{b\alpha_0\phi} \end{aligned} \qquad b \in \mathbf{Z} \tag{4.69}$$

Similarly, we can start with $e^{\gamma\phi}$ and get a whole family of dimension–one operators:

$$e^{i\frac{bQ}{2}\eta}e^{(\gamma+\frac{bQ}{2})\phi} \quad b \in \mathbf{Z} \tag{4.70}$$

It is not difficult to find all solutions to (4.66) giving all possible screening operators. Now the conditions for screening the charge in correlation functions are

$$\begin{aligned} \sum \alpha_i &= 2\alpha_0 \\ \sum \beta_i &= Q \end{aligned} \tag{4.71}$$

and in principle there may be several ways of satisfying them, depending on the choice of screening operators. Although this is a sensible prescription, it is not clear whether it agrees with other methods of defining Liouville correlation functions.

The arguments of this lecture can be extended to the supersymmetric case. This was done originally using $SL(2|1)$ current algebra by Polyakov and Zamolodchikov [57] and by Distler, Hlousek and Kawai in the superconformal gauge [58].

Exercise. Following the steps in lecture 3, compute the fusion rules for minimal models coupled to gravity.

This concludes our study of the minimal (p, p') models coupled to gravity in the continuum. The basic formulas to remember from this lecture are (4.44), (4.49), (4.50), (4.52) and (4.57), which give the string susceptibility Γ and the critical exponents $h_{m,n}$ in the presence of gravity.

Appendix

We collect here a few details of the computation of determinants and conformal anomalies. Further details can be found in the literature. The identity

$$\log \frac{a}{b} = -\int_0^\infty \frac{dt}{t}\left(e^{-ta} - e^{-tb}\right) \tag{4A.1}$$

suggests the definition of the determinant of an operator $\mathcal{O}$:

$$\log \det \mathcal{O} = -\int_\varepsilon^\infty \frac{dt}{t} \operatorname{tr} e^{-t\mathcal{O}} \tag{4A.2}$$

where we assume ε very small. If the operator $\mathcal{O}$ has no zero modes, the trace in (4A.2) is unrestricted. When the operator has zero modes, however, we must modify this prescription. The basic example to study is the determinant of the scalar laplacian on a compact

Riemann surface Σ with metric g_{ij}. We showed in lecture 3 that the correct treatment of the zero modes leads to the computation of

$$\left(\frac{\det' \Delta}{\int \sqrt{g}}\right) \tag{4A.3}$$

with the laplacian operator given by $\Delta = -(\sqrt{g})^{-1}\partial_i(\sqrt{g}g^{ij}\partial_j)$. Using local isothermal co-ordinates, the metric can be written as

$$ds^2 = e^\sigma dz d\overline{z} \qquad g_{z\overline{z}} = \frac{1}{2}e^\sigma \tag{4A.4}$$

The non-zero Christoffel symbols are

$$\Gamma^z{}_{zz} = \partial_z \sigma \qquad \Gamma^{\overline{z}}_{\overline{z}\overline{z}} = \partial_{\overline{z}}\sigma \tag{4A.5}$$

The curvature tensor is

$$R^z{}_{zz\overline{z}} = -\partial_z\partial_{\overline{z}}\sigma \tag{4A.6}$$

The Ricci tensor is

$$R_{z\overline{z}} = -\partial_z\partial_{barz}\sigma = R_{\overline{z}z} \tag{4A.7}$$

and the scalar curvature is, finally,

$$R = 2g^{z\overline{z}}R_{z\overline{z}} = -4e^{-\sigma}\partial_z\partial_{\overline{z}}\sigma \tag{4A.8}$$

In these co-ordinates, the laplacian of a scalar function takes the form

$$\Delta\sigma = -\frac{1}{\sqrt{g}}\partial_i\left(g^{ij}\sqrt{g}\partial_j\sigma\right) = -2e^{-\sigma}\partial_\sigma\partial_{\overline{z}}\sigma \tag{4A.9}$$

and therefore

$$R = 2\Delta\sigma \tag{4A.10}$$

Under a one-parameter family of conformal transformations

$$g(t) = e^{t\sigma}g \tag{4A.11}$$

we can compute the changes in Γ and R from the above formulae and obtain

$$\begin{aligned} \Gamma^z{}_{zz}(t) &= \Gamma^z{}_{zz} + t\partial_z\sigma \\ R^z{}_{zz\overline{z}}(t) &= R^z{}_{zz\overline{z}} - t\partial_z\partial_{\overline{z}}\sigma \\ R_{z\overline{z}}(t) &= R_{z\overline{z}} - t\partial_z\partial_{\overline{z}}\sigma \\ R(t) &= e^{-t\sigma}R - 2te^{-t\sigma}g^{z\overline{z}}\partial_z\partial_{\overline{z}}\sigma \\ &= e^{-t\sigma}R + te^{-t\sigma}\Delta\sigma \end{aligned} \tag{4A.12}$$

Then we compute

$$\log\frac{\det\mathcal{O}(e^\sigma g)}{\det\mathcal{O}(g)} = \int_0^1 dt\frac{d}{dt}\log\det\mathcal{O}(e^{t\sigma}g) \tag{4A.13}$$

$$\frac{d}{dt}\log\det\mathcal{O}_t = -\int_\varepsilon^\infty dy\, y^{-1}\,\mathrm{tr}\frac{d}{dt}e^{-y\mathcal{O}_t} = \int_\varepsilon^\infty dy\,\mathrm{tr}\frac{d\mathcal{O}_t}{dt}e^{-y\mathcal{O}_t} \tag{4A.14}$$

In our case,

$$\frac{d}{dt}\log\det{}'\Delta_t = \int_\varepsilon^\infty dy\,\mathrm{tr}'\frac{d}{dt}\Delta_t e^{-y\Delta_t} \tag{4A.15}$$

where the prime in $\det'$ and tr' means that the zero–mode is removed. Since $\Delta_t = e^{-t\sigma}\Delta$, we have

$$\begin{aligned} \frac{d}{dt}\log\det{}'\Delta_t &= -\int_\varepsilon^\infty dy\,\mathrm{tr}'\sigma\Delta_t e^{-y\Delta_t} = \int_\varepsilon^\infty dy\frac{d}{dy}\,\mathrm{tr}'\sigma e^{-y\Delta_t} = -\mathrm{tr}'\sigma e^{-\varepsilon\Delta_t} \\ &= -\,\mathrm{tr}\sigma e^{-\varepsilon\Delta_t} + \int d^2\xi\sqrt{g_t}\phi_0^{(t)}\sigma\phi_0^{(t)} \end{aligned} \tag{4A.16}$$

Now tr has no constraints and $\phi_0^{(t)} = (\int\sqrt{g_t})^{-1/2}$ is the normalized zero mode. Thus

$$\frac{d}{dt}\log\det{}'\Delta_t = -\mathrm{tr}\sigma e^{-\varepsilon\Delta_t} + \frac{d}{dt}\log\int\sqrt{g_t}d^2\xi \tag{4A.17}$$

and we obtain

$$\frac{d}{dt}\log\frac{\det'\Delta_t}{\int\sqrt{g_t}} = -\mathrm{tr}\sigma e^{-\varepsilon\Delta_t} \tag{4A.18}$$

We can now use the Seeley–DeWitt coefficients to evaluate the right–hand side:

$$\mathrm{tr}\sigma e^{-\varepsilon\Delta_t} = \frac{1}{4\pi\varepsilon}\int\sqrt{g_t}\sigma + \frac{1}{244\pi}\int\sqrt{g_t}R_t\sigma +)(\varepsilon) \tag{4A.19}$$

Therefore, if we want to compute the infinitesimal form of the conformal anomaly for the determinant of the laplacian

$$\delta \log \frac{\det' \Delta}{\int \sqrt{g}} = \frac{-1}{4\pi\varepsilon} \int \sqrt{g}\delta\sigma - \frac{1}{24\pi} \int \sqrt{g} R \delta\sigma + O(\varepsilon) \tag{4A.20}$$

The effective action for d scalar fields was written in lecture 4 as

$$\left(\frac{\det' \Delta}{\int \sqrt{g}} \right)^{-d/2} \quad (4.8)$$

Therefore,

$$\int \mathcal{D}_{(1+\delta\sigma)g} X e^{-S(X,(1+\delta\sigma)g)} = \int \mathcal{D}_g X e^{-S(X,g)} . \exp \left[\frac{d}{8\pi\varepsilon} \int \sqrt{g}\delta\sigma + \frac{d}{48\pi} \int \sqrt{g} R \delta\sigma + O(\varepsilon) \right] \tag{4A.21}$$

If we want to compute instead the integrated change:

$$\log \left(\frac{\frac{\det' \Delta_{e^\sigma g}}{\int e^\sigma \sqrt{g}}}{\frac{\det' \Delta_g}{\int \sqrt{g}}} \right)^{-d/2} = \frac{d}{8\pi\varepsilon} \int_0^1 dt \int d^2\xi \sqrt{g_t}\sigma + \frac{d}{48\pi} \int_0^1 dt \int d^2\xi \sqrt{g_t} R_t \sigma + O(\varepsilon) \tag{4A.22}$$

Using (4A.12), we obtain

$$\left(\frac{\det' \Delta_{e^\sigma g}}{\int e^\sigma \sqrt{g}} \right)^{-d/2} = \left(\frac{\det' \Delta_g}{\int \sqrt{g}} \right)^{-d/2} \exp \left[\frac{d}{8\pi\varepsilon} \int d^2\xi e^\sigma \sqrt{g} + \frac{d}{48\pi} \int d^2\xi \sqrt{g} \left(\frac{1}{2} g^{ab} \partial_a \sigma \partial_b \sigma + R\sigma \right) \right. \tag{4A.23}$$

which is the result quoted in section 4.1.

Similar results can be obtained for the (b,c) ghost system or for any set of anticommuting fields (b,c) with spins $(j, 1-j)$ respectively. See for example [21, 22, 43].

Lecture 5. Random Surfaces and the Large N Limit in Field Theory

5.1. Introduction and Examples

Previous lectures have dealt with the study in the continuum of conformal field theories and their coupling to two–dimensional gravity. We would like to find now some explicit ways of computing effectively the integration over metrics on a two–dimensional surface, which was reduced in lecture 4 to integrating over modular parameters and solving the Liouville theory. We want to write the sum over geometries modulo diffeomorphisms on a fixed topology in terms of discrete random triangulations (or mixed tesselations with irregular polygons) of a surface. Furthermore, we want to include some statistical variables on the sites or links of the lattice and study the critical behaviour of the combined matter plus gravity system.

The model of discrete strings and two–dimensional gravity presented were introduced in [59, 60, 61]. They were inspired by Regge calculus [62]. This work generated a good deal of activity [63, 64, 65, 66, 67, 68, 69] together with some numerical work to explore the non–perturbative properties of string theory. These studies also included in some cases the contribution of the terms in (1.50) describing the extrinsic curvature. Since we are not going to review the results of numerical simulations, a partial list of investigations and reviews which could help the reader find her/his way through the literature is [70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81].

Before the results of KPZ appeared [2], there were several lattice models out of which exact critical information was obtained which agree with the later work in [2]. These include the pure gravity case [59, 60], the Ising model on a random planar lattice [82, 83], the Q–state Potts model [84, 85], and an $O(n)$ σ–model [86]. The $D = -2$ string theory in the planar limit was solved in [65, 67, 87, 88]. After [2], exact results were also obtained for other values of c of conformal field theories in random planar lattices, for example [89, 90, 91, 5, 9, 92, 86, 93].

Obviously, we cannot give a thorough account of all these developments. To illustrate the main ideas and technique, we have selected as representative examples in the rest of these lectures the cases of $c = 0$ (pure gravity), the Kazakov critical points [5], $c = 1/2$ (Ising model), and $c = 1$ string theory [91]. Even for these examples we shall not give a full account of their properties. Further details can be found in the literature.

After this long digression on references, we start with the main theme of this lecture: how to simulate random triangulations using large N field theory methods.

We begin by exploring the rudiments of simplicial geometry. Let S be a triangulation of a surface with the topology of a sphere S^2. We define $V(S) = \{\text{vertices of } S\}$, $L(S) = \{\text{links in } S\}$, $F(S) = \{\text{faces, or triangles in } S\}$. If n_V (respectively n_L, n_F) is the number of vertices (respectively links, faces) in S, the Euler number (a topological invariant) is defined by $\mathcal{X}(S) = n_v - n_L + n_F = 2$, independent of S. If we change the topology of the underlying surface, then $\mathcal{X}(S)$ changes. A given simplicial complex S may have a non–trivial symmetry group $G(S)$: the group of permutations of lines and vertices leaving S unchanged. In continuum geometry, $G(S)$ is the isometry group of the manifold. The order of $G(S)$ is denoted by $|G(S)|$, a standard notation in finite group theory. We define an intrinsic metric on S by assigning the same length (=1) to each link of S, namely by considering all triangles to be equilateral. If N_i is the total number of triangles meeting at site i, one can define the analogue of $\sqrt{g}$ as $\sigma_i = N_i/3$. The total area of the surface S is

$$|S| = \frac{1}{3} \sum_{i \in V(S)} N_i = \text{number of triangles} \tag{5.1}$$

The intrinsic curvature is concentrated at the vertices, and it is equal to the deficit angle. At vertex i, the curvature is

$$R_i = \pi \frac{6 - N_i}{N_i} \tag{5.2}$$

For a regular triangular lattice, $N_i = 6$ for any i, and therefore $R_i = 0$. In this case, we can certainly draw the lattice on a flat plane. The deficit angle at vertex i is clearly

$$\sqrt{g}R_i = 2\pi\left(1 - \frac{N_i}{6}\right) \tag{5.3}$$

and the discrete form of the Gauss–Bonnet theorem becomes

$$\sum 2\pi\left(1 - \frac{N_i}{6}\right) = 4\pi \tag{5.4}$$

We can discretize the free string by including an action

$$H(S,X) = \sum_{\ell_{ij}\in L(S)} \left(\vec{X}_i - \vec{X}_j\right)^2 \tag{5.5}$$

where the sum runs over all the links of S. Hence, the discrete form of string theory is

$$Z = \sum_S e^{-\beta|S|}\frac{1}{|G(S)|}\int\prod_i \frac{d^dX_i}{(2\pi)^{d/2}} e^{-\sum_{\langle i,j\rangle\in S}(X_i-X_j)^2} \tag{5.6}$$

We sum over all the simplices S and in the last exponent the nearest neighbour pairs $\langle i,j\rangle$ depend on the simplex S. This can also be written in terms of the connectivity (or adjacency, or incidence) matrix $G_{ij}(S)$ of the simplex S:

$$G_{ij}(S) = \begin{cases} 1 & i,\, j \text{ nearest neighbours on } S \\ 0 & \text{otherwise} \end{cases} \tag{5.7}$$

Other examples are easy to construct. For instance, the Ising model on a random planar lattice is

$$\begin{aligned} Z(\Lambda,\beta) &= \sum_S e^{-\beta|S|}\frac{1}{|G(S)|}\sum_{\{\sigma_i\}} e^{\beta\sum_{\langle i,j\rangle}\sigma_i\sigma_j} \\ &= \sum_S e^{-\beta|S|}\frac{1}{|G(S)|}\sum_{\{\sigma_i\}}(\cosh\beta)^{|L(S)|}\prod_{\langle i,j\rangle\in S}(1+\tanh\beta\sigma_i\sigma_j) \end{aligned} \tag{5.8}$$

The sum over S can be defined by summing first over simplices with n sites and then summing over n, in principle with some other chemical potential for the number of sites. In the thermodynamic limit, we want $n \to \infty$. Later, we will also remove the constraint on the fixed topology of the simplices and this will bring us into string theory. For more details and references on this subject, see [89, 94, 95, 96].

Another example which is important in the study of polymers is the $O(n)$ model. In the limit $n \to 0$, one obtains the statistical mechanics of self–avoiding random walks. In this model, one assigns spins $\vec{s}_i$ to each site, subject to the constraint $s_i^2 = n$. On a regular honeycomb lattice (see figure 5.1), the partition function is

$$Z_{O(n)} = \int \prod_i d\vec{s}_i \prod_{\langle i,j \rangle} (1 + k\vec{s}_i \cdot \vec{s}_j) \tag{5.9}$$

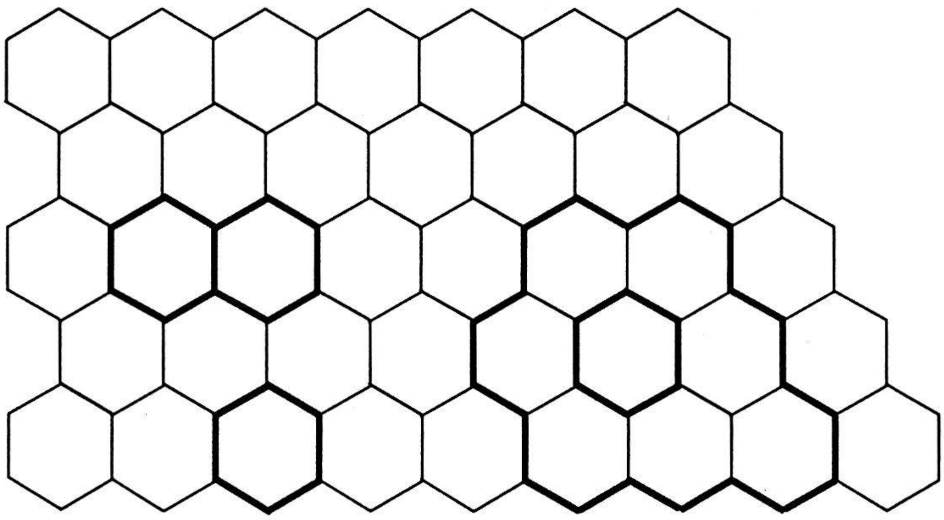


Figure 5.1. A possible configuration contributing to (5.9).

The spins $\vec{s}_i$ have each n components, and each link appears only once in (5.9). It is clear from (5.9) that only closed loop configurations can contribute, otherwise the ds_i integral over the boundary spins would vanish. Furthermore, if three links meet at a vertex, again the $O(n)$ integration with respect to the vertex spin will vanish. Hence, the closed loops cannot intersect and (5.9) is an expansion in closed non–intersecting loops.:

$$Z = \sum_{\text{loops}} k^{N_{\text{links}}} n^{N_{\text{loops}}} \tag{5.10}$$

One can consider correlation functions in this case of fields $\Phi_p(x)$ representing a source of p infinite non–intersecting lines at a point x. If we consider a random lattice with co-ordination number 3 (only three lines at each vertex), for any graph S we can define the $O(n)$ partition function as in (5.9), and the total partition function becomes

$$Z_n(\beta, k) = \sum_S \frac{1}{|G(S)|} e^{-\beta|S|} Z_{O(n)}(S) \tag{5.11}$$

A configuration contributing to (5.11) appears in figure 5.2. The parameter β is the cosmological constant, the variable conjugate to the area of the surface. The system has two fugacities: k, related to the inverse temperature of the model, and $e^{-\beta}$, the lattice fugacity.

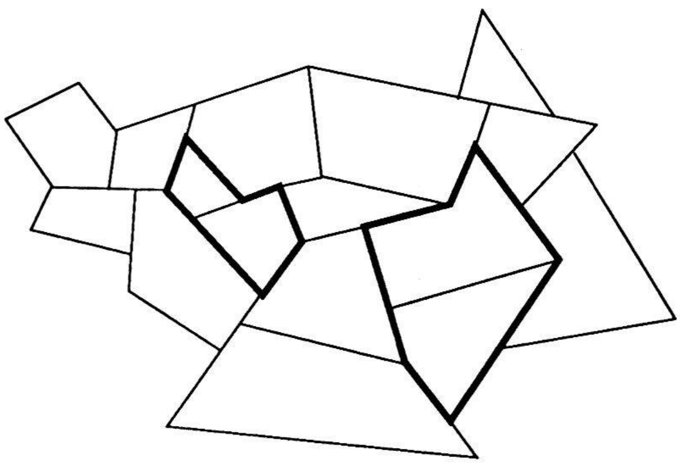


Figure 5.2. A configuration contributing to (5.11).

One would like to compute the partition functions and critical exponents of the models considered so far and many others as well. Anyone familiar with the solution to the Ising model or other solvable models on a regular lattice may be ready at this point to give up and abandon a problem where in principle one would have to solve first the Ising model (say) for an arbitrary simplex S, and then sum over simplices. Since, generically, S will have no symmetries, we do not worry now about computing $|G(S)|$. Actually, the apparent curse of having to sum over simplices is a blessing in disguise. There are very powerful techniques in quantum field theory to enumerate systematically graphs of arbitrary topology and complexity. they are based on the large N limit of matrix field theories. Before returning to models such as (5.6), (5.8) or (5.11), we present the tools in large N technology that will be needed. These methods are so powerful that it is possible to solve the Ising model on a random planar lattice in the presence of an arbitrary constant magnetic field. We will later analyze in some detail the correlation functions and the type of exponents used in these theories.

5.2. Large N Expansions. Orthogonal Polynomials

The use of large N expansions in field theory is a rather large subject. It was started by the seminal paper of G. 't Hooft [97]. The application to 0+0 and 0+1 dimensional models was first carried out in [98]. A good reference on the the theory and applications of random surfaces is [99]. The use of orthogonal polynomials to solve this problem appears in [100, 101, 102]. We will follow mainly the very lucid presentation in [100]. The case of two– and more– matrix models in the large N limit appears in [101, 103, 104, 105, 106]. Many details and references on orthogonal polynomials can be found in [107].

To make the arguments as transparent as possible, let us study the simplest possible case: a matrix field theory in $0+0$ dimensions. The "field" is just an $N \times N$ matrix M_{ij} $(i, j = 1, \cdots N)$. We take M to be hermitean, although the arguments can be generalized easily to M complex, symmetric or antisymmetric. The partition function is just an ordinary integral

$$
\begin{aligned}
Z &= \int d^{N^2}M \ \exp[-\beta\, \mathrm{tr} V(M)] \\
V(M) &= \frac{1}{2}M^2 + \sum_{k\geq 3} g_k M^k
\end{aligned}
\tag{5.12}
$$

We can compute Z using Feynman rules. For this, it is useful to introduce a double line notation corresponding to the two indices of M. The lines are oriented because M is hermitean and from a group–theoretical point of view, M can be thought of as a quark–

antiquark pair. The Feynman rules will look as in figure 5.3.

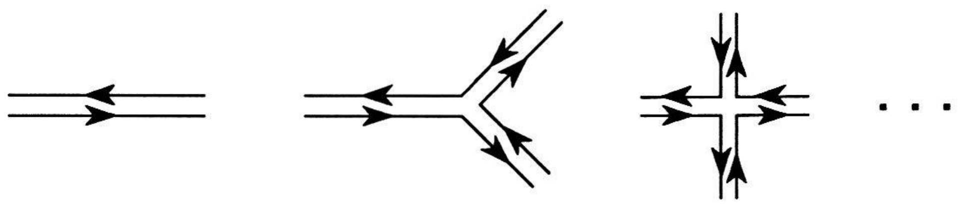


Figure 5.3. Feynman rules for (5.12).

The advantage of the double line notation is that we do not have to write explicitly all the indices on the matrices. To follow a particular index, all we have to do is to follow the arrows. In perturbation theory we expand the interaction term in (5.12) and compute the gaussian integrals:

$$Z = \int d^{N^2}M\ e^{-\frac{1}{2}\beta \mathrm{tr} M^2} \prod_{p\geq 3} \left(\sum_{n_p=0}^{\infty} \frac{1}{n_p!} g_p^{n_p} (\mathrm{tr}\, M^p)^{n_p} \right) \tag{5.13}$$

The propagator is

$$\left\langle M_i{}^j M_k{}^\ell \right\rangle = \beta^{-1} \delta_i^\ell \delta_k^j \tag{5.14}$$

A generic graph for Z will have V_p vertices of type p (g_p tr M^p). Since there are no external lines, all lines will close to form a loop and for every loop the traces appearing in the vertices will produce a factor of N (some graphs are shown in figure 5.4).

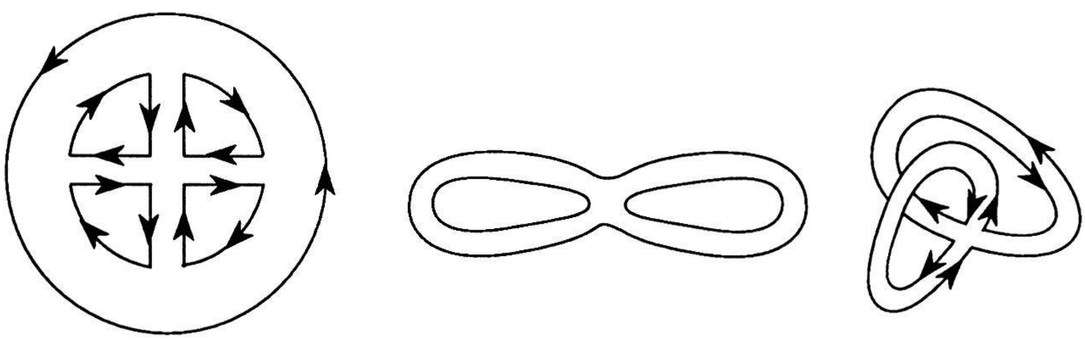


Figure 5.4. Some graphs appearing in (5.13).

We want to show now that in the large N limit, if we expand Z in inverse powers of N we obtain a topological expansion. A graph with V_3 vertices of type 3, V_4 vertices of type 4,... will give a contribution proportional to

$$\left(g_3^{V_3} g_4^{V_4} \cdots\right) N^I \beta^{-P+V} \tag{5.15}$$

where I is the number of "index" loops, P is the number of propagators and $V = \sum V_p$ is the total number of vertices. Now think of the diagram geometrically, as an abstract simplex where the vertices are the vertices of the graph, the edges are the propagators and the faces are the index loops. The Euler characteristic of this particular simplex is

$$\mathcal{X} = 2 - 2h = V - P + I \tag{5.16}$$

where h is the number of handles of the surface. For every closed graph, the total number of lines emanating from all the vertices must be equal to twice the number of propagators:

$$2P = \sum_p V_p \tag{5.17}$$

Then (5.15) becomes

$$\left(g_3^{V_3} g_4^{V_4} \cdots\right) N^{2-2h+P-V} \beta^{\sum_p V_p(1-\frac{p}{2})} = \prod_{p\geq 3} \left(g_p N^{\frac{p}{2}-1}\right)^{V_p} N^{2-2h} \beta^{\sum_p V_p(1-\frac{p}{2})} \tag{5.18}$$

If we define $g_p = \overline{g}_p N^{1-p/2}$ *i.e.*, if the potential $V(M)$ takes the form

$$V(M) = \sum_{p>2} \frac{\overline{g}_p}{N^{\frac{p}{2}-1}} \operatorname{tr} M^p \tag{5.19}$$

we can write the partition function as

$$Z = \sum_h N^{2-2h} \sum_{S_h} \beta^{\sum_p V_p(1-\frac{p}{2})} \frac{1}{|G(S_h)|} \prod \overline{g}_p^{\,-V_p} \tag{5.20}$$

Equivalently, since we have not fixed the behaviour of β as $N \to \infty$, we can write

$$Z = \sum_h N^{2-2h} \sum_{S_h} \frac{1}{|G(S_k)|} \left(\frac{N}{\beta}\right)^{\sum_p V_p(\frac{p}{2}-1)} \prod g_p^{-V_p} \tag{5.20}$$

Several things are worth pointing out about this formula:

1) For a fixed order in N (fixed h) we sum over all graphs S_h in the theory. We know from standard field theory that the factorial factors in (5.13) do not cancel completely if the graph S_h has some symmetry. What remains in the denominator is the order of the symmetry group of the graph. This factor, needed in (5.6) and subsequent formulae is an outcome of the large N computation.

2) The large N expansion is an expansion in genera. Thus if we want to keep only spherical (planar) topology, it suffices to keep the leading term $h = 0$ in (5.20).

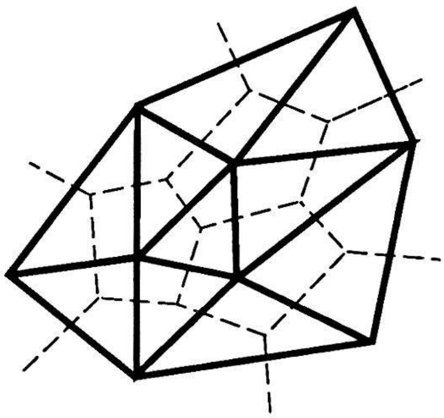


Figure 5.5. The solid line is a portion of a triangulation. The broken lines represent the dual simplex.

3) There remains to understand the exponent in β in (5.20) and why this expansion is counting metrics modulo diffeomorphisms. This is clarified by noticing that the large N expansion gives us the dual simplices to those considered in the previous section. Take $g_4 = g_5 = \ldots = 0$. Now we only have ϕ^3 vertices. If we consider an arbitrary triangulation of a surface (as in figure 5.5), its dual graph is a ϕ^3 diagram. In this case, the exponent of β in (5.20) is $-\frac{1}{2}\sum V_p$. However, the number of vertices in the ϕ^3 graph is equal to the number of triangles in the dual triangulation, and therefore it represents the area of the surface if we let the length of every side of the triangles to be equal to 1. Thus the factor $\sum_p V_p(p/2-1) = |S|$ is the area of the triangulation. We can also define a distance between triangulations. If $G_{ij}(S)$ and $G_{ij}(S')$ are the adjacency matrices for the graphs

S, S' we define their distance as $\frac{1}{2}\sum_{ij}|G_{ij}(S)-G_{ij}(S')|^2$. Configurations differing by the flip of a single link as in figure 5.6 have $d(S,S')=1$. By flipping links, one can move in the space of triangulations. This procedure has been implemented in Monte Carlo algorithms to simulate random surfaces numerically.

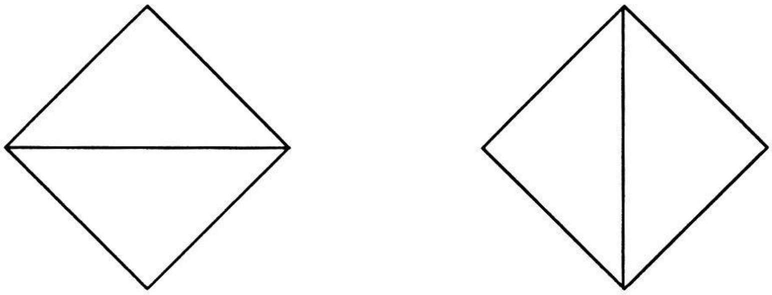


Figure 5.6. Two configurations differing by the flip of a single link.

Finally, we should mention that the factor $1/|G(S)|$ is expected also in the space of metrics modulo diffeomorphisms. This is due to the existence of orbifold points in the moduli space of Riemann surfaces. In other words, some diffeomorphisms may have fixed points for some metrics. If we have some manifold M and a group G acting on it, the quotient space M/G, *i.e.*, the space of orbits of M in G is again a a manifold, as long as G acts freely on M. If G has fixed points, the quotient space will have singularities at these points. Think for instance of M as the two–dimensional plane and G as the group with two elements $\{1,P\}$ where $P(\vec{x})=-\vec{x}$. The quotient M/G is a cone with a 30° opening angle. The origin of the plane is a fixed point under G and this generates a conical singularity when modding with respect to G.

There are also multi-matrix models, with a similar interpretation. In (5.12), there is a single matrix field M. Take instead M_α $\alpha=1,2,\ldots,n$. later, we will need also models of the form

$$Z=\int\prod_{\alpha=1}^{n}d^{N^2}M_\alpha\exp\left[-\sum_\alpha V_\alpha(M)-\sum_{\alpha=1}^{p-1}c_\alpha\,\mathrm{tr}\,M_\alpha M_{\alpha+1}\right]\tag{5.21}$$

The Ising model on a random lattice, for example, is given by (5.21) with $p = 2$.

The most efficient method to evaluate (5.12) and (5.21) is in terms of orthogonal polynomials. The theory has $U(N)$ invariance, and the observables are simply the operators $\mathrm{tr}M^p$. Therefore, we would like to transform the integral in (5.12) over M into an integral over the eigenvalues of M. Since M is hermitean, it can be diagonalized by unitary transformations. We can parametrize M as

$$M = U^{-1}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} U = U^{-1}DU \tag{5.22}$$

An infinitesimal change in M is

$$UdMU^{-1} = dD + [D, duU^{-1}] \tag{5.23}$$

or, in components,

$$\begin{aligned} \left(UdMU^{-1}\right)_{ij} &= d\lambda_i\delta_{ij} + (\lambda_i - \lambda_j)\omega_{ij}(U) \\ \omega_{ij}(U) &= \left(dUU^{-1}\right)_{ij} \end{aligned} \tag{5.24}$$

The differential $\omega_{ij}(U)$ involves only angular variables. If E_{ij} denotes the matrix unit whose single non–vanishing entry is a 1 in position (i,j), we can write (5.24) as

$$d\lambda_i E_{ii} + (\lambda_i - \lambda_j)\omega_{ij}E_{ij} \tag{5.25}$$

Then the volume element $d^{N^2}M$ will take the form [99]

$$d^{N^2}M = \prod_{i=1}^{N} d\lambda_i \mu(U) \prod_{i<j}(\lambda_i - \lambda_j)^2 \tag{5.26}$$

The measure over the angular variables $\mu(U)$ will be ignored in the computation of Z and in the computation of $U(N)$–invariant expectation values. We obtain

$$\begin{aligned} Z &= \int \prod d\lambda_i \Delta^2(\lambda_i) e^{-\beta\sum_i V(\lambda_i)} \\ V(\lambda) &= \frac{1}{2}\lambda^2 + \sum_{p\geq 3}\overline{g}_p\lambda^p \\ \Delta(\lambda_i) &= \prod_{i<j}(\lambda_i - \lambda_j)^2 \end{aligned} \tag{5.27}$$

The free energy or ground-state energy $E_N(g)$ is defined by

$$Z = e^{-N^2 E_N(g)} \tag{5.28}$$

where we have factored out a power of N^2 in the exponent to normalize the planar (genus zero) diagrams:

$$\exp -N^2 E_n(g) = \lim_{N\to\infty} \int \prod d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp -\beta \sum V(\lambda_i) \tag{5.29}$$

In the large N limit, we can use steepest descent to evaluate the integral. Equation (5.29) should be thought of as the statistical mechanics of N charged particles on a line in the presence of a potential:

$$\exp -N^2 E_n(g) = \int \prod d\lambda_i \exp 2 \sum_{i\neq j} \log |\lambda_i - \lambda_j| - \beta \sum V(\lambda_I) \tag{5.30}$$

The steepest descent equations are

$$\beta V'(\lambda_i) = 2 \sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} \tag{5.31}$$

This problem can be solved in the large N limit by going to a continuum approximation [98]. We will use a different method to find Z. This is the method of orthogonal polynomials [100, 101].

The interpretation of (5.30) as a Coulomb gas is due to Dyson. We have a Coulomb gas at a temperature β^{-1} in an external potential $V(\lambda_i)$. At very low temperature $\beta^{-1} \sim 1/N$, $N \to \infty$, the charges are very weak and the attractive potential balances the Coulomb repulsion. In this situation, the Dyson gas freezes into a solid whose equilibrium positions are given by the minimum of the energy $E(g)$ and satisfy (5.31).

To evaluate (5.27), we introduce orthogonal polynomials with respect to the measure

$$d\mu(\lambda) = d\lambda e^{-\beta V(\lambda)} \tag{5.32}$$

By Gramm–Schmidt orthogonalization, we start with the basic monomials 1, λ, λ^2, $\cdots$ and construct monic polynomials (a polynomial is monic if the coefficient of the leading term is 1):

$$P_n(\lambda) = \lambda^n + a_{n,n-1}\lambda^{n-1} + a_{n,n-2}\lambda^{n-2} + \cdots \tag{5.33}$$

Then

$$\int d\mu(\lambda)P_n(\lambda)P_m(\lambda) = h_n\delta_{n,m} \tag{5.34}$$

It is clear then that $P_n(\lambda)$ is orthogonal to λ^m $(m < n)$. These polynomials satisfy a two–step recursion relation

$$\lambda P_n = P_{n+1} + S_nP_n + R_nP_{n-1} \tag{5.35}$$

To prove this, we write

$$\lambda P_n = P_{n+1} + S_nP_n + R_nP_{n_1} + \sum_{\alpha=0}^{n-2} A_\alpha P_\alpha \tag{5.36}$$

Multiplying by P_α $(\alpha < n-2)$ and integrating with $d\mu(\lambda)$, the integrals of $P_{n+1}P_\alpha$, P_nP_α, $P_{n-1}P_\alpha$ vanish, and $\lambda P_nP_\alpha = P_n(\lambda^{\alpha+1} + \cdots)$. Since $\alpha \leq n-2$ and P_n is orthogonal to λ^m $m \leq n-1$, we obtain $A_\alpha = 0$, $\alpha = 0, 1, \ldots, n-2$. An explicit representation of $P_n(\lambda)$ once R_n and S_n are known is

$$P_{n+1}(\lambda) = \begin{vmatrix} \lambda - S_0 & 1 & 0 & \cdots & & \\ R_1 & \lambda - S_1 & 1 & 0 & \cdots & \\ 0 & R_2 & \lambda - S_2 & 1 & 0 & \cdots \\ & & & \ddots & & \\ & & & R_{n-1} & \lambda - S_{n-1} & 1 \\ & & & & R_n & \lambda - S_n \end{vmatrix} \tag{5.37}$$

Expanding in minors with respect to the last row,

$$\begin{aligned} P_{n+1} =& (\lambda - S_n)\begin{vmatrix} \lambda - S_0 & 1 & & \\ R_1 & \lambda - S_1 & 1 & \\ & & \ddots & \\ & & R_{n-1} & \lambda - S_{n-1} \end{vmatrix} \\ & - R_n\begin{vmatrix} \lambda - S_0 & 1 & & \\ R_1 & \lambda - S_1 & 1 & \\ & & \ddots & \\ & & R_{n-2} & \lambda - S_{n-2} \end{vmatrix} \\ =& (\lambda - S_n)P_n - R_nP_{n-1} \end{aligned} \tag{5.38}$$

Next, notice that $\Delta(\lambda_i)$ is a Van der Monde determinant

$$\prod_{i<j}(\lambda_i - \lambda_j) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_N \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{N-1} & \lambda_2^{N-1} & \cdots & \lambda_N^{N-1} \end{vmatrix} \tag{5.39}$$

From (5.33), we can write the above expression (5.39) as

$$\Delta(\lambda) = \det_{i,j=1,\cdots,N} P_{i-1}(\lambda_j) \tag{5.40}$$

Define the "one–body" wave functions

$$\psi_i(\lambda) = P_i(\lambda)e^{-\beta V(\lambda)/2} \tag{5.41}$$

Then we can construct the N–body fermionic wave–function

$$\Psi_N(\lambda_i) = \det \psi_{i-1}(\lambda_j) \tag{5.42}$$

corresponding to the ground state of N fermions. Then Z is the norm of Ψ_N:

$$Z = \langle \Psi_N | \Psi_N \rangle = \int \prod d\lambda_i \psi(\lambda_i)\psi(\lambda_i) \tag{5.43}$$

In this form, Z is the norm of a Slater determinant:

$$Z = N! \prod_{i=0}^{N-1} h_i \tag{5.44}$$

Using

$$h_n = \int d\mu(\lambda) P_{n-1}\lambda P_n = \int d\mu(\lambda) P_{n-1}(P_{n+1} + S_n P_n + R_n P_{n-1}) = R_n h_{n-1} \tag{5.45}$$

we obtain

$$\begin{aligned} h_n &= R_n R_{n-1} \cdots R_2 R_1 h_0 \\ Z &= N! h_0^N \prod_{i=1}^{N-1} R_i^{N-1} \end{aligned} \tag{5.46}$$

Finally, we obtain two basic identities:

$$0 = \int d\mu(\lambda) P_n(\lambda) \frac{d}{d\lambda} P_n(\lambda) = \frac{1}{2} \int d\mu(\lambda) \frac{d}{d\lambda} P_n^2(\lambda) = \beta \int d\mu(\lambda) V'(\lambda) P_n^2(\lambda) \tag{5.47}$$

and

$$\int d\mu(\lambda) P_{n-1} \frac{d}{d\lambda} P_n = n h_{n-1} \tag{5.48}$$

Integrating by parts and using the orthogonality of P_n and $dP_{n-1}/d\lambda$, we obtain

$$\int d\mu(\lambda) P_{n-1} V'(\lambda) P_n = \frac{n}{\beta} h_{n-1} \tag{5.49}$$

The two equations

$$\begin{aligned} \int d\mu(\lambda) V'(\lambda) P_n^2 &= 0 \\ \int d\mu(\lambda) P_{n-1} V'(\lambda) P_n &= \frac{n}{\beta} h_{n-1} \end{aligned} \tag{5.50}$$

are known as the string equations [6, 7, 8]. Combining (5.35) and (5.50) we can obtain a set of equations for R_n and S_n known as the staircase equations [100]. Let us consider some examples first.

In the simplest non–trival case, $V(\lambda) = \lambda^2/2$ and P_n are the Hermite polynomials:

$$\int d\mu(\lambda) P_{n-1} V'(\lambda) P_n = \int d\mu(\lambda) P_{n-1} \lambda P_n = R_n h_{n-1}$$

and hence for the purely Gaussian theory

$$R_n = \frac{n}{\beta} \tag{5.51}$$

Also, $S_n = 0$, which is generic for the case of even potentials $V(\lambda) = V(-\lambda)$.

Consider next the potential $V(\phi) = g_2\phi^2 + g_4\phi^4$, which is also even and thus $S_n = 0$. We need

$$\int d\mu(\lambda) \frac{1}{h_{n-1}} P_{n-1} V'(\lambda) P_n = \int d\mu(\lambda) \frac{1}{h_{n-1}} P_{n-1} (2g_2\lambda + 4g_4\lambda^3) P_n \tag{5.52}$$

The first term is easy to compute and it yields $2g_2 R_n$. For the second one, we have to use (5.35) three times; we always have to start with P_n and end with P_{n-1}, so as to make (5.52) different from zero. The answer is

$$4g_4 R_n (R_{n+1} + R_n + R_{n-1}) \tag{5.53}$$

and hence

$$2g_2R_n + 4g_4R_n(R_{n+1} + R_n + R_{n-1}) = \frac{n}{\beta} \tag{5.54}$$

The general pattern can be understood if we define an operator $\hat{n}$ such that $\hat{n}P_n = nP_n$ and shift operators. Similarly, we also define $R(\hat{n})$ and $S(\hat{n})$. If $U_\pm |n\rangle = |n \pm 1\rangle$, then (5.35) can be written as

$$\lambda = (U_+ + U_- R(\hat{n})) \tag{5.55}$$

Notice that the operator is defined acting on the monic polynomials P_n. We can define the operator also with respect to an orthonormal basis

$$\mathcal{P}_n = h_n^{-1/2} P_n \tag{5.56}$$

In this case,

$$\begin{aligned} \hat{\lambda}\mathcal{P}_n &= h_n^{-1/2}(P_{n+1} + S_nP_n + R_nP_{n-1}) \\ &= \sqrt{\frac{h_{n+1}}{h_n}}\mathcal{P}_{n+1} + S_n\mathcal{P}_n + R_n\sqrt{\frac{h_{n-1}}{h_n}}\mathcal{P}_{n-1} \\ &= \sqrt{R_{n+1}}\mathcal{P}_{n+1} + S_n\mathcal{P}_n + \sqrt{R_n}\mathcal{P}_{n-1} \end{aligned} \tag{5.57}$$

Hence, in this basis

$$\hat{\lambda} = \sqrt{R(\hat{n})}U_+ + S(\hat{n}) + U_-\sqrt{R(\hat{n})} \tag{5.58}$$

and the string equations become

$$\begin{aligned} \int d\mu(\lambda)\mathcal{P}_{n-1}V'(\lambda)\mathcal{P}_n &= \frac{n}{\beta\sqrt{R_n}} \\ \int d\mu(\lambda)V'(\lambda)\mathcal{P}_n^2(\lambda) &= 0 \end{aligned} \tag{5.59}$$

Define the states $|n)$ and $|n\rangle$ according to

$$\begin{gathered} (\lambda|n) = P_n(\lambda) \qquad\qquad \langle\lambda|n\rangle = \mathcal{P}_n(\lambda) \\ |n\rangle = h_n^{-1/2}|n) \end{gathered} \tag{5.60}$$

Therefore, we can write (5.50) as

$$\begin{aligned} \langle n|\, V'(\hat{\lambda})\, |n\rangle &= 0 \\ \langle n-1|\, V'(\hat{\lambda})\, |n\rangle &= \frac{n}{\beta\sqrt{R_n}} \end{aligned} \tag{5.61}$$

Graphically, these expressions can be computed in terms of a "staircase" diagram. Think of a staircase where each step is labelled in order 1,2,3 $\ldots, n, n+1, \ldots$ To compute the contribution $\langle n-1|\,\hat{\lambda}^p\,|n\rangle$, we start from step n and apply $\hat{\lambda}$ p times in such a way as to end up at the $(n-1)$–st step. The rules of this ladder game are

$$n \nearrow^{n+1} = \sqrt{R_{n+1}} \qquad {}^{n}\searrow_{n+1} = \sqrt{R_n} \qquad n \longrightarrow n = S_n \tag{5.62}$$

For even potentials, the last term is not there.

Acting instead on the basis $|n\rangle$, we have equivalently

$$n \nearrow^{n+1} = 1 \qquad {}^{n}\searrow_{n+1} = R_n \qquad n \longrightarrow n = S_n \tag{5.63}$$

Consider for example the graphical computation of $\langle n-1|\,\hat{\lambda}^3\,|n\rangle$ with even potentials. The contributing diagrams (paths) are shown in figure 5.7.

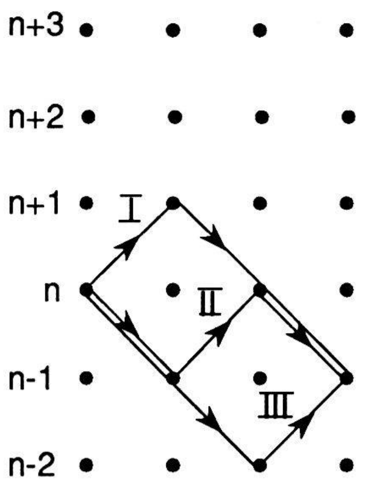


Figure 5.7. Staircase paths contributing to $\langle n-1|\,\hat{\lambda}^3\,|n\rangle$.

Using the rules (5.62), we find the contributions

$$\begin{aligned} \mathrm{I}: &\quad \sqrt{R_{n+1}}\sqrt{R_{n+1}}\sqrt{R_n} \\ \mathrm{II}: &\quad \sqrt{R_n}\sqrt{R_n}\sqrt{R_n} \\ \mathrm{III}: &\quad \sqrt{R_n}\sqrt{R_{n-1}}\sqrt{R_n} \end{aligned} \tag{5.64}$$

Inserting these expressions into (5.61) leads immediately to (5.54).

Our third and last example is based on the cubic potential $V(\lambda) = g_2\lambda^2 + g_3\lambda^3$; $V'(\lambda) = 2g_2\lambda + 3g_3\lambda^2$. Now we have to use (5.62) with $S_n \neq 0$. As before, the $2g_2\lambda$ term contributes $2g_2\sqrt{R_n}$. The staircase graphs contributing to $\hat{\lambda}^2$ are shown in figure 5.8.

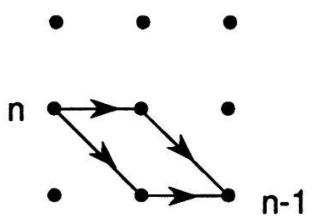


Figure 5.8. Paths contributing to $\langle n-1|\,\hat{\lambda}^2\,|n\rangle$.

The two of them together yield $6\overline{g}_3 S_n\sqrt{R_n}$, so the total is

$$(2\overline{g}_2 + 6\overline{g}_3 S_n)\sqrt{R_n} = \frac{n}{\beta\sqrt{R_n}} \tag{5.65}$$

We also need to compute $\langle n|\,V'(\lambda)\,|n\rangle$. The staircase paths now start and end at level n:

$$\langle n|\,V'(\lambda)\,|n\rangle = 2\overline{g}_2 S_n + 3\overline{g}_3(S_n^2 + R_{n+1} + R_n) = 0 \tag{5.66}$$

In general, if we take $V(\lambda) = V(-\lambda)$, the form of the string equation is

$$\frac{n}{\beta} = 2\overline{g}_2 R_n + \sum_{p>1} 2p\overline{g}_{2p} \sum_{\substack{\text{staircase}\\ \text{paths}}} R_{s_1}\cdots R_{s_p} \tag{5.67}$$

The staircase paths are obtained from $\langle n-1|\,\hat{\lambda}^{2p-1}\,|n\rangle$ or from $h_{n-1}^{-1}(n-1|\hat{\lambda}^{2p-1}|n)$. Using (5.63):

$$\hat{\lambda}^{2p-1} = (U_+ + U_- R(\hat{n}))^{2p-1} \tag{5.68}$$

In order that the expectation value $\langle n-1|\,\hat{\lambda}^{2p-1}\,|n\rangle$ be non-vanishing, p steps must be taken downwards and $(p-1)$ upwards. The number of contributing terms is $\binom{2p-1}{p} =$

$(2p-1)!/p!(p-1)!$. If we can solve the staircase equations for R_n, then the free energy follows from

$$\begin{aligned} E_N &= \frac{-1}{N^2}\log\frac{Z_N(g)}{Z_N(0)} \\ &= \frac{-1}{N}\sum_{k=1}^{N}\left(1-\frac{k}{N}\right)\log\frac{R_k(g)}{R_k(0)} - \frac{1}{N}\log\frac{h_0(g)}{h_0(0)} \end{aligned} \tag{5.69}$$

where the argument $g=0$ means the restriction to the gaussian model.

Before concluding this lecture, we quote a result due to Mehta which allows us to work also with multi-matrix models of the form (5.21). (The proof can be found in [103] and [104], although the result was also found in a different form in [101].) The result is

$$\int d^{N^2}B e^{-V(B)+2c\,\mathrm{tr}\,AB} = \text{const.}\times\int d\lambda_1\cdots d\lambda_N\frac{\Delta(\lambda)}{\Delta(\mu)}\exp\left(-V(\lambda_i)+2c\sum_i\lambda_i\mu_i\right) \tag{5.70}$$

where the λ_i's (respectively the μ_i's) are the eigenvalues of B (respectively of A), and

$$\Delta(\lambda)=\prod_{i<j}(\lambda_i-\lambda_j) \qquad \Delta(\mu)=\prod_{i<j}(\mu_i-\mu_j) \tag{5.71}$$

Now we have the tools to analyze statistical models on random triangulated surfaces.

Lecture 6. Pure Gravity. Kazakov Critical Points. A First Look at the Double Scaling Limit

After all the machinery built in the last lecture, we would like to obtain a number of interesting results concerning pure gravity in two dimensions and also verify some of the formulae of Knizhnik, Polyakov and Zamolodchikov (KPZ). The arguments in this lecture will be rather elementary, but conceptually they are important for understanding how the continuum two-dimensional gravity theory is obtained from the sum over random triangulations. In the following lecture, we will explain in more detail the critical exponents and ambiguities in the construction of the coupling of matter to 2D-gravity.

6.1. Remarks on Asymptotic Estimates

In lecture 4 we introduced the string susceptibility γ_{st} in terms of the "microcanonical" partition function with fixed area A. This coefficient also determines the divergence of the partition function $Z(\beta)$ as the cosmological constant approaches a critical value β_c. For later convenience and also to agree with standard notation, we define the bare cosmological constant Λ_B as

$$\beta = e^{\Lambda_B} \tag{6.1}$$

Since we have set the length of the elementary links to one, there are no units in the exponent in (6.1). The full partition function can be written as a sum:

$$Z_h(\Lambda_B) = \sum_{N=0}^{\infty} Z_h(N) e^{-\Lambda_B N} \tag{6.2}$$

where $Z_h(N)$ is the partition function for fixed area A, and h is the genus of the surface considered. In general, (6.2) will be convergent for some values $\Lambda_B \neq \Lambda_c$, and it will diverge at the critical cosmological constant Λ_c. The asymptotic behaviour of (6.2) is determined by the behaviour of $Z_h(N)$ as $N \to \infty$:

$$Z_h(N) \sim_{N\to\infty} A^N N^{\gamma - 3} \tag{6.3}$$

Then $A = e^{\Lambda_c}$ and the behaviour of Z_h as $\Lambda_B \to \Lambda_c$ is

$$\begin{aligned} Z_h(\Lambda_B) &\simeq \sum_N A^N N^{\gamma-3} e^{-\Lambda_B N} = \sum_N N^{\gamma-3} e^{-(\Lambda_B-\Lambda_c)N} \\ &\sim \int^\infty dx\, x^{\gamma-3} e^{-(\Lambda_B-\Lambda_c)x} \sim (\Lambda_B - \Lambda_c)^{2-\gamma_{st}} \end{aligned} \tag{6.4}$$

The string susceptibility is defined as

$$\mathcal{X} = \frac{\partial^2 Z}{\partial \Lambda_B^2} \sim (\Lambda_B - \Lambda_c)^{-\gamma_{st}} \tag{6.5}$$

The average area of the string world–sheet behaves as

$$\langle \text{Area} \rangle = -\frac{\partial}{\partial \Lambda_B} Z_h \sim (\Lambda_B - \Lambda_c)^{1-\gamma_{st}} \tag{6.6}$$

If instead we look at the two–point function of some operator Φ, in the limit of large area N:

$$e^{-\Lambda_B N} Z_{\Phi\Phi}(N) \sim N^{2-2h_\Phi+\gamma_{st}-3} e^{-(\Lambda_B-\Lambda_c)N} \tag{6.7}$$

and the behaviour of $Z_{\Phi\Phi}(\Lambda_B)$ near Λ_c is

$$Z_{\Phi\Phi}(\beta) \sim (\Lambda_B - \Lambda_c)^{2h_\Phi - \gamma_{st}} \tag{6.8}$$

where h_Φ is the dressed dimension of the operator Φ according to the KPZ definition. For pure gravity ($D = 0$) the formulae of lecture 4 imply

$$\gamma_{st} = \frac{D - 1 - \sqrt{(D-1)(D-25)}}{12}\bigg|_{D=0} = -\frac{1}{2} \tag{6.9}$$

Hence, the string susceptibility is directly related to the divergence of the partition function. We want to derive (6.9) starting with one–matrix models in the large N limit.

6.2. Φ^4 Planar Theory

To derive (6.8), we begin with the staircase equations for a quartic potential:

$$\frac{n}{\beta} = 2\overline{g}_2 R_n + 4\overline{g}_4 R_n (R_{n+1} + R_n + R_{n-1}) \tag{6.10}$$

Dividing by N as $N \to \infty$, using $\overline{g}_2 = g_2$, $\overline{g}_4 = g_4/N$ and redefining $R_n \to R_n/N$, with $x = n/N$ and $\varepsilon = 1/N$, we find

$$\frac{x}{\beta} = 2g_2R(x) + 4g_4R(x)[R(x+\varepsilon) + R(x) + R(x-\varepsilon)] \tag{6.11}$$

In the large N limit, we can neglect ε and we end up with an algebraic equation for the coefficient $R(x)$:

$$e^{-\Lambda_B}x = 2g_2R(x) + 12g_4R^2(x) \tag{6.12}$$

whose solution is

$$R(x) = \frac{-2g_2 \pm \sqrt{4g_2^2 + 48g_4e^{-\Lambda_B}x}}{24g_4} \tag{6.13}$$

We can normalize $g_2 = 1$, and we see that the critical behaviour is found for $g_4 = -1/12$, $\Lambda_c = 0$. The scaling region appears for $x \sim 1$:

$$R = 1 - \sqrt{1 - e^{-\Lambda_B}x} \tag{6.14}$$

The sign choice in (6.14) is due to the fact that R is always positive. This solution follows by requiring the polynomial in (6.12) to have a double zero at $x = 1$ for $\Lambda_B \sim 0$. Substituting (6.14) into (5.69) we obtain the free energy [59, 60]:

$$E = \frac{1}{(5/2)(3/2)}(\Lambda_B - \Lambda_c)^{2+\frac{1}{2}} + \text{analytic} \tag{6.15}$$

and we disregard the analytic terms, which can be accounted for by the appropriate counterterms. We have also included Λ_c in the final answer (6.15), from which we can read immediately $\gamma_{st}(D = 0) = -1/2$ in agreement with the KPZ formula.

6.3. Kazakov Multi-critical Points

We can consider now an arbitrary potential $V(\phi)$ and study other possible critical behaviours as we tune the couplings [5]. In the naive large N (planar) limit for an even potential of order $2k$, the staircase equations will give in (6.12) a polynomial of order k.

From (5.67) we can write

$$\frac{n}{\beta} = \sum_{p=1}^{k} 2p\overline{g}_{2p} \binom{2p-1}{p} R^p \tag{6.16}$$

Again redefining $R \to R/N$, $x = n/N$, $\overline{g}_{2p} = g_{2p}/N^{p-1}$, $\beta = e^{\Lambda_B}$, we obtain

$$\begin{aligned} e^{-\Lambda_B} x &= W_k(R) \\ W_k(R) &= \sum_{p=1}^{k} g_{2p} \binom{2p-1}{p} R^p \end{aligned} \tag{6.17}$$

We can write $W(R)$ in terms of the original potential via an integral representation, which the reader is invited to verify:

$$W(R) = \oint \frac{dz}{2\pi i} V'(z + \frac{R}{z}) \tag{6.18}$$

and its inverse

$$V(\phi^2) = \int_0^1 \frac{du}{u} W(u(1-u)\phi^2) \tag{6.19}$$

In the pure gravity case, $x \sim 1$ is the critical point, where $W(R)$ has a double zero. Tuning the couplings, we can require $W(R)$ to have a zero of order k. Hence, the k–th multi-critical point appears when

$$W(R) = \left[1 - (1-R)^k\right] F(R) \tag{6.20}$$

Since $F(R)$ is analytic and non–vanishing in the critical region, for all practical purposes we may ignore the explicit form of this effectively constant term. Now we have

$$e^{-\Lambda_B} x = 1 - (1-R)^k \tag{6.21}$$

$$R(x) = 1 - (1 - e^{-\Lambda_B} x)^{1/k} \tag{6.22}$$

Note that we can absorb $F(0)$ into the definition of Λ_B. Substituting (6.22) into (5.69), we obtain

$$E = \frac{1}{(2+k^{-1})(1+k^{-1})} (\Lambda_B - \Lambda_c)^{2+k^{-1}} + \text{ analytic} \tag{6.23}$$

Therefore, in this case we obtain

$$\gamma_k = -\frac{1}{k} \tag{6.24}$$

When $W(R)$ takes the form (6.20), we can compute the potential $U_k(\phi^2)$ according to (6.19):

$$U_{2k}(\phi) = \sum_{n=1}^{k}(-1)^{n-1}\frac{k!(n-1)!}{(k-n)!(2n)!}\phi^{2n} \tag{6.25}$$

For instance,

$$\begin{aligned} U_2(\phi) &= \frac{1}{2}\phi^2 \\ U_4(\phi) &= \phi^2 - \frac{1}{12}\phi^4 \\ U_6(\phi) &= \frac{3}{2}\phi^2 - \frac{1}{4}\phi^4 + \frac{1}{60}\phi^6 \end{aligned} \tag{6.26}$$

Several comments are now in order:

1) It si very important to realize that the sum over triangulations is playing the role of a dynamical variable, namely the world–sheet metric g_{ij}, and also that our main goal is to reproduce the sum over geometries in the continuum limit. Therefore, it is not sufficient to compute the partition function in the large N limit, we have to look for critical points. Near them, we find scaling behaviour and the onset of the continuum limit. This is an illustration of the arguments advanced in lecture 2 on the renormalization group. Near a second–order phase transition, we find scaling behaviour and the memory of the discrete structures is lost. If we look only for $W(R)$ or $R = R(x)$, then we are only counting graphs in the large N limit. This is an important problem solved by Bessis, Itzykson and Zuber [88] using orthogonal polynomials, but it is not the problem we wish to solve. We are not only using the large N methods to simulate triangulations, we also want the sum to become critical in order to recover the original continuum theory we were interested in.

2) We have not yet specified how Λ_B approaches Λ_c. This would require introducing some scale, a renormalized cosmological constant and various scaling variables. Often, we are not interested in any relation between the two limit $\Lambda_B \to \Lambda_c$ and $N \to \infty$; thus we take $N \to \infty$, restrict ourselves to planar graphs and then look for possible continuum limits. For the applications to statistical mechanics this suffices. One is most often interested only in planar topology for the fluctuating surfaces. If one also wants to obtain solutions to string theories, then one has to sum the partition functions for each genera to obtain the full string partition function. This is the beauty of the double scaling limit. By tuning carefully the two limits $N \to \infty$ and $\Lambda_B \to \Lambda_c$, we obtain the sum over all genera in a single step. The next section is devoted to a first exploration of the double scaling limit.

6.4. A Primer on the Double Scaling Limit

Before taking the $N \to \infty$ limit, recall from the previous lecture that the free energy has an expansion

$$E(\Lambda_B; g) = N^2 E_0 + E_1 + N^{-2} E_2 + \cdots + N^{2-2h} E_h + \cdots \tag{6.27}$$

where E_h is the sum over all graphs with the topology of a Riemann surface of genus h. If we consider the dependence on a single coupling, for instance Λ_B, the singularity nearest to the the origin occurs at the same location μ_c at every order in $1/N$. Its position depends only on the potential V. This is the behaviour we found in lecture 4, where the partition function for fixed area A and fixed genus h in the limit $A \to \infty$ behaves according to

$$e^{-\Lambda_B A} Z_h(A) = A^{-1+\frac{Q}{\gamma}(1-h)} e^{-(\Lambda_B - \Lambda_c)A} \tag{6.28}$$

Hence the singularity in $(\Lambda_B - \Lambda_c)$ for $Z_h(\Lambda_B)$ is

$$\begin{aligned} Z_h(\Lambda_B) &\sim \int^{\infty} dx\, x^{-1-\frac{Q}{\gamma}(1-h)} e^{-(\Lambda_B - \Lambda_c)x} \\ &\sim(\Lambda_B - \Lambda_c)^{\frac{Q}{\gamma}(1-h)} \\ &= (\Lambda_B - \Lambda_c)^{(2-\gamma_0)(1-h)} \\ \gamma_0 &= \gamma_{\rm st} = 2 - Q/\gamma \quad , \quad Q = \sqrt{\frac{25-d}{3}} \quad , \quad \gamma = \frac{Q - \sqrt{Q^2 - 8}}{2} \end{aligned} \tag{6.29}$$

In the discrete case, the behaviour

$$2 - \gamma_h = (2 - \gamma_0)(1 - h) \tag{6.30}$$

has been verified in several explicit examples. Hence the behaviour of a generic term in

(6.27) for large N and small $(\Lambda_B - \Lambda_c)$ is

$$\begin{aligned} E_h(\Lambda_B) &= N^{2-2h}(\Lambda_B - \Lambda_c)^{(2-\gamma_{st})(1-h)} \\ &= \left[\frac{1}{N(\Lambda_B - \Lambda_c)^{1-\gamma_{st}/2}}\right]^{2h-2} \end{aligned} \tag{6.31}$$

The term in brackets is therefore the string coupling constant *i.e.*, the parameter which counts the number of string loops in

$$Z_{\text{string}} = \sum_h \kappa^{2h-2} Z_h \tag{6.32}$$

$$\kappa = \frac{1}{N(\Lambda_B - \Lambda_c)^{1-\gamma_{st}/2}} \tag{6.33}$$

Thus if we want to obtain a sum of this form, we have to take the limits $N \to \infty$ and $\Lambda_B \to \Lambda_c$ in such a way as to leave the string coupling constant κ different from zero.

To take the continuum limit, we introduce an explicit constant a with dimensions of length to play the role of the cut–off. This means that the length of the basic links in the triangulations are taken to be all equal to a a instead of 1. Next, we introduce a in the expressions above and then take the dominant terms as $a \to 0$. The renormalized cosmological constant is defined according to

$$\Lambda_R = \frac{\Lambda_B - \Lambda_c}{a^2} \tag{6.34}$$

and Λ_R is kept fixed by tuning $(\Lambda_B - \Lambda_c)$ as $a \to 0$. We can think of Λ_B as the action per polygon and a^2 as the area per polygon, so that Λ_R represents the action per unit area. The continuum limit is taken to be the first non–analytic term in μ_R which satisfies the correct scaling relations. There are some analytic contributions in μ_R to E_0. These are the remnants of the cut–off and can be subtracted if desired in the definition of the renormalized free energy. The double scaling limit is obtained by requiring [6, 7, 8]

$$\kappa^{-1} = N(a^2)^{1-\gamma_{st}/2}\Lambda_R^{1-\gamma_{st}/2} \tag{6.35}$$

to remain constant, *i.e.*,

$$Na^{2-\gamma_{st}} = \lambda^{-1} = \text{constant} \tag{6.36}$$

For critical strings, the string coupling constant is dimensionless (it is just the dilaton expectation value), while here it has a well-defined dimension. This is one of the many features distinguishing between critical and sub-critical strings. It also makes clear the fact that we cannot expand about $\Lambda_R = 0$. Since physical quantities will depend on κ, we see that a change in the string coupling constant λ can be compensated by a change in Λ_R. To obtain the equations governing the continuum limit, we first blow up the scaling region ($x \sim 1$):

$$e^{-a^2\Lambda_R}x = e^{-\Lambda_B}x \equiv 1 - a^2 t \tag{6.37}$$

Now, for $x \in [0,1]$, $t \in [a^{-2}, \Lambda_R]$, and the derivatives with respect to x become

$$\frac{1}{N}\frac{d}{dx} \simeq -\frac{1}{Na^2}\frac{d}{dt} = -\lambda a^{-\gamma_{\rm st}}\frac{d}{dt} \tag{6.38}$$

In the planar limit, we would have

$$W = 1 - (1-R)^k = e^{-\Lambda_B}x = 1 - a^2 t \tag{6.39}$$

It is convenient to introduce the scaling function

$$1 - R = a^{2/k} f(t) \tag{6.40}$$

and now we solve the string equation to leading order for the case of pure gravity, $k = 2$:

$$\begin{aligned} xe^{-\Lambda_B} &= R + 4g_4 R(x)(R(x+\varepsilon) + R(x) + R(x-\varepsilon)) \\ &= R + 12g_4R^2 + 4g_4R\left(\varepsilon\frac{d}{dx}\right)^2 R(x) + \cdots \end{aligned} \tag{6.41}$$

(recall $\varepsilon = 1/N$). For $k = 2$, we have

$$\begin{aligned} &R(x) = 1 - af(t) \\ &\varepsilon\frac{d}{dx} = -\lambda a^{1/2}\frac{d}{dt} \quad , \quad g_4 = -\frac{1}{12} \end{aligned} \tag{6.42}$$

$$1 - a^2 t = 1 - a^2 f^2(t) + \frac{1}{3}\lambda^2 a^2 \frac{d^2 f}{dt^2} + O(a^3) \tag{6.43}$$

Finally,

$$t = f^2(t) - \frac{1}{3}\lambda^2\frac{d^2 f}{dt^2} \tag{6.44}$$

we obtain the Painlevé equation of the first kind, or Painlevé–I for short. This is a remarkable result [6, 7, 8]. In quantum field theory, we should not have any right to expect such a simple answer. It should be wonderful to be able to compute the $(g-2)$ factor of the electron as a function of the fine structure constant α in terms of the solution to a non–linear ordinary differential equation. The double scaling limit has shown us, however, that the free energy of string theory can be obtained from (6.44).

In the derivation of (6.44) we have made some simplifications. For example, we have taken the k–th Kazakov critical point to be of the form $W_k(R) = 1-(1-R)^k$. We could instead choose a more general parametrization $W_k(R) = 1-\alpha(R_c - R)^k$, but by redefining f and t the final result would again be (6.44). This is expected from universality. It is a good exercise to verify it. We can solve (6.44) as a power series expansion. Since the expansion parameter is κ^2, from (6.35) it follows that $\kappa^2 = \lambda^2 \Lambda_R^{-5/2}$ and, in the planar limit $f(t) = t^{1/2}$ we can write

$$f(t) = t^{1/2} \sum_{h=0}^{\infty} A_h \left(\lambda^2 t^{-5/2}\right)^h \tag{6.45}$$

Upon substitution into (6.44), we obtain a recurrence relation for the coefficients A_h:

$$A_{h+1} = \frac{1}{24}(25h^2-1)A_h - \frac{1}{2} \sum_{\substack{m+n=h+1 \\ m,n>0}} A_m A_n \tag{6.46}$$

and asymptotically $A_h \approx (24/25)^{-h}(2h)!$. This series is divergent, and its behaviour is such that non–perturbative effects are expected to be stronger than in field theory. In field theory, large orders of perturbation theory grow like $h!$, which can be traced to singularities of the form $\exp{-A/g^2}$, with g the coupling constant. The behaviour $(2h)!$ leads, on the other hand, to singularities like $\exp{-A/g}$. For small g, this is much stronger than $\exp{-A/g^2}$, and hence one should suspect that non–perturbative effects in string theory play a very important role [108]. In the next two lectures we will pursue in more detail the properties of the double scaling limit. Since the series (6.45) is divergent and not Borel–summable, it is an important question whether 2d–gravity can be defined non–perturbatively. Several attempts towards this goal can be found in [109, 110, 111, 112, 113, 114, 115, 116, 117, 118]. The universality properties of the double scaling limit for the one–matrix models are studied using heat kernel methods in [119]. We will have more to say about some of these topics in the last lecture.

Lecture 7. Statistical Systems on Random Surfaces

7.1. General Considerations. Critical Exponents

In this lecture we want to explore in more detail the properties of statistical systems coupled to random triangulations. In lecture 5, we presented the basic definitions but did not study in detail either the critical exponents or the many subtleties associated with the integration measure, or the role of irrelevant operators, etc. The first part of this lecture is devoted to these issues: we follow closely [64]. Later we whall turn to some examples other than pure gravity.

We begin with an arbitrary triangulation S with adjacency matrix $G_{ij}(S)$. Recall that we have analogues of several geometrical quantities in the continuum limit. If at site i there are N_i incident triangles, the volume element $\sqrt{g}$ and curvature R are defined at i according to

$$\begin{aligned}
&\sqrt{g} \to \sigma_i = \frac{N_i}{3} \\
&R \to R_i = \pi \frac{6 - N_i}{N_i} \\
&\sqrt{g}R \to \sigma_i R_i = 2\pi \frac{6 - N_i}{6} \\
&\sqrt{g}R^2 \to \frac{\pi^2}{3N_i}(6 - N_i)^2
\end{aligned} \tag{7.1}$$

The discrete action for a D–dimensional string on the triangulation S is

$$S = \frac{1}{2} \sum_{\langle i,j \rangle} (X_i - X_j)^2 + \Lambda \sum_i \sigma_i \tag{7.2}$$

If instead of triangulations we look at more complicated simplicial approximations to the surface, N_i in (7.1) has to be replaced by the co-ordination number q_i of the point i. For triangulations, $q_i = N_i$ obviously. The pure gravity action may contain terms other than the surface area:

$$S_{\text{gravity}} = \Lambda |S| + \mu \mathcal{X} + \sum_i \frac{1}{2} \sigma_i R_i^2 + \cdots \tag{7.3}$$

In this expression, $|S|$ is the area of the triangulation and $\mathcal{X}$ its Euler number. The measure over X_i is

$$\prod_i \sigma_i^{D/2} \frac{d^D X_i}{(2\pi)^{D/2}} \tag{7.4}$$

The term $\sigma_i^{D/2}$ has to be included in the quantization of a scalar field in the presence of a gravitational background to account for the correct conformal properties. As pointed out by David, we can combine the measure factor and the higher curvature terms in (7.3) into an arbitrary exponent for σ_i. We can replace $D/2$ by some exponent α, and up to constants we may write

$$-\sum_i \log \sigma_i/2 = \sum_i \sigma_i(1+\frac{R_i}{\pi})\log(1+\frac{R_i}{\pi}) \tag{7.5}$$

The first term in the expansion of (7.5) is proportional to the area of the triangulation. The second term gives a correction proportional to the Euler character. Next, we have $\sigma_i R_i^2$, etc. One expects that the continuum limit should not depend on curvature squared and higher-order terms. This is difficult to test analytically, but there is some evidence based on strong coupling expansions and Monte Carlo simulations. We can write for the partition function

$$\begin{aligned} Z(\alpha,\beta,D) &= \sum_S \frac{1}{G|S|} \prod_{i\in S} \sigma_i^{\alpha} e^{-\beta\sigma_i} Z(S,D) \\ &= \sum_{N=1}^{\infty} e^{-2\beta N} Z_N(\alpha,D) \\ Z(S) &= \int \prod_{i\in S}{}' \frac{d^D X_i}{(2\pi)^{D/2}} \exp\left[-\frac{1}{2}\sum_{\langle ij\rangle}(X_i - X_j)^2\right] \end{aligned} \tag{7.6}$$

The prime in the last measure indicates that we should remove the zero mode $\delta X_i = C$. We found a similar phenomenon in the continuum version in lecture 4. In lecture 5, we argued that we can replace the sum over triangulations by a large N ϕ^3 or ϕ^4 field theory. Let us make these arguments more concrete now.

By a duality transformation, we can write $Z(S)$ in terms of the dual lattice S^* which is a ϕ^3 diagram. This is done as follows. First introduce a new set of link variables

$$V^{\mu}_{\langle ij\rangle} = X^{\mu}_i - X^{\mu}_j \tag{7.7}$$

For any triangle $\langle ijk\rangle$ in S, the variables $V^{\mu}_{\langle ij\rangle}$ satisfy

$$V^{\mu}_{\langle ij\rangle} + V^{\mu}_{\langle jk\rangle} + V^{\mu}_{\langle ki\rangle} = 0 \tag{7.8}$$

Denoting by i^*, j^*, etc. the sites of the dual graph S^*, we can define link variables in the dual graph by the identification $V^{\mu}_{\langle i^*j^*\rangle} = V^{\mu}_{\langle ij\rangle}$ whenever the links $\langle ij\rangle$ and $\langle i^*j^*\rangle$ are dual to each other. Since the three dual links to those appearing in (7.8) meet at the point dual to the tringle $\langle ijk\rangle$, (7.8) can be written in terms of a set of Lagrange multipliers $X^{\mu}_{i^*}$ associated to each site of the dual lattice:

$$\delta\left(\sum_{j^*} V^{\mu}_{\langle i^*j^*\rangle}\right) = \frac{1}{(2\pi)^D}\int d^D X_{i^*} e^{iX_{i^*}\cdot\sum_j V_{\langle i^*j^*\rangle}} \tag{7.9}$$

Now (7.6) can be written as an integral over the link variables $V^{\mu}_{\langle ij\rangle}$ (or $V^{\mu}_{\langle i^*j^*\rangle}$) subject to the constraint (7.9). Performing the gaussian integral over V_{ij}, we obtain a gaussian action on the X_{i^*} variables *i.e.,* a free field action on the dual lattice. Thus the string partition function becomes

$$\lim_{N\to\infty}\int \mathcal{D}X \exp -N\,\mathrm{Tr}\left\{\frac{1}{2}\int d^Dx d^Dy e^{(x-y)^2/2}\Phi(x)\Phi(y) - \frac{1}{3}g\int d^Dx\Phi^3(x)\right\} \tag{7.10}$$

Similar arguments can be carried out for other simplicial lattices. The difference between the theory defined on the Φ^3 graphs and the original one (7.6) stems from the measure contribution $\prod_i \sigma_i^{\alpha}$ or more concretely, in the higher curvature terms. Hence the theories we can solve using large N techniques are those with $\alpha = 0$ in (7.3). To include an α–dependence, other methods have to be invoked.

Another example of the same duality transformation is the Ising model on a triangulated (or "quadrangulated") surface. The duality transformation then transforms the theory into an Ising model on a lattice with fixed co-ordination number. Suppose we have a lattice with n vertices and planar topology, built out of irregular squares. The partition function is

$$Z(\beta,S)=\sum_{\substack{\sigma_i=\pm1\\ i\in V(S)}} e^{\beta\sum_{ij}G_{ij}(S)\sigma_i\sigma_j} \tag{7.11}$$

Using the strong coupling expansion, we find

$$Z(\beta,S)=\sum_{\{\sigma_i\}}\prod_{\langle ij\rangle}(\cosh\beta+\sinh\beta\sigma_i\sigma_j) \tag{7.12}$$

The product can be expanded in terms of closed loops, where each link is traversed only once. Then

$$\begin{aligned} Z(\beta,S) &= \sum_{\{\sigma_i\}}(\cosh\beta)^{N_L}\prod_{\langle ij\rangle}(1+\tanh\beta\sigma_i\sigma_j)\\ &= 2^{N_V}(\cosh\beta)^{N_L}\sum_{\text{loops}}(\tanh\beta)^{\text{length}} \end{aligned} \tag{7.13}$$

In the dual simplex S^* we introduce spin $\mu_{i^*}=\pm1$, and we assign the relative value of two nearest neighbours μ_{i^*}, μ_{j^*} to be -1 if the link dual to $\langle i^*j^*\rangle$ is occupied by the contours and $+1$ otherwise. Then we can write

$$\text{length}=\sum_{\langle i^*j^*\rangle}\frac{1}{2}(1-\mu_{i^*}\mu_{j^*}) \tag{7.14}$$

and hence

$$\begin{aligned} Z(\beta,S) &= 2^{N_V}(\cosh\beta)^{N_L}(\tanh\beta)^{N_L/2}\sum_{\{\mu_{i^*}\}} e^{-\frac{1}{2}\log\tanh\beta\sum_{\langle i^*j^*\rangle}\mu_{i^*}\mu_{j^*}}\\ &= 2^{N_V}\left(\frac{1}{2}\sinh 2\beta\right)^{N_L/2}\sum_{\{\mu_i\in S^*\}} e^{\beta^*\sum_{\langle ij\rangle}\mu_i\mu_j} \end{aligned} \tag{7.15}$$

with

$$\tanh\beta=e^{-2\beta^*}\qquad;\qquad \sinh 2\beta\sinh 2\beta^*=1 \tag{7.16}$$

Then using $N_V=N_{F^*}$, $N_L=N_{L^*}$, $N_F=N_{V^*}$, and the formula for the Euler number $\mathcal{X}=N_V-N_L+N_F=N_{V^*}-N_{L^*}+N_{F^*}$, and since the dual graph is a ϕ^4 graph

$4N_{V^*} = 2N_{L^*}$, we can cancel 2^{N_V} and $2^{-N_L/2}$, leading to

$$Z(\beta, S) = (\sinh\beta^*)^{-N_L/2} \sum_{\substack{\mu_i=\pm1 \\ i\in S^*}} e^{\beta^* \sum_{(ij)} \mu_i\mu_j} \tag{7.17}$$

again showing that (7.12) is equivalent to a theory on a simplicial lattice with co-ordination number four.

The correlation functions for a string moving in D–dimensional space are defined by pinning down N points to pre-assigned positions $X_1, X_2, \ldots, X_n$. We can carry out the argument in the continuum because it is essentially the same as in the lattice. By definition then

$$G(X_1, \ldots, X_n) = \left\langle \int d^2\xi_1\sqrt{g(\xi_1)}\cdots d^2\xi_n\sqrt{g(\xi_n)} \prod_{i=1}^{n} \delta^D\left(X(\xi_1) - X_i\right) \right\rangle \tag{7.18}$$

Assuuming the D–dimensional space to be flat euclidean (or Minkowski) space, we compute the Fourier transform of $F(X_1, \ldots, X_n)$. The integration over the zero mode of the $X^\mu(\xi)$ field enforces momentum conservation. Hence

$$\begin{aligned} G_n(P_1, \ldots, P_n) &= \int \prod dX_i e^{iP_i\cdot X_i} G(X_1, \ldots, X_n) \\ &= (2\pi)^D \delta(\sum P_i) \left\langle \int d^2\xi_1\sqrt{g(\xi_1)}\cdots d^2\xi_n\sqrt{g(\xi_n)} e^{i\sum_k P_k\cdot X(\xi_k)} \right\rangle \\ &= (2\pi)^D \delta(\sum P_i) \left\langle \prod_{i=1}^{n} V(P_i) \right\rangle \end{aligned} \tag{7.19}$$

Where we have introduced the operators

$$V(P_i) = \int d^2\xi\sqrt{g(\xi)} e^{iP\cdot X(\xi)} \tag{7.20}$$

which are known in string theory as vertex operators. For example, the two–point function takes the form

$$G(P, \Lambda_B) = \sum_S \frac{1}{G|S|} e^{-\Lambda|S|} \prod_{i\in V(S)} \int \frac{\sigma_i^\alpha d^D X_i}{(2\pi)^{D/2}} e^{-S(X)} \sum_{k,\ell} \sigma_k\sigma_\ell e^{iP\cdot(X_k - X_\ell)} \tag{7.21}$$

Since $\sum \sigma_k = |S|$, we learn that the string susceptibility $\mathcal{X}$ is given by

$$\mathcal{X}(\Lambda_B) = G(P, \Lambda_B)\Big|_{P=0} = \frac{\partial^2 Z(\Lambda_B)}{\partial \Lambda_B^2} \tag{7.22}$$

Another important quantity is the m,ass gap $m(\Lambda_B)$. It determines the behaviour of the two–point function at large X:

$$G_2(X) \underset{|X| \to \infty}{\sim} e^{-m(\Lambda_B)|X|} \tag{7.23}$$

At the critical point, the mass gap may or may not vanish. If it does, then we define the critical exponent ν accodring to

$$m(\Lambda) \underset{\Lambda \to \Lambda_c}{\sim} (\Lambda - \Lambda_c)^\nu \tag{7.24}$$

Then for $\Lambda_B \sim \Lambda_c$, $m(\Lambda)$ is small, and for $|X|$ large we expect

$$G_2(X) \sim |X|^{2-D-\eta} e^{-m(\Lambda)|X|} \tag{7.25}$$

which defines the critical exponent η.

Finally, other interesting quantities are the mean square extent of the surface and its Hausdorff dimension. The mean square extent of the surface is defined as

$$\overline{X_S^2} = \frac{1}{|S|^2} \sum_{i,j \in V(S)} \sigma_i \sigma_j \overline{(X_i - X_j)^2} \tag{7.26}$$

where the average is taken with respect to the matter action. The sum over X with fixed S amounts to averaging over the embeddings of S in space–time. To eliminate the zero mode, the center of mass of the embedded surface is fixed *i.e.*, $X_{\rm CM} = |S|^{-1} \sum \sigma_i X_i = 0$:

$$\overline{X_S^2} = \frac{1}{|S|^2} \frac{\int \prod_i \frac{d^D X_i}{(2\pi)^{D/2}} e^{-S(X,S)} \delta(X_{\rm CM}) \left[\sum_{i,j} \sigma_i \sigma_j (X_i - X_j)^2\right]}{\int \prod_i \frac{d^D X_i}{(2\pi)^{D/2}} e^{-S} \delta(X_{\rm CM})} \tag{7.27}$$

Next, define a generating function

$$Z(\Lambda, \overline{X^2}) = \sum_S \frac{1}{|G(S)|} e^{-\Lambda_B |S|} \int \prod_i \sigma_i^\alpha \frac{d^D X_i}{(2\pi)^{D/2}} e^{-S(X,S)} \, \overline{X_S^2} \tag{7.28}$$

which we can expand in terms of the area,

$$Z(\Lambda, \overline{X^2}) = \sum_n e^{-n\Lambda_B} Z_n(\overline{X^2}) \tag{7.29}$$

For large area n:

$$Z_n(\overline{X^2}) \sim A^n n^{\gamma'-3} \tag{7.30}$$

Considering the ratio of the asymptotic behaviour (7.30) to that of the partition function

$$\overline{X_S^2} \sim_{n\to\infty} n^{\gamma'-\gamma} \tag{7.31}$$

we find

$$\overline{X_S^2} \sim_{|S|\to\infty} |S|^{2/d_H} \quad , \quad d_H = \frac{2}{\gamma' - \gamma} \tag{7.32}$$

with d_H the Hausdorff dimension of the surface. If we define correlation functions for fixed area $|S|$, then $\overline{X_S^2}$ can be constructed from

$$d_H = \frac{1}{2} \lim_{|S|\to\infty} \frac{\log \langle X^2 \rangle_S}{\log |S|} \tag{7.33}$$

$$\overline{X_S^2} = \frac{\int d^D X X^2 G_{|S|}(X)}{\int d^D X G_{|S|}(X)} = -D \frac{\partial}{\partial P^2} \log G_{|S|}(P^2) \Big|_{P=0} \tag{7.34}$$

where $G_{|S|}(X)$ is the two–point function for fixed area.

We can derive a scaling relation among η, γ and ν as follows:

$$\begin{aligned} G_2(P)\Big|_{P=0} &= \int d^D X G_2(X) \\ &\sim \int^\infty r^{D-1} dr\, r^{2-D-\eta} e^{-m(\Lambda) r} \\ &\sim m(\Lambda)^{\eta-2} \end{aligned} \tag{7.35}$$

Near Λ_c,

$$G_2(P)\Big|_{P=0} = \mathcal{X} = (\Lambda_B - \Lambda_c)^{-\gamma} \sim (\Lambda_B - \Lambda_c)^{\nu(\eta-2)} \tag{7.36}$$

and therefore

$$\gamma = \nu(2 - \eta) \tag{7.37}$$

which is one of the standard scaling relations. Similarly,

$$\frac{\partial}{\partial P^2} G_2(P)\bigg|_{P=0} = \int d^D X |X|^2 G_2(X) \sim (\Lambda_B - \Lambda_c f)^{\nu(\eta-4)} \sim (\Lambda_B - \Lambda_c)^{-\gamma'} \qquad (7.38)$$

and

$$\begin{aligned} \gamma' &= \nu(r-\eta) \\ \gamma &= \nu(2-\eta) \\ d_H &= \frac{2}{\gamma-\gamma'} = \frac{1}{\nu} \end{aligned} \qquad (7.39)$$

These relations are independent of α.

Next, we will become more acquainted with the large N methods developed so far. In lecture 4, we learned that $c = 1$ is a very special point. It is the boundary between two very different regions. For $c < 1$, we have a reasonably good understanding of the coupling of conformal field theories to gravity. For $c > 1$, on the other hand, it seems that the analytic methods presented are incapable of leading to any physical insight. We know from rather simple arguments that an instability must set in beyond $c = 1$. In string theory, the lowest-lying state has a square mass proportional to $(1 - d)$. For $d < 1$ this state is massive. At $d = 1$, the state becomes massless (signal of a Kosterlitz–Thouless phase transition) and for $d > 1$ the state is tachyonic. This is a signal for an instability. In spontaneously broken field theories we have learned how to deal with similar situations. When a scalar field has a potential energy $V(\phi) = \lambda(\phi^2 - a^2)^2$, expanding about $\phi = 0$ would imply the appearance of a tachyon in the spectrum. No one in her/his right mind would insist on defining the full theory about the $\phi = 0$ unstable state. The true ground state appears at $|\phi| = a$. If we prepared an initial state concentrated about $\phi = o$ in a large space–time region, it would decay to the true vacuum via the emission of some radiation. Our understanding of string theory is so incomplete at the moment that for $d > 1$ we do not know the stable ground state. However, it seems likely that the picture of a world-sheet looking like a smooth surface will have to be given up. What are the correct variables describing the $c > 1$ phase is a deep outstanding problem which remains to be solved. Since we cannot yet venture into the uncharted $c > 1$ territory, we will consider in some detail the $c = 1$ theories.

7.2. Strings at $c = 1$. Planar Limit

In many respects, this theory is simpler than the $c < 1$ theories and the Kazakov critical points for one–matrix models. As we showed in the previous section, at $D = 1$ we have to consider a matrix–valued field $\Phi(t)$ with a gaussian propagator simulating the string action $\sqrt{g} g^{ij} \partial_i X \partial_j X$. In this form, the model cannot be solved. If instead of a gaussian propagator $G(X) \sim \exp -X^2$ corresponding in momentum space to $G(p) \sim \exp p^2$, we choose $G(X) = \exp -|X|$ (*i.e.*, $G(p) = (1+p^2)^{-1}$ in momentum space), the discrete action would contain a sum over links of $|X_i - X_j|$, whose continuum form is $|g^{ij}\partial_i X \partial_j X|^{1/2}$. This change should not affect the critical properties. Since field theories in $0+1$ dimensions are well–behaved in the ultra-violet, only the short distance non–universal behaviour of the theory will be affected. It should be legitimate to invoke universality to believe that at the critical point the two actions lead to the same physics. In spite of these plausibility arguments and the compelling use of universality, there is no proof of equivalence, and although unlikely, surprises might arise. The large N analysis of this model was carried out in [98], and its critical properties in the planar limit were first studied in [91]. For a detailed recent study of the $c = 1$ theory in the planar and double scaling limits, with references, see [120].

The partition function is taken to be

$$\begin{aligned} Z(g, g_i) &= \int \mathcal{D}\Phi(t) \exp - \int_0^T dt \frac{N}{g} \mathrm{tr}\left(\frac{1}{2}\dot{\Phi}^2 + U(\Phi)\right) \\ U(g_i, \Phi) &= \sum_{p \geq 2} g_p \Phi^p \end{aligned} \tag{7.40}$$

To write Z in a tractable way, we diagonalize Φ:

$$\begin{aligned} &\Phi = \Omega^\dagger \lambda \Omega \\ &\Omega\Omega^\dagger = 1 \qquad \lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \end{aligned} \tag{7.41}$$

Then

$$\begin{aligned}
&\Omega\dot{\Phi}\Omega^{\dagger} = \dot{\lambda} + [\lambda, \dot{\Omega}\Omega^{-1}] \\
&\mathrm{tr}\dot{\Phi}^2 = \sum_i \dot{\lambda}_i^2 + \sum_{i,j}(z_i - z_j)^2 |A_{ij}|^2 \\
&A_{ij} = -(\dot{\Omega}\Omega^{-1})_{ij} \qquad \dot{f} = \frac{df}{dt}
\end{aligned} \tag{7.42}$$

The change of variables in $\mathcal{D}\Phi(t)$ again takes the form of a van der Monde determinant:

$$\mathcal{D}\Phi(t) = \prod_i \mathcal{D}\lambda_i(t)\mathcal{D}\Omega(t) \prod_t \prod_{i<j} (\lambda_i(t) - \lambda_j(t))^2 \tag{7.43}$$

The invariant measure $\mathcal{D}\Omega(t)$ is just an integration over A_{ij} (an anti-hermitean $N \times N$ matrix):

$$\begin{aligned}
Z(g, g_i) = \int \mathcal{D}\lambda(t)\mathcal{D}\Omega \prod_t \prod_{i<j} (\lambda_i(t) - \lambda_j(t))^2 \\
\exp -\frac{N}{g} \int_0^T dt \left(\frac{1}{2} \sum_i \dot{\lambda}_i^2 + \frac{1}{2} \sum_{i,j} (\lambda_i(t) - \lambda_j(t))^2 |A_{ij}|^2 \right)
\end{aligned} \tag{7.44}$$

If we naively carry out the integration over A_{ij}, it seems that all the van der Monde determinants drop out of sight. To obtain the right answer, however, we should discretize t. The kinetic term will have a contribution of the form

$$\sum_n \mathrm{tr}\Phi(n)\Phi(n+1) \tag{7.45}$$

Diagonalizing $\Phi(n)$ at each time step,

$$\Phi(n) = \Omega^{-1}(n)\lambda(n)\Omega(n) \tag{7.46}$$

and then

$$\mathrm{tr}\,\Phi(n)\Phi(n+1) = \mathrm{tr}\,\Big(\lambda(n)\Omega(n)\Omega^{-1}(n+1)\lambda(n+1)\Omega(n+1)\Omega^{-1}(n)\Big) \tag{7.47}$$

Near the continuum limit, $T = M\varepsilon$ with $M \to \infty$, $\varepsilon \to 0$, and we can define the gauge field $A(n)$ as

$$\Omega(n)\Omega^{-1}(n+1) \simeq 1 - A(n) + \frac{1}{2}A^2(n) + \cdots \tag{7.48}$$

with $A(n)$ anti-hermitean. Substituting into $\mathrm{tr}\Phi(n)\Phi(n+1)$, we find

$$\begin{aligned}\mathrm{tr}\,\lambda(n)\left(1-A(n)+\frac{1}{2}A(n)^2+\cdots\right)\lambda(n+1)\left(1+A(n)+\frac{1}{2}A(n)^2+\cdots\right)\\ =\mathrm{tr}\,\lambda(n)\lambda(n+1)+\mathrm{tr}\,[-\lambda(n)A(n)\lambda(n+1)+\lambda_n\lambda(n+1)A(n)]\\ +\mathrm{tr}\left[-\lambda(n)A(n)\lambda(n+1)A(n)+\frac{1}{2}\lambda(n)A(n)^2\lambda(n+1)+\frac{1}{2}\lambda(n)\lambda(n+1)A(n)^2\right]\\ =\mathrm{tr}\,\lambda(n)\lambda(n+1)+\sum_{i,j}[\lambda_i(n)-\lambda_j(n)][\lambda_i(n+1)-\lambda_j(n+1)]|A_{ij}|^2\end{aligned} \tag{7.49}$$

As $\varepsilon \to 0$, $M \to \infty$, we can write $\lambda_i(n) = \lambda_i(t)$, $\lambda_i(n+1) = \lambda_i(t+\varepsilon)$. The coefficient of $|A_{ij}|^2$ is then

$$\sum_{i,j}[\lambda_i(t)-\lambda_j(t)][\lambda_i(t+\varepsilon)-\lambda_j(t+\varepsilon)]|A_{ij}|^2 \tag{7.50}$$

Therefore, integrating over A_{ij} produces $\Delta(\lambda(t)^{-1}\Delta(\lambda(t+\varepsilon))^{-1}$. But from the measure we have a factor $\prod_t \Delta(\lambda(t))^2$, so all the van der Monde determinants cancel except for the ends, at $t=0$ and $t=T$:

$$Z_N(g,g_i)=\mathrm{const.}\int \mathcal{D}\lambda(t)\Delta(\lambda(0))\Delta(\lambda(T))\exp-\frac{N}{g}\int_o^T dt\left(\frac{1}{2}\sum_i\dot{\lambda}_i^2(t)+\sum_i\sum_p g_p\lambda_i^p\right) \tag{7.51}$$

Two things should be noticed in (7.51):

1) The eigenvalues $\lambda_i(t)$ decouple completely, so the system described by $Z_N(g,g_i)$ is equivalent to a gas of non–interacting particles subject to the potential $V(\lambda)$.

2) The statistics of these particles is fermionic as a consequence of the initial and final van der Monde determinants, which are totally antisymmetric.

A simple consequence of 1) and 2) is that there is no need to use orthogonal polynomials to solve (7.51). We have to study a non–interacting Fermi gas where every "electron" is subject to the same one–body potential $V(\lambda)$. The lagrangian for each particle is

$$L=\frac{N}{g}\left(\frac{1}{2}\left(\frac{d\lambda}{dt}\right)^2-V(\lambda)\right) \tag{7.52}$$

with hamiltonian

$$H=\frac{g}{2N}p^2+\frac{N}{g}V(\lambda)\quad,\quad p=\frac{N}{g}\dot{\lambda} \tag{7.53}$$

If $e_1, \ldots, e_N$ are the N lowest energy levels of H, the ground state energy is simply $\sum e_i$, and $e_N = e_F$ is the Fermi energy ($e_1 \leq e_2 \leq \cdots \leq e_N = e_F$). For large T and large N, the leading term in $\log Z$ is $-N^2 E_o(g)$:

$$N^2 E_o = \sum_k e_k \theta(e_F - e_k)$$
$$N = \sum_k \theta(e_F - e_k) \tag{7.54}$$

where $\theta(x)$ is the Heaviside step function.

Next we want to evaluate (7.54) and find the values of the coupling constants leading to non–analytic behaviour of the partition function and to the continuum limit. This large N problem was first solved in the pioneering paper of Brézin *et al.*[98] and their result was used to study a $c = 1$ theory coupled to gravity by V. Kazakov and A.A. Migdal [91]. Notice that in writing (7.40), we have changed the conventions of previous lectures slightly. There is no need to rescale the couplings g_p with powers of N, and a graph with $V = \sum V_p$ vertices, P propagators and F index loops has automatically a factor

$$N^{2\mathcal{X}} g^{\sum_p V_p(\frac{p}{2}-1)} \prod_p g_p^{V_p} \times \text{propagators} \tag{7.55}$$

In the large N limit the potential energy becomes large, the characteristic length scale becomes of order $N^{1/2}$ and the enrgy scale is of order N. This is a typical situation in which the WKB approximation can be used. To find the critical behaviour, we proceed as follows. Define the hamiltonian

$$\hat{h} = -\frac{1}{2\beta^2}\frac{\partial^2}{\partial\lambda^2} + V(\lambda) \tag{7.56}$$

The N first energy levels of $\hat{h}$ are $e_i \leq e_2 \leq \ldots \leq e_N = \mu_F$ (notice that $E_i = \beta e_i$). Introducing the density of states

$$g(e) = \frac{1}{\beta}\sum_i \delta(e_i - e) \tag{7.57}$$

The constraints (7.54) become

$$g = \frac{N}{\beta} = \int_0^{\mu_F} \rho(e)de$$
$$E_o = \beta^2 \int_0^{\mu_F} e\rho(e)de \tag{7.58}$$

As in previous cases, the signal for reaching the continuum limit is that E_o becomes a singular function of the cosmological constant. This can be achieved by adjusting the couplings of the model. If $V(\lambda)$ has a local maximum $\mu_c = V_{\max}$, off criticality we start with a Fermi level below μ_c. As $N \to \infty$, the probability of tunnelling through the barrier is suppressed with an exponential proportional to N, hence in the planar limit the system is stable under tunnelling. If we let $\mu = \mu_c - \mu_F \to 0^+$, in the limit the state can simply roll to the other side of the barrier and we should expect the renormalized coupling constant $\Delta \equiv g_{\text{critical}} = g$ to be a singular function of μ. Choosing (without loss of generality) $g_c = 1$, we find $g^{\text{Area}} \simeq (1-\Delta)^{\text{Area}} \simeq e^{-\Delta \text{Area}}$ as expected. Differentiating (7.58) with respect to μ:

$$\frac{\partial g}{\partial \mu} = -\rho(\mu_F)$$
$$\frac{\partial E_o}{\partial \mu} = -\beta^2 \mu_F \tag{7.59}$$
$$\rho(\mu_F) = \beta^2 \mu_F \frac{\partial g}{\partial \mu_F}$$

In the WKB limit the total number of states with energy smaller than or equal to E is given by

$$N(E) = \int \frac{dxdp}{2\pi} \theta(E - H(p,x)) \tag{7.60}$$

and the density of states follows from $\partial N(E)/\partial E$. In our case,

$$\rho(e) = \frac{1}{\pi} \int_{-\lambda_c}^{\lambda_c} \frac{d\lambda}{\sqrt{2(e - V(\lambda))}} \tag{7.61}$$

We are very near to the top of the barier μ_c, close to the continuum limit, and $\pm\lambda_c$ represent the turning points *i.e.*, $\mu_c = V(\pm\lambda_c)$. The singularity at $e = \mu_F$ occurs when $\mu \to 0$. Near the top of the barier, we have generically $V(\lambda) \sim \mu_c - 2(\lambda_c - \lambda)^2$ (we normalize

$U''(\lambda_c) = -4$ as in [121]). Then

$$\rho(\mu_F) = -\frac{1}{2\pi}\log\mu + \text{regular} \tag{7.62}$$

Therefore $\partial g/\partial\mu \sim \log\mu$, $\Delta \sim -\mu\log\mu$. Then as $N \to \infty$ we get the estimates

$$\begin{aligned} \rho(\mu_F) &\sim -\frac{1}{2\pi}\log\mu \\ E_\text{o} &\sim -\frac{\beta^2\Delta^2}{\log\Delta} = -\frac{N^2\Delta^2}{\log\Delta} \end{aligned} \tag{7.63}$$

consistent with the KPZ relation $E_\text{o} \sim^2 \Delta^{2-\gamma_{st}}$ for $\gamma_{st} = 0$. The logarithmic singularity is probably due to the existence of a massless mode in the continuum string theory, and one expects real infra-red divergences to appear in higher topologies when massless tadpoles can be coupled to tori or other surfaces. Working by analogy with the pure gravity case, we can identify the string coupling constant from the leading term (7.63) as

$$g^2_{\text{string}} = \frac{\log\Delta}{2\pi\beta^2\Delta^2} \tag{7.64}$$

and the genus h contribution is then expected to have a leading behaviour like $g^{2h}_{\text{string}}(\log\Delta)^h$. Hence, the perturbative expansion (topological expansion) will have severe infra-red divergences and one may wonder whether we have a sensible theory at $c = 1$. It was found by Gross and Miljković [121] that the theory can nevertheless be constructed in a strong coupling expansion in the the double scaling limit. This could correspond to a new stable non–perturbative ground state whose physical meaning remains to be elucidated. The result (7.63) is universal as long as the potential is a local maximum $V = \mu_c - 2(\lambda_c - \lambda)^2$. If we tune the couplings so that the first $(k-1)$ derivatives vanish at the maximum, then one easily shows that $\partial g/\partial\mu \sim \mu^{-\frac{1}{2}+\frac{1}{k}}$ and this leads to a string susceptibility $\gamma_{st}- = (k-2)/(k+2)$ because

$$\begin{gathered} \frac{\partial}{\partial\mu}(g-1) \sim \mu^{-\frac{1}{2}+\frac{1}{k}} \quad , \quad \frac{\partial}{\partial\mu}\Delta \sim \mu^{-\frac{1}{2}+\frac{1}{k}} \quad , \quad \Delta \sim \mu^{\frac{1}{2}+\frac{1}{k}} \\ \frac{\partial E}{\partial\Delta} \sim \Delta^{\frac{2k}{k+2}} \quad , \quad \frac{\partial^2 E}{\partial\Delta^2} \sim \Delta^{-\gamma_{st}} = \Delta^{\frac{k-2}{k+2}} \end{gathered} \tag{7.65}$$

The meaning of these critical points is not yet clear.

7.3. Ising Model on a Random Planar Lattice

Using the KPZ formulas, we know what to expect in this case for the string susceptibility and the gravitationally–dressed dimensions:

$$\gamma_{\rm st} = -\frac{1}{3} \qquad \Delta_{(2,1)} = \frac{2}{3} \qquad \Delta_{(1,2)} = \frac{1}{6} \tag{7.66}$$

These results were first derived on planar lattices [82, 83].

One of the advantages of working with a random planar lattice is that the Ising model can be solved exactly, including a constant external magnetic field. We consider a random lattice with co-ordination number four (a general ϕ^4 graph) which we know is dual to a covering of the surface with irregular squares. The partition function for a lattice with n sites and co-ordination number four is

$$Z_n(\beta, H) = \sum_{\{S^n\}} \sum_{\{\sigma\}} e^{\frac{\beta}{2} \sum_{i,j=1}^{n} G_{ij}(S)\sigma_i\sigma_j + H \sum_{i=1}^{n} \sigma_i} \tag{7.67}$$

Since the graphs are those of a ϕ^4 theory, the number of vertices is related to the number of propagators $2P = 4n$, $P = 2n$, and the area of S^n according to the definitions in lecture 5 is n. Hence, $Z_n(\beta, H)$ can be thought of as the area microcanonical ensemble. Summing over n with a chemical potential equal to the cosmological constant generates the full partition function. We have thus three parameters (λ, β, H) in the theory, and depending on how the continuum limit is approached we may have a point in parameter space where two–dimensional gravity becomes critical (yielding the exponents of pure gravity) or another one where both the lattice and the spin system become critical. This is the case we are interested in to reproduce (7.66).

The computation of (7.67) can be carried out in terms of a two–matrix model in the large N limit. Define the partition function as the integral

$$Z(g, c, H) = \int d^{N^2}\Phi_+ d^{N^2}\Phi_- \exp\left\{ {\rm tr} \left[-\Phi_+^2 - \Phi_-^2 + 2c\Phi_+\Phi_- - g\frac{e^H}{N}\Phi_4^4 - g\frac{e^{-H}}{N}\Phi_-^4 \right] \right\} \tag{7.68}$$

There are two types of Φ^4 vertices: we identify Φ_+ with spin up and Φ_- with spin down. For a graph with $n = n_+ + n_-$ vertices, we will have a contribution

$$g^{n_+ + n_-} e^{(n_+ - n_-)H} = g^n e^{(n_+ - n_-)H} \tag{7.69}$$

There are three types of propagators:

$$\begin{aligned} P(\Phi_+, \Phi_+) &= \left\langle \frac{1}{N} \mathrm{tr} \Phi_+^2 \right\rangle = \frac{1}{1-c^2} = P_{++} \\ P(\Phi_-, \Phi_-) &= \left\langle \frac{1}{N} \mathrm{tr} \Phi_-^2 \right\rangle = \frac{1}{1-c^2} = P_{--} \\ P(\Phi_+, \Phi_-) = P(\Phi_-, \Phi_+) &= \frac{c}{1-c^2} = P_{+-} = P_{-+} \end{aligned} \tag{7.70}$$

We may identify the propagators with the link factors $e^{-\beta E_{++}}$, $e^{-\beta E_{--}}$, $e^{-\beta E_{+-}} = e^{-\beta E_{-+}}$, the Boltzmann factors associated to links. In the low temperature expansion of (7.67) the lowest energy state is the one with all spins pointing along the magnetic field H. Comparing the low temperature expansion of the Ising model and the perturbative expansion of (7.68) we find

$$Z(g, \beta, H) = \sum_n \left(\frac{-4gc}{(1-c^2)^2} \right) Z_n(\beta, H) \tag{7.71}$$

with

$$c = e^{-2\beta} \tag{7.72}$$

Using (5.70), we can evaluate (7.68) in terms of orthogonal polynomials. First write $Z(g, \beta, H)$ as

$$Z = \int \prod_i \left(dx_i dy_i w(x_i, y_i) \right) \Delta(x) \Delta(y) \tag{7.73}$$

where $\{x_i\}$, $\{y_i\}$ are the eigenvalues of Φ_+ and Φ_-, respectively, $\Delta(x)$ and $\Delta(y)$ are van der Monde determinants and $w(x, y)$ is the measure:

$$\begin{aligned} w(x, y) &= \exp \left[-x^2 - y^2 + 2cxy - \frac{g}{N} e^H x^4 - \frac{g}{N} e^{-H} y^4 \right] \\ \Delta(x) &= \prod_{i<j} (x_i - x_j) \end{aligned} \tag{7.74}$$

We can define monic polynomials

$$P_i(x) = x^i + p_{i-1}^{(i)} x^{i-1} + \cdots$$
$$Q_i(y) = y^i + q_{i-1}^{(i)} y^{i-1} + \cdots \tag{7.75}$$

orthogonal with respect to the measure (7.74):

$$\int dx dy w(x,y) P_i(x) Q_j(y) = h_i \delta_{ij} \tag{7.76}$$

Since

$$\Delta(x) = \det_{i,j=1,\ldots,N} P_{i-1}(x_j)$$
$$\Delta(y) = \det_{i,j=1,\ldots,N} Q_{i-1}(y_j) \tag{7.77}$$

we obtain as in the one–matrix model

$$Z = N! \prod_{i=0}^{N-1} h_i(\beta, g, H) \tag{7.78}$$

The recursion relation for P_i, Q_j takes the form

$$xP_i = P_{i+1} + r_i P_{i-1} + s_i P_{i-3} + \cdots$$
$$yQ_i = Q_{i+1} + t_i Q_{i-1} + u_i Q_{i-3} + \cdots \tag{7.79}$$

Since the measure satisfies $w(-x,-y) = w(x,y)$, the polynomials P_i and Q_j have well–defined parity. If $H = 0$, then $w(x,y) = w(y,x)$ and $P_i(x) = Q_i(x)$, but when $H \neq 0$ this is no longer true. The analogues of the string equations follow from the identities

$$\int dx dy w(x,y) \frac{dP_{i-1}(x)}{dx} Q_i(y) = 0$$
$$\int dx dy w(x,y) \frac{d}{dx}[P_i(x) - x^i] Q_{i-1}(y) = 0 \tag{7.80}$$
$$\int dx dy w(x,y) \frac{dP_{i-3}(x)}{dx} Q_i(y) = 0$$

and the equations obtained by exchanging $x \leftrightarrow y$ and $P \leftrightarrow Q$. The equations (7.80) lead to the recursion relations

$$h_i \left[1 + 2g\frac{e^H}{N}(r_{i+1} + r_i + r_{i-1})\right] - ct_i h_{i-1} = 0$$
$$h_{i-1}\left\{r_i + 2g\frac{e^h}{N}[s_{i+2} + s_{i+1} + s_i + r_i(r_{i+1} + r_i + r_{i-1})]\right\} - ch_i = \frac{1}{2} i h_{i-1} \tag{7.81}$$
$$2g\frac{e^H}{N} h_i - cu_i h_{i-3} = 0$$

and three other equations obtained by exchanging $H \leftrightarrow -H$, $r_i \leftrightarrow t_i$, $s_i \leftrightarrow u_i$. defining $f_i = h_i/H_{i-1}$ and taking the large N limit,

$$x_i = \frac{i}{N} \quad f(x) \simeq \frac{f_i}{N} \quad r(x) \simeq \frac{r_i}{N} \quad s(x) \simeq \frac{s_i}{N} \quad t(x) \simeq \frac{t_i}{N} \quad u(x) \simeq \frac{u_i}{N} \tag{7.82}$$

we are led to a set of algebraic equations

$$\begin{aligned} &f\left(1+6ge^{H}r\right) - ct = 0 \\ &f\left(1+6ge^{-H}t\right) - cr = 0 \\ &\left(2ge^{H}/c\right) f^3 - u = 0 \\ &\left(2ge^{-H}/c\right) f^3 - s = 0 \\ &r + 6ge^{H}(s+r^2) - cf - \frac{1}{2}x = 0 \\ &t + 6ge^{-H}(u+t^2) - cf - \frac{1}{2}x = 0 \end{aligned} \tag{7.83}$$

Boulatov and Kazakov [83] introduced the parameter $z = 6gf/c$ and found an explicit representation for the free energy:

$$F(z,c,H) = \log\frac{z}{g(z)} - \frac{1}{2g^2(z)}\int_0^{z/3}\frac{dz'}{z'}g^2(z') + \frac{1}{g(z)}\int_0^{z/3}\frac{dz'}{z'}g(z') \tag{7.84}$$

with

$$\begin{aligned} g(z) &= \frac{1}{9}c^2z^3 + \frac{z}{3}\left[\frac{1}{(1-z)^2} - c^2 + \frac{zB}{(1-z^2)^2}\right] \\ B &= 2(\cosh H - 1) \end{aligned} \tag{7.85}$$

When $H=0$ (pure Ising) the singularities in the free energy are determined by the zeroes of $g'(z)$. They are given by

$$g'(z, H=0) = 0 \Rightarrow \begin{cases} z_0 = -\frac{1}{3} \\ z_{1,2} = \frac{1}{3}(1 \mp i/\sqrt{c}) \\ z_{3,4} = \frac{1}{3}(1 \mp i/\sqrt{c}) \end{cases} \tag{7.86}$$

In the physical interval of temperatures $0 < c < 1$, only z_0 and z_1 define the asymptotics of F. The critical couplings

$$\begin{aligned} g_o &= -\frac{1}{12} + \frac{2}{9}c^2 \\ g_1 &= -\frac{2}{9}\sqrt{c}(\sqrt{c}-1)^2(\sqrt{c}+2) \end{aligned} \tag{7.87}$$

become equal at $c = 1/4$. This is the critical temperature of the spin–ordering phase transition. As z changes from 0 to $-\infty$, $0 < c < 1/4$, we encounter first the singularity at g_0. For $1/4 < c < 1$ the singularities g_0 and g_1 are interchanged. The asymptotics of $Z_n(c)$ are

$$\begin{aligned} Z_n(c) &\sim n^{-b}\left[-\frac{4cg_0(c)}{(1-c^2)^2}\right]^{-n} \qquad 0 \le c \le 1/4 \\ Z_n(c) &\sim n^{-b}\left[-\frac{4cg_1(c)}{(1-c^2)^2}\right]^{-n} \qquad 1/4 \le c \le 1 \end{aligned} \tag{7.88}$$

The critical temperature $\beta^* = \log 2$ corresponds to $c = 1/4$. When $c \neq 1/4$, the string susceptibility is $\gamma_{\mathrm{st}} = -1/2 = -b + 3$ whereas at $c = 1/4$, $\gamma_{\mathrm{st}} = -1/3$ as expected from (7.66). Similar arguments can be carried out when $H \neq 0$, and one obtains the dressed dimensions in (7.66). Details can be found in the original paper by D.V. Boulatov and V.A. Kazakov [83].

This concludes our analysis of models on planar lattices. Many other examples can be found in the literature. Next, we analyze these models and some others in the double scaling limit, where they can be identified as non–critical strings.

Lecture 8. Double Scaling Limit. Selected Topics

In this last lecture we will explore a number of results obtained during the last year in the exploration of the properties of the double scaling limit and sub-critical strings. It is virtually impossible to give a comprehensive review of the vast literature on this subject, and therefore choices must be made according to the taste and expertise of the author. The only explicit examples we present are the one–matrix models, the two–dimensional Ising model and the $c = 1$ theory. We will present the loop equation approach to two–dimensional gravity and the Virasoro constraints on the non–critical string partition functions. This result is quite mysterious and it is likely that important progress on the subject will come from an understanding of these constraints. There is also an important connection between $D < 1$ non–critical strings and integrable systems, and we have decided to present the double scaling limit of the Kazakov critical points from this viewpoint. This approach gives an alternative understanding of the Douglas equations [9] and the appearance of the reduced KP (Kadomtsev–Petyashvily) hierarchies in the double scaling limit.

8.1. Discrete Integrable Systems

The aim of this section is to show that the one–matrix models are equivalent to well–known discrete integrable systems together with some special initial conditions. These integrable systems are the Toda and Volterra latices [122, 123, 124, 125]. We will follow in this section the presentation in [123]. If the potential $V(\Phi)$ is generic *i.e.*, $V(\Phi) = \sum_{k\geq 2} g_k \Phi^k$ we are led to the Toda hierarchy [124, 125]. If, on the other hand, we consider only even potentials, $V(\Phi) = \sum_{k\geq 1} g_k \Phi^{2k}$ we obtain instead the Volterra hierarchy. We believe tat there are some pedagogical advantages to studying the even potential case, and we will concentrate in what follows mostly on this case. The generalization to Toda hierarchies is straightforward. All these systems are completely integrable by inverse scattering methods. Some useful general references on integrable systems and KdV and KP hierarchies are [126, 127, 128, 129, 130, 131, 132].

Let $H(q_i, p_i)$ $i = 1, \dots, n$ be a hamiltonian describing the dynamics of n particles. A classic theorem of Liouville states that the system is completely integrable if one can find n integrals of motion $I_i(p,q)$ in involution *i.e.,* with vanishing Poisson brackets. This means that we can find the action–angle variables in the problem and write the general solution to the equations of motion. A useful way to implement integrability was found by Lax. Suppose that we can find two matrices L and B such that the equations of motion are equivalent to

$$\frac{dL}{dt} = [B, L] \tag{8.1}$$

(For example, L and B are $n \times n$ matrices, although this theory generalizes to the case when L and B are operators.) If L is a symmetric matrix, then B is antisymmetric. In this cases, the conserved quantities are given by

$$I_k = \frac{1}{k} \mathrm{tr}\, L^k \tag{8.2}$$

and the eigenvalues of L are time–independent. We can construct an orthogonal matrix U, $U^T U = 1$, defined by

$$\frac{dU}{dt} = BU \tag{8.3}$$

such that

$$L(t) = UL(0)U^{-1} \tag{8.4}$$

Then

$$L_n \phi_n(t) = \lambda_n \phi_n(t) \tag{8.5}$$

and λ_n are t–independent. Since $\phi_n(t) = B\phi_n(0)$, we can write the Lax pair in terms of an auxiliary iso-spectral problem

$$\begin{aligned} L(t)\phi(t) &= \lambda\phi(t) \\ \frac{d\phi}{dt} &= B\phi(t) \end{aligned} \tag{8.6}$$

Requring that the spectrum of λ be time–independent we are led to (8.1). For each case we have to check whether the eigenvalues λ_i have commuting Poisson brackets. When this is the case, we have complete integrability. The basic non-trivial example is the Toda lattice [130], defined by the hamiltonian

$$\begin{aligned} H &= \frac{1}{2}\sum_n p_n^2 + \sum_n \left(e^{-(q_{n+1}-q_n)} - 1\right) \\ \dot{q}_n &= p_n \\ \dot{p}_n &= e^{-(q_n-q_{n-1})} - e^{-(q_{n+1}-q_n)} \end{aligned} \tag{8.7}$$

Following Flaschka, we introduce the variables (for references see [130])

$$\begin{aligned} a_n &= \frac{1}{2}e^{\frac{1}{2}(q_n-q_{n+1})} \\ b_n &= \frac{1}{2}p_n \end{aligned} \tag{8.8}$$

Then the equations of motion becomes

$$\begin{aligned} \dot{a}_n &= a_n(b_n - b_{n+1}) \\ \dot{b}_n &= 2(a_{n-1}^2 - a_n^2) \end{aligned} \tag{8.9}$$

and they admit a Lax pair representation:

$$L = \begin{pmatrix} b_1 & a_1 & & & & \\ a_1 & b_2 & a_2 & & & \\ 0 & a_2 & b_3 & a_3 & & \\ & & & \ddots & & \\ & & & & b_{n-1} & a_{n-1} \\ & & & & a_{n-1} & b_n \end{pmatrix} \tag{8.10}$$

$$B = \begin{pmatrix} 0 & -a_1 & & & & \\ a_1 & 0 & -a_2 & & & \\ 0 & a_2 & 0 & a_{-3} & & \\ & & & \ddots & & \\ & & & & 0 & -a_{n-1} \\ & & & & a_{n-1} & 0 \end{pmatrix} \tag{8.11}$$

This is the case with free boundary conditions. If we want to impose periodic boundary conditions, then L and B become

$$L = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_n \\ 0 & & & & 0 \\ \vdots & & \ddots & & \vdots \\ a_n & 0 & \cdots & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_n \\ 0 & & & & 0 \\ \vdots & & \ddots & & \vdots \\ -a_n & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{8.12}$$

We can write L and B in terms of matrix units,

$$\begin{aligned} L &= \sum_{i=1}^{n-1} a_i E_{i,i+1}^{+} + \sum_{i=1}^{n} b_i E_{ii} \\ B &= -\sum_{i\leq n-1} a_i E_{i,i+1}^{-} \end{aligned} \tag{8.13}$$

where

$$\left(E_{ij}\right)_{mn} = \delta_{im}\delta_{jn}$$
$$E^{\pm}_{ij} = E_{ij} \pm E_{ji} \tag{8.14}$$

It is possible to show that in this case the eigenvalues of L Poisson–commute, and it is possible to write the exact solution for $a_i(t)$ and $b_i(t)$ as functions of the initial conditions. Since we will not need the explicit form of the solution, we shall not write it down.

To understand the analogy between the Toda lattice and the one–matrix models, we write the auxiliary linear problem (8.6) in components:

$$\begin{aligned} \lambda\phi_n &= (L\phi)_n = a_{n+1}\phi_{n+1} + b_n\phi_n + a_{n-1}\phi_{n-1} \\ \frac{d\phi_n}{dt} &\equiv \dot{\phi}_n = a_{n-1}\phi_{n-1} - a_n\phi_{n+1} \end{aligned} \tag{8.15}$$

which is reminiscent of the recursion relation for orthogonal polynomials. We will see that this analogy is not a mere coincidence. Although we have written the simplest time evolution equations (8.7,8.8,8.9), after noticing that the hamiltonian $H = H_2 = \operatorname{tr} L^2/2$, we could instead define the time evolution in terms of the higher conservation laws H_k. These evolution equations are known as the "higher flows" for the Toda hierarchy

$$\begin{aligned} \frac{\partial a_n}{\partial t_k} &= \{H_k, a_n\} \\ \frac{\partial b_n}{\partial t_k} &= \{H_k, b_n\} \end{aligned} \tag{8.16}$$

In the case of even potentials, $b_n = 0$ and (8.15) is inconsistent. This means that we must instead consider the flow equations with respect to a higher hamiltonian. To get some familiarity with the structures involved, consider the $b_n = 0$ case and borrow the notation from the orthogonal polynomials in lecture 5:

$$\lambda\phi_n = (L\phi)_n = \sqrt{R_{n+1}}\phi_{n+1} + \sqrt{R_n}\phi_{n-1} \tag{8.17}$$

that is to say

$$L = \sum_{k\geq 1} \sqrt{R_k} E^{+}_{k,k+1} \tag{8.18}$$

Now $\mathrm{tr}L^{2p+1} = 0$, and the simplest Lax pair representation for an integrable evolution is obtained from

$$B = \sum_k \sqrt{R_k R_{k+1}} E^-_{k,k+2} \tag{8.19}$$

This form of B follows from the second Toda flow where one can consistently set $b_n = 0$ and still obtain non-trivial equations of motion:

$$\frac{\partial}{\partial t} R_m = R_m(R_{m+1} - R_{m-1}) \tag{8.20}$$

An important property of (8.20) is that in the naive continuum limit (and up to a galilean transformation on t), $R_m \sim 1 - a^2 u(x)$, and (8.20) becomes the KdV equation

$$\frac{\partial u}{\partial t} = 6u \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} \tag{8.21}$$

which is a well-known classical integrable system. We would expect that many of its properties also hold for the system (8.18,8.19), known as the Volterra equation (or Volterra hierarchy). The first three conserved quantites are

$$\begin{aligned} H_1 &= \frac{1}{2}\mathrm{tr}\, L^2 = \sum_n R_n \\ H_2 &= \frac{1}{4}\mathrm{tr}\, L^4 = \sum_n \frac{1}{2} R_n^2 + R_n R_{n+1} \\ H_3 &= \sum_n \left(\frac{1}{3} R_n^3 + R_n^2 R_{n+1} + R_n R_{n+1}^2 + R_n R_{n+1} R_{n+2} \right) \end{aligned} \tag{8.22}$$

The hamiltonian structure which out of H_1 produces the Volterra equation (8.20) is

$$\{R_n, R_m\}_1 = R_n R_m \left(\delta_{n,m+1} - \delta_{n,m-1}\right) \tag{8.23}$$

It is left as an exercise to verify the Jacobi identity for (8.23).

To compute the explicit form of H_k and other quantities we shall need later, we must have a procedure to evaluate the matrix elements of L_k. This is done by generalizing the staircase quations. Since $L_{ij} \neq 0$ only for $|i-j| \leq 1$ and L is symmetric, we say that L is a Jacobi matrix of locality one. L involves only $E^+_{k,k+1}$, but L^2 will involve $E^+_{k,k+2}$

and E^+_{kk}, and L^3 will contain pieces proportional to $E^+_{k,k+3}$, $E^+_{k,k+1}$. In general, L^p has a matrix structure proportional to $E^+_{k,k+p}$, $^+_{k,k+p-2}$, $\ldots$, $E^+_{k,k+1}$ (for p odd), or E_{kk} (for p even). To compute the coefficient of $E^+_{k,k+n}$ in L^p, we draw all staircase paths from k to $k+n$ in p steps according to the rules (5.62).

Using these rules, one easily obtains

$$L = \sum_k \sqrt{R_k} E^+_{k,k+1} \tag{8.24}$$

$$L^2 = \sum_k \sqrt{R_k R_{k+1}} E^+_{k,k+2} + \sum_k (R_k + R_{k-1}) E_{kk} \tag{8.25}$$

$$L^3 = \sum_k \sqrt{R_k R_{k+1} R_{k+2}} E^+_{k,k+3} + \sum_k \sqrt{R_k}(R_{k+1} + R_k + R_{k-1}) E^+_{k,k+1} \tag{8.26}$$

$$\begin{aligned} L^4 = \sum_k \Big\{ & (R_k R_{k+1} R_{k+2} R_{k+3})^{\frac{1}{2}} E^+_{k,k+4} \\ & + (R_k R_{k+1})^{\frac{1}{2}} (R_{k+2} + R_{k+1} + R_k + R_{k-1}) E^+_{k,k+2} \\ & + (R_k R_{k+1} + R_{k-1} R_{k-2} + R_k^2 + R_{k-1}^2 + 2 R_k R_{k-1}) E_{kk} \Big\} \end{aligned} \tag{8.27}$$

etcetera. Now we prove the following

Theorem 1 [123]. The k–th flow of the Volterra hierarchy can be written as a Lax pair

$$\frac{\partial L}{\partial t_k} = \left[\left(L^{2k} \right)_+, L \right] \tag{8.28}$$

where $(L^{2k})_+$ is the antisymmetric matrix whose upper triangular part coincides with L^{2k}.

A first check that this theorem is reasonable is to verify that the only non–vanishing entries in the right–hand side of (8.28) are $[(L^{2k})_+, L]_{m,m\pm 1}$ as required by the fact that $\partial L/\partial t_k$ only contains matrices of the form $E^+_{m,m+1}$. The proof goes as follows:

$$\begin{aligned} \frac{\partial}{\partial t_k} R_m &= -\left\{ R_m, \frac{1}{2k} \operatorname{tr} L^{2k} \right\}_1 \\ &= -\frac{1}{2k} \sum_{\ell_1,\ldots,\ell_{2k}} \left\{ R_m, (R_{\ell_1} \cdots R_{\ell_{2k}})^{\frac{1}{2}} \right\} \operatorname{tr} E^+_{\ell_q,\ell_1+1} \cdots E^+_{\ell_{2n},\ell_{2n+1}} \end{aligned} \tag{8.29}$$

Using the general properties of Poisson brackets, equation (8.23), and the cyclicity of the trace, equation (8.29) becomes

$$\begin{aligned}\frac{\partial R_m}{\partial t_k} &= \frac{1}{2} R_m \operatorname{tr}\left(R_{m+1}^{\frac{1}{2}} E_{m+1,m+2}^{+} - R_{m-1}^{\frac{1}{2}} E_{m-1,m}^{+}\right) L^{2k-1} \\ &= \operatorname{tr} E_{m+1,m}\left[\left(L^{2k}\right)_{+}, L\right]\end{aligned} \tag{8.30}$$

where the last equality uses only the definition of $(L^{2k})_{+}$. We can also write

$$\begin{aligned}\frac{\partial R_m}{\partial t_k} &= \frac{1}{2} R_m \operatorname{tr} \delta_m^{(1)} L\, L^{2k-1} \\ \delta_m^{(1)} L &\equiv R_{m+1}^{\frac{1}{2}} E_{m+1,m+1}^{+} - R_{m-1}^{\frac{1}{2}} E_{m-1,m}^{+}\end{aligned} \tag{8.31}$$

Theorem 2. There is a second hamiltonian structure [131] for the Volterra hierarchy compatible with the first one:

$$\begin{aligned}\{R_n, R_m\}_2 =& R_n R_m (R_n + R_m)(\delta_{n.,m+1} - \delta_{n,m-1}) \\ &+ R_n R_m (\delta_{n,m+2} R_{m+1} - \delta_{n,m-2} R_{m-1})\end{aligned} \tag{8.32}$$

Since (8.32) is antisymmetric, we have to check the Jacobi identity. This is an unpleasant algebraic exercise. Using that the only non–trivial brackets are

$$\begin{aligned}\{R_{m+1}, R_m\}_2 &= R_m R_{m+1}(R_m + R_{m+1}) \\ \{R_{m-1}, R_m\}_2 &= -R_m R_{m-1}(R_m + R_{m-1}) \\ \{R_{m+2}, R_m\}_2 &= R_m R_{m+1} R_{m+2} \\ \{R_{m-2}, R_m\}_2 &= -R_m R_{m-1} R_{m-2}\end{aligned} \tag{8.33}$$

it is not difficult to check that the non–trivial relations implied by the Jacobi identity such as

$$\{R_m, \{R_{m+1}, R_{m+2}\}_2\}_2 + \text{cyclic permutations} = 0 \tag{8.34}$$

are satisfied. The compatibility between the two Poisson structures is equivalent to the statement that

$$\{R_n, R_m\}_{\lambda,\mu} = \lambda\{R_n, R_m\}_1 + \mu\{R_n, R_m\}_2 \tag{8.35}$$

again satisfies the Jacobi identity. Its proof follows after a lengthy algebraic exercise.

Theorem 3 [123]. The following identity (analogous to the Gel'fand–Dikĭi [133] relation for the KdV hierarchy) is satisfied:

$$\{H_n, R_m\}_2 = \{H_{n+1}, R_m\}_1 \tag{8.36}$$

Define the flows with respect to the second hamiltonian structure according to

$$\begin{aligned} \frac{\partial R_m}{\partial T_k} &= -\{R_m, H_n\}_2 \\ &= \frac{1}{2} R_m \operatorname{tr} \Big[R^{\frac{1}{2}}_{m+1}(R_m + R_{m+1}) E^+_{m+1,m+2} \\ &- R^{\frac{1}{2}}_{m-1}(R_m + R_{m-1}) E^+_{m-1,m} + R_m R_{m+1} R^{\frac{1}{2}}_{m+2} E^+_{m+2,m+3} \\ &- R_m R_{m-1} R^{\frac{1}{2}}_{m-2} E^+_{m-2,m-1} \Big] L^{2n-1} \end{aligned} \tag{8.37}$$

Writing

$$\frac{\partial}{\partial t_{n+1}} R_m = \operatorname{tr} \delta^{(1)}_m L\, L^{2n+1} = \operatorname{tr} L\, \delta^{(1)}_m L\, L\, L^{2n-1} \tag{8.38}$$

one obtains after some simple computations

$$2\left(R^{-1}_m \frac{\partial R_m}{\partial T_n} - R^{-1}_m \frac{\partial R_m}{\partial t_{n+1}} \right) = 2 R^{\frac{1}{2}}_m \left([L^2, L^{2n-1}] \right)_{m,m+1} = 0 \tag{8.39}$$

Theorems 1, 2 and 3 capture completely the integrability property of the model. Notice that given H_1 and assuming homogeneity of H_2 as a function of the R_n's, we can use (8.36) to compute H_2 from H_1. Once H_2 is known, we can use (8.36) again to compute H_3 and so on. These hamiltonians are all commuting, again as a consequence of the theorems. For instance,

$$\begin{aligned} &\{H_2, H_1\}_1 = \{H_1, H_1\}_2 = 0 \\ &\{H_3, H_1\}_1 = \{H_2, H_1\}_2 = -\{H_1, H_2\}_2 = -\{H_2, H_2\}_1 = 0 \end{aligned} \tag{8.40}$$

Hence two compatible Poisson structures satsifying the Gel'fand–Dikĭi relation (8.36) generate all the higher flows from H_1. These properties will play an important role later, when we analyze the continuum limit. We only point out now that the continuum limit of the second Poisson structure gives the Virasoro algebra, an observation made some years ago by Fadeev and Takhtajan in the context of classical Liouville theory [53]. Similar results can be worked out for the Toda theory.

8.2. One–matrix Models as Integrable Systems

Given some even potential $V(\lambda) = \sum g_p \lambda^{2p}$ and a basis of orthonormal polynomials $P_n(\lambda)$ with respect to the measure $e^{-V(\lambda)}$, the operation of multiplication by λ is represented by a Jacobi matrix L of the same form as (8.18). As we change the couplings g_i, the polynomials $P_n(\lambda)$ also change:

$$\frac{\partial P_n}{\partial g_i} = (M_i)_{n\ell} P_\ell \tag{8.41}$$

Since λ is independent of the couplings, differentiating $\lambda P_n = L_{nk} P_k$ we obtain

$$\frac{\partial L}{\partial g_i} = [M_i, L] \tag{8.42}$$

The couplings g_i are all independent and therefore, we can vary them independently, implying that the g_i–flows commute. In terms of the matrices M_i this implies

$$\frac{\partial}{\partial g_i} M_j - \frac{\partial}{\partial g_j} M_i - [M_i, M_j] = 0 \tag{8.43}$$

In the space of couplings $A = M_i dg^i$ represents a flat $O(n)$ gauge field and (8.42) means that L changes by parallel transport in the presence of the gauge connection $A(g)$. The matrices M_i are antisymmetric and they should be local to have a continuum limit. In this context, locality means that $(M_i)_{mn} = 0$ for $|m-n| > p$ finite. We could modify M_i by adding a symmetric matrix X. However, the symmetry of $\partial L/\partial g_i$ requires $[X, L] = 0$. If L is generic (all its eigenvalues are distinct) then X has to be a polynomial in L and therefore it is irrelevant in the construction of the flows. The matrices M_i are determined by the conditions (1) M_i is antisymmetric, (2) $[M_i, L]$ is a Jacobi matrix of locality one, and (3) M_i is homogeneous of degree $2i$ in the matrix elements of L. This is so because $V = \sum g_p \lambda^{2p}$ is unchanged under $\lambda \to \alpha\lambda$ accompanied by $g_p \to g_p \alpha^{-2p}$. These three conditions fix M_i to be given by L_+^{2i}, the matrices defined in the previous section:

$$\frac{\partial L}{\partial g_i} = [L_+^{2i}, L] \tag{8.44}$$

The flows are generated only by the even powers L_+^{2i}. We know that the only non–vanishing matrix elements of $[L_+^{2i}, L]$ are $(m, m \pm 1)$. If instead we considered $[L_+^{2i+1}, L]$, only the diagonal matrix elements would be different from zero. Consequently, L_+^{2i} generates flows and L_+^{2i+1} provides initial conditions. This canot be seen so clearly in the continuum limit, where the operators generated by $[L_+^{2n}, L]$ and $[L_+^{2n+1}, L]$ differ only by a total derivative.

In conclusion, $L(g_i)$ is an orbit of the Volterra flows. What is left to do is to determine the initial conditions for the flows. If $V(\lambda)$ is a polynomial, then the operator $d/d\lambda$ is also local (in the Jacobi sense), and it can be chosen antisymmetric. Write

$$\frac{d}{d\lambda} P_k = \sum N_{k,r} P_r \tag{8.45}$$

Then

$$\begin{aligned} 0 &= \int d\lambda \frac{d}{d\lambda}\left(e^{-V} P_k P_\ell\right) \\ &= -\int d\lambda e^{-V(\lambda)} V'(\lambda) P_k P_\ell + N_{k\ell} + N_{\ell k} \end{aligned} \tag{8.46}$$

and hence $N - V'(L)/2$ is antisymmetric, and if n =degree(V), then $N_{kr} = 0$ unless $|k-r| \leq n-1$. We can choose N to be antisymmetric. Therefore, we can expand N as a linear combination of L_+^p. Since $[N, L] = \mathbf{1}$, this means that only odd powers of p appear, and therefore $[N, L] = \mathbf{1}$ becomes a set of initial conditions for the Volterrra hierarchy

$$\sum_{j=1}^{n} 2j g_j^{(0)} [L_+^{2j-1}, L] = \mathbf{1} \tag{8.47}$$

In conclusion, the generic one–matrix model is equivalent to a Volterra hierarchy with a particular initial condition (8.47). It is possible to show that this initial condition does not intersect any of the multi-soliton sectors of the Volterra equation. To study the mechanical properties of this system, we may define a Volterra equation with a finite number of points, instead of considering the equation on the semi-infinite line. For a finite number of particles the initial condition (8.47) becomes

$$[N, L] = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ \vdots & & & \ddots & \\ 0 & & & & -n+1 \end{pmatrix} \tag{8.48}$$

The converse of this result can also be proven. Starting with the Volterra hierarchy and taking into account that $R_n > 0$, we can use Favard's theorem [107] which guarantees that, given a set $\phi_n(\lambda)$ $(N \geq 0)$ satisfying $\lambda\phi_n = \phi_{n+1} + R_n\phi_{n-1}$ with $R_n > 0$, we can always find a measure $d\lambda e^{-V(\lambda)}$ making the ϕ_n orthogonal. This measure is unique up to an overall constant depending on the normalization of the ϕ_n's. The initial condition (8.47) implies that $V(\lambda) = \sum g_p^{(0)} \lambda^{2p}$. The string equations can be written in hamiltonian form:

$$\sum_j 2jg_jR_m \frac{\partial H_j}{\partial R_m} = m\beta^{-1} \tag{8.49}$$

where, as in previous lectures, we have taken the potential to be $V(\lambda) = \beta \sum g_p \lambda^{2p}$ with $\beta = e^{\Lambda_B}$, and Λ_B the bare cosmological constant. In terms of the partition function $Z = \prod_n R_n^{N-n}$, equation (8.49) becomes

$$\begin{gathered} \sum_j jg_j \frac{\partial \log Z}{\partial u_m} = m\beta^{-1} \\ u_m = \log R_m \end{gathered} \tag{8.50}$$

Next, we want to show that in the double scaling limit the Volterra hierarchy turns into the KdV hierarchy. In this way, we will reproduce directly the results in the original papers on the double scaling limit. The strategy is to use the fact that in the KdV hierarchy we also have a Gel'fand–Dikĭi [133] relation giving recursion relations among different members of the hierarchy. We recapitulate these results here. The original KdV equation is defined in terms of a Schrödinger operator

$$L = \frac{d^2}{dx^2} - u(x) \tag{8.51}$$

and a Lax pair representation

$$\frac{\partial L}{\partial t} = [B, L] \tag{8.52}$$

with B a third-order differential operator

$$\begin{aligned} B &= D^3 - \frac{3}{2}uD - \frac{3}{4}u' \\ D &\equiv \frac{\partial}{\partial x} \qquad U' \equiv \frac{\partial u}{\partial x} \end{aligned} \tag{8.53}$$

For the higher flows it can be shown in general that

$$\frac{\partial L}{\partial t_n} = [L_+^{n+\frac{1}{2}}, L] \tag{8.54}$$

where $L^{\frac{1}{2}}$ is defined as the pseudo-differential operator with expansion

$$L^{\frac{1}{2}} = D + q_1 D^{-1} + q_2 D^{-2} + \cdots \tag{8.55}$$

and for any pseudo-differential operator Q, Q_+ means that we keep only its differential part. Notice that in the scaling limit the operator of multiplication by λ becomes a second-order differential operator. The two-step recursion relation is a discrete version of the Schrödinger operator (8.51). The equations (8.54) can also be derived from a collection of commuting hamiltonians H_n. There are again two Poisson structures. The first one is

$$\{u(x), u(y)\}_1 = \frac{1}{2}\left(\partial_x - \partial_y\right)\delta(x-y) \tag{8.56}$$

and the second one is the classical version of the Virasoro algebra:

$$\{u(x), u(y)\}_2 = \left(\frac{1}{2}D^3 + 2uD + u'\right)\delta(x-y) \tag{8.57}$$

These two Poisson structures are compatible and we can construct the commuting hamiltonians using the analogue of the Gel'fand–Dikĭi relation. Using the first hamiltonian structure,

$$\frac{\partial u}{\partial t_n} = D\frac{\delta H_n}{\delta u(x)} \equiv DR_{n+1}[u] \tag{8.58}$$

and the Gel'fand–Dikĭi relation now implies

$$DR_{k+1} = \left(\frac{1}{2}D^3 - 2uD - u'\right)R_k(u) \tag{8.59}$$

Starting with the simplest possible $R_0 = 1$, we obtain all others $R_1 = -u$, $R_2 = \frac{3}{2}u^2 - \frac{1}{2}u''$, etc. From lecture 6 we know that R_m is not a good scaling variable. For the k–th multi-critical point we had

$$\begin{aligned} x = \frac{n}{\beta} &= 1 - \beta^{-\frac{2k}{2k+1}} t \\ R &= 1 - \beta^{-\frac{2}{2k+1}} f(t) \end{aligned} \tag{8.60}$$

In terms of the variable $u_m = \log R_m$ and setting $\lambda = \beta^{-1/2k+1}$, the first Poisson structure

$$\{u_n, u_m\} = \delta_{n,m+1} - \delta_{n,m-1} \tag{8.61}$$

becomes in the continuum limit $u_m \to -\lambda^2 f_m \to -\lambda^2 f(t)$:

$$\{f(t), f(t')\}_1 = \lim_{\lambda \to 0} \lambda^2 \{f_n, f_m\} \tag{8.62}$$

Next we construct the scaling limit of the second Poisson structure. To obtain a second Poisson structure with good scaling properties, we have to subtract the first Poisson structure with a coefficient depending appropriately on the scaling parameter λ. This is legitimate because the two hamiltonian structures are compatible for any value of λ. thus the second hamiltonian structure becomes in the continuum limit:

$$\{f(t), f(t')\}_2 = \left(\frac{1}{2}D^3 - 2fD - \dot{f}\right) \delta(t - t') \tag{8.63}$$

and now we can immediately construct the combination of hamiltonians H_n with the good scaling properties (with respect to (8.60)). These hamiltonians are the ones which will give the string equations at each of the multi-critical points:

$$\mathcal{H}_k = \sum_{j=1}^{k} 2j \overline{g}_j^{(k)} H_j \tag{8.64}$$

where $\overline{g}_j^{(k)}$ are the couplings chosen to give good scaling properties. We actually do not need to do any work to find the explicit form of $\mathcal{H}_k$. It is given automatically by the Gel'fand–Dikiĭ relation in the continuum limit. For the k–th multi-critical point the string equation (8.49) is written as

$$\sum_j 2j \overline{g}_j^{(k)} \frac{\partial H_j}{\partial u_m} = m\beta^{-1} = \frac{\partial \mathcal{H}_k}{\partial u_m} \tag{8.65}$$

which in the scaling limit becomes, up to some constants,

$$\delta\mathcal{H}_k/\delta f(t)$$

and $\mathcal{H}_k$ is the KdV hamiltonian containing a term proportional to f^k. Hence the string equation for the k–th Kazakov multi-critical point becomes

$$R_k[f] = t \tag{8.66}$$

Defining now $D \equiv d/dt$, we can also differentiate (8.66) to obtain

$$\frac{\partial f}{\partial t_k} = DR_k[f] = 1 \tag{8.67}$$

and using the Lax pair representation of the KdV hierarchy this implies

$$\frac{\partial L}{\partial t_k} = [L_+^{k+\frac{1}{2}}, L] = 1 \tag{8.68}$$

Hence, in the continuum limit the operator $d/d\lambda$ becomes $L_+^{k+\frac{1}{2}}$. If we write

$$Q \equiv L \qquad P \equiv L_+^{k+\frac{1}{2}} \tag{8.69}$$

then (8.68) takes the form of a Heisenberg algebra

$$[P, Q] = 1 \tag{8.70}$$

The string equations were written in this form for all multi-matrix models in a beautiful paper by M. Douglas [9]†. Once we know the scaling operators $\mathcal{H}_k$, we can perturb the critical point by adding them with arbitrary couplings

$$\sum_{\ell,k} \mu_\ell \alpha_\ell^{(k)}(\beta)\frac{\partial\mathcal{H}_k}{\partial u_m} = m\beta^{-1} \tag{8.71}$$

where the constants $\alpha_\ell^{(k)}(\beta)$ have a well–defined scaling limit. The answer one obtains is then

$$\sum_{n=1}^{\infty}(2n+1)t_n R_n[u] = -t \tag{8.72}$$

† The string equations (8.70) can be derived from an action principle [134].

where $t_k = -1/(2k+1)$, $t_n = 0$ for $n \neq k$, defines the k–th critical point.

Before we write explicitly the form of (8.66) for the first few values of k, we sould like to comment on Douglas's derivation of (8.70) [9]. The starting point is a chain of p $N \times N$ matrix partition functions:

$$Z_p = \int \prod_{t=1}^{p} M(t) \exp \left(-\sum_{i=1}^{p} \operatorname{tr} V_t(M(t)) + \sum_{t=1}^{p-1} e_t \operatorname{tr} M(t)M(t+1) \right) \qquad (8.73)$$

Using (5.70), Z_p becomes

$$\begin{gathered} Z_p = \int \prod_{i,t} d\lambda_i(t) \Delta(\lambda(1)) \Delta(\lambda_p)) \exp - \sum_{i,t} S_t[\lambda_i] \\ S_t[\lambda] = V_t(\lambda(t)) + e_t \lambda(t) \lambda(t+1) \end{gathered} \qquad (8.74)$$

In the basis of orthogonal polynomials $P_n(x)$, $Q_m(y)$

$$\begin{gathered} \int d\mu(\lambda_1, \ldots, \lambda_p) P_n(\lambda_1) Q_m(\lambda_p) = h_n \delta_{n,m} \\ d\mu(\lambda_i) = \left(\prod d\lambda_i \right) \exp - \sum_t S_t[\lambda] \end{gathered} \qquad (8.75)$$

we can again construct the operators of multiplication by λ and differentiation with respect to it, $d/d\lambda$. For polynomial potentials $V_t(M)$ we will obtain local Jacobi matrix representations for λ and $d/d\lambda$, and in the continuum limit they will become differential operators satisfying (8.70). If we choose two relatively prime integers (p, q) with $p > q$, the operator λ will be represented by an operator

$$Q = D^q + u_{q-2} D^{q-2} + \cdots + u_0 \qquad (8.76)$$

and we can take

$$P = \left(Q^{p/q} \right)_+ \qquad (8.77)$$

since

$$\left[Q_+^{p/q}, Q \right] = 1 \qquad (8.78)$$

This gives a set of non–linear differential equations in the coefficients $u_0, \ldots u_{q-2}$, which should represent the string equations for the (q, p) minimal models coupled to two–dimensional gravity.

There is a subtlety here which is important to stress. The string susceptibility is the ratio of the scaling dimension of u_{q-2} to that of the string equation involving t, equation (8.66). If we take $p = q+1$ as in the minimal unitary models, the answer agrees with the KPZ formula $\gamma_{\rm st} = -1/q$. However, if we take $p \neq q+1$, matrix models lead us to

$$\gamma_{\rm st} = -\frac{2}{p+q-1} \tag{8.79}$$

which disagrees with the KPZ formula

$$\gamma_{\rm st} = -\frac{1}{q}(p-q) \tag{8.80}$$

The two values agree under the assumption that the most relevant parameter controlling shifts of t couples to the operator with the most negative dimension. When $p = q+1$, the lowest dimension operator is the identity and there is no problem. For non–unitary theories, however, a better understanding of the origin of the disagreement between (8.79) and (8.80) would be most welcome. Very recently, I. Kostov [93] has solved the planar limit of strings embedded in Dynkin diagrams (the Pasquier IRF or RSOS models [135] coupled to two–dimensional gravity) and has been able to show independently of matrix models that the KPZ table of dressed conformal dimensions for the (p,q) models is reproduced exactly. Hence, in this construction of the non–unitary (p,q)–models, the cosmological constant is coupled to the identiyy and not to the operator of minimal dimension. It would be very interesting to work out these models over non–planar topologies to see if any instabilities set in.

To finish this long section, we give a few examples of string equations. The computation of the coefficients $R_k[f]$ can be carried out by iterating the Gel'fand–Dikiĭ relation starting with the simplest case. They can also be obtained in terms of the coefficients of the heat–kernel expansion of the operator $-d^2/dt^2 + f(t)$. We give only the answer, details can be found in the literature:

$$\begin{aligned} k=1 \quad & t=f \\ k=2 \quad & t=f^2-\frac{1}{3}\ddot{f} \\ k=3 \quad & t=f^3-f\ddot{f}-\frac{1}{2}\dot{f}^2+\frac{1}{10}f^{(4)} \\ k=4 \quad & t=f^4-2f\dot{f}^2-2f^2\ddot{f}+\frac{3}{5}\ddot{f}^2+\frac{4}{5}\dot{f}f^{(3)}+\frac{2}{5}ff^{(4)}-\frac{1}{35}f^{(6)} \end{aligned} \tag{8.81}$$

A very intriguing property of the partition function of multi-critical points is that it satisfies a set of Virasoro constraints. They were found by Fukuma, Kawai and Nakayama [136], and by Dijkgraaf, Verlinde and Verlinde [137]. Without going into much detail, the statement is that $Z(t_0, t_1, t_2, \ldots)$, *i.e.*, the partition function including all scaling operations, is the square of the τ–function for the KdV hierarchy:

$$Z(t_0, t_1, \ldots) = \tau^2(t_0, t_1, \ldots) \qquad t_0 \equiv t \tag{8.82}$$

A function $\tau(t_0, \ldots)$ is said to be a τ–function for the KdV hierarchy if $u = 2D^2 \log \tau$ ($D = d/dt_0$) is a solution to the KdV flows $\partial u/\partial t_n = DR_{n+1}[u]$. Hence

$$D^2 \frac{\partial}{\partial t_{n+1}} \log \tau = \left(\frac{1}{2}D^3 + 2uD + Du\right) D \frac{\partial}{\partial t_n} \log \tau \tag{8.83}$$

The string equations can be written as

$$\begin{aligned} & L_{-1}\tau = 0 \\ & L_{-1} = \sum_{m=1}^{\infty} (m+\frac{1}{2}) t_m \frac{\partial}{\partial t_{m-1}} + \frac{1}{8} t_0^2 \end{aligned} \tag{8.84}$$

Using the fact that τ satisfies (8.83), it is possible furthermore to show that

$$L_n \tau = 0 \qquad n \geq -1 \tag{8.85}$$

with the L_n operators defined as

$$\begin{aligned} L_{-1} &= \sum_{m=1}^{\infty} (m+\frac{1}{2}) t_m \frac{\partial}{\partial t_{m-1}} + \frac{1}{8} t_0^2 \\ L_0 &= \sum_{m=0}^{\infty} (m+\frac{1}{2}) t_m \frac{\partial}{\partial t_{m-1}} + \frac{1}{16} \\ L_n &= \sum_{m=0}^{\infty} (m+\frac{1}{2}) t_m \frac{\partial}{\partial t_{m-1}} + \frac{1}{2} \sum_{m=1}^{n} \frac{\partial^2}{\partial t_{m-1} \partial t_{n-m}} \end{aligned} \tag{8.86}$$

It is an easy exercise to verify that

$$[L_n, L_m] = (n-m)L_{n+m} \qquad n, m \geq 1 \tag{8.87}$$

and the conditions (8.85) are due to the fact that the KdV relation (8.83) implies the recursion relation

$$D^2\left(\frac{L_{n+1}\tau}{\tau}\right) = \left(\frac{1}{2}D^3 + 2uD + Du\right) D\left(\frac{L_n\tau}{\tau}\right) \tag{8.88}$$

This, together with the string equation (8.84), implies (8.85). The expressions (8.86) have the same form as the Virasoro generators of a $\mathbf{Z}_2$–twisted free scalar field $\phi(z)$ *i.e.*, satisfying $\phi(e^{2\pi i}z) = -\phi(z)$. Expanding ϕ in modes,

$$\partial\phi(z) = \sum_{n\in\mathbf{Z}} \alpha_{n+\frac{1}{2}} z^{-n-\frac{3}{2}} \tag{8.89}$$

The corresponding conformal energy–momentum tensor is

$$T(z) =: \frac{1}{2}\partial\phi(z)^2 : + \frac{1}{16z^2} = \sum L_n z^{-n-2} \tag{8.90}$$

Making the correspondences

$$\begin{aligned} \alpha_{-n-\frac{1}{2}} &\rightarrow (n+\frac{1}{2})t_n \\ \alpha_{n+\frac{1}{2}} &\rightarrow \frac{\partial}{\partial t_n} \end{aligned} \tag{8.91}$$

we obtain (8.86) from (8.90). For multi–matrix models, the Virasoro constraints seem to be replaced by W–algebra constraints [136, 137, 138]. It seems that the correct interpretation of the twisted field $\phi(z)$ is a string field for $d < 1$ strings. A lot of effort is now being directed towards the understanding of the Virasoro constraints [139, 140, 141, 15, 142].

The double scaling limit for $c = 1$ and the Ising model are presented in the next two sections. Later, we sill study in more detail the Painlevé equation for pure gravity and the loop space approach to string theory, which will allow us to gain some understanding of the non–perturbative constraints fixing the solution to this equation.

8.3. Double Scaling Limit for $c = 1$

The literature on the $c = 1$ theory in the double scaling limit is large and growing. THe planar limit [91] was presented in the previous lecture. A probably incomplete list of investigations on this particular subject is [143, 144, 121, 145], [146, 147, 148, 149], [150, 151, 152, 153], [154, 155, 156, 157].

After the discovery of the double scaling limit for matrix models, several papers appeared dealing with the double scaling limit of the $c = 1$ theory. Continuing the analysis of the previous lecture, we have to take $N \to \infty$ and $\Delta \to 0$ so that $g_{\rm st}$ is kept fixed. From the pure gravity case we expect the genus h contribution to behave according to $N^{-2h}\Delta^{-(2-\gamma_{\rm st})h}$ up to logarithmic corrections.

In the planar limit we found the singularity in the density of states to be sensitive only to the behaviour of the potential near its maximum. Now we represent the density of states by

$$\rho(\mu_F) = \frac{1}{\pi\beta}{\rm Im}\frac{1}{\hat{h} - \mu_F - i\varepsilon} \tag{8.92}$$

where $\hat{h}$ is defined in (7.56). Expanding the denominator about the maximum of the potential, $y = \lambda_c - \lambda$, we find

$$\hat{h} - \mu_F \sim -\frac{1}{2\beta^2}\frac{\partial^2}{\partial y^2} + \mu - 2y^2 + O(y^3) \tag{8.93}$$

In the scaling limit $\beta \to \infty$, $\mu \sim 1/\beta$ and hence for $y^2 \sim 1/\beta$ the cubic and higher terms may be neglected and we are left with the computation of $\rho(\mu_F)$ for an inverted harmonic oscillator. A simple way of obtaining $\rho(\mu_F)$ is to analytically continue in frequency the answer for a standard harmonic oscillator. With the convention $U''(\lambda_c) = -4$ we have

$$\rho(\mu_F) = \frac{1}{\pi}{\rm Re}\sum_n \frac{1}{2n + 1 + i\beta\mu} \tag{8.94}$$

Even though this expression is divergent, its divergence is μ–independent and we can choose a finite renormalization such that as $\beta \to \infty$, $\rho(\mu_F) \sim -(2\pi)^{-1}\log\mu$ as in the planar case. The answer is then

$$\rho(\mu_F) = \frac{1}{2\pi}{\rm Re}\left[\zeta\left(1, \frac{1 + i\beta\mu}{2}\right) - \infty\right] \tag{8.95}$$

where

$$\zeta(z,q) = \sum_{n=0}^{\infty}(n+q)^{-z} \tag{8.96}$$

Expanding in $1/\beta\mu$ and defining the divergent series $\sum(2n+1)^k$ in terms of ζ–functions, we have

$$\rho(\mu_F) = \frac{1}{2\pi}\left(-\log\mu + 2\sum_{p=1}^{\infty}(-1)^k\left[\frac{1}{\beta\mu}\right]^{2k}(2^{2k-1}-1)\zeta(1-2k)\right) \tag{8.97}$$

which can be expressed in terms of Bernouilli numbers:

$$\rho(\mu_F) = \frac{1}{2\pi}\left[-\log\mu + \sum_{m=1}^{\infty}\left(2^{2m-1}-1\right)\frac{|B_{2m}|}{m}\frac{1}{(\beta\mu)^{2m}}\right] \tag{8.98}$$

Since $|B_{2m}| \sim (2m)!$ as $m \to \infty$, we obtain a divergent behaviour similar to the one found for pure gravity in lecture 6. Using

$$\begin{aligned}\frac{\partial\Delta}{\partial\mu} &= \rho(\mu_F)\\ \frac{\partial E_o}{\partial\Delta} &= \beta^2\mu + \text{regular}\end{aligned} \tag{8.99}$$

one obtains

$$\Delta = \frac{\mu}{2\pi}\left[-\log\mu - \sum_{m=1}^{\infty}\left(2^{2m-1}-1\right)\frac{|B_{2m}|}{m(2m1)}\frac{1}{(\beta\mu)^{2m}}\right] \tag{8.100}$$

Writing $\mu = -2\pi\Delta/\log\mu + f(\beta\mu)/\beta\log\mu$ iteratively, we can solve for μ as a function of Δ:

$$\begin{aligned}\mu &= -\frac{2\pi\Delta}{\log\Delta}\left[1+\sum_{n=1}^{\infty}\sum_{m=1}^{n}c_{n,m}\left(\frac{-\log\Delta}{\beta^2\Delta^2}\right)^n(-\log\Delta)^m\right]\\ E_o &= \frac{1}{g^2_{\text{string}}}\left[1+\sum_{n=1}^{\infty}\sum_{m=1}^{n}\epsilon_{n,m}g^{2n}_{\text{string}}(-\log\Delta)^m\right]\end{aligned} \tag{8.101}$$

and the coefficients $c_{n,m}$, $\varepsilon_{n,m}$ can be computed iteratively by inverting the representation

$\Delta = \Delta(\mu)$.

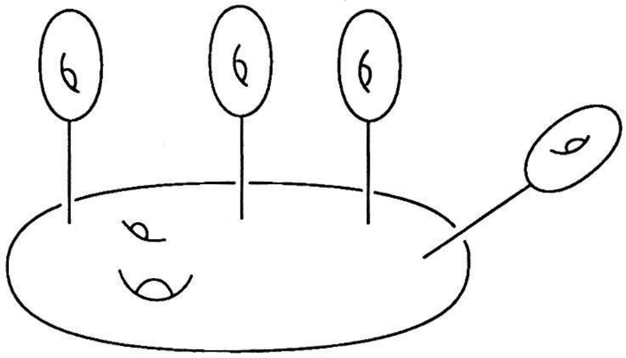


Figure 8.1. The propagator in the tadpoles behaves as $G(p=0) \sim \log\Delta$ as $\delta \to 0$ in the continuum limit.

From the point of view of string theory we can identify the infra-red divergences with graphs having multiple tadpoles (see figure 8.1). The weak coupling expansion is ill–defined. It cannot be used as an asymptotic expansion because no matter how small $g_{\rm st}$ is chosen, $g_{\rm st}\log\Delta$ blows up in the continuum limit. Recall that $\rho(\mu_F)$ was expanded in powers of $1/\beta\mu$, but in the scaling limit $(\beta\mu)^{-1} \sim (\log\Delta)/\beta\Delta\, g_{\rm st}\sqrt{-\log\Delta}$ and there is no region where this parameter can be taken to be small. One way to deal with this problem was found by Gross and Miljković, who realized that the exact representation of $\rho(\mu_F)$ can be used to explore the theory for large $\beta\mu$. Expanding $\rho(\mu_F)$ as a power series in $\beta\mu$,

$$\rho(\mu_F) = -\frac{1}{2\pi}\log\mu + \frac{1}{\pi}\mathrm{Re}\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{1}{1+\frac{i\beta\mu}{2n+1}} \tag{8.102}$$

yields

$$\frac{\partial\Delta}{\partial\mu} = \rho(\mu_F) = -\frac{1}{2\pi}\log\mu + \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^k\left(1-2^{-2k-1}\right)\zeta(2k+1)(\beta\mu)^{2k} \tag{8.103}$$

Since $\zeta(z) \to 1$ when $z \to \infty$, this series is well–behaved and absolutely convergent for $|\beta\mu| < 1$. In the continuum limit we expect $\beta\mu$ to be small. We can invert the previous series to find $\mu = \mu(\Delta)$ and $E_o = E_o(\Delta)$:

$$\begin{aligned} \mu &= -\frac{2\pi\Delta}{\log\Delta}\left[1+\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty} d_{n,m}\left(\frac{\beta^2\Delta^2}{\log\Delta}\right)^n(\log\Delta)^{-m}\right] \\ E_circ &= \frac{1}{g^2_{\text{string}}}\left[1+\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\varepsilon_{n,m}\frac{1}{g^2_{\text{string}}\log^m\Delta}\right] \end{aligned} \tag{8.104}$$

Both series are very well–behaved. thus, we have found a consistent non–perturbative solution with no infra-red divergences at $c = 1$. This solution does not admit, nevertheless, a weak coupling expansion nor, consequently, an interpretation as a genus expansion. The interpretation in terms of two–dimensional surfaces is probably not appropriate either. Note also that the original weak coupling series is not Borel–summable: $\sum A^m(2m)!$. Nevertheless, the strong coupling expansion is convergent. Hence, in the strong coupling expansion we have found a prescription to analytically continue the Borel transform to the physical region. The meaning of this prescription remains to be elucidated.

When we look at the double scaling limit of theories with a k–th order maximum, a procedure similar to the one outlined here yields an expansion of E_o in terms of $G^2_{\text{string}}\Delta^{(k-2)/(k+2)}$ which vanishes in the continuum limit $\Delta \to 0$, and thus these theories become trivial. The physical interpretation of these models is very unclear.

8.4. Ising Model in the Double Scaling Limit

We will be rather brief in the description of the double scaling limit of the Ising model. This limit was constructed in [158, 159, 160]. From the previous lecture we obtain, before taking $N \to \infty$, the equations

$$\begin{aligned} cR(x) &= f(x)\left[1+2g(R(x-\varepsilon)+R(x)+R(x+\varepsilon))\right] \\ cf(x) &= -\frac{x}{2}+R(x)\left[1+2g(R(x-\varepsilon)+R(x)+R(x+\varepsilon))\right] \\ &\quad + 2g\left(S(x)+S(x+\varepsilon)+S(x+2\varepsilon)\right) \\ cS(x) &= 2gf(x)f(x-\varepsilon)f(x-2\varepsilon) \end{aligned} \tag{8.105}$$

where $\varepsilon = 1/N$ and f is related to the free energy in the scaling limit $\partial^2 F/\partial g^2 = N^2 f(1)$ for $g \sim g_c$. Introducing

$$\begin{aligned}
\Lambda_R &= (g - g_c)/a^2 \\
f &= \frac{3}{5}\left(1 + a^{2/3}\tilde{f}\right) \\
R &= \frac{6}{5}\left(1 + a^{2/3}\tilde{R}\right) \\
S &= \frac{5}{9}\left(\frac{3}{5}\right)^2\left(1 + a^{2/3}\tilde{S}\right) \\
1 - a^2 z &= e^{-(\Lambda_B - \Lambda_c)}x
\end{aligned} \tag{8.106}$$

The renormalized strong coupling becomes $\lambda = a^{-7/3}/N$, which is kept fixed as $a \to 0$ and $N \to \infty$. Substituting (8.106) into (8.105) and rescaling λ one obtains

$$z = 6\tilde{f}^3 + 9\tilde{f}\tilde{f}''' + \frac{9}{2}\tilde{f}'^2 + \tilde{f}'''' \tag{8.107}$$

Solving (8.107), one finds again a behaviour $\sim (2h)!$ for genus h, as for pure gravity. This is a generic feature of all these theories. Equations like (8.107), $R_k[f] = t$, and their generalizations can be studied in general using the theory of iso-monodromy deformations of differential operators ([161] and references therein). Since we know the genus zero result (lecture 7), we can linearize (8.107) around this solution. This yields a fourth–order linear equation with two exponentially growing and two exponentially decreasing solutions, therefore it would seem that there are two free non–perturbative parameters. In the case of pure gravity, we could have carried out the same argument and we would have found a free non–perturbative parameter. If true, this would be very important: only non–perturbative effects could fix these parameters. The way to analyze this question is to study the loop equation formulation of two–dimensional gravity [162]. This is in fact the way the Kazakov multi-critical points were found [5]. The Painlevé I equation (or generalizations thereof) is only a small part of the theory. We want to first write down the Schwinger–Dyson equations for the theory and see what type of solutions to the string equations are implied by them. We are also forced to look beyond the string equations because the non–Borel summability of the weak coupling expansion implies that we must add non–perturbative prescriptions to define the theory.

8.5. Non–perturbative Effects in Pure Gravity

The arguments of this section apply in principle to many models, but for simplicity and definiteness we will restrict our attention to the simplest pure gravity case. We follow the analysis of F. David [114, 115, 163]. We have concerned ourselves almost exclusively with partition functions until now. In terms of matrix models we can naturally think of microscopic and macroscopic loop operators. The operator $\mathrm{tr}\Phi^p$ for fixed p represents a loop (in the direct lattice) made of p links. In the scaling limit this loop has zero length and $\mathrm{tr}\Phi^p$ will be proportional to the puncture operator P. If on the other hand we let $p \to \infty$ and $N \to \infty$, $a \to 0$, we can obtain an operator creating a loop of fixed length ℓ. Since $Na^{2-\gamma_{st}} =$ constant, this implies $\ell = pN^{-1/5} =$ constant. We may consider operators of the form

$$W(L) = \mathrm{tr}\, e^{L\Phi} \tag{8.108}$$

or its Laplace transform

$$\hat{W}(p) = \mathrm{tr}\frac{1}{p-\Phi} = \int_0^\infty dLe^{-pL}W(L) \tag{8.109}$$

The expectation value of the product of $\hat{W}(p)'s$ is

$$\left\langle \hat{W}(p_1)\cdots\hat{W}(p_n)\right\rangle = \frac{1}{Z}\int\prod_{i=1}^{N}d\lambda_i\Delta(\lambda)^2e^{-\beta V(\lambda)}\prod_{I=1}^{n}\left(\sum_{i=1}^{N}\frac{1}{p_I-\lambda_i}\right) \tag{8.110}$$

Banks *et al.*[164] introduced a non–relativistic many–body fermionic field

$$\hat{\psi}(\lambda) = \sum a_n\mathcal{P}_n(\lambda)e^{-\frac{1}{2}\beta V(\lambda)} = \sum a_n\psi_n(\lambda) \tag{8.111}$$

where $\mathcal{P}_n(\lambda)$ are orthonormal polynomials. Then the ground state is the fixed Fermi sea $|F\rangle$ with N fermions, and (8.110) can be written as

$$\left\langle \hat{W}(p_1)\cdots\hat{W}(p_n)\right\rangle = \langle F|\,\psi^\dagger\frac{1}{p_1-L}\psi\cdots\psi^\dagger\frac{1}{p_n-L}\psi\,|F\rangle \tag{8.112}$$

and L stands as usual for the operation of multiplication by λ. In the matrix model the correlators of loop operators satisfy the Schwinger–Dyson equation which can be deduced directly from the representation

$$\langle W(L)\rangle = \int d\Phi e^{-\beta V} \operatorname{tr} e^{L\Phi} \tag{8.113}$$

For a single loop one obtains

$$V'\left(\frac{\partial}{\partial L}\right)\langle W(L)\rangle = \int_0^L dL_1 \langle W(L_1)W(L-L_1)\rangle \tag{8.114}$$

This equation was written down by Kazakov from phenomenological arguments, thus leading to the original discovery of the Kazakov multi-critical points. Geometrically, (8.114) has a simple interpretation which is shown in figure 8.2: a local perturbation at a point on the boundary of the loop of length ℓ is represented by $\hat{V}$. In principle, this deformation may lead to a self-touching of the loop, and this is represented by the right-hand side of (8.114).

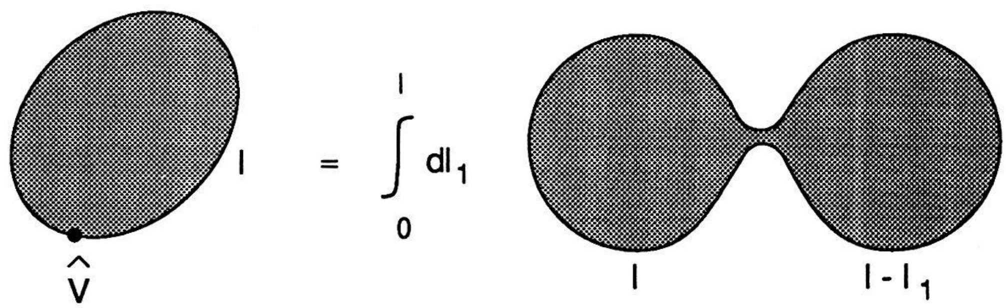


Figure 8.2. Geometrical interpretation of the loop equation (8.114).

After taking the double scaling limit, the Laplace transform $\hat{W}(p)$ satisfies

$$ap^3 - bpz + \langle P\rangle = \left\langle \hat{W}(p)\right\rangle^2 + \left\langle \hat{W}(p)\hat{W}(p)\right\rangle_c \tag{8.115}$$

with a and b constants, and $\langle P\rangle = \partial F/\partial z$ the expectation value of the puncture operator. The subscript c means connected correlation function. One can derive similar equations for arbitrary n-point correlators. These Schwinger–Dyson equations should be expected to be valid beyond perturbation theory. They are the "quantum equation of motion" of the system. Hence, to understand what solutions to the string equations are physically acceptable, we should require them to be compatible with the Schwinger-Dyson equations. Recall that for pure gravity we have the Painlevé I equation. Rescaling the cosmological constant, we must study

$$u^2 - \frac{1}{3}u'' = x \qquad u(x) \to \sqrt{x} \quad , \quad x \to \infty \tag{8.116}$$

This equation has the Painlevé property *i.e.,* its movable singularities are poles. Since the free energy should be real, we should like to look for real solutions to (8.116). Any real solution to (8.116) has an infinite series of double poles on the real axis, accumulating at $x = -\infty$ (see figure 8.3).

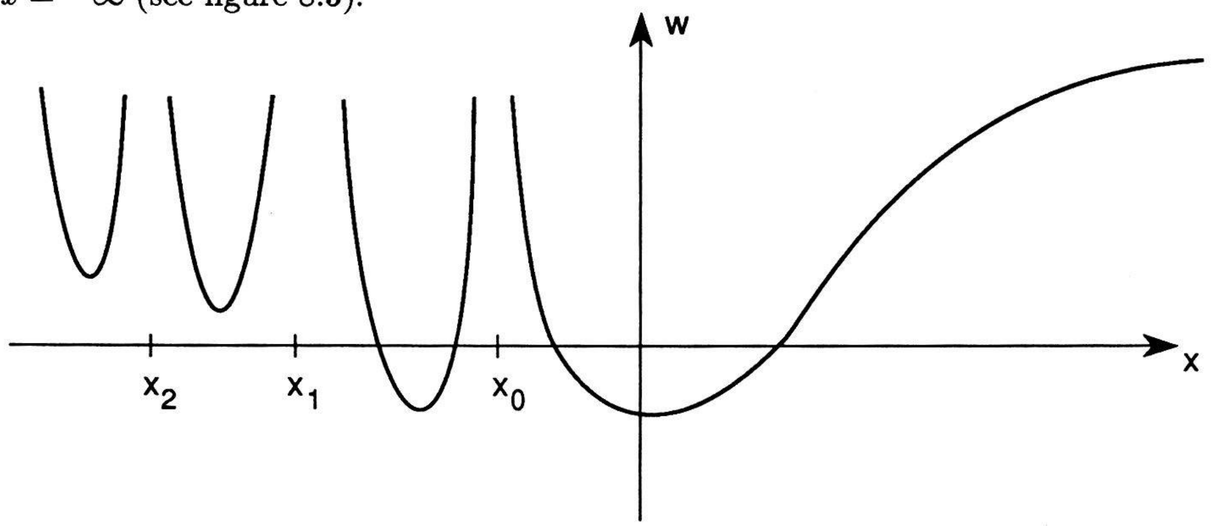


Figure 8.3. Schematic behaviour of $u(x)$.

Near a double pole x_0,

$$u(x) \simeq \frac{2}{(x - x_0)^2} + O\left((x - x_0)^2\right) \tag{8.117}$$

and hence the partition function has a double zero at each pole. This is a very strange behaviour in statistical mechanics, where all Boltzmann weights give a positive contribution. Since we know that $u(x) \sim \sqrt{x}$ when $x \to \infty$, the position of the largest pole completely determines the solution. Solutions differing by the position of the first pole will differ as

$x \to \infty$ by exponentially small terms. From the point of view of the Schrödinger operator $H = -\frac{d}{dx^2} + u(x)$, the singularity of u is strong enough to prevent tunnelling from one side of the pole to the other. The eigenfunctions of H behave as $(x - x_0)^2$ near the pole. One might be tempted to define the correlation functions (8.112) in terms of the propagator $\langle x| (p - H)^{-1} |y\rangle$ which vanishes at x_0. Since there is no tunnelling through the pole, everything would seem to work well. Unfortunately, it was shown by F. David [114] that these real solutions do not satisfy the loop equations. The only way out is to have a potential $u(x)$ with no poles and a behaviour at infinity smooth enough that H has a continuous spectrum. No real solution of (8.116) exists which satisfies this requirement. This can be done only if we look for complex solutions. This is not completely surprising. In pure gravity we are looking at integrals of the form

$$I(g) = \int_{-\infty}^{+\infty} dx \exp\left(-\frac{1}{2}x^2 + gx^4\right) \tag{8.118}$$

for Im$g = 0$, $g > 0$. In order to define $I(g)$, we have to evaluate the integral in the complex g–plane and then try to analytically continue back to the real, positive g–axis. One should expect in general that this process leads to non–vanishing imaginary parts.

Introducing the variables $X = \frac{4}{5}x^{5/4}$, $V(X) = x^{-\frac{1}{2}}u(x)$, the Painlevé equation becomes

$$V^2 - \frac{1}{3}V'' = 1 + \frac{V'}{2x} - \frac{4}{25x^2}V \tag{8.119}$$

For large X we obtain the differential equation satisfied by the Weierstraß $\wp$–function, $V^2 - V''/3 = 1$. We have three possibilites:

i) The general solution to $V^2 - V''/3 = 1$ has double poles on an infinite two–dimensional latttice on the complex plane. This lattice defines a two–torus.

ii) If one of the periods of the lattice vanishes, then we obtain a solution with a single string of poles.

iii) The trivial solution $V^2 = 1$ is always good.

It was shown by Boutroux that these families of solutions to the Weierstrass equation have close analogues for the Painlevé equation [165]. If $u(x)$ is a solution to Painlevé, then $u(x) = e^{-4\pi i/5}u(e^{2\pi i/5}x)$ is also a solution. The analogue of (i) has an infinite number of poles in the complex x–plane. There is a family of truncated solutions analogous to (ii) having no poles at infinity in two out of five sectors (say for $-2\pi/5 < \operatorname{Arg} x < 2\pi/5$), but an infinite number of poles in the other sectors. There is finally a "triply truncated solution" which asymptotically contains poles only in one sector out of five. Using the $2\pi/5$ rotation, it corresponds to two complex conjugate solutions $U_{\pm}$ satisfying

$$\begin{aligned} u_{\pm}(x) &\sim_{x\to\infty} \sqrt{x} \\ u_{\pm}(x) &\sim_{x\to-\infty} \pm i\sqrt{x} \end{aligned} \tag{8.120}$$

A reasonable conjecture formulated by David is that the loop amplitudes constructed with this solution satisfy the Schwinger–Dyson equation. Intensive work on settling this conjecture is in progress.

References

[1] A.M. Polyakov, *Mod. Phys. Lett.* **A2** (1987) 893.

[2] V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, *Mod. Phys. Lett.* **A3** (1988) 819.

[3] F. David, *Mod. Phys. Lett.* **A3** (1988) 1651.

[4] J. Distler and H. Kawai, *Nucl. Phys.* **B321** (1989) 509.

[5] V.A. Kazakov, *Mod. Phys. Lett.* **A4** (1989) 2125.

[6] M.R. Douglas and S.H. Shenker, *Nucl. Phys.* **B335** (1990) 635.

[7] E. Brézin and V.A. Kazakov, *Phys. Lett.* **236B** (1990) 144.

[8] D.J. Gross and A.A. Migdal, *Phys. Rev. Lett.* **64** (1990) 127; *Nucl. Phys.* **B340** (1990) 333.

[9] M.R. Douglas, *Phys. Lett.* **238B** (1990) 176.

[10] E. Witten, *On the structure of the topological phase of two-dimensional gravity*, Princeton preprint IASSNS-HEP-89/66 (December 1989).

[11] E. Verlinde and H. Verlinde, *Conformal field theory and geometric quantization*, Princeton preprint IASSNS-HEP-89/58.

[12] E. Verlinde and H. Verlinde, *A solution of two-dimensional topological quantum gravity*, Princeton preprint IASSNS-HEP-90/40.

[13] Some general references for the topological field theory approach are E. Witten, *Nucl. Phys.* **B340** (1990) 281; J. Distler, Princeton preprint PUPT-1161 (1990), R. Dijkgraaf and E. Witten, Princeton preprint PUPT-1166 (1990).

[14] K. Li, *Recursion relations in topological gravity with minimal matter*, Pasadena preprint CALT-68-1670.

[15] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Notes on topological string theory and 2d quantum gravity*, Princeton preprint IASSNS-HEP-90/80.

[16] F. Distler, *Two-dimensional quantum gravity, topological field theory and multi-critical matrix models*, Princeton preprint PUPT-1161.

[17] M. Green, J. Schwarz and E. Witten, *String Theory*, vols. 1,2 (1986) Cambridge University Press.

[18] M. Kaku, *String Theory* (1988) Springer.

[19] D. Lüst and S. Theisen, *Lectures on String Theory*, Lecture Notes in Physics (1989) Springer.

[20] A. Pressley and G.B. Segal, *Loop Groups* (1986) Oxford University Press.

[21] A.M. Polyakov, *Phys. Lett.* **103B** (1981) 207; *ibid.*, 211.

[22] D. Friedan, in Les Houches 1984 Summer School, ed. R. Stora *et al.*(1985) North-Holland.

[23] See for example J.M. Drouffe and C. Itzykson, *Field Theory and Statistical Mechanics*, vol. I, II (1989) Cambridge University Press.

[24] F. David, *Phys. Reports* **184** (1989) .

[25] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Nucl. Phys.* **B241** (1984) 333.

[26] C. Itzykson, H. Saleur and J.B. Zuber, eds., *Conformal Invariance and Applications to Statistical Mechanics* (1988) World Scientific.

[27] P. Ginsparg, in Les Houches 1988 Summer School, E. Brézin and J. Zinn-Justin, eds. (1989) North-Holland.

[28] J. Cardy, in Les Houches 1988 Summer School, E. Brézin and J. Zinn-Justin, eds. (1989) North-Holland.

[29] L. Alvarez–Gaumé, C. Gomez and G. Sierra in *Knizhnik Memorial Volume* (1989) World Scientific.

[30] S.K. Ma, *Modern Theory of Critical Phenomena* (1976) Benjamin.

[31] K. Wilson and J. Kogut, *Phys. Reports* **C12** (1974) 75.

[32] R. Baxter, *Exactly Solved Models in Statistical Mechanics,* (1982) Academic Press.

[33] V.S. Dotsenko, V.A. Fateev, *Nucl. Phys.* **B240** (1984) 312; **B251** (1985) 691, *Phys. Lett.* **154B** (1985) 291.

[34] G. Moore and N. Seiberg, *Phys. Lett.* **212B** (1988) 451; *Nucl. Phys.* **B313** (1989) 16; *Comm. Math. Phys.* **123** (1989) 177.

[35] E. Verlinde, *Nucl. Phys.* **B300** (1988) 360.

[36] V. Kac. *Lecture Notes in Physics* **94** (1979) 441.

[37] B.L. Feigin and D.B. Fuks, *Funct. Anal. Appl.* **16** (1982) 114.

[38] D. Friedan, Z. Qiu and S. Shenker, *Phys. Rev. Lett.* **51** (1984) 1575.

[39] D. Gepner and E. Witten, *Nucl. Phys.* **B278** (1986) 493.

[40] A. Rocha–Caridi, in *Vertex Operators in Mathematical Physics*, MSRI Publication # 3 (1984) Springer, p. 451.

[41] V. Fateev and A.B. Zamolodchikov, *Nucl. Phys.* **B280** (1987) 644.

[42] L. Alvarez–Gaumé, C. Gomez and G. Sierra, *Nucl. Phys.* **B330** (1990) 347.

[43] O. Alvarez, *Nucl. Phys.* **B216** (1983) 125.

[44] N.E. Mavromatos and J.L. Miramontes, CERN preprint TH–5428/89.

[45] E. d'Hoker, P.S. Kurzepa, Los Angeles preprint UCLA/90/TEP/15.

[46] A. Gupta, S.P. Trivedi and M.B. Wise, Caltech preprint CALT–68–1593.

[47] A. Dhar, T. Jayaraman, K.S. Narain and S.R. Wadia, Tata (Bombay) preprint TIFR–TH–89–55.

[48] J.L. Gervais, *Critical dimensions for non–critical strings*, Paris preprint LPTENS 90/4 (March 1990).

[49] T. Curtright and C.B: Thorn, *Phys. Rev. Lett.* **48** (1982) 1309.

[50] E. Braaten, T. Curtright and C.B. Thorn, *Phys. Lett.* **118B** (1982) 115; *Ann. Phys.* **147** (1983) 365.

[51] E. Braaten, T. Curtright, G. Ghandour and C.B. Thorn, *Phys. Rev. Lett.* **51** (1983) 19; *Ann. Phys.* **153** (1984) 147.

[52] J.L. Gervais and A. Neveu, *Nucl. Phys.* **B199** (1982) 59 **B209** (1982) 125 **B224** (1983) 329 **B238** (1984) 125, 396, *Phys. Lett.* **151B** (1985) 271.

[53] L.D. Fadeev and L. Takhtajan, *Liouville theory on the lattice*, in *Meudon Meeting on Integrable Systems*, eds. de Vega *et al.*(1986) Springer.

[54] F. Smirnov and L. Takhtajan, *Towards a quantum Liouville theory with* $c > 1$, Boulder preprint.

[55] C. Gomez and G. Sierra, *Integrability and uniformization in Liouville theory: the geometrical origin of quantized symmetries*, Geneva preprint UGVA–DPT–1990–08–694.

[56] N. Seiberg, *Notes on quantum Liouville theory and quantum gravity*, Rutgers preprint RU–90–29.

[57] A.M. Polyakov and A.B. Zamolodchikov, *Mod. Phys. Lett.* **A3** (1988) 1213.

[58] J. Distler, Z. Hlousek and H. Kawai, *Int. J. Mod. Phys* **A5** (1990) 391.

[59] F. David, *Nucl. Phys.* **B257** (1985) 45.

[60] V. Kazakov, *Phys. Lett.* **150B** (1985) 282.
[61] J. Ambjorn, B. Durhuus and J. Fröhlich, *Nucl. Phys.* **B259** (1985) 433.
[62] T. Regge, *Nuovo Cim.* **19** (1961) 558.
[63] J. Fröhlich, Lecture Notes in Physics, vol. 216, ed. L. Garrido (1985) Springer.
[64] F. David, *Nucl. Phys.* **B257** (1985) 543.
[65] V.I. Kazakov, I.K. Kostov and A.A. Migdal, *Phys. Lett.* **157B** (1985) 295.
[66] B. Durhuus, J. Fröhlich and J. Jonsson, *Nucl. Phys.* **B240** (1984) 453.
[67] D. Boulatov, V.A. Kazakov, I.K. Kostov and A.A. Migdal, *Nucl. Phys.* **B275** (1986) 641.
[68] J. Ambjorn, B. Durhuus, J. Fröhlich and P. Orland, *Nucl. Phys.* **B270** (1986) 457.
[69] I. Kostov and A. Krzywicki, *Phys. Lett.* **187B** (1987) 149.
[70] A. Billoire and F. David, *Phys. Lett.* **168B** (1986) 279.
[71] A. Billoire and F. David, *Nucl. Phys.* **B275** (1986) 617.
[72] J. Jurkiewicz, A. Krzywicki and B. Petersson, *Phys. Lett.* **168B** (1986) 273, *Phys. Lett.* **177B** (1986) 89.
[73] D. Boulatov and V.A. Kazakov, *Phys. Lett.* **184B** (1987) 247.
[74] F. David, J. Jurkiewicz, B. Krzywicki and B. Petersson, *Nucl. Phys.* **B290** (1987) 218.
[75] F. David and E Guitter, *Nucl. Phys.* **B295** (1988) 332.
[76] J.M. Alberty, *Monte Carlo simulations on and of two–dimensional dynamical lattices. A theoretical overview*, CERN preprint TH–5534/89, to appear in *Phys. Rep.*
[77] M.E. Agishtein and A.A. Migdal, Princeton preprints PUPT–1171, 1180.
[78] L. Jacobs, in *Selected Topics on Quantum Field Theory and Mathematical Physics*, MIT preprint MIT–CTP–1811 (November 1989).
[79] C.F. Baillie, D.A. Johnston and R.D. Williams, *Non–universality in dynamically triangulated random surfaces with extrinsic curvature*, Caltech preprint CALT–68–1624/C3P–868.
[80] C.F. Baillie, D.A. Johnston and R.D. Williams, *Crumpling and dynamically triangulated random surfaces in higher dimensions*, Pasadena preprint CALT–68–1623/C3P–867.
[81] M.E. Agishtein, L. Jacobs and A.A. Migdal, *Mod. Phys. Lett.* **A5** (1990) 965.
[82] V.A. Kazakov, *Phys. Lett.* **119B** (1986) 140.
[83] D. Boulatov and V.A. Kazakov, *Phys. Lett.* **186B** (1987) 379.
[84] I. Kostov, *Nucl. Phys.* **B** *(Proc. Suppl.)* **10A** (1989) 295.
[85] V.A. Kazakov, *Nucl. Phys. (Proc. Suppl.)* **B4** (1988) 93.
[86] I.K. Kostov, *Mod. Phys. Lett.* **A4** (1989) 217.
[87] I.K. Kostov and M.L. Mehta, *Phys. Lett.* **189B** (1987) 118.
[88] F. David, *Phys. Lett.* **159B** (1985) 303.
[89] B. Duplantier and I. Kostov, *Phys. Rev. Lett.* **61** (1988) 1433.
[90] M. Staudacher, *Nucl. Phys.* **B336** (1990) 349.
[91] V.A. Kazakov and A.A. Migdal, *Nucl. Phys.* **B311** (1988/89) 171.
[92] I. Kostov, *Nucl. Phys.* **B326** (1989) 583.
[93] I. Kostov, *Strings embedded in Dynkin diagrams*, Saclay preprint SPhT/90–133 (December 1990).
[94] B. Duplantier, *Phys. Reports* **184** (1989) .
[95] B. Duplantier, in *Fundamental Problems in Statistical Mechanics VII*, H. van Beijeren ed.

[96] B. Nienhuis, in *Fundamental Problems in Statistical Mechanics VII*, H. van Beijeren ed.

[97] G. 't Hooft, *Nucl. Phys.* **B72** (1974) 461.

[98] E. Brézin, C. Itzykson, G. Parisi and J.B. Zuber, *Comm. Math. Phys.* **59** (1979) 35.

[99] M.L. Mehta, *Random Matrices,* (1967) Academic Press.

[100] O. Bessis, C. Itzykson and J.B. Zuber, *Adv. App. Math.* **1** (1980) 109.

[101] C. Itzykson, J.B. Zuber, *J. Math. Phys.* **21** (1980) 411.

[102] D. Bessis, *Comm. Math. Phys.* **69** (1979) 147.

[103] M.L. Mehta, *Comm. Math. Phys.* **79** (1981) 327.

[104] S. Chadha, G. Mahoux and M.L. Mehta, *J. Phys.* **A14** (1981) 579.

[105] M.L. Mehta and G. Mahoux, *A method of integration over matrix variables: III*, Saclay preprint SPhT/90–084.

[106] C. Itzykson and J.B. Zuber, *Matrix integration and combinatorics of modular groups*, Saclay preprint SPhT/90–004.

[107] T.S. Chihara *An Introduction to Orthogonal Polynomials,* (1978) Gordon and Breach.

[108] S. Shenker, in *Proceedings of Cargèse Workshop on 2d Gravity*, eds. O. Alvarez, E. Marinari and P. Windey.

[109] E. Marinari and G. Parisi, *The supersymmetric one–dimensional string*, Roma preprint ROM2F–90–05.

[110] E. Brézin, E. Marinari and G. Parisi, *Phys. Lett.* **242B** (1990) 35.

[111] M. Douglas, N. Seiberg and S. Shenker, Rutgers preprint RU–90–19 (April 1990).

[112] M. Karliner and A.A. Migdal, *Non–perturbative two–dimensional quantum gravity via supersymmetric string*, Princeton preprint PUPT–1191 (July 1990).

[113] F. David, *Phases of the large N matrix model and non–perturbative effects in two–dimensional gravity*, Saclay preprint SPhT/90–090.

[114] F. David, *Loop equations and non–perturbative effects in two–dimensional gravity*, Saclay preprint SPhT/90–043.

[115] F. David, *Matrix models and two–dimensional gravity*, Saclay preprint SPhT/90–127.

[116] V. Periwal and D. Shevitz, *Phys. Rev. Lett.* **64** (1990) 1326.

[117] E. Marinari and G. Parisi, *Phys. Lett.* **240B** (1990) 375.

[118] E. Marinari and G. Parisi, *A non–perturbative definition of two–dimensional gravity*, Roma preprint ROM2F–90–29 (June 1990).

[119] O. Alvarez and P. Windey, *Universality in two–dimensional quantum gravity*, preprint PAR–LPTHE–90–12.

[120] V.A. Kazakov, *Bosonic strings and string field theories in one–dimensional target space*, to appear in *Proceedings of Cargèse Workshop on 2d Gravity*, eds. O. Alvarez, E. Marinari and P. Windey.

[121] D.J. Gross and N. Miljković, *A non–perturbative solution of $D = 1$ string theory*, Princeton preprint PUPT–1160 (January 1990).

[122] E. Witten, *Two–dimensional gravity and intersection theory on moduli space*, Princeton preprint IASSNS–HEP–90/45 (May 1990).

[123] L. Alvarez–Gaumé, C. Gómez and J. Lacki, *Integrability in random matrix models*, CERN preprint TH–5875/90.

[124] A. Gerasimov, A. Marshakov, A. Mironov, A. Morozov and A. Orlov, *Matrix models of two–dimensional gravity and Toda theory*, Lebedeev Institute (Moscow) preprint.

[125] E. Martinec, *On the origin of integrability in matrix models*, Chicago preprint EFI–90–67.

[126] A. Das, *Integrable Models* (1989) World Scientific.

[127] G. Segal and G. Wilson, *Publ IHES* **61** (1985) 1.

[128] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, *Proc. Jap. Acad.* **57A** (1981) 342, 387; *J. Phys. Soc. Japan* **50** (1981) 3806, 3813; *Physica* **4D** (1982) 343; *Publ. RIMS* **18** (1982) 1077, 1111.

[129] V.G. Drinfel'd and V.V. Sokolov, *Sov. Math. Dokl.* **23** (1981) 457.

[130] M. Toda, *Theory of Non–Linear Lattices* (1984) Springer.

[131] L.D. Faddeev and L. Takhtajan, *Hamiltonian Theory of Solitons* (1984) Springer.

[132] M. Wadati, T. Deguchi and Y. Akutsu, *Phys. Reports* **180** (1989) 247.

[133] I.M. Gel'fand and L.A. Dikĭi, *Soviet Mathematical Surveys* **30** (1975) 1977.

[134] P. Ginsparg, M. Goulan, M.R. Plesser and J. Zinn–Justin, Harvard/Saclay preprint HUTP–90/A015, SPhT/90–049.

[135] V. Pasquier, *Nucl. Phys.* **B285** (1987) 162, *J. Phys.* **35** (1987) 5707.

[136] M. Fukuma, H. Kawai and R. Nakayama, KEK preprint KEK–TH–251 (May 1990).

[137] R. Dijkgraaf, E. Verlinde and H. Verlinde, Princeton preprint IASSNS–HEP–90/48 (May 1990).

[138] J. Goeree, *W-constraints in two–dimensional quantum gravity*, Utrecht preprint THU–19 9/90.

[139] J. Ambjorn, J. Jurkiewicz and Yu.M. Make'enko, Niels Bohr preprint NBI–HE–90–41 (August 1990).

[140] H.S. La, *Method of orbits and Virasoro Ward identities*, preprint UPR–0413T.

[141] H. Itoyama and Y. Matsuo, *Non–critical Virasoro algebra of* $d < 1$ *matrix model and quantized string theory*, Stony Brook preprint ITP–SB–90–75.

[142] C. Imbimbo and S. Mukhi, *Space–time interpretation of string field theory in minimal model backgrounds*, CERN preprint TH–5910/90.

[143] E. Brézin, V. Kazakov and Al.B. Zamolodchikov, *Nucl. Phys.* **B338** (1990) 673.

[144] I. Kostov, *Phys. Lett.* **215B** (1988) 499.

[145] P. Ginsparg and J. Zinn–Justin, *Two–dimensional gravity and one–dimensional matter*, Harvard preprint HUTP–90/A004.

[146] D.J. Gross and I. Klebanov, *String theory on a circle*, Princeton preprint PUPT–1172.

[147] J. Gonzalez and M.A.H. Vozmediano, preprint IEM–FT–18/90 (April 1990).

[148] Z. Yang, *Compactified* $D = 1$ *string models*, Austin preprint UTTG–10–90.

[149] D.V. Boulatov, *Phys. Lett.* **237B** (1990) 202.

[150] D. J. Gross, I.R. Klebanov and M.J. Newman, *The two–point correlation function of the one–dimensional matrix model*, Princeton preprint PUPT–1192 (July 1990).

[151] G. Parisi, *String theory on the one–dimensional lattice*, Roma preprint ROM2F–90/2.

[152] D.J. Gross and I.R. Klebanov, *Fermionic string field theory of* $c = 1$ *two–dimensional gravity*, Princeton preprint PUPT–1198 (September 1990).

[153] G. Parisi, *On the spectrum of the one–dimensional string*, Roma preprint ROM2F–90/1.

[154] J. Gonzalez, preprint IEM–FT–25/90 (August 1990).

[155] S.R. Das, A. Dhar and S. Wadia, *Mod. Phys. Lett.* **A5** (1990) 799.

[156] J. Polchinski, Austin preprint UTTG–15–90.

[157] S.R. Das and A. Jevicki, Brown preprint HET–750 (1990).

[158] C. Crnković, P. Ginsparg and G. Moore, *Phys. Lett.* **237B** (1990) 196.

[159] E. Brézin, M.R. Douglas, V.A. Kazakov and S.H. Shenker, *Phys. Lett.* **B237** (1990) 43.

[160] D. Gross and A. Migdal, *Phys. Rev. Lett.* **64** (1990) 717.

[161] G. Moore, *Geometry of the string equations*, preprint YCTP–84–90.

[162] A.A. Migdal, *Phys. Reports* **102** (1983) 199.

[163] F. David, *Matrix models and two–dimensional gravity*, Lectures at the 1990 Trieste Spring School on String Theory and Quantum Gravity, Saclay preprint SPhT/90–127.

[164] T. Banks, M.E. Douglas, N. Seiberg and S.H. Shenker, *Phys. Lett.* **238B** (1990) 279.

[165] E. Hille, *Ordinary Differential Equations in the Complex Domain* (1976) Wiley.