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Autor: Feldmann, Joel / Trubowitz, Eugene

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The Flow of an Electron-Phonon System to the Superconducting State

Joel Feldman*
Department of Mathematics
University of British Columbia
Vancouver, B.C.
CANADA V6T 1Z2

Eugene Trubowitz
Mathematik
ETH-Zentrum
CH-8092 Zürich
SWITZERLAND

(14. I. 1991)

This paper is dedicated to the memory of Res Jost.

Abstract

We consider the standard model for an interacting system of screened electrons and phonons in dimension $d \geq 2$. A localization operator $\mathbf{L}$ acting on the effective potential $\mathcal{G}$ is introduced. It is proven that $(\mathbf{1}-\mathbf{L})\mathcal{G}$ is irrelevant. The relevant part $\mathbf{L}\mathcal{G}$ is analyzed by a renormalization group flow. It is shown that, when perturbation theory is truncated at any finite order and the particle number symmetry is broken, to exclude the Goldstone boson, the flow converges to a nontrivial fixed point determined by a BCS gap equation.

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I. Introduction and Overview

In this paper we consider the standard model for an interacting system of screened electrons and phonons in dimension $d \geq 2$ given by the Hamiltonian

$$
\begin{aligned}
H = & \int \frac{d^d\mathbf{k}}{(2\pi)^d} \in (\mathbf{k}) \ \left[a^+_{\mathbf{k}\uparrow} a_{\mathbf{k}\uparrow} + a^+_{\mathbf{k}\downarrow} a_{\mathbf{k}\downarrow}\right] \\
& + \int \frac{d^d\mathbf{q}}{(2\pi)^d} \hbar\omega(\mathbf{q}) \left[c^+_{\mathbf{q}} c_{\mathbf{q}} + \frac{1}{2}\right] \\
& + \frac{1}{2} \sum_{\alpha,\beta\in\{\uparrow,\downarrow\}} \int \prod_{i=1}^{4} \frac{d^d\mathbf{k}_i}{(2\pi)^d} (2\pi)^d \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \hat{U}(\mathbf{k}_3 - \mathbf{k}_1) a^+_{\mathbf{k}_1,\alpha} a^+_{\mathbf{k}_2,\beta} a_{\mathbf{k}_4,\beta} a_{\mathbf{k}_3,\alpha} \\
& + \gamma \sum_{\alpha\in\{\uparrow,\downarrow\}} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{d^d\mathbf{q}}{(2\pi)^d} \left(\frac{\hbar\omega(\mathbf{q})}{2}\right)^{1/2} \theta(\omega_D - \omega(\mathbf{q})) a^+_{\mathbf{k}+\mathbf{q},\alpha} a_{\mathbf{k},\alpha} [c_{\mathbf{q}} + c^+_{-\mathbf{q}}]
\end{aligned}
\qquad (I.1)
$$

Here, $\in (\mathbf{k}) = \frac{\mathbf{k}^2}{2m}$ is the dispersion relation for a free electron gas, $\omega(\mathbf{q})$ is the jellium phonon dispersion relation and U is the two body electron-electron potential. The last term represents the electron-phonon interaction, in which $\theta(\omega) = 1, 0 < \omega < \frac{1}{2}, = 0, \omega > 2$, smoothly restricts the interaction to phonons with frequency $\omega(\mathbf{q})$ less than the Debye frequency ω_D.

The d and $d+1$ dimensional Fourier transforms are defined by

$$\hat{U}(\mathbf{k}) = \int d^d\mathbf{x} e^{i\mathbf{k}\cdot\mathbf{x}} U(\mathbf{x})$$

and

$$\tilde{\psi}(k) = \int d\tau d^d\mathbf{x} e^{i\langle k,(\tau,\mathbf{x})\rangle_-} \psi(\tau,\mathbf{x})$$

$$\text{with } k = (k_0, \mathbf{k}) \in \mathbf{R}^{d+1}$$

$$\langle k, (\tau,\mathbf{x})\rangle_- = -k_0\tau + \mathbf{k}\cdot\mathbf{x}$$

respectively. We will frequently omit ˆ and ˜ when their presence is clearly indicated by the context.

The model above is also formally characterized by the effective potential for the external fields $\psi^e, \bar{\psi}^e$ and π^e

$$\mathcal{G}(\psi^e, \bar{\psi}^e, \pi^e) = \log \frac{1}{Z} \int e^{-\mathcal{U}(\psi+\psi^e, \bar{\psi}, +\bar{\psi}^e, \pi+\pi^e)} d\mu_C(\psi, \bar{\psi}) d\nu_D(\pi). \qquad (I.2a)$$

where the interaction

$$\mathcal{U}(\psi,\bar{\psi},\pi)=\frac{1}{2}\sum_{\alpha,\beta\in\{\uparrow,\downarrow\}}\int d^d\mathbf{x}d\tau d^d\mathbf{y}d\sigma\bar{\psi}(\tau,\mathbf{x},\alpha)\psi(\tau,\mathbf{x},\alpha)\delta(\tau-\sigma)U(\mathbf{x}-\mathbf{y})\bar{\psi}(\sigma,\mathbf{y},\beta)\psi(\sigma,\mathbf{y},\beta)$$
$$+\gamma\sum_{\alpha\in\{\uparrow,\downarrow\}}\int d^d\mathbf{x}d\tau\bar{\psi}(\tau,\mathbf{x},\alpha)\psi(\tau,\mathbf{x},\alpha)\Omega(-i\nabla_{\mathbf{x}})\pi(\tau,\mathbf{x}) \tag{I.2b}$$

and Ω is the pseudodifferential operator given by

$$(\Omega\pi)\check{}(q)=\left(\frac{\hbar\omega(\mathbf{q})}{2}\right)^{1/2}\theta(\omega_D-\omega(\mathbf{q}))\tilde{\pi}(q).$$

Here, $d\mu_C(\psi,\bar{\psi})$ is the fermionic Gaussian measure in the Grassmann variables $\{\psi(\xi),\bar{\psi}(\xi)|\xi=(\tau,\mathbf{x},\sigma),\tau\in\mathbf{R},\mathbf{x}\in\mathbf{R}^d,\sigma\in\{\uparrow,\downarrow\}\}$ with covariance

$$\begin{aligned}C(\xi_1,\xi_2)&=\langle\psi(\xi_1)\bar{\psi}(\xi_2)\rangle\\&=\delta_{\sigma_1,\sigma_2}\int\frac{d^{d+1}k}{(2\pi)^{d+1}}\frac{e^{i\langle k,\xi_1-\xi_2\rangle_-}}{ik_0-e(\mathbf{k})}\end{aligned} \tag{I.3}$$

where

$$e(\mathbf{k})=\frac{\mathbf{k}^2}{2m}-\mu. \tag{I.4}$$

For $(\tau_1-\tau_2)\neq 0$ the integral (I.3) is conditionally convergent. The special case $\tau_1-\tau_2=0$ is defined as the limit $\tau_1-\tau_2\to 0$ with $\tau_1-\tau_2<0$. The chemical potential μ in $e(\mathbf{k})=\in(\mathbf{k})-\mu$ determines the electron density of the model. Also, $d\nu_D$ is the Gaussian measure (for the free phonons) with covariance

$$D(\xi_1,\xi_2)=\int\frac{d^{d+1}q}{(2\pi)^{d+1}}e^{i\langle q,\xi_1-\xi_2\rangle_-}\frac{2\omega(\mathbf{q})}{q_0^2+\omega(\mathbf{q})^2}.$$

We now integrate out the phonon field, set $\hbar=1$ and suppress the external phonon field π^e, which plays no role here. One obtains

$$\mathcal{G}(\psi^e,\bar{\psi}^e)=\log\frac{1}{Z}\int e^{-\lambda\mathcal{V}(\psi+\psi^e,\bar{\psi}+\bar{\psi}^e)}d\mu_C(\psi,\bar{\psi})|_{\lambda=1}, \tag{I.5a}$$

$$\mathcal{V}(\psi,\bar{\psi})=\frac{1}{2}\sum_{a_i\in\{\uparrow,\downarrow\}}\int\prod_{i=1}^{4}\frac{d^{d+1}k_i}{(2\pi)^{d+1}}(2\pi)^{d+1}\delta(k_1+k_2-k_3-k_4)\delta_{\alpha_1,\alpha_3}\delta_{\alpha_2,\alpha_4}$$
$$\langle k_1,k_2|V|k_3,k_4\rangle\bar{\psi}(k_1,\alpha_1)\bar{\psi}(k_2,\alpha_2)\psi(k_4,\alpha_4)\psi(k_3,\alpha_3) \tag{I.5b}$$

$$\langle k_1, k_2|V|k_3, k_4\rangle = \hat{U}(\mathbf{k}_3 - \mathbf{k}_1) - \gamma^2\theta(\omega_D - \omega(\mathbf{k}_3 - \mathbf{k}_1))^2 \frac{\omega(\mathbf{k}_3 - \mathbf{k}_1)^2}{(k_3 - k_1)_0^2 + \omega(\mathbf{k}_3 - \mathbf{k}_1)^2} \qquad (I.5c)$$

Note that the second term, due to the phonon-electron interaction, is attractive.

We shall consider a general two-body interaction $\langle k_1, k_2|V|k_3, k_4\rangle$ such that, on the support of $\delta(k_1 + k_2 - k_3 - k_4)$,

S1.

$$\begin{aligned}\langle k_1, k_2|V|k_3, k_4\rangle &= \langle -k_3, k_2|V| - k_1, k_4\rangle \\ &= \langle k_1, -k_4|V|k_3, -k_2\rangle \\ &= \langle -k_3, -k_4|V| - k_1, -k_2\rangle \\ &= \langle k_2, k_1|V|k_4, k_3\rangle \\ &= \langle -k_1 - k_2|V| - k_3 - k_4\rangle \\ &= \langle k_3, k_4|V|k_1, k_2\rangle\end{aligned}$$

S2. $\langle k_1, k_2|V|k_3, k_4\rangle$ is real

S3. $\langle Rk_1, Rk_2|V|Rk_3, Rk_4\rangle = \langle k_1, k_2|V|k_3, k_4\rangle$,where R is any element of $O(d)$ acting on spatial components

S4. $\langle Tk_1, Tk_2|V|Tk_3, Tk_4\rangle = \langle k_1, k_2|V|k_3, k_4\rangle$, where T is time reversal i.e. $T(k_0, \mathbf{k}) = (-k_0, \mathbf{k})$

$(I.6)$

The interaction of (I.5) has all these symmetries for any real even rotation invariant two-body potential U and phonon dispersion relation ω. Later we will make a further assumption that ensures that V is sufficiently attractive in the zero angular momentum sector. See, Theorem I.3.

Our long term goal is to construct the effective potential $\mathcal{G}$ for the class of interactions V described above. In this paper we consider the small field part of the construction. Specifically, we define a localization operator $\mathbf{L}$ (I.99) and prove in Theorem I.1 that $(\mathbf{1}-\mathbf{L})\mathcal{G}$ is irrelevant. The relevant part $\mathbf{L}\mathcal{G}$ is analyzed by a renormalization group flow (I.102), (I.124a). The content of Theorem I.2 is that, excluding the Goldstone boson (I.82), the flow converges to a nontrivial superconducting fixed point determined by the (non-perturbative) BCS gap equation (I.75), (I.80).

The rest of the introduction is divided into two parts. The first motivates the definition of **L** and discusses the associated flow near the "normal" ground state. The second introduces a superconducting formalism, discusses the flow near the superconducting fixed point and gives statements of our main results.

Ia. Background and Motivation

We will ultimately do perturbation theory around a covariance that reflects properties of the superconducting ground state. However, to motivate the definitions and constructions of Section Ib, we begin by analyzing the effective potential (I.5). This is done by decomposing $\psi(k), \bar{\psi}(k)$ into scales that reflect the distance of the vector momentum $\mathbf{k}$ from the Fermi surface

$$\{\mathbf{k} \,|\, |\mathbf{k}| = k_F\} \tag{I.7a}$$

where

$$k_F = \sqrt{2m\mu}. \tag{I.7b}$$

Namely,

$$C = \sum_{j=-\infty}^{0} C^{(j)} \tag{I.8a}$$

where

$$\begin{aligned} C^{(j)}(\xi) &= \delta_{\sigma_1,\sigma_2} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{e^{i<k,\xi>_-}}{ik_0 - e(\mathbf{k})} f(M^{-2j} e(\mathbf{k})^2) \\ &= \delta_{\sigma_1,\sigma_2} \int \frac{d^d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}} e^{-|e(\mathbf{k})\tau|} f(M^{-2j} e(\mathbf{k})^2) \begin{cases} -\chi(e(\mathbf{k})) & \tau > 0 \\ \chi(-e(\mathbf{k})) & \tau \leq 0 \end{cases} \end{aligned} \tag{I.8b}$$

for $j > 0$ and

$$C^{(0)}(\xi) = \delta_{\sigma_1,\sigma_2} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{e^{i<k,\xi>_-}}{ik_0 - e(\mathbf{k})} h(e(\mathbf{k})^2). \tag{I.8c}$$

Here

$$1 = h(r) + \sum_{j=-\infty}^{-1} f(M^{-2j} r) \tag{I.8d}$$

for all $r \geq 0$. Roughly speaking $f(M^{-2j}e(\mathbf{k})^2)$ forces $||\mathbf{k}| - k_F| \approx M^j$. For the precise definitions of h and f see (II.1) and the figure following (II.2). There is a corresponding decomposition

$$d\mu_C(\psi, \bar{\psi}) = \prod_{j=-\infty}^{0} d\mu_{C^{(j)}}(\psi^{(j)}, \bar{\psi}^{(j)}). \tag{I.9}$$

The decomposition (I.8) differs from that of [FT§2] in that k_0 is not localized near zero. This has the advantage that $C^{(j)}$ retains physical positivity (Osterwalder-Schrader positivity).

Let us define the effective interaction at scale $h, -1 \geq h \geq -\infty$, by

$$\mathcal{G}^{(h)}(\phi^{(\leq h)}) := \log \frac{1}{Z_h} \int e^{-\lambda \mathcal{V}(\phi^{\leq 0})} \prod_{j>h} d\mu_{C^{(j)}}(\phi^{(j)}) \tag{I.10}$$

where

$$\phi^{(\leq h)} = (\psi^e + \sum_{j \leq h} \psi^{(h)}, \bar{\psi}^e + \sum_{j \leq h} \bar{\psi}^{(h)}) \tag{I.11}$$

and the constant Z_h is chosen so that $\mathcal{G}^h(0) = 0$. We have

$$\mathcal{G}^{(h-1)}(\phi^{(\leq h-1)}) = \log \int \exp \mathcal{G}^{(h)}(\psi^{(\leq h)}) d\mu_{C^{(h)}}(\phi^{(h)}) + \log \frac{Z_h}{Z_{h-1}} \tag{I.12}$$

for $h \leq 0$ and

$$\mathcal{G}^{(0)}(\phi^{(\leq 0)}) = -\lambda \mathcal{V}(\phi^{(\leq 0)}).$$

$$\mathcal{G}^{(-\infty)}(\phi^e) = \mathcal{G}(\phi^e).$$

As expected the coefficients in the formal expansion of $\mathcal{G}^{(h)}$ in powers of λ diverge as h tends to $-\infty$. (See [FT§1].) A well-defined expansion is generated by a λ-dependent chemical potential, which we now explain.

The two-point function G_2, defined by

$$\mathcal{G}(\phi^e) = \int d\xi_1 d\xi_2 G_2(\xi_2, \xi_2) \bar{\psi}^e(\xi_1) \psi^e(\xi_2) + O((\phi^e)^4) \tag{I.13a}$$

where

$$\int d\xi := \int d^d \mathbf{x} d\tau \sum_{\sigma \in \{\uparrow, \downarrow\}},$$

is the connected, amputated 2-point Green's function and is related to the two point Schwinger function by

$$S_2 = C + CG_2C = [ik_0 - e(\mathbf{k}) - \Sigma]^{-1} \tag{I.13b}$$

and to the proper self-energy Σ by

$$G_2 = \Sigma[\mathbf{1} + \sum_{n=1}^{\infty} (C\Sigma)^n]. \tag{I.13c}$$

In perturbation theory, G_2 is the sum of all connected two-legged Feynman diagrams, except the trivial $\longrightarrow\!\!\!\!\!-\!\!\!-$, whose external free propagators are amputated and Σ is the sum of all two-legged, one-particle irreducible (with respect to electron lines) diagrams.

The renormalized effective potential, which by abuse of notation we continue to call $\mathcal{G}$, is defined by replacing the interaction $-\lambda\mathcal{V}$ of (I.5) by $-\lambda\mathcal{V}+\delta\mathcal{V}$ where

$$\delta\mathcal{V}(\psi,\bar{\psi}) = \delta\mu(\lambda,\mu)\int d\xi\bar{\psi}(\xi)\psi(\xi)$$

to obtain

$$\mathcal{G}(\psi^e,\bar{\psi}^e) = \log\frac{1}{Z}\int \exp[(-\lambda\mathcal{V}+\delta\mathcal{V})(\psi+\psi^e,\bar{\psi}+\bar{\psi}^e)]d\mu_C(\psi,\bar{\psi}) \tag{I.14}$$

Here $\delta\mu(\lambda,\mu)$ is a formal power series in λ, uniquely determined by the condition that to all orders in perturbation theory

$$\tilde{\Sigma}(k,\mu,\lambda)|_{k_0=0,|\mathbf{k}|=\sqrt{2m\mu}} = 0 \tag{I.15}$$

Hence the bare chemical potential μ_0, which determines the position of the free Fermi surface, has been rewritten as $\mu_0 = \mu + \delta\mu(\lambda,\mu)$ where $\delta\mu$, which determines the position of the interacting Fermi surface, has been placed in the interaction. Now the coefficients in the expansion of the renormalized $\mathcal{G}^{(h)}$ converge as h tends to $-\infty$. This is the content of [FT, Theorem VII.4]. <u>We will not refer to the unrenormalized effective potential again</u>.

We now view (I.12) as defining a map from $\mathcal{G}^{(h)}$ to $\mathcal{G}^{(h-1)}$ whose iterates flow in the space of effective potentials. The renormalization condition (I.15) becomes a boundary condition at $h=-\infty$ for this flow.

The map (I.12) may be rewritten as a difference equation

$$\mathcal{G}^{(h-1)}(\phi^{(\leq h-1)}) = \mathcal{G}^{(h)}(\phi^{(\leq h-1)}) + \mathcal{E}^{(h)}(\mathcal{G}^{(h)})(\phi^{(\leq h-1)}) + \log\frac{Z_h}{Z_{h-1}} \tag{I.16}$$

where

$$\begin{aligned}\mathcal{E}^{(h)}(U)(\phi^e) : &= \log\int \exp U(\phi^{(h)}+\phi^e)d\mu_{C^{(h)}}(\phi^{(h)}) - U(\phi^e)\\ &= \int U(\phi^{(h)}+\phi^e)d\mu_{C^h}(\phi^{(h)}) - U(\phi^e)\\ &\qquad + \sum_{n=2}^{\infty}\frac{1}{n!}\mathcal{E}_n^{(h)}(U)(\phi^e)\\ &= \sum_{n=1}^{\infty}\frac{1}{n!}\mathcal{E}_n^{(h)}(U)(\phi^e)\end{aligned} \tag{I.17a}$$

where

$$\mathcal{E}_1^{(h)}(U)(\phi^e) = \int U(\phi^{(h)} + \phi^e) d\mu_{C^{(h)}}(\phi^{(h)}) - U(\phi^e) \tag{I.17b}$$

and, for $n \geq 2$,

$$\mathcal{E}_n^{(h)}(U) = \mathcal{E}_n^{(h)}(U, ..., U), \tag{I.17c}$$

$$\mathcal{E}_n^{(h)}(U_1, ..., U_n)(\phi^e) = \frac{\partial^n}{\partial\lambda_1 ... \partial\lambda_n} \log \int \exp[\sum \lambda_i U_i(\phi^{(h)} + \phi^e)] d\mu_{C^{(h)}}(\phi^e)|_{\lambda_i=0}$$

is the usual truncated expectation value. Our boundary value problem may be solved by converting (I.16) into an "integral" equation in which (I.15) appears explicitly.

To do this we introduce a localization operator ℓ. It is a projection defined for monomials by

$$\begin{gathered}\ell \int d\xi_1 d\xi_2 K(\xi_1 - \xi_2)\bar{\psi}(\xi_1)\psi(\xi_2) = \tilde{K}(k_0 = 0, |\mathbf{k}| = \sqrt{2m\mu}) \int d\xi \bar{\psi}(\xi)\psi(\xi) \\ \ell \int d\xi_1 ... d\xi_n K(\xi_1, ..., \xi_n) \overset{(-)}{\psi}(\xi_1) ... \overset{(-)}{\psi}(\xi_n) = 0 \text{ for } n > 2 \\ \ell \text{ const} = \text{const}\end{gathered} \tag{I.18}$$

and extended by linearity to all formal power series in $\psi, \bar{\psi}$. We ought to define ℓ to be an <u>orthogonal</u> projection by Wick ordering the monomials $\overset{(-)}{\psi}(\xi_1) ... \overset{(-)}{\psi}(\xi_n)$. For pedagogical purposes we delay doing so until Section III. In terms of ℓ the final value condition (I.15) is

$$\begin{aligned}0 = \lim_{h \to -\infty} \ell \mathcal{G}^{(h)}(\psi^e) &= \lim_{h \to -\infty} \ell \int d\xi_1 d\xi_2 G_2^{(h)}(\xi_1 - \xi_2)\bar{\psi}^e(\xi_1)\psi^e(\xi_2) \\ &= \lim_{h \to -\infty} \tilde{G}_2^{(h)}(k_0 = 0, |\mathbf{k}| = \sqrt{2m\mu}) \int \bar{\psi}^e(\xi)\psi^e(\xi) d\xi \\ &= \lim_{h \to -\infty} \tilde{\Sigma}^{(h)}(k_0 = 0, |\mathbf{k}| = \sqrt{2m\mu}) \int \bar{\psi}^e(\xi)\psi^e(\xi) d\xi \\ &= \tilde{\Sigma}(k_0 = 0, |\mathbf{k}| = \sqrt{2m\mu}) \int \bar{\psi}^e(\xi)\psi^e(\xi) d\xi\end{aligned} \tag{I.19}$$

since $\tilde{\mathbf{C}}^{(\geq h)}(k_0 = 0, |\mathbf{k}| = \sqrt{2m\mu}) = \sum_{j \geq h} \tilde{\mathbf{C}}^{(j)}(k_0 = 0, |\mathbf{k}| = \sqrt{2m\mu}) = 0$ by construction.

The corresponding "integral" equation is

$$\begin{aligned}\mathcal{G}^{(h)}(\phi^e) = -\lambda\mathcal{V}(\phi^e) - \sum_{i \leq h} \ell \mathcal{E}^{(i)}(\mathcal{G}^{(i)}(\phi^{(i)} + \phi^e)) + \sum_{i > h} (\mathbf{1} - \ell)\mathcal{E}^{(i)}(\mathcal{G}^{(i)}(\phi^{(i)} + \phi^e)) \\ + \sum_{i \leq h} \mathcal{E}^{(i)}(\mathcal{G}^{(i)}(\phi^{(i)}))\end{aligned} \tag{I.20}$$

Any solution of (I.20) obeys the difference equation (I.16). Furthermore, the boundary conditions (I.19) and

$$\mathcal{G}^{(0)}(\phi^e) = -\lambda\mathcal{V}(\phi^e) + \delta\mu(\lambda,\mu)\int d\xi\bar{\psi}^e(\xi)\psi^e(\xi), \tag{I.21a}$$

where $\delta\mu(\lambda,\mu)$ is defined by

$$\delta\mu(\lambda,\mu)\int d\xi\bar{\psi}^e(\xi)\psi^e(\xi) = -\sum_{i\le 0}\ell\mathcal{E}^{(i)}(\mathcal{G}^{(i)}(\phi^{(i)}+\phi^e)). \tag{I.21b}$$

are satisfied.

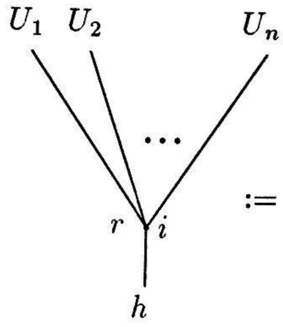
The integral equation (I.20) is solved to order n in λ by iterating n times beginning with $\mathcal{G}^{(h)}(\phi^e) = -\lambda\mathcal{V}(\phi^e)$ for all h. At the same time (I.21b) yields a closed expression for the coefficient of λ^n in the formal power series expansion of $\delta\mu(\lambda,\mu)$.

To elaborate, a single iteration of (I.20) produces terms in $\mathcal{G}^{(h)}(\phi^e)$ of, for example, the form

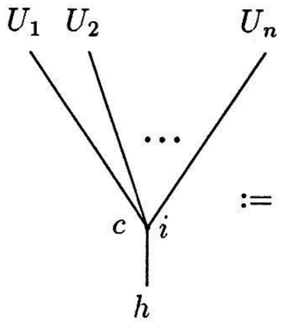
$$\begin{aligned}
&\sum_{i>h}(\mathbf{1}-\ell)\mathcal{E}^{(i)}(\mathcal{G}^{(i)}(\phi^{(h)}+\phi^e)) \\
&= \sum_{i>h}(\mathbf{1}-\ell)\mathcal{E}^{(i)}\left(-\lambda\mathcal{V}(\phi^{(i)}+\phi^e) - \sum_{j\le i}\ell\mathcal{E}^{(j)}(\mathcal{G}^{(j)}) + \sum_{j>i}(\mathbf{1}-\ell)\mathcal{E}^{(j)}(\mathcal{G}^{(j)}) + \text{const}\right) \\
&= \dots + \sum_{i>h}(\mathbf{1}-\ell)\frac{1}{4!}\sum_{\substack{j_1\le i\\ j_2\le i\\ j_3>i}}\mathcal{E}_4^{(i)}\Big(-\lambda\mathcal{V}(\phi^{(i)}+\phi^e), -\ell\mathcal{E}^{(j_1)}(\mathcal{G}^{(j_1)}), -\ell\mathcal{E}^{(j_2)}(\mathcal{G}^{(j_2)}), \\
&\qquad\qquad (\mathbf{1}-\ell)\mathcal{E}^{(j_3)}(\mathcal{G}^{(j_3)})\Big) + \dots \\
&= \dots + \sum_{i>h}(\mathbf{1}-\ell)\frac{1}{4!}\sum_{\substack{j_1\le i\\ j_2\le i\\ j_3>i}}\mathcal{E}_4^{(i)}\Big(-\lambda\mathcal{V}, -\frac{1}{2!}\ell\mathcal{E}_2^{(j_1)}(-\lambda\mathcal{V},-\lambda\mathcal{V}), -\frac{1}{2!}\ell\mathcal{E}_2^{(j_2)}(-\lambda\mathcal{V},-\lambda\mathcal{V}), \\
&\qquad\qquad \frac{1}{4!}(\mathbf{1}-\ell)\mathcal{E}^{(j_3)}(-\lambda\mathcal{V},-\lambda\mathcal{V},-\lambda\mathcal{V},-\lambda\mathcal{V}(\phi^{(j_3)}+\phi^{(i)}+\phi^e))\Big) + \dots
\end{aligned} \tag{I.22}$$

The multilinearity of $\mathcal{E}_n^{(h)}(U_1,...,U_n)$ is used in the third and fourth lines. Repeated interation of (I.20) produces terms of arbitrary depth (that is, $\mathcal{E}$'s within $\mathcal{E}$'s within $\mathcal{E}$'s ...) but with an obvious tree structure, that is conveniently expressed by introducing the following notation.

Define



$$:= \frac{1}{n!}(\mathbf{1}-\ell)\chi(i>h)\mathcal{E}_h^{(i)}(U_1(\phi^{(i)}+\phi^e),\ldots,U_n(\phi^{(i)}+\phi^e)) \qquad (I.23a)$$

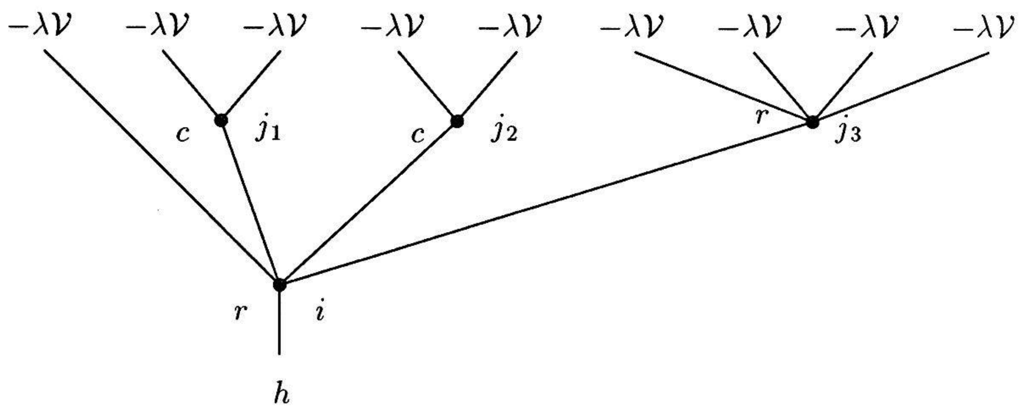


$$:= \frac{1}{n!}(-\ell)\chi(i\le h)\mathcal{E}_h^{(i)}(U_1(\phi^{(i)}+\phi^e),\ldots,U_n(\phi^{(i)}+\phi^e)) \qquad (I.23b)$$

$$:= \chi(0\ge h)U(\phi^e) \qquad (I.23c)$$

In this notation the explicit term in (I.22) becomes

$$\sum_{i,j_1,j_2,j_3}$$



The solution $\mathcal{G}^{(h)}$ of the integral equation (I.20) is just the sum over all planar trees (including the trivial tree (I.23c)) constructed from the r and c forks given above, with root scale h and "leaves" - $\lambda\mathcal{V}$. Similarly, $\delta\mu(\lambda,\mu)$ is the coefficient of $\int d\xi\bar{\psi}(\xi)\psi(\xi)$ in the sum of all trees whose lowest fork is of type (I.23b) with $h=0$.

It is shown in [FT§7] that (I.21b) is a well-defined formal powers series in λ, that is, all the coefficients are finite. With this choice of $\delta\mu(\lambda,\mu)$ the terms in the expansion of $\mathcal{G}^{(h)}$ converge as h tends to $-\infty$. More precisely, if

$$\mathcal{G}^{(h)}(\phi^e)=\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(2p)!}\lambda^n\prod_{k=1}^{2p}\Big(\int d\xi_k\phi^e(\xi_k)\Big)G^{(h)}_{2p,n}(\xi_1,...,\xi_{2p}) \tag{I.24a}$$

then there exist constants K_p and a such that

$$\|G^{(h)}_{2p,n}(\xi_1,...,\xi_{2p})\|_{1,\infty}\le K_pa^nn! \tag{I.24b}$$

where K_p is independent of n and h and a is independent of p, h and n. Here

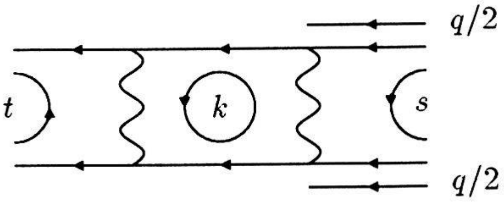
$$\|U(\xi_1,...,\xi_p)\|_{1,\infty}:=\sup\{\int d\xi_1...d\xi_p|U_p(\xi_1,...,\xi_p)f_1(\xi_1)...f_p(\xi_p)|:\|f_i\|_{L^1}\le 1,\|f_i\|_{L^\infty}\le 1\}.$$

The limit of $G^{(h)}_{2p,n}$ as $h\to-\infty$ exists and obeys the same bound. We shall give a simplified self-contained derivation of this result in Section III.

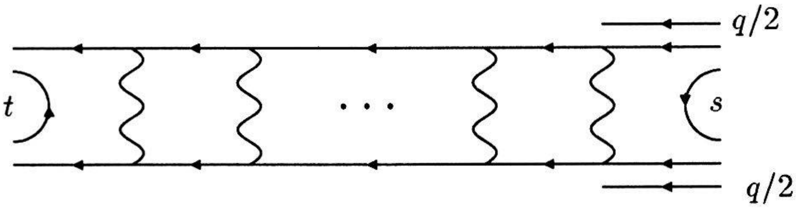
We wish to construct $G_{2p}:=G^{(-\infty)}_{2p}$ as a tempered distribution, not only as a formal power series in λ. The $n!$ in (I.24) prevents us from directly controlling the sum over n. It is therefore necessary to investigate exactly how this factor arises. There are two sources.

First, we did not, in our proof of (I.24), attempt to exploit the cancellations between the roughly $(2n)!$ Feynman graphs contributing to $G^{(h)}_{2p,n}$. Indeed, due to the Grassmann nature of the measure $d\mu_C$, each graph is one term in a $2n\times 2n$ determinant. We have grossly overestimated the determinant by taking the absolute value of each term. To eliminate this problem one exploits the antisymmetry of the determinant to estimate it as a whole.

The second, more subtle, source already appears in the simple graph



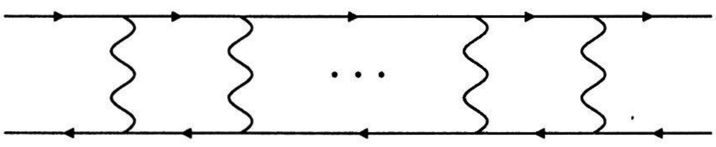
whose value, for small transfer momentum q, has the logarithmic singularity $\log\{|\mathbf{q}|+\frac{iq_0}{2\sqrt{2m\mu}}\}$. Similarly, for small q, the (electron-electron) ladder



$$(I.25)$$

grows like $[\log\{|\mathbf{q}| + \frac{iq_0}{2\sqrt{2m\mu}}\}]^n$, where n is the number of momentum loops. So, the value of a single graph containing a ladder is of order $n!$ due to the integral over q.

Somewhat surprisingly, however, the (electron-hole) ladder

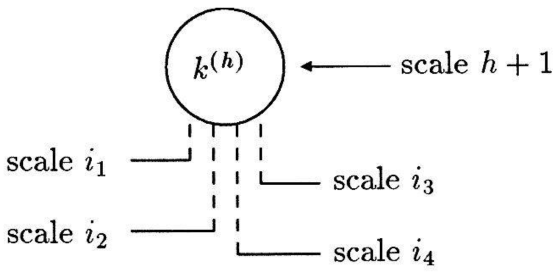


$$(I.26)$$

is exponentially bounded. The example above is discussed in more detail in [FT§5]. In this paper we show how to isolate and then remove the logarithmic singularity responsible for anomously large graphs.

We start by introducing a more refined localization process that acts nontrivially on quartic monomials. The quartic part of $\mathcal{G}^{(h)}$

$$\sum_{i_1,\cdots,i_4\le h}$$



$$= \sum_{\substack{i_1,\cdots,i_4\le h\\ \alpha,\beta,\lambda,\mu\in\{\uparrow,\downarrow\}}} \int \frac{d^{d+1}t}{(2\pi)^{d+1}}\frac{d^{d+1}s}{(2\pi)^{d+1}}\frac{d^{d+1}q}{(2\pi)^{d+1}}\, \bar\psi^{(i_1)}_{t+\frac{q}{2},\beta}\bar\psi^{(i_2)}_{-t+\frac{q}{2},\mu}\psi^{(i_4)}_{-s+\frac{q}{2},\lambda}\psi^{(i_3)}_{s+\frac{q}{2},\alpha}$$

$$\frac{1}{2}[k^{(h)}(t,s,q)\delta_{\alpha,\beta}\delta_{\lambda,\mu} - k^{(h)}(-t,s,q)\delta_{\alpha,\mu}\delta_{\lambda,\beta}] \qquad (I.27)$$

has external fields of scale at most h and a kernel $k^{(h)}$ produced by integrating out internal fields of scale at least $h+1$. Here, t, s and $2q$ are three independent momenta and the kernel has been written in an explicitly antisymmetric, spin independent form where, necessarily,

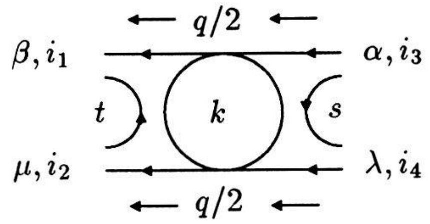
$$k^{(h)}(t,s,q) = k^{(h)}(-t,-s,q).$$

The localization process takes into account the separation between the scale h of the kernel and the scales $i_1,...,i_4$ of the external legs.

Let

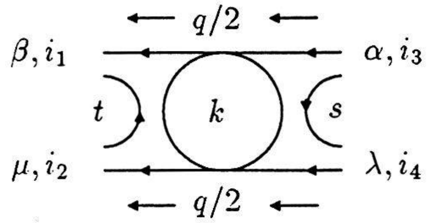
$$\int \frac{d^{d+1}t}{(2\pi)^{d+1}}\frac{d^{d+1}q}{(2\pi)^{d+1}}\frac{1}{2}[k(t,s,q)\delta_{\alpha,\beta}\delta_{\lambda,\mu} - k(-t,s,q)\delta_{\alpha,\mu}\delta_{\lambda,\beta}]$$

$$\bar\psi^{(i_1)}_{t+\frac{q}{2},\beta}\;\bar\psi^{(i_2)}_{-t+\frac{q}{2},\mu}\;\psi^{(i_4)}_{-s+\frac{q}{2},\lambda}\;\psi^{(i_3)}_{s+\frac{q}{2},\alpha}$$

= 

$(I.28)$

be a general quartic monomial. We define $L^{(h)}, h \leq 0$, by

$L^{(h)}$ 

$(I.29)$

$$= \int \frac{d^{d+1}t}{(2\pi)^{d+1}} \frac{d^{d+1}s}{(2\pi)^{d+1}} \frac{d^{d+1}q}{(2\pi)^{d+1}} \rho(|\mathbf{q}| M^{-\frac{1}{2}[i^*+h]}) \bar{\psi}^{(i_1)}_{t+\frac{q}{2},\beta} \, \bar{\psi}^{(i_2)}_{-t+\frac{q}{2},\mu} \, \psi^{(i_4)}_{-s+\frac{q}{2},\lambda} \, \psi^{(i_3)}_{s+\frac{q}{2},\alpha}$$

$$\frac{1}{2}[k((t', s', 0)\delta_{\alpha,\beta}\delta_{\lambda,\mu} - k(-t', s', 0)\delta_{\alpha,\mu}\delta_{\lambda,\beta}]$$

where

$$i^* = \max(i_1, i_2, i_3, i_4)$$

$$\rho(r) = 1 - h(r)$$

and

$$t' = (0, \tfrac{\mathbf{t}}{|\mathbf{t}|} k_F) = \text{projection of } t \text{ onto the Fermi surface.}$$

The localization operator evaluates the kernel k at zero transfer momentum and so isolates the logarithmic singularity discussed above.

As before, set

$$L^{(h)}\text{const} = \text{const} \qquad (I.29b)$$

$$L^{(h)}\int d\xi_1 d\xi_2 K(\xi_1-\xi_2)\bar\psi^{(i_1)}(\xi_1)\psi^{(i_2)}(\xi_2)=\ell\int d\xi_1 d\xi_2 K(\xi_1-\xi_2)\bar\psi^{(i_1)}(\xi_1)\psi^{(i_2)}(\xi_2)$$
$$=\tilde K(0,|\mathbf{k}|=\sqrt{2m\mu})\int d\xi\bar\psi^{(i_1)}(\xi)\psi^{(i_2)}(\xi) \qquad (I.29c)$$

$$L^{(h)}\int d\xi_1\dots d\xi_n K(\xi_1,\dots,\xi_n)\overset{(-)}{\psi}{}^{(i_1)}(\xi_1)\cdots\overset{(-)}{\psi}{}^{(i_n)}(\xi_n)=0 \text{ for } n\geq 6 \qquad (I.29d)$$

and extend by linearity to all formal power series in $\psi,\bar\psi$. Observe that

$$L^{(h)}L^{(h')}=L^{(h)}.$$

Once again, we ought to define $L^{(h)}$ to be an orthogonal projection by a scale dependent Wick ordering of the monomials $\overset{(-)}{\psi}{}^{(i_1)}(\xi_1)\cdots\overset{(-)}{\psi}{}^{(i_n)}(\xi_n)$. This is done in Section III.

The local part of a quartic monomial is not as complicated as it looks. Observe that the function ρ restricts the transfer momentum to a ball of radius $\sim M^{\frac12(i^*+h)}$ that shrinks to zero in the infrared limit $h\to-\infty$ thus effectively localizing $\mathbf{q}$ at zero. For technical convenience we have cut $\mathbf{q}$ off at the scale midway between that of the kernel and the highest field. This is somewhat arbitrary.

Let $q^0=(q_0,0)$. If we approximate

$$\bar\psi^{(i_1)}_{t+q/2}\,\bar\psi^{(i_2)}_{-t+q/2}\,\psi^{(i_4)}_{-s+q/2}\,\psi^{(i_3)}_{s+q/2} \text{ by } \bar\psi^{(i_1)}_{t+q^0/2}\,\bar\psi^{(i_2)}_{-t+q^0/2}\,\psi^{(i_4)}_{-s+q^0/2}\,\psi^{(i_3)}_{s+q^0/2}$$

the local part becomes

$$\sum_{\alpha,\beta,\lambda,\mu}\frac1V\int\frac{dq_0}{2\pi}\frac{d^{d+1}t}{(2\pi)^{d+1}}\frac{d^{d+1}s}{(2\pi)^{d+1}}\frac12[F(t',s')\delta_{\alpha,\beta}\delta_{\lambda,\mu}-F(-t',s')\delta_{\alpha,\mu}\delta_{\lambda,\beta}]$$
$$\bar\psi^{(i_1)}_{t+q^0/2,\beta}\,\bar\psi^{(i_2)}_{-t+q^0/2,\mu}\,\psi^{(i_4)}_{-s+q^0/2,\lambda}\,\psi^{(i_3)}_{s+q^0/2,\alpha}$$
$$V=\left[\int\frac{d^d\mathbf{q}}{(2\pi)^d}\rho(|\mathbf{q}|M^{-\frac12(i^*+h)})\right]^{-1}$$
$$F(t',s')=k(t',s',0).$$

As we will later explain, the most important case is $i_1=i_2=i_3=i_4=h$. Then the last expression reduces to

$$\sum_{\alpha,\beta,\lambda,\mu}\frac1V\int\frac{dq_0}{2\pi}\frac{d^{d+1}t}{(2\pi)^{d+1}}\frac{d^{d+1}s}{(2\pi)^{d+1}}F(t',s')\delta_{\alpha,\beta}\delta_{\lambda,\mu}\bar\psi^{(h)}_{t+q^0/2,\beta}\bar\psi^{(h)}_{-t+q^0/2,\mu}\psi^{(h)}_{-s+q^0/2,\lambda}\psi^{(h)}_{s+q^0/2,\alpha}$$

$$V = \text{const} M^{-dh}.$$

The local part of a quartic may be regarded as [(-1) a reduced interaction]. The discussion above demonstrates that it is the analog in the functional integral setting of the usual BCS reduced Hamiltonian (See [S], p. 37,(2-17))

$$\sum_{\mathbf{k},s} e(\mathbf{k}) a^+_{\mathbf{k},s} a_{\mathbf{k},s} + \sum_{\mathbf{k}_1,\mathbf{k}_2} \langle \mathbf{k}_2, -\mathbf{k}_2 | V | \mathbf{k}_1, -\mathbf{k}_1 \rangle a^+_{\mathbf{k}_2\uparrow} a^+_{-\mathbf{k}_2\downarrow} a_{-\mathbf{k}_1\downarrow} a_{\mathbf{k}_1\uparrow}.$$

To this point we have not discussed spin pairing.

The kernel $F(t', s')$ defines an operator on $L^2(k_F S^{d-1})$. By (I.6, S3) the operator F commutes with the action of $SO(d)$. Therefore the eigenspaces of F coincide with the $SO(d)$ irreducible invariant subspaces of $L^2(k_F S^{d-1})$. Recall that the space H^n of homogeneous harmonic polynomials of degree n is an $SO(d)$ irreducible invariant subspace and that

$$L^2(k_F S^{d-1}) = \otimes_{n \geq 0} H^n.$$

It follows that

$$F(t', s') = \sum_{n \geq 0} \lambda_n \pi_n(t', s')$$

where π_n is the orthogonal projection onto H^n and $\lambda_n, n \geq 0$ is the spectrum of F. For example when $d = 3, \pi_n(t', s') = (2n+1) k_F^{-2-n} P_n(\langle t', s' \rangle)$ where P_n is the Legendre polynomial of degree n. Substituting we obtain

$$\begin{aligned}
&\frac{1}{V} \sum_{n \geq 0} \sum_{\alpha,\beta,\lambda,\mu} \lambda_n \int \frac{dq_0}{2\pi} \frac{d^{d+1}t}{(2\pi)^{d+1}} \frac{d^{d+1}s}{(2\pi)^{d+1}} \pi_n(t', s') \delta_{\alpha,\beta} \delta_{\lambda,\mu} \bar{\psi}^{(h)}_{t+\frac{q^0}{2},\beta} \bar{\psi}^{(h)}_{-t+\frac{q^0}{2},\mu} \psi^{(h)}_{-s+\frac{q^0}{2},\lambda} \psi^{(h)}_{s+\frac{q^0}{2},\alpha} \\
&= \frac{1}{V} \sum_{n \text{ even}} \lambda_n \int \frac{dq_0}{2\pi} \frac{d^{d+1}t}{(2\pi)^{d+1}} \frac{d^{d+1}s}{(2\pi)^{d+1}} \pi_n(t', s') 2 \bar{\psi}^{(h)}_{t+\frac{q^0}{2}\uparrow} \bar{\psi}^{(h)}_{-t+\frac{q^0}{2}\downarrow} \psi^{(h)}_{-s+\frac{q^0}{2}\downarrow} \psi^{(h)}_{s+\frac{q^0}{2}\uparrow} \\
&+ \frac{1}{V} \sum_{\text{odd}} \lambda_n \int \frac{dq_0}{2\pi} \frac{d^{d+1}t}{(2\pi)^{d+1}} \frac{d^{d+1}s}{(2\pi)^{d+1}} \pi_n(t', s') \Big\{ \bar{\psi}^{(h)}_{t+\frac{q^0}{2}\uparrow} \bar{\psi}^{(h)}_{-t+\frac{q^0}{2}\uparrow} \psi^{(h)}_{-s+\frac{q^0}{2}\uparrow} \psi^{(h)}_{s+\frac{q^0}{2}\uparrow} \\
&\qquad + \bar{\psi}^{(h)}_{t+\frac{q^0}{2}\downarrow} \bar{\psi}^{(h)}_{-t+\frac{q^0}{2}\downarrow} \psi^{(h)}_{-s+\frac{q^0}{2}\downarrow} \psi^{(h)}_{s+\frac{q^0}{2}\downarrow} \\
&+ \frac{1}{2} (\bar{\psi}^{(h)}_{t+\frac{q^0}{2}\uparrow} \bar{\psi}^{(h)}_{-t+\frac{q^0}{2}\downarrow} + \bar{\psi}^{(h)}_{t+\frac{q^0}{2}\downarrow} \bar{\psi}^{(h)}_{-t+\frac{q^0}{2}\uparrow}) (\psi^{(h)}_{-s+\frac{q^0}{2}\downarrow} \psi^{(h)}_{s+\frac{q^0}{2}\uparrow} + \psi^{(h)}_{-s+\frac{q^0}{2}\uparrow} \psi^{(h)}_{s+\frac{q^0}{2}\downarrow}) \Big\}
\end{aligned} \tag{I.30}$$

since

$$\pi_n(-t', s') = (-1)^n \pi_n(t', s')$$
$$\text{and } \pi_n(t', -s') = (-1)^n \pi_n(t', s').$$

The eigenvalue λ_n is the coupling constant for the interaction in the n^{th} angular momentum sector. In this paper we will ultimately consider a class of interactions, including (I.5c), in which $\lambda_0 > 0, \lambda_0 \geq |\lambda_n|$, $n \geq 1$. See Theorem I.3, for the precise condition. Then the, dominant, angular momentum zero contribution to the interaction is

$$(-1)2\frac{\lambda_0}{V}\int \frac{dq_0}{2\pi}\frac{d^{d+1}t}{(2\pi)^{d+1}}\frac{d^{d+1}s}{(2\pi)^{d+1}}\bar{\psi}^{(h)}_{t+q^0/2\uparrow}\bar{\psi}^{(h)}_{-t+q^0/2\downarrow}\psi^{(h)}_{-s+q^0/2\downarrow}\psi^{(h)}_{s+q^0/2\uparrow}. \qquad (I.31)$$

As h approaches minus infinity particles of scale h have their momenta restricted to a shell about the Fermi surface and "see" a spatial volume $V \sim M^{-dh}$ since their covariance $|C^{(h)}(x,y)| \leq \text{const}[1+M^h|x-y|]^{-N}$. Thus, the local part of the interaction is effectively the familiar BCS interaction (See [FW], p. 333, (37,43)). The factor $\frac{1}{V}$ should not be interpreted as a small coupling constant even though it tends to zero as $h \to -\infty$. Rather, it maintains the power counting neutrality, i.e. dimensionlessness, of the interaction (I.31). Operationally it compensates for the lack of decay between $\mathbf{x}$ and $\mathbf{y}$ in

$$\int \frac{dq_0}{2\pi}\frac{d^{d+1}t}{(2\pi)^{d+1}}\frac{d^{d+1}s}{(2\pi)^{d+1}}\bar{\psi}^{(h)}_{t+q^0/2\uparrow}\bar{\psi}^{(h)}_{-t+q^0/2\downarrow}\psi^{(h)}_{-s+q^0/2\downarrow}\psi^{(h)}_{s+q^0/2\uparrow}$$

$$= \int d\tau \int d^d\mathbf{x}\bar{\psi}^{(h)}(\mathbf{x},\tau,\uparrow)\bar{\psi}^{(h)}(\mathbf{x},\tau,\downarrow)\int d^d\mathbf{y}\psi^{(h)}(\mathbf{y},\tau,\downarrow)\psi^{(h)}(\mathbf{y},\tau,\uparrow)$$

As usual we have omitted the symbol for the Fourier transform on $\bar{\psi}^{(h)}_{t+q^0/2\uparrow}$ etc.

We have seen that the solution $\mathcal{G}^{(h)}$ of the difference equation (I.16)

$$\mathcal{G}^{(h-1)} = \mathcal{G}^{(h)} + \mathcal{E}^{(h)}(\mathcal{G}^{(h)}) + \log\frac{Z_h}{Z_{h-1}}$$

obeying the final value condition (I.19)

$$\lim_{h\to-\infty} \ell\mathcal{G}^{(h)} = 0$$

may be constructed as the sum of all trees built from the r,c forks (I.23a,b). It is clear that the construction above is unchanged when ℓ is replaced everywhere by $L^{(h)}$. Doing so however does not quite yield the desired model. The quadratic part of $\mathcal{G}^{(h)}$ will satisfy the final value condition $\lim_{h\to-\infty} \ell\mathcal{G}^{(h)} = 0$, as required. But the quartic part of $\mathcal{G}^{(h)}$ will also satisfy a final condition rather than the initial value condition

$$\mathcal{G}^{(0)}(\phi^e)_{\text{quartic}} = -\frac{\lambda}{2}\sum_{\alpha_i\in\{\uparrow,\downarrow\}}\int\prod\frac{dk_i}{(2\pi)^{d+1}}(2\pi)^{d+1}\delta(k_1+k_2-k_3-k_4)\delta_{\alpha_1,\alpha_3}\delta_{\alpha_2,\alpha_4}$$

$$\langle k_1, k_2 | V | k_3, k_4 \rangle \bar{\psi}(k_1, \alpha_1) \bar{\psi}(k_2, \alpha_2) \psi(k_4, \alpha_4) \psi(k_3, \alpha_3) \tag{I.32}$$

This defect is easily eliminated by changing the integral equation (I.20) to

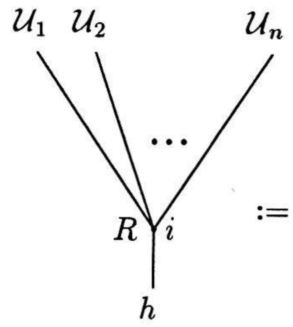
$$\begin{aligned} \mathcal{G}^{(h)}(\phi^e) = &- \lambda L^{(0)} \mathcal{V}(\phi^e) - \sum_{i \le h} \ell \mathcal{E}^{(i)}\big(\mathcal{G}^{(i)}(\phi^{(i)} + \phi^e)\big) + \sum_{i > h} (L^{(i-1)} - \ell) \mathcal{E}^{(i)}\big(\mathcal{G}^{(i)}(\phi^{(i)} + \phi^e)\big) \\ &- \lambda(\mathbf{1} - L^{(0)}) \mathcal{V}(\phi^e) + \sum_{i > h} (\mathbf{1} - L^{(i-1)}) \mathcal{E}^{(i)}\big(\mathcal{G}^{(i)}(\phi^{(i)} + \phi^e)\big) + \sum_{i \le h} \mathcal{E}^{(i)}\big(\mathcal{G}^{(i)}(\phi^{(i)})\big) \end{aligned} \tag{I.33}$$

Any solution of (I.33) obeys the difference equation (I.16), the initial condition (I.32) and the final condition (I.19). Equation (I.33) is obtained from (I.20) by adding and subtracting the quartic effective interaction at scale h

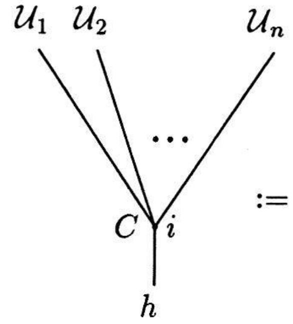
$$\sum_{i > h} (L^{(i-1)} - \ell) \mathcal{E}^{(i)}\big(\mathcal{G}^{(i)}(\phi^{(i)} + \phi^e)\big),$$

thus leaving $\mathcal{G}^{(h)}$ unchanged.

Let



$$:= \frac{1}{n!} (\mathbf{1} - L^{(i-1)}) \chi(i > h) \mathcal{E}_n^{(i)}\big(\mathcal{U}_1(\phi^{(i)} + \phi^e), ..., \mathcal{U}_n(\phi^{(i)} + \phi^e)\big) \tag{I.34a}$$



$$:= \frac{1}{n!} [-\ell \chi(i \le h) + (L^{(i-1)} - \ell) \chi(i > h)] \mathcal{E}_n^{(i)}(\mathcal{U}_1, ..., \mathcal{U}_n)) \tag{I.34b}$$

$\mathcal{U}$

$|C$

$$:= \chi(0 \geq h) L^{(0)} \mathcal{U}(\phi^e) \qquad (I.34c)$$

h

$\mathcal{U}$

$|R$

$$:= \chi(0 \geq h)(\mathbf{1} - L^{(0)}) \mathcal{U}(\phi^e) \qquad (I.34d)$$

h

The solution $\mathcal{G}^{(h)}$ of the integral equation (I.33) is the sum of all planar trees (including the trivial trees (I.34c,d)) constructed from the R and C forks with root scale h and "leaves" $-\lambda\mathcal{V}$. The new and old trees differ only in that an effective interaction $\sum_{j>1}(L^{(j-1)} - \ell)\mathcal{E}^{(j)}(\mathcal{G}^{(j)})$ at scale i is added and subtracted at each fork.

The operator $L^{(h)}$ isolates the logarithmic singularity that produces anomolously large values (of order $n!$) for graphs containing ladders in the C forks of trees. To make this precise let $g^{(h)}(\phi^e)$ be the sum of all planar trees (including the trivial trees (I.34c,d)) constructed from R (I.34a) and c ((I.34b) forks with root scale h and "leaves" $-\lambda\mathcal{V}$. We remark that the functional $g^{(h)}(\phi^e)$ is the solution of the integral equation

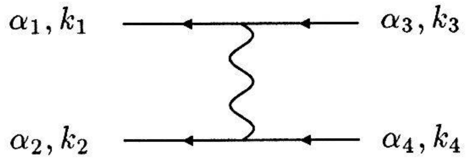
$$g^{(h)}(\phi^e) = -\lambda\mathcal{V}(\phi^e) - \sum_{i \leq h} \ell \mathcal{E}^{(i)}\big(g^{(h)}(\phi^{(i)} + \phi^e)\big)$$
$$+ \sum_{i>h}(\mathbf{1} - L^{(i-1)})\mathcal{E}^{(i)}\big(g^{(i)}(\phi^{(i)} + \phi^e)\big) + \sum_{i \leq h} \mathcal{E}^{(i)}\big(g^{(i)}(\phi^{(i)})\big) \qquad (I.35)$$

obtained from (I.33) by discarding the term $\sum_{i>h}(L^{(i-1)} - \ell)\mathcal{E}^{(i)}(\mathcal{G}^{(i)})$. Write

$$g^{(h)}(\phi^e) = \sum_{p=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(2p)!}\lambda^n \prod_{k=1}^{2p}\Big(\int d\xi_k \phi^e(\xi_k)\Big) g^{(h)}_{2p,n}(\xi_1, \ldots, \xi_{2p}). \qquad (I.36)$$

The idea is that, in contrast to $G^{(h)}_{2p,n}$, (I.24), every graph contributing to $g^{(h)}_{2p,n}$ should be exponentially bound in n.

We now recall how a tree is expressed as a sum of graphs. Consider a tree $\mathcal{T}$ contributing to $g^{(h)}_{2p,n}$. Such a tree has n "leaves" $-\lambda\mathcal{V}$. Introduce a vertex



$$= \delta_{\alpha_1,\alpha_3}\delta_{\alpha_2,\alpha_4}\delta(k_1 + k_2 - k_3 - k_4)$$
$$\langle k_1, k_2|V|k_3, k_4\rangle \qquad (I.37a)$$

for each leaf. Then, form graphs by connecting outgoing legs $\bullet\!\!\longrightarrow\!\!-$ to incoming legs $\bullet\!\!-\!\!\longleftarrow$ pairwise in all possible ways leaving p outgoing and p incoming external legs

$$\bullet\!\!\longrightarrow\!\!-^{k} = \bar{\psi}^e(k)$$
$$\bullet\!\!\longleftarrow\!\!-^{k} = \psi^e(k) \qquad (I.37b)$$

To simplify the combinatorics regard all legs as distinguishable so that the graphs above are distinct.

Next, we assign scales to the internal lines of a graph G consistent with the structure of $\mathcal{T}$. Let

$$s(\mathcal{T}) = \{i_f | f \text{ a fork of } \mathcal{T}\} \qquad (I.38)$$

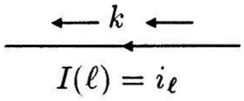
and I a map from the internal lines of G to $s(\mathcal{T})$ such that, for each fork $f \in \mathcal{T}$, the subgraph

$$G^I_f = \{\text{lines } \ell \in G \text{ and connecting vertices } | I(\ell) = i_{f'}, f' \geq f\} \qquad (I.39)$$

is connected. We denote the graph G with scale assignments I by G^I. The set of subgraphs G^I_f, when ordered by inclusion, form a tree isomorphic to $\mathcal{T}$. The set of all consistently labelled graphs G^I is denoted $\Gamma_{2p}(\mathcal{T})$.

The value $\mathrm{Val}(G^I)$ of the labelled graph G^I in momentum space is computed by the following rules.

(i) To each internal line ℓ assign the covariance



$$= \tilde{C}^{(i_\ell)}(k) \tag{I.37c}$$

of scale i_ℓ. Assign vertices and external legs the values given in (I.37a,b). $(I.40i)$

(ii) For each c, r, C, R fork f we respectively apply the operator

$$-\frac{1}{n_f!}\chi(i_f \le i_{\pi(f)})\ell$$
$$\frac{1}{n_f!}\chi(i_f > i_{\pi(f)})(\mathbf{1}-\ell)$$
$$\frac{1}{n_f!}\{\chi(i_f \le i_{\pi(f)})(-\ell) + (L^{(i_f-1)} - \ell)\chi(i_f > i_{\pi(f)})\}$$
$$\frac{1}{n_f!}\chi(i_f > i_{\pi(f)})(\mathbf{1} - L^{(i_f-1)})$$

to the subgraph G_f^I. Here $\pi(f)$ is the fork immediately below f in the tree and n_f is the upward branching number of f. Of course trees contributing to $g_{2p,n}^{(h)}$ do not contain r, C forks. $(I.40ii)$

(iii) Integrate $\int \frac{dk_\ell}{(2\pi)^{d+1}}$ for all internal lines ℓ and multiply by

$$(-1)^{\text{number of fermion loops}}.$$

$(I40iii)$

These rules are derived and discussed in greater detail in [FT§VI]. Finally

$$g_{2p,n}^{(h)} = \sum_{\text{trees } \mathcal{T} \text{ with } n \text{ leaves}} \sum_{\Gamma_{2p}(\mathcal{T})} \mathrm{Val}(G^I). \tag{I.41}$$

We now indicate why a graph contributing to $g_{2p,n}^{(h)}$ should be exponentially bounded. It follows from (I.29a) that

$$(\mathbf{1} - L^{(h)})\Big\{ \sum_{\alpha,\beta\lambda,\mu} \int \frac{d^{d+1}t}{(2\pi)^{d+1}} \frac{d^{d+1}s}{(2\pi)^{d+1}} \frac{d^{d+1}q}{(2\pi)^{d+1}} \bar\psi^{(i_1)}_{t+\frac{q}{2},\beta} \bar\psi^{(i_2)}_{-t+\frac{q}{2},\mu} \psi^{(i_4)}_{-s+\frac{q}{2},\lambda} \psi^{(i_3)}_{s+\frac{q}{2},\alpha}$$
$$\frac{1}{2}[k(t,s,q)\delta_{\alpha,\beta}\delta_{\lambda,\mu} - k(-t,s,q)\delta_{\alpha,\mu}\delta_{\lambda,\beta}]\Big\}$$

$$= \sum_{\alpha,\beta,\lambda,\mu} \int \frac{d^{d+1}t}{(2\pi)^{d+1}} \frac{d^{d+1}s}{(2\pi)^{d+1}} \frac{d^{d+1}q}{(2\pi)^{d+1}} \bar{\psi}^{(i_1)}_{t+\frac{q}{2},\beta} \bar{\psi}^{(i_2)}_{-t+\frac{q}{2},\mu} \psi^{(i_4)}_{-s+\frac{q}{2},\lambda} \psi^{(i_3)}_{s+\frac{q}{2},\alpha}$$

$$\frac{1}{2}[f^{(h)}(t,s,q)\delta_{\alpha,\beta}\delta_{\lambda,\mu} - f^{(h)}(-t,s,q)\delta_{\alpha,\mu}\delta_{\lambda,\beta}] \qquad (I.41a)$$

where

$$f^{(h)}(t,s,q) = k(t,s,q)[1-\rho(|\mathbf{q}|M^{-\frac{1}{2}[i^*+h]})] \qquad (I.41b)$$

$$+[k(t,s,q)-k(t,s,0)]\rho(|\mathbf{q}|M^{-\frac{1}{2}[i^*+h]}) \qquad (I.41c)$$

$$+[k(t,s,0)-k(t',s',0)]\rho(|\mathbf{q}|M^{-\frac{1}{2}[i^*+h]}). \qquad (I.41d)$$

We shall see that when at least one of $(\pm t+\frac{q}{2})_0$, $(\pm s+\frac{q}{2})_0$ is bigger than $M^{\frac{1}{2}[i^*+h]}$ the corresponding field $\bar{\psi},\psi$ acts as if it were of scale $\frac{1}{2}[i^*+h]$ rather than i_j to produce an extra factor

$$M^{-\frac{1}{2}[i^*+h]}M^{i_j} \le M^{-\frac{1}{2}(h-i^*)}.$$

So consider $|t_0|, |s_0|, |q_0| \le M^{\frac{1}{2}[i^*+h]}$. Then the last term is small since $\big||\pm\mathbf{t}+\mathbf{q}/2|-k_F\big|$, $\big||\pm\mathbf{s}+\mathbf{q}/2|-k_F\big| \le O(M^{i^*})$ and $|\mathbf{q}| \le O(M^{\frac{1}{2}[i^*+h]})$ imply $|t-t'|, |s-s'| \le O(M^{\frac{1}{2}[i^*+h]})$. Estimating the difference by $|\nabla_t k||t-t'| + |\nabla_s k||s-s'|$, the gradients produce an M^{-h} which combines with $M^{\frac{1}{2}[i^*+h]}$ to yield the exponentially small $M^{-\frac{1}{2}[h-i^*]}$. Just as in the discussion following (I.31), $\rho(|\mathbf{q}|M^{-\frac{1}{2}[i^*+h]})$, does not produce an additional small factor even though it severely restricts the domain of integration for $\mathbf{q}$. That the term (I.41c) is small is also seen by Taylor expanding.

The first term (I.41b) is more subtle. By construction $\pm\mathbf{t}+\mathbf{q}/2$ and $\pm\mathbf{s}+\mathbf{q}/2$ lie in a shell of thickness $O(M^{i^*})$ about the Fermi surface. However, $1-\rho(|\mathbf{q}|M^{-\frac{1}{2}[i^*+h]})$ forces the transfer momentum $\mathbf{q}$ to be relatively large with the result that s and t are constrained to small regions of momentum space. Precisely, it is shown in Lemma IV.2 that

$$\mathrm{vol}\{\mathbf{t}\in\mathbf{R}^d : \big||\pm\mathbf{t}+\mathbf{q}/2|-k_F\big| \le O(M^{i^*})\}$$

$$\le O(M^{-\frac{1}{2}[h-i^*]})\mathrm{vol}\{\mathbf{t}\in\mathbf{R}^d : \big||\mathbf{t}|-k_F\big| \le O(M^{i^*})\}$$

when $|\mathbf{q}| \ge O(M^{\frac{1}{2}[h+i^*]})$.

To give a physical interpretation to the last estimate consider a pair of particles (or holes) with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$. As a composite particle the pair has momentum $\mathbf{q} = \mathbf{k}_1 + \mathbf{k}_2$. In the physically interesting case $\mathbf{k}_1$ and $\mathbf{k}_2$ are in a thin shell about the Fermi surface and the estimate is a quantitative statement that $\mathbf{q}$ is usually small, or equivalently $\mathbf{k}_2 \approx -\mathbf{k}_1$. In other words the composite particle is a Cooper pair.

The intuition above is exploited in the superconducting context and formalized as Theorem I.1 below. Thus, the mechanism responsible for anomolously large graphs is localized in the quartic contributions to the C forks of trees, which, incidentally justifies approximating the full interaction by the BCS reduced interaction

$$\sum_{\substack{\mathbf{k}_1,\mathbf{k}_2\\ \alpha,\beta,\lambda,\mu}} \langle \mathbf{k}_2, -\mathbf{k}_2 | V | \mathbf{k}_1, -\mathbf{k}_1 \rangle \delta_{\alpha,\beta}\delta_{\lambda,\mu} a^{+}_{\mathbf{k}_2,\alpha} a^{+}_{\mathbf{k}_2,\lambda} a_{-\mathbf{k}_1,\mu} a_{\mathbf{k}_1,\beta}.$$

Thus g (I.35) should now be constructed nonperturbatively by exploiting cancellations arising from the Grassman antisymmetry. The quartic contributions to the C forks are treated by means of a renormalization group flow.

Our discussion of $\mathbf{1} - L^{(h)}$ is finished for the moment. We now treat the quartic contributions to the C forks by means of a flow that nonperturbatively resums anomolously large graphs to an exponentially bounded effective interaction. Set

$$\begin{aligned}
\mathcal{F}^{(h)}(\bar\psi^{(i)}, \psi^{(i)}; i \le h) &:= (L^{(h)} - \ell)\mathcal{G}^{(h)}\Big(\sum_{i\le h} \bar\psi^{(i)}, \sum_{i\le h} \psi^{(i)}\Big) \\
&:= \sum_{\substack{i_1,i_2,i_3,i_4\le h\\ \alpha_1,\alpha_2,\alpha_3,\alpha_4\in\{\uparrow,\downarrow\}}} \frac{1}{4}\int \frac{d^{d+1}s}{(2\pi)^{d+1}}\,\frac{d^{d+1}t}{(2\pi)^{d+1}}\,\frac{d^{d+1}q}{(2\pi)^{d+1}}\,\rho(|\mathbf{q}|M^{-\frac{1}{2}(i^*+h)}) \\
&\qquad [F^{(h)}(t',s')\delta_{\alpha_1,\alpha_3}\delta_{\alpha_2,\alpha_4} - F^{(h)}(-t',s')\delta_{\alpha_1,\alpha_4}\delta_{\alpha_2,\alpha_3}] \\
&\qquad \bar\psi^{(i_1)}_{t+q/2,\alpha_1}\bar\psi^{(i_2)}_{-t+q/2,\alpha_2}\psi^{(i_4)}_{-s+q/2,\alpha_4}\psi^{(i_3)}_{s+q/2,\alpha_3}. \qquad (I.42)
\end{aligned}$$

In particular

$$\mathcal{F}^{(0)} = -\frac{\lambda}{4}\sum_{i_j,\alpha_j}(L^{(0)} - \ell)\int \frac{d^{d+1}s}{(2\pi)^{d+1}}\,\frac{d^{d+1}t}{(2\pi)^{d+1}}\,\frac{d^{d+1}q}{(2\pi)^{d+1}}\,\bar\psi^{(i_1)}_{t+q/2,\alpha_1}\bar\psi^{(i_2)}_{-t+q/2,\alpha_2}\psi^{(i_4)}_{-s+q/2,\alpha_4}\psi^{(i_3)}_{s+q/2,\alpha_3}$$

$$\Big\{\langle t+\frac{q}{2}, -t+\frac{q}{2}|V|s+\frac{q}{2}, -s+\frac{q}{2}\rangle\delta_{\alpha_1,\alpha_3}\delta_{\alpha_2,\alpha_4} - \langle -t+\frac{q}{2}, t+\frac{q}{2}|V|s+\frac{q}{2}, -s+\frac{q}{2}\rangle\delta_{\alpha_1,\alpha_4}\delta_{\alpha_2,\alpha_3}\Big\}$$

$$= \sum_{i_1,i_2,i_3,i_4\le 0} -\frac{\lambda}{4}\int \frac{d^{d+1}s}{(2\pi)^{d+1}}\frac{d^{d+1}t}{(2\pi)^{d+1}}\frac{d^{d+1}q}{(2\pi)^{d+1}}\,\rho(|\mathbf{q}|M^{-\frac{1}{2}i^*})$$
$$\{\langle t',-t'|V|s',-s'\rangle\delta_{\alpha_1,\alpha_3}\delta_{\alpha_2,\alpha_4} - \langle -t',t'|V|s',-s'\rangle\delta_{\alpha_1,\alpha_4}\delta_{\alpha_2,\alpha_3}\}$$
$$\bar\psi^{(i_1)}_{t+q/2,\alpha_1}\bar\psi^{(i_2)}_{-t+q/2,\alpha_2}\psi^{(i_4)}_{-s+q/2,\alpha_4}\psi^{(i_3)}_{s+q/2,\alpha_3}. \qquad (I.43)$$

Note that

$$\begin{aligned} &F^{(h)}(t',s') = F^{(h)}(-t',-s')\\ &F^{(h)}(Rt',Rs') = F^{(h)}(t',s') \text{ for all } R\in S0(d) \qquad (I.44)\\ &F^{(h)}(t',s') = \overline{F^{(h)}(s',t')}\end{aligned}$$

The first two parts of (I.44) follow immediately from antisymmetry and rotation invariance. The third is proven following (I.95).

Apply $L^{(h-1)}-\ell$ to (I.16) to obtain

$$\mathcal{F}^{(h-1)} = (L^{(h-1)}-\ell)\mathcal{F}^{(h)} + (L^{(h-1)}-\ell)\mathcal{E}^{(h)}(\mathcal{F}^{(h)} + (\mathbf{1}-L^{(h)}+\ell)\mathcal{G}^{(h)}) \qquad (I.45)$$

since

$$(L^{(h-1)}-\ell)\mathcal{G}^{(h)} = (L^{(h-1)}-\ell)(L^{(h)}-\ell)\mathcal{G}^{(h)}.$$

It follows that $F^{(h)}$ is the solution of the difference (flow) equation

$$F^{(h-1)} = F^{(h)} + \frac{4}{\rho(|\mathbf{q}|M^{-1/2(i^*+h-1)})}\ [\text{antisymmetric kernel of}$$
$$(L^{(h-1)}-\ell)\mathcal{E}^{(h)}(\mathcal{G}^{(h)}) \text{ evaluated at } \alpha_1=\alpha_3=\uparrow, \alpha_2=\alpha_4=\downarrow] \qquad (I.46a)$$

with initial value

$$F^{(0)}(t',s') = -\lambda\langle t',-t'|V|s',-s'\rangle. \qquad (I.46b)$$

We remark that $(L^{(h-1)}-\ell)\mathcal{E}^{(h)}(\mathcal{G}^{(h)})$ has a unique kernel antisymmetric under the exchange $\begin{pmatrix} i_1\\ \alpha_1\\ t\end{pmatrix} \leftrightarrow \begin{pmatrix} i_2\\ \alpha_2\\ -t\end{pmatrix}$ and under the exchange $\begin{pmatrix} i_3\\ \alpha_3\\ s\end{pmatrix} \leftrightarrow \begin{pmatrix} i_4\\ \alpha_4\\ -s\end{pmatrix}$.

Substituting

$$(L^{(i-1)}-\ell)\mathcal{E}^{(i)} = \mathcal{F}^{(i-1)} - (L^{(i-1)}-\ell)\mathcal{F}^{(i)}$$

into (I.33) and telescoping the sum

$$
\begin{aligned}
&-\lambda L^{(0)}\mathcal{V} + \sum_{i=h+1}^{0} (L^{(i-1)} - \ell)\mathcal{E}^{(i)} \\
&= -\lambda L^{(0)}\mathcal{V} + \sum_{i=h+1}^{0} [\mathcal{F}^{(i-1)} - (L^{(i-1)} - \ell)\mathcal{F}^{(i)}] \\
&= -\lambda L^{(0)}\mathcal{V} + \mathcal{F}^{(h)} + \sum_{i=h+1}^{0} (L^{(i)} - L^{(i-1)})\mathcal{F}^{(i)} - (L^{(0)} - \ell)\mathcal{F}^{(0)} \\
&= \mathcal{F}^{(h)} + \sum_{i=h+1}^{0} (L^{(i)} - L^{(i-1)})\mathcal{F}^{(i)}
\end{aligned}
$$

we obtain the coupled system

$$
\begin{aligned}
\mathcal{G}^{(h)} = \mathcal{F}^{(h)} &+ \sum_{i>h}(L^{(i)} - L^{(i-1)})\mathcal{F}^{(i)} - \sum_{i\leq h} \ell\mathcal{E}^{(i)}(\mathcal{G}^{(i)}) - \lambda(\mathbf{1} - L^{(0)})\mathcal{V} \\
&+ \sum_{i>h}(\mathbf{1} - L^{(i-1)})\mathcal{E}^{(i)}(\mathcal{G}^{(i)}) + \log\frac{Z_{h+1}}{Z_h} \qquad (I.47a)
\end{aligned}
$$

$$
\mathcal{F}^{(h-1)} = (L^{(h-1)} - \ell)\mathcal{F}^{(h)} + (L^{(h-1)} - \ell)\mathcal{E}^{(h)}(\mathcal{G}^{(h)}). \qquad (I.47b)
$$

with boundary conditions

$$
(\text{projection onto nonquadratic part of } \mathcal{G}^{(0)}) = -\lambda\mathcal{V} \qquad (I.48a)
$$

$$
F^{(0)}(t', s') = -\lambda\langle t', -t'|V|s', -s'\rangle \qquad (I.48b)
$$

$$
\lim_{h\to-\infty} \ell\mathcal{G}^{(h)} = 0 \qquad (I.48c)
$$

Equation (I.47a) is a decomposition of $\mathcal{G}^{(h)}$ into a quadratic piece $(-\sum_{i\leq h}\ell\mathcal{E}^{(i)})$, a quartic effective interaction $\left(\mathcal{F}^{(h)} + \sum_{j>h}(L^{(j)} - L^{(j-1)})\mathcal{F}^{(j)}\right)$ and, by the discussion of $\mathbf{1} - L^{(h)}$ above,an "irrelevant" piece $\left\{-\lambda(\mathbf{1} - L^{(0)})\mathcal{V} + \sum_{i>h}(\mathbf{1} - L^{(i-1)})\mathcal{E}^{(i)} + \log\frac{Z_{h+1}}{Z_h}\right\}$. The term $\sum_{j>h}(L^{(j)} - L^{(j-1)})\mathcal{F}^{(j)}$ arises from the scale dependent nature of our localization process and does not appear in standard field theory models. It may be helpful to visualize

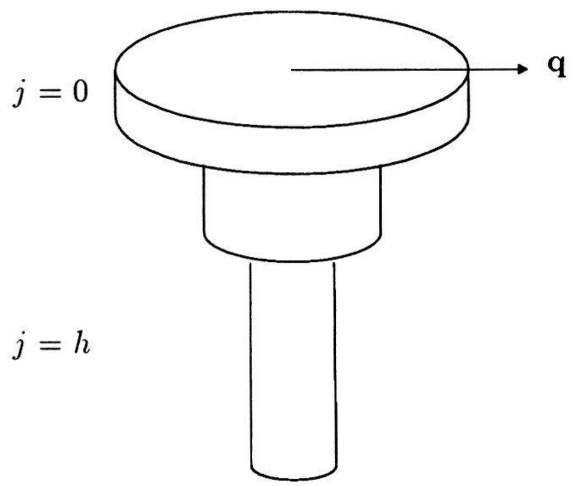
$$\mathcal{F}^{(h)} + \sum_{j>h}(L^{(j)} - L^{(j-1)})\mathcal{F}^{(j)}$$

$$= \sum_{\substack{i_j \le h \\ a_j \in \{\uparrow,\downarrow\}}} \frac{1}{4}\int \frac{d^{d+1}s}{(2\pi)^{d+1}} \frac{d^{d+1}t}{(2\pi)^{d+1}} \frac{d^{d+1}q}{(2\pi)^{d+1}} \bar\psi^{(i_1)}_{t+\frac{q}{2},\alpha_1}\bar\psi^{(i_2)}_{-t+\frac{q}{2},\alpha_2}\psi^{(i_4)}_{-s+\frac{q}{2},\alpha_4}\psi^{(i_3)}_{s+\frac{q}{2},\alpha_3}$$

$$\Big\{\rho(|\mathbf{q}|M^{-\frac{1}{2}[i^*+h]})[F^{(h)}(t',s')\delta_{\alpha_1,\alpha_3}\delta_{\alpha_2,\alpha_4} - F^{(h)}(-t',s')\delta_{\alpha_1,\alpha_4}\delta_{\alpha_2,\alpha_3}]$$

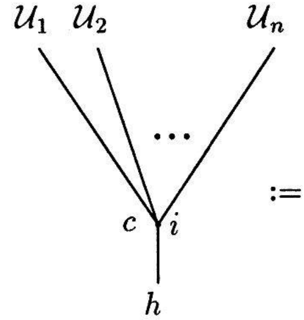
$$\sum_{j=h+1}^{0} \rho_{i^*+j}(|\mathbf{q}|)[F^{(j)}(t',s')\delta_{\alpha_1,\alpha_3}\delta_{\alpha_2,\alpha_4} - F^{(j)}(-t',s')\delta_{\alpha_1,\alpha_4}\delta_{\alpha_2,\alpha_3}]\Big\} \qquad (I.49)$$

where $\rho_k(\mathbf{q}) = \rho(|\mathbf{q}|M^{-k/2}) - \rho(|\mathbf{q}|M^{-(k-1)/2})$, as a spy-glass.

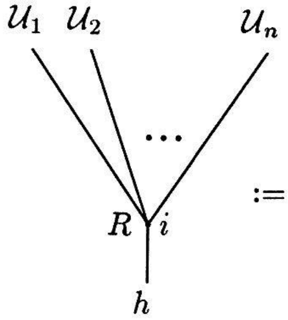


When h is decreased by one the outside shell of $\rho(|\mathbf{q}|M^{-\frac{1}{2}(i^*+h)})[F^{(h)}(t',s')...]$ detaches, is added to $\sum_{j=h+1}^{0}$ ..., and stops flowing. Iterating, we obtain an infinitely extended spy-glass.

The structure of the coupled system (I.47) allows us to express $\mathcal{G}^{(h)}$ as a formal power series in the infinitely many variables (running coupling "constants") $\mathcal{F}^{(h)}, h \le 0$. Precisely, $\mathcal{G}^{(h)}$ is the sum of all trees of root scale h constructed from the forks



$$:= \frac{1}{n!}(-\ell)\chi(i \le h)\mathcal{E}_n^{(i)}\left(\mathcal{U}_1(\phi^{(i)} + \phi^e), ..., \mathcal{U}_n(\phi^{(i)} + \phi^e)\right) \qquad (I.50a)$$



$$:= \frac{1}{n!}(\mathbf{1} - L^{(i-1)})\chi(i > h)\mathcal{E}_n^{(i)}\left(\mathcal{U}_1(\phi^{(i)} + \phi^e), ..., \mathcal{U}_n(\phi^{(i)} + \phi^e)\right) \qquad (I.50b)$$

$$:= -\chi(h \le 0)(\mathbf{1} - L^{(0)})\lambda\mathcal{V}(\phi^e) \qquad (I.50c)$$

$$:= \chi(h \le 0)[\mathcal{F}^{(h)}(\phi^e) + \sum_{i>h}(L^{(i)} - L^{(i-1)})\mathcal{F}^{(i)}(\phi^e)] \qquad (I.50d)$$

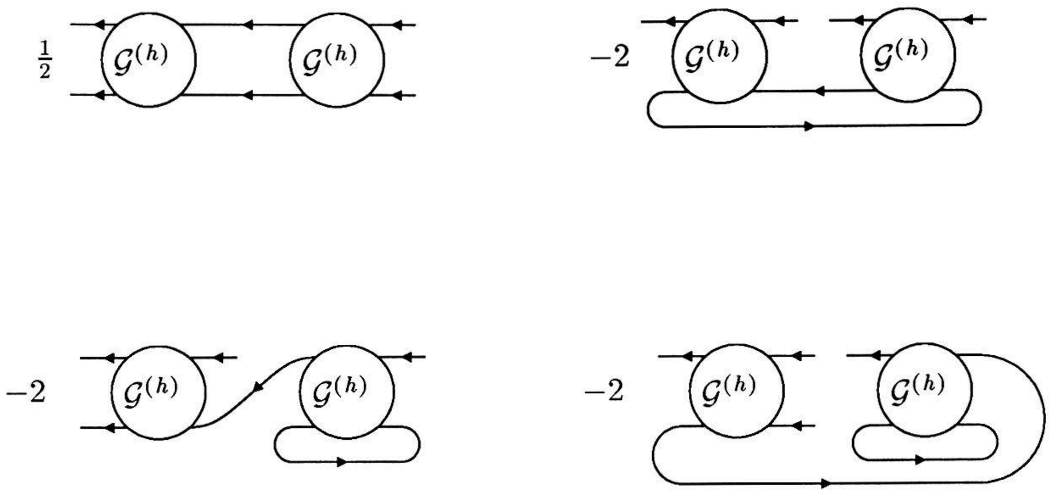
One expects to prove that $\mathcal{G}(\lambda, \mathcal{F}^{(i)}, i \le 0) = \lim_{h\to-\infty} \mathcal{G}^{(h)}(\lambda, \mathcal{F}^{(0)}, ..., \mathcal{F}^{(h)})$ is holomorphic on a suitable infinite polydisc. Here, by abuse of notation $(\lambda, \mathcal{F}^{(0)}, \mathcal{F}^{(1)}, ...)$ is any vector in the polydisc - not necessarily the solution of (I.47b). If (I.47b) has a solution $(\lambda, \mathcal{F})$ in the polydisc, then the effective potential is $\mathcal{G}(\lambda, \mathcal{F})$.

We now investigate the solvability of (I.47b), or equivalently (I.46a). To second order

$$(L^{(h-1)} - \ell)\mathcal{E}^{(h)}(\mathcal{G}^{(h)}) = (L^{(h-1)} - \ell)[\mathcal{E}_1^{(h)}(\mathcal{G}^{(h)}) + \frac{1}{2!}\mathcal{E}_2^{(h)}(\mathcal{G}^{(h)}, \mathcal{G}^{(h)})].$$

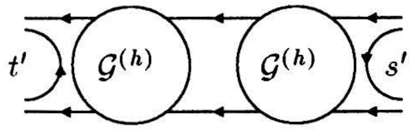
Taking Wick ordering into account $\mathcal{E}_1^{(h)}(\mathcal{G}^{(h)}) = 0$. To evaluate the second term observe that

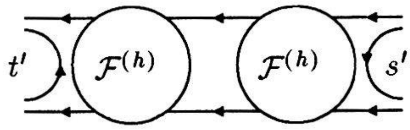
$$[\text{the quartic part of } \frac{1}{2!}\mathcal{E}_2^{(h)}(\mathcal{G}^{(h)}, \mathcal{G}^{(h)})] =$$



$(I.51)$

where $\frac{1}{2}$ $\mathcal{G}^{(h)}$ is the antisymmetric kernel of the quartic part of $\mathcal{G}^{(h)}$. In Section IV we show (again using Lemma IV.2) that the last three diagrams are irrelevant for all kernels that are suitably smooth and uniformly bounded in h. For this reason we begin by considering the ladder approximation and retain only the first diagram.

All quartic terms on the right hand side of (I.47a) other than the first vanish for $q = 0$ so that antisymmetric kernel of $\frac{1}{2}$ 

$=$ antisymmetric kernel of $\frac{1}{2}$ 

$$= \frac{1}{4}\int \frac{d^{d+1}p}{(2\pi)^{d+1}}[F^{(h)}(t',p')F^{(h)}(p',s')\delta_{\alpha_1,\alpha_3}\delta_{\alpha_2,\alpha_4} - F^{(h)}(-t',p')F^{(h)}(p',s')\delta_{\alpha_1,\alpha_4}\delta_{\alpha_2,\alpha_3}]$$
$$[C^{(\le h)}(p)C^{(\le h)}(-p) - C^{(<h)}(p)C^{(<h)}(-p)]. \qquad (I.52)$$

Here $C^{(\le h)}(C^{(<h)})$ denotes $\sum\limits_{\substack{j\le h\\(j<h)}} C^{(j)}$. The effect of Wick ordering on the evaluation of the electron lines is to replace $C^{(h)}(p)C^{(h)}(-p)$ by the last bracket. Thus, in the ladder approximation, (I.46a) becomes

$$\begin{aligned} F^{(h-1)}(t',s') &= F^{(h)}(t',s') + \int \frac{d^{d+1}p}{(2\pi)^{d+1}} F^{(h)}(t',p')F^{(h)}(p',s')[|C^{(\le h)}(p)|^2 - |C^{(h)}(p)|^2] \\ &= F^{(h)}(t',s') + \beta^{(h)} \int dp' F^{(h)}(t',p')F^{(h)}(p',s') \end{aligned} \qquad (I.53)$$

where

$$\begin{aligned} \beta^{(h)} &= \int \frac{dp_0\, d|\mathbf{p}|}{(2\pi)^{d+1}} \left[\frac{|\mathbf{p}|}{k_F}\right]^{d-1} \left[|C^{(\le h)}(p)|^2 - |C^{(<h)}(p)|^2\right] \\ &= \int \frac{d|\mathbf{p}|}{(2\pi)^d} \left[\frac{|\mathbf{p}|}{k_F}\right]^{d-1} \frac{1}{2e(\mathbf{p})} \left[\rho^2\left(M^{-2(h+1)}e(p^2)\right) - \rho^2\left(M^{-2h}e(p^2)\right)\right] \end{aligned} \qquad (I.54a)$$

is nonnegative, independentof p', and approaches the limit

$$\beta = \frac{m}{(2\pi)^d k_F}\int_0^\infty dy \frac{1}{y}\left[\rho^2\left(M^{-2}y^2\right) - \rho^2(y^2)\right] \qquad (I.54b)$$

as $h \to -\infty$.

By (I.44) $F^{(h)}$ is a self-adjoint rotation invariant operator and may be decomposed into spherical harmonics

$$F^{(h)}(t',s') = \sum_{n\geq 0} \lambda_n^{(h)} \pi_n(t',s')$$

with real eigenvalues $\lambda_n^{(h)}, n \geq 0$. Recall that $\pi_n(t',s')$ is the projection in $L^2(k_F S^{d-1})$ onto the space H^n of homogeneous harmonic polynomials of degree n. This decomposition converts (I.53) into the equivalent decoupled system

$$\lambda_n^{(h-1)} = \lambda_n^{(h)} + \beta^{(h)}(\lambda_n^{(h)})^2, n \geq 0 \tag{I.55}$$

with initial data determined by

$$\sum_{n\geq 0} \lambda_n^{(0)} \pi_n(t',s') = -\lambda\langle t', -t'|V|s', -s'\rangle.$$

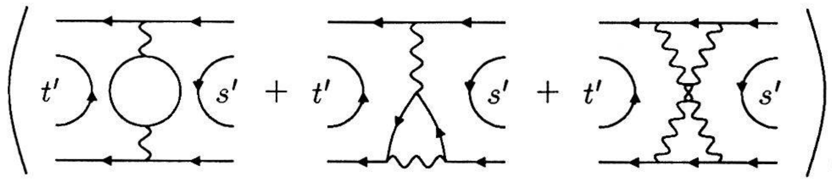
If $\lambda_n^{(0)} > 0$ the iterates $\lambda_n^{(h)}, h \leq 0$, generated by (I.55) diverge to infinity faster than $\lambda_n^{(0)}(1+\beta_n^{(0)})^{|h|}$. This paper is devoted to a class of models in which $\lambda_0^{(0)} > 0$ but $\lambda_0^{(0)} \geq \sup_{n>0} |\lambda_n^{(0)}|$, for example (I.5c) with γ, ω_D large enough. They are driven by the full flow (I.47b) to a nontrivial superconducting fixed point.

On the other hand if $\lambda_n^{(0)} \leq 0$ then the iterates

$$\lambda_n^{(0)} = \frac{\lambda_n^{(0)}}{1 + \lambda_n^{(0)}\beta h + O(\ell n|h|)}$$

converge to zero as $h \to -\infty$. In particular, the vector $(\lambda_n^{(h)}, n \geq 0)$ tends to zero as $h \to -\infty$ when $\langle t', -t'|V|s', -s\rangle$ is the kernel of a positive definite operator on $L^2(k_F S^{d-1})$. However, for a smooth kernel $\lambda_n^{(0)} = O(n^{-N})$ for all $N > 0$, which can be a source of instability . Coupling a higher order term to (I.53) may cause some $\lambda_n^{(h)}$, with n very large, to change sign after a few iterations. In that event the leading term takes over and drives $\lambda_n^{(h)}$ to infinity. This effect may be seen by starting (I.55) from the effective interaction

$-\lambda\langle t', -t'|V|s', -s'\rangle +$



$q = 0$

Ib) Superconducting formalism and statement of results.

The rest of the introduction concerns the superconducting state. It is useful, following Nambu [N], to make the change of variables from $\{\bar\psi_{k\uparrow}, \bar\psi_{k\downarrow}, \psi_{k\uparrow}, \psi_{k\downarrow}, k \in \mathbf{R}^{d+1}\}$ to

$$\mathbf{\Psi}_k = \begin{bmatrix} \mathbf{\Psi}_{k,1} \\ \mathbf{\Psi}_{k,2} \end{bmatrix} = \begin{bmatrix} \psi_{k\uparrow} \\ \bar\psi_{-k\downarrow} \end{bmatrix} \tag{I.56a}$$

$$\bar{\mathbf{\Psi}}_k = \left[\bar{\mathbf{\Psi}}_{k,1}, \bar{\mathbf{\Psi}}_{k,2}\right] = \left[\bar\psi_{k\uparrow}, \psi_{-k\downarrow}\right]. \tag{I.56b}$$

Let

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \; \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \; \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{I.57}$$

be the Pauli matrices. We have

$$\begin{aligned} \bar{\mathbf{\Psi}}_k \sigma^0 \mathbf{\Psi}_k &= \bar\psi_{k\uparrow}\psi_{k\uparrow} - \bar\psi_{-k\downarrow}\psi_{-k\downarrow} \\ \bar{\mathbf{\Psi}}_k \sigma^1 \mathbf{\Psi}_k &= \bar\psi_{k\uparrow}\bar\psi_{-k\downarrow} + \psi_{-k\downarrow}\psi_{k\uparrow} \\ \bar{\mathbf{\Psi}}_k \sigma^2 \mathbf{\Psi}_k &= -i[\bar\psi_{k\uparrow}\bar\psi_{-k\downarrow} - \psi_{-k\downarrow}\psi_{k\uparrow}] \\ \bar{\mathbf{\Psi}}_k \sigma^3 \mathbf{\Psi}_k &= \bar\psi_{k\uparrow}\psi_{k\uparrow} + \bar\psi_{-k\downarrow}\psi_{-k\downarrow} \end{aligned} \tag{I.58}$$

The covariance $C = [ik_0 - e(\mathbf{k})]^{-1}$ correspond to the quadratic form

$$\begin{aligned} &\int \frac{d^{d+1}k}{(2\pi)^{d+1}} [ik_0 - e(\mathbf{k})][\bar\psi_{k\uparrow}\psi_{k\uparrow} + \bar\psi_{k\downarrow}\psi_{k\downarrow}] \\ =& \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \bar{\mathbf{\Psi}}_k [ik_0\mathbf{1} - e(\mathbf{k})\sigma^3] \mathbf{\Psi}_k \end{aligned}$$

so that, in the new variables, the Grassman Gaussian measure becomes $d\mu_{\mathbf{C}_0}(\bar{\mathbf{\Psi}}, \mathbf{\Psi})$ with covariance matrix

$$\begin{aligned} \mathbf{C}_0(\xi_1, \xi_2) &= \langle \mathbf{\Psi}(\xi_1) \bar{\mathbf{\Psi}}(\xi_2) \rangle \\ &= \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{i\langle k, \xi_1 - \xi_2 \rangle_-} [ik_0\mathbf{1} - e(\mathbf{k})\sigma^3]^{-1} \\ &= \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{i\langle k, \xi_1 - \xi_2 \rangle_-} (-1) \frac{ik_0\mathbf{1} + e(\mathbf{k})\sigma^3}{k_0^2 + e(\mathbf{k})^2} \end{aligned} \tag{I.59}$$

and $\langle \Psi(\xi_1)\Psi(\xi_2)\rangle = \langle \bar{\Psi}(\xi_1)\bar{\Psi}(\xi_2)\rangle = 0$. Once again the special case $\tau_1 - \tau_2 = 0$ is defined by a limit, namely, $\tau_1 - \tau_2 \to 0-$ for $(\mathbf{C}_0)_{1,1}$ and $\tau_1 - \tau_2 \to 0+$ for $(\mathbf{C}_0)_{2,2}$.

The interaction becomes

$$\frac{1}{2}\sum_{\alpha_1\in\{\uparrow,\downarrow\}}\int \frac{d^{d+1}t}{(2\pi)^{d+1}}\frac{d^{d+1}s}{(2\pi)^{d+1}}\frac{d^{d+1}q}{(\pi)^{d+1}}\langle t+\frac{q}{2},-t+\frac{q}{2}|V|s+\frac{q}{2},-s+\frac{q}{2}\rangle\delta_{\alpha_1,\alpha_3}\delta_{\alpha_2,\alpha_4}$$
$$\bar{\psi}_{t+q/2,\alpha_1}\bar{\psi}_{-t+q/2,\alpha_2}\psi_{-s+q/2,\alpha_4}\psi_{s+q/2,\alpha_3}$$
$$=\frac{1}{2}\int \frac{d^{d+1}t}{(2\pi)^{d+1}}\frac{d^{d+1}s}{(2\pi)^{d+1}}\frac{d^{d+1}q}{(2\pi)^{d+1}}(\bar{\Psi}_{t+q/2}\sigma^3\Psi_{s+q/2})\langle t+\frac{q}{2},-t+\frac{q}{2}|V|s+\frac{q}{2},-s+\frac{q}{2}\rangle$$
$$(\bar{\Psi}_{-t+q/2}\sigma^3\Psi_{-s+q/2}) \qquad (I.60a)$$

and the (renormalized) effective potential (I.14)

$$\mathcal{G}(\Psi^e,\bar{\Psi}^e) = \log\frac{1}{Z}\int \exp[(-\lambda\mathcal{V}+\delta\mathcal{V})(\Psi+\Psi^e,\bar{\Psi}+\bar{\Psi}^e)]d\mu_{\mathbf{C}_0}(\Psi,\bar{\Psi}) \qquad (I.60b)$$

where

$$\delta\mathcal{V} = \delta\mu(\lambda,\mu)\int d^{d+1}\xi\,\bar{\Psi}(\xi)\sigma^3\Psi(\xi). \qquad (I.60c)$$

The particle number symmetry

$$\begin{aligned}\psi &\to e^{i\theta}\psi\\ \bar{\psi} &\to e^{-i\theta}\bar{\psi}\end{aligned} \qquad (I.61a)$$

or equivalently

$$\begin{aligned}\Psi &\to \begin{bmatrix} e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{bmatrix}\Psi = e^{i\theta\sigma_3}\Psi\\ \bar{\Psi} &\to \bar{\Psi}\begin{bmatrix} e^{-i\theta} & 0\\ 0 & e^{i\theta}\end{bmatrix} = \bar{\Psi}e^{-i\theta\sigma_3}\end{aligned} \qquad (I.61b)$$

that forces $(\mathbf{C}_0)_{12} = (\mathbf{C}_0)_{21} = 0$ is broken in the superconducting state. Therefore we attempt to construct a superconducting model by perturbing about the Grassman Gaussian measure with covariance

$$\begin{aligned}\mathbf{C} = \mathbf{C}_\triangle &= [ik_0\mathbf{1} - e(\mathbf{k})\sigma^3 - \triangle_1\sigma^1 - \triangle_2\sigma^2]^{-1}\\ &= -\frac{ik_0\mathbf{1} + e(\mathbf{k})\sigma^3 + \triangle_1\sigma^1 + \triangle_2\sigma^2}{k_0^2 + E(\mathbf{k})^2}\end{aligned} \qquad (I.62a)$$

where $\triangle = \triangle_1 + i\triangle_2$ and

$$E(\mathbf{k})^2 = e(\mathbf{k})^2 + |\triangle|^2. \qquad (I.62b)$$

The corresponding quadratic form

$$\int \frac{d^{d+1}k}{(2\pi)^{d+1}} \bar{\Psi}(k)[ik_0 - e(\mathbf{k})\sigma^3 - \triangle_1\sigma^1 - \triangle_2\sigma^2]\Psi(k)$$

is the mean field approximation to (minus) the action of the BCS model (see (I.31))

$$\int \frac{d^{d+1}k}{(2\pi)^{d+1}}[ik_0 - e(\mathbf{k})][\bar{\psi}_{k\uparrow}\psi_{k\uparrow} + \bar{\psi}_{k\downarrow}\psi_{k\downarrow}]$$

$$-2\frac{\lambda_0}{V}\int \frac{dq_0}{2\pi}\frac{d^{d+1}s}{(2\pi)^{d+1}}\frac{d^{d+1}s}{(2\pi)^{d+1}}\bar{\psi}_{t+\frac{q_0}{2}\uparrow}\bar{\psi}_{-t+\frac{q_0}{2}\downarrow}\psi_{-s+\frac{q_0}{2}\downarrow}\psi_{s+\frac{q_0}{2}\uparrow}.$$

Observe that

$$\langle e^{i\theta\sigma_3}\Psi\bar{\Psi}e^{-i\theta\sigma_3}\rangle_\triangle = \langle\Psi\bar{\Psi}\rangle_{e^{2i\theta}\triangle}. \tag{I.63}$$

In other words the $U(1)$ symmetry (I.61) moves us around a circle of equivalent states. We may therefore assume, without loss of generality that $\triangle$ is real i.e. $\triangle_2 = 0$ and positive.

The effective potential (I.60b) is expressed in terms of $\mathbf{C}_\triangle$ by

$$\mathcal{G}(\Psi^e, \bar{\Psi}^e) = \log\frac{1}{Z}\int \exp[(-\lambda\mathcal{V} + \delta\mathcal{V})(\Psi + \Psi^e, \bar{\Psi} + \bar{\Psi}^e)]d\mu_{\mathbf{C}_0}$$

$$= \log\frac{1}{Z}\int \exp\left[(-\lambda\mathcal{V} + \delta\mathcal{V})(\Psi + \Psi^e, \bar{\Psi} + \bar{\Psi}^e) - \triangle\int \frac{d^{d+1}k}{(2\pi)^{d+1}}\bar{\Psi}(k)\sigma^1\Psi(h)\right]$$

$$\exp\left[\triangle\int \frac{d^{d+1}k}{(2\pi)^{d+1}}\bar{\Psi}(k)\sigma^1\Psi(k)\right] d\mu_{\mathbf{C}_0}$$

$$= \log\frac{1}{Z'}\int \exp\left[(-\lambda\mathcal{V} + \delta\mathcal{V})(\Psi + \Psi^e, \bar{\Psi} + \bar{\Psi}^e) - \triangle\int \frac{d^{d+1}k}{(2\pi)^{d+1}}\bar{\Psi}(k)\sigma^1\Psi(h)\right] d\mu_{\mathbf{C}_\triangle} \tag{I.64}$$

It is more convenient to use the effective potential

$$\mathcal{G}(\Psi^e, \bar{\Psi}^e) = \log\frac{1}{Z}\int \exp\Big[(-\lambda\mathcal{V} + \delta\mathcal{V})(\Psi + \Psi^e, \bar{\Psi} + \bar{\Psi}^e)$$

$$-\triangle\int \frac{d^{d+1}k}{(2\pi^{d+1}}(\bar{\Psi} + \bar{\Psi}^e)\sigma^1(\Psi + \Psi^e)\Big] d\mu_{\mathbf{C}_\triangle} \tag{I.65}$$

By abuse of notation, we drop the $'$ on Z and continue to call the effective potential $\mathcal{G}$. Expanding (I.60b) in powers of $\Psi^e, \bar{\Psi}^e$ generates the connected Green's functions amputated by $\mathbf{C}_0$. Expanding (I.65) generates the same connected Green's functions but now amputated by $\mathbf{C}_\triangle$.

To understand perturbation theory about $d\mu_{\mathbf{C}_\triangle}$ we begin by considering the model whose effective potential is

$$\log \frac{1}{Z} \int \exp[-\lambda \mathcal{V} + \delta \mathcal{V}] d\mu_{\mathbf{C}_\triangle}. \qquad (I.66)$$

The perturbation expansion in powers of λ is well-defined since $[ik_0 - e(k)\sigma^3 - \triangle\sigma^1]^{-1} = O\left(\frac{1}{\triangle}\right)$ when $k_0 = 0$ and $|\mathbf{k}| = k_F$. However, each terms diverges as $\triangle$ tends to zero. This is unsatisfactory because $\triangle$ must ultimately be chosen nonperturbatively small.

To produce an expansion that is uniform in $\triangle$ we must renormalize. Let

$$\delta \mathcal{V} = \delta\mu(\lambda, \mu, \triangle) \int d\xi \bar{\Psi}(\xi) \sigma^3 \Psi(\xi) \qquad (I.67a)$$

$$\mathcal{D} = D(\lambda, \mu, \triangle) \int d\xi \bar{\Psi}(\xi) \sigma^1 \Psi(\xi). \qquad (I.67b)$$

Also let

$$S = (S_{ij})_{i,j \in \{1,2\}} = \langle \Psi \bar{\Psi} \rangle \qquad (I.68a)$$

and

$$\Sigma = \mathbf{C}_\triangle^{-1} - S^{-1} \qquad (I.68b)$$

be, respectively, the two point Schwinger function and proper self-energy for the, possibly non-physical, effective potential

$$\mathcal{W}(\Psi^e, \bar{\Psi}^e) = \log \frac{1}{Z} \int \exp[(-\lambda\mathcal{V} + \delta\mathcal{V} + \mathcal{D})(\Psi + \Psi^e, \bar{\Psi} + \bar{\Psi}^e)] d\mu_{\mathbf{C}_\triangle}(\Psi, \bar{\Psi}). \qquad (I.69)$$

The proper self-energy is a linear combination

$$\tilde{\Sigma}(k) = r_0(k)\mathbf{1} + r_1(k)\sigma^1 + r_2(k)\sigma^2 + r_3(k)\sigma^3 \qquad (I.70)$$

of the Pauli matrices.

The measure $d\mu_{\mathbf{C}_\triangle}$ and interaction $-\lambda\mathcal{V} + \delta\mathcal{V} + \mathcal{D}$ are invariant under

$$\Psi_k \to i(\bar{\Psi}_k)^t, \bar{\Psi}_k \to i(\Psi_k)^t. \qquad (I.71a)$$

Therefore

$$\langle \Psi_p \bar{\Psi}_k \rangle = -\langle (\bar{\Psi}_p)^t (\Psi_k)^t \rangle = \langle \Psi_k \bar{\Psi}_p \rangle^t \qquad (I.71b)$$

and energy momentum conservation further implies

$$\tilde{S}(k) = \tilde{S}(k)^t.$$

It follows from (I.68b) that $\tilde{\Sigma}(k)^t = \tilde{\Sigma}(k)$, or equivalently

$$\tilde{\Sigma}(k) = r_0(k)\mathbf{1} + r_1(k)\sigma^1 + r_3(k)\sigma^3.$$

The measure $d\mu_{\mathbf{C}_\triangle}$ and interation $-\lambda\mathcal{V} + \delta\mathcal{V} + \mathcal{D}$ are also invariant under

$$\mathbf{\Psi}_k \to \sigma^2(\bar{\mathbf{\Psi}}_{-k})^t, \bar{\mathbf{\Psi}}_k \to -(\sigma^2\mathbf{\Psi}_{-k})^t \tag{I.72a}$$

so that

$$\langle \mathbf{\Psi}_p \bar{\mathbf{\Psi}}_k \rangle = -\langle \sigma^2(\bar{\mathbf{\Psi}}_{-p})^t(\sigma^2\mathbf{\Psi}_{-k})^t \rangle = -\sigma^2 \langle \mathbf{\Psi}_{-k} \bar{\mathbf{\Psi}}_{-p} \rangle^t \sigma^2 \tag{I.72b}$$

and hence

$$r_0(k) = -r_0(-k)$$
$$r_1(k) = r_1(-k)$$
$$r_3(k) = r_3(-k).$$

Denote by

$$\hat{S}(\mathbf{k},\tau) = \int d^d\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} S((\mathbf{x},\tau),(0,0))$$

the partial Fourier transform of S. Since $\hat{\mathbf{C}}_\triangle(\mathbf{k},\tau)$ is real and invariant under $\mathbf{k} \to R\mathbf{k} := -\mathbf{k}$ and the interaction satisfied (S2), (S3) and (S4) we find that $\hat{r}_0(\mathbf{k},\tau), \hat{r}_1(\mathbf{k},\tau)$ and $\hat{r}_3(\mathbf{k},\tau)$ are real and invariant under $\mathbf{k} \to -\mathbf{k}$. Finally

$$\tilde{\Sigma}(k_0 = 0, \mathbf{k}) = r_1(0,\mathbf{k})\sigma^1 + r_3(0,\mathbf{k})\sigma^3 \tag{I.73}$$

with $r_1(0,\mathbf{k})$ and $r_3(0,\mathbf{k})$ real.

We are now in a position to renormalize. Identity (I.73) and an "integral" equation completely analogous to (I.20) ensure that $\delta\mu(\lambda,\mu,\triangle)$ and $D(\lambda,\mu,\triangle)$ are uniquely determined as formal power series in λ by the renormalization condition

$$\tilde{\Sigma}(k_0 = 0, |\mathbf{k}| = k_F) = r_1(0, |\mathbf{k}| = k_F)\sigma^1 + r_3(0, |\mathbf{k}| = k_F)\sigma^3 = 0. \tag{I.74}$$

(See (I.15) and (I.21).) We shall show in Section III that the coefficients in the expansion of the counterterms $\delta\mu$ and D are finite, uniformly bounded in $\triangle$ and converge as $\triangle \to 0$.

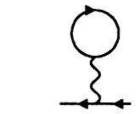
Furthermore, the obvious analogue of (I.24b) holds uniformly in $\triangle$. Let us remark that the renormalization condition (I.74) determined k_F as the zero of $Tr(\sigma^3 S^{-1}(k_0 = 0, |\mathbf{k}|))$ and $\triangle$ as $-\frac{1}{2}Tr(\sigma^1 S^{-1}(k_0 = 0, |\mathbf{k}| = k_F))$.

We now have a perturbation expansion that is uniform in $\triangle$ for the effective potention $\mathcal{W}$. In order to recover the physical effective potential $\mathcal{G}$, (I.65), from $\mathcal{W}$ we impose the constraint

$$\triangle = -D(\lambda, \mu, \triangle). \tag{I.75}$$

To first order in λ

$$D(\lambda,\mu,\triangle) = -\frac{\lambda}{2}Tr[\sigma^1(\quad k \;-\; k \quad)|_{k_0=0,|\mathbf{k}|=k_F}]$$



$$\begin{aligned}
&= \frac{\lambda}{2}Tr[\sigma^1 \int \frac{d^{d+1}p}{(2\pi)^{d+1}}\langle k,p|V|p,k\rangle \sigma^3 \mathbf{C}_\triangle(p)\sigma^3|_{k_0=0,|\mathbf{k}|=k_F}] \\
&= \lambda \int \frac{d^{d+1}p}{(2\pi)^{d+1}}\langle k,p|V|p,k\rangle \frac{\triangle}{p_0^2 + E(\mathbf{p})^2}|_{k_0=0,|\mathbf{k}|=k_F} \\
&= \lambda \int \frac{d^{d+1}p}{(2\pi)^{d+1}}\langle k',-k'|V|p,-p\rangle \frac{\triangle}{p_0^2 + E(\mathbf{p})^2}
\end{aligned} \tag{I.76}$$

where $k' = \left(0, k_F \frac{\mathbf{k}}{|\mathbf{k}|}\right)$ and (S1) (I.6) is used in the last line. Therefore, to first order, the constraint (I.75) is

$$\triangle = -\lambda \int \frac{d^{d+1}p}{(2\pi)^{d+1}}\langle k',-k'|V|p,-p\rangle \frac{\triangle}{p_0^2 + E(\mathbf{p})^2} \tag{I.77}$$

which one recognizes as the BCS gap equation. Here it appears as the Hartree-Fock approximation.

For each $p_0, |\mathbf{p}|$ the kernel $\langle k',-k'|V|p,-p\rangle$ defines a rotation invariant, self-adjoint operator on $L^2(k_F S^{d-1})$. Expanding

$$-\lambda\langle k',-k'|V|p,-p\rangle = \sum_{n\geq 0} \lambda_n(p_0,|\mathbf{p}|)\pi_n(k',p')$$

one finds

$$-\lambda \int \frac{d^{d+1}p}{(2\pi)^{d+1}}\langle k',-k'|V|p,-p\rangle \frac{\triangle}{p_0^2 + E(\mathbf{p})^2} = \int \frac{d^{d+1}p}{(2\pi)^{d+1}}\lambda_0(p_0,|\mathbf{p}|)\frac{\triangle}{p_0^2 + E(\mathbf{p})^2}.$$

If $\lambda_0(0,k_F) > 0$ and, for example, $\int \frac{d^{d+1}p}{(2\pi)^{d+1}} |\lambda_0(p_0,\mathbf{p})| < \infty$, then, as is well-known, (I.77) has a unique solution $\triangle > 0$ provided $|\lambda|$ is sufficiently small. Moreover $\triangle \sim \exp - \left[\frac{\text{const}}{|\lambda|}\right]$. This follows from the simple observation that $\int_{\substack{|p_0|<\epsilon \\ ||\mathbf{p}|-\mathbf{k}_F|<\epsilon}} \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{1}{p_0^2+E(\mathbf{p})^2}$ is mononotonically decreasing and logarithmically divergent at $\triangle = 0$. Later in this section, (I.75) is solved, in conjunction with the relevant flow, to any order in λ.

The dispersion relation $E(k) = \sqrt{e(k)^2+\triangle^2}$ of the free action

$$\int \frac{d^{d+1}k}{(2\pi)^{d+1}} \bar{\Psi}(k)[ik_0 - e(\mathbf{k})\sigma^3 - \triangle\sigma^1]\Psi(k) \tag{I.78}$$

is bounded below by $\triangle$. In particular, there is no spectrum in the interval $(0,\triangle)$. Observe that (I.78) is not invariant under the continuous particle number symmetry (I.61).

On the other hand, the full action of (I.69) with constraint (I.75) is invariant under (I.61) with the result that the gap $(0,\triangle)$ in the energy spectrum typically disappears, due to the presence of a "Goldstone boson". However, the gap should persist for the Coulomb interaction because of the Higgs mechanism. Here we do not treat screening and spontaneous mass generation. They will be discussed in another paper. Rather we mimic the Higgs mechanism by adding the external field

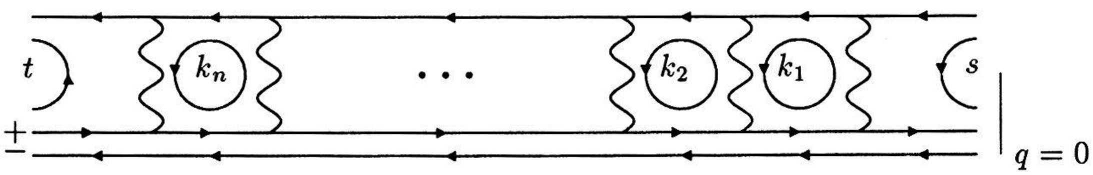
$$J\int \frac{d^{d+1}k}{(2\pi)^{d+1}} \bar{\Psi}(k)\sigma^1\Psi(k), \tag{I.79}$$

which breaks the symmetry. Equivalently, we replace the constraint (I.75) by

$$\triangle - J = -D(\lambda,\mu,\triangle). \tag{I.80}$$

To explain the difficulties that can be caused by Goldstone bosons and to understand the effect of (I.80) we evaluate the ladders

$\wedge_n^{\pm}(t,s) =$



(I.81)

for an interaction obeying $\langle t,-t|V|s,-s\rangle = w(\mathbf{t})w(\mathbf{s})$, where $w(\mathbf{t}) = \chi(|e(\mathbf{t})| \le \omega)$. Under appropriate hypotheses we will flow to interactions of essentially this type. We remark that in the Nambu formalism Λ_n^+ is no more regular than the usual Λ_n^- of (I.25c), because the direction of the particle lines is given a new meaning. We have

$$\Lambda_n^{\pm}(t,s) = -\lambda\omega(\mathbf{t})(\Lambda^{\pm})^n\sigma^3 \otimes \sigma^3 w(\mathbf{s})$$

where

$$\Lambda^{\pm} = -\lambda\int \frac{d^{d+1}k}{(2\pi)^{d+1}} w(\mathbf{k})^2 \left[\sigma^3\frac{ik_0 + e(\mathbf{k})\sigma^3 + \triangle\sigma^1}{k_0^2 + E(\mathbf{k})^2}\right] \otimes \left[\sigma^3\frac{\pm ik_0 + e(\mathbf{k})\sigma^3 + \triangle\sigma^1}{k_0^2 + E(\mathbf{k})^2}\right].$$

Let

$$\mathcal{E}_{\pm}(k) = \frac{\pm E(k) + e(k)\sigma^3 + \triangle\sigma^1}{2E(\mathbf{k})}.$$

Further calculation yields

$$\begin{aligned}\Lambda^{\pm} &= -\lambda\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{w(\mathbf{k})^2}{2E(\mathbf{k})}\sigma^3\otimes\sigma^3\{\mathcal{E}_+(\mathbf{k})\otimes\mathcal{E}_{\mp}(\mathbf{k}) + \mathcal{E}_-(\mathbf{k})\otimes\mathcal{E}_{\pm}(\mathbf{k})\}\\ &= -\lambda\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{w(\mathbf{k})^2}{2E(\mathbf{k})}\sigma^3\otimes\sigma^3\frac{1}{2}\left\{\mp\mathbf{1}\otimes\mathbf{1} + \left(\frac{\triangle}{E}\sigma^1 + \frac{e}{E}\sigma^3\right)\otimes\left(\frac{\triangle}{E}\sigma^1 + \frac{e}{E}\sigma^3\right)\right\}\\ &= -\lambda\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{w(\mathbf{k})^2}{2E(\mathbf{k})}\frac{1}{2}\left\{\mp\sigma^3\otimes\sigma^3 + \left(\frac{e}{E}\mathbf{1} + i\frac{\triangle}{E}\sigma^2\right)\otimes\left(\frac{e}{E}\mathbf{1} + i\frac{\triangle}{E}\sigma^2\right)\right\}\end{aligned}$$

The matrix

$$\frac{e}{E}\mathbf{1} + i\frac{\triangle}{E}\sigma^2 = \frac{1}{E}\begin{bmatrix} e & \triangle \\ -\triangle & e\end{bmatrix}$$

has eigenvalues $\frac{e}{E} \pm i\frac{\triangle}{E}$ with corresponding eigenvectors $\begin{bmatrix}1\\ \pm i\end{bmatrix}$. Therefore the tensor product $\left(\frac{e}{E}\mathbf{1} + i\frac{\triangle}{E}\sigma^2\right)\otimes\left(\frac{e}{E}\mathbf{1} + i\frac{\triangle}{E}\sigma^2\right)$ has eigenvalues $\left(\frac{e}{E} + i\frac{\triangle}{E}\right)^2, 1, 1, \left(\frac{e}{E} - i\frac{\triangle}{E}\right)^2$ with corresponding eigenvectors $\begin{bmatrix}1\\ i\end{bmatrix}\otimes\begin{bmatrix}1\\ i\end{bmatrix}$, $\begin{bmatrix}1\\ i\end{bmatrix}\otimes\begin{bmatrix}1\\ -i\end{bmatrix}$, $\begin{bmatrix}1\\ -i\end{bmatrix}\otimes\begin{bmatrix}1\\ i\end{bmatrix}$ and $\begin{bmatrix}1\\ -i\end{bmatrix}\otimes\begin{bmatrix}1\\ -i\end{bmatrix}$. Also,

$$b_{\pm} = \begin{bmatrix}1\\ i\end{bmatrix}\otimes\begin{bmatrix}1\\ -i\end{bmatrix} \pm \begin{bmatrix}1\\ -i\end{bmatrix}\otimes\begin{bmatrix}1\\ i\end{bmatrix} \tag{I.82}$$

are eigenvectors of $\sigma^3\otimes\sigma^3$ of eigenvalue ±1. It follows that $b_{\pm}$ is an eigenvector of eigenvalue 1 for $\frac{1}{2}\left\{\mp\sigma^3\otimes\sigma^3 + \left(\frac{e}{E}\mathbf{1} + i\frac{\triangle}{E}\sigma^2\right)\otimes\left(\frac{e}{E}\mathbf{1} + i\frac{\triangle}{E}\sigma^2\right)\right\}$ and of eigenvalue

$$\gamma = -\lambda\int_{|e(\mathbf{k})|\le\omega}\frac{d^dk}{(2\pi)^d}\frac{1}{2E(\mathbf{k})} \tag{I.83}$$

for $\Lambda^\pm$. Furthermore all other eigenvalues of $\Lambda^\pm$ have magnitude strictly smaller than γ.

The sum of the ladders is given by

$$\sum_{n=0}^{\infty} \Lambda_n^\pm(t,s) = -\lambda\chi(|e(\mathbf{t})| < \omega)[\mathbf{1} - \Lambda^\pm]^{-1}\sigma^3 \otimes \sigma^3 \chi(|e(\mathbf{s})| < \omega). \qquad (I.84)$$

When $\triangle$ is determined by (I.77), $\gamma = 1$ and (I.84) is marginally divergent. On the other hand when $\triangle$ is determined by the first order approximation to (I.80), $\gamma = \frac{\triangle - J}{\triangle} < 1$ and (I.84) converges for $\triangle >> J > 0$. One can show directly, by means of a Ward identity, that, under the constraint (I.75), the particle number symmetry (I.61) forces $\Lambda^\pm$ to have an eigenvector of eigenvalue one ([N]§4, [S] p. 235-236).

To make the formal discussion above precise we return to the flow. As before we decompose the covariance

$$\mathbf{C} = \mathbf{C}_\triangle = \sum_{j=-\infty}^{0} \mathbf{C}_\triangle^{(j)} \qquad (\triangle \text{ real and positive}) \qquad (I.85a)$$

where

$$\begin{aligned} \mathbf{C}_\triangle^{(j)}(\xi) &= \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{i\langle k,\xi\rangle_-} [ik_0\mathbf{1} - e(\mathbf{k})\sigma^3 - \triangle\sigma^1]^{-1} f(M^{-2j}E(\mathbf{k})^2) \\ &= -\int \frac{d^d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}} e^{-E(\mathbf{k})|\tau|} \frac{\operatorname{sgn}(\tau)E(\mathbf{k})\mathbf{1} + e(\mathbf{k})\sigma^3 + \triangle\sigma^1}{2E(\mathbf{k})} f(M^{-2j}E(\mathbf{k})^2) \end{aligned} \qquad (I.85b)$$

$$\begin{aligned} \mathbf{C}_\triangle^{(0)}(\xi) &= \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{i\langle k,\xi\rangle_-} [ik_0\mathbf{1} - e(\mathbf{k})\sigma^3 - \triangle\sigma^1]^{-1} h(M^{-2j}E(\mathbf{k})^2) \\ &= -\int \frac{d^d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}} e^{-E(\mathbf{k})|\tau|} \frac{\operatorname{sgn}(\tau)E(\mathbf{k})\mathbf{1} + e(\mathbf{k})\sigma^3 + \triangle\sigma^1}{2E(\mathbf{k})} h(M^{-2j}E(\mathbf{k})^2) \end{aligned} \qquad (I.85c)$$

$$1 = h(r) + \sum_{j=-\infty}^{-1} f(M^{-2j}r)$$

for all $r \geq 0$ (see (II.2)) and

$$E(\mathbf{k})^2 = e(\mathbf{k})^2 + \triangle^2. \qquad (I.62b)$$

Note that $f(M^{-2h}E^2(\mathbf{k})) = 0$ unless $M^h \geq \triangle$. In other words, $\triangle$ imposes a deep infrared cutoff. From this point of view there are only finitely many scales. But the number of scales grows unboundedly as λ tends to zero since $\triangle \sim \exp\left(-\frac{\text{const}}{|\lambda|}\right)$ (see the discussion following (I.77)). To be precise the scales are restricted to the interval

$$0 \geq j \geq \text{const } \log\triangle = -\frac{\text{const}}{|\lambda|} \qquad (I.85d)$$

Let

$$\mathcal{W}^{(h)}(\Psi^{(\le h)},\bar\Psi^{(\le h)}) = \log \frac{1}{Z_h}\int \exp[-\lambda\mathcal{V}+\delta\mathcal{V}+\mathcal{D}](\Psi^{(\le 0)},\bar\Psi^{(\le 0)}) \prod_{j>h} d\mu_{\mathbf{C}^{(j)}_{\triangle}}(\Psi^{(j)},\bar\Psi^{(j)}), \tag{I.86}$$

where $\delta\mathcal{V},\mathcal{D}$ are given by (I.67a,b) and

$$\Psi^{(\le h)} = \Psi^{(-\infty)} + \sum_{-\infty<j\le h} \Psi^{(h)}$$

with $\Psi^{(-\infty)} := \Psi^e$, an external field.

The analog of (I.18) is evident. This is not the case for (I.29) because the quartic Nambu monomials

$$(\bar\Psi_{k_1}\sigma^i\Psi_{k_3})(\bar\Psi_{k_2}\sigma^j\Psi_{k_4}) \qquad 0\le i,j\le 3 \tag{I.87}$$

obscure the dependence on the physical fields $\psi_{k\uparrow},\psi_{-k\downarrow},\bar\psi_{k\uparrow},\bar\psi_{-k\downarrow}$. For this reason we, temporarily, work with the physical fields.

A priori there are $4^4 = 256$ independent quartic monomials in the fields

$$\bar\psi_{k\uparrow},\ \bar\psi_{-k\downarrow},\ \psi_{k\uparrow},\ \psi_{-k\downarrow}.$$

The superconducting effective potential (I.86) is not invariant under (I.61). It is, however, invariant under

$$\Psi_k \to e^{i\theta}\Psi_k \quad , \quad \bar\Psi_k \to e^{-i\theta}\bar\Psi_k. \tag{I.88}$$

Therefore the quartic monomials that appear are necessarily the 16 possible products of

$$\begin{aligned}
\bar\psi_{p\uparrow}\psi_{k\uparrow} &= \frac{1}{2}\bar\Psi_p(\sigma^0+\sigma^3)\Psi_k := \bar\Psi_p\tau^0\Psi_k\\
\psi_{-p\downarrow}\bar\psi_{-k\downarrow} &= \frac{1}{2}\bar\Psi_p(\sigma^0-\sigma^3)\Psi_k := \bar\Psi_p\tau^3\Psi_k\\
\bar\psi_{p\uparrow}\bar\psi_{-k\downarrow} &= \frac{1}{2}\bar\Psi_p(\sigma^1+i\sigma^2)\Psi_k := \bar\Psi_p\tau^1\Psi_k\\
\psi_{-p\downarrow}\psi_{k\uparrow} &= \frac{1}{2}\bar\Psi_p(\sigma^1-i\sigma^2)\Psi_k := \bar\Psi_p\tau^2\Psi_k.
\end{aligned} \tag{I.89}$$

(Note that τ^0,τ^1,τ^2 and τ^3 are 2×2 matrices. Tau's with subscripts will denote various times.) It follows that a quartic monomial $\mathcal{M}$ in $\mathcal{W}$ has a representation

$$\mathcal{M} = \frac{1}{2}\sum_{m,n=0}^{3}\int \frac{d^{d+1}k_1}{(2\pi)^{d+1}}\,\frac{d^{d+1}k_2}{(2\pi)^{d+1}}\,\frac{d^{d+1}k_3}{(2\pi)^{d+1}}\,\frac{d^{d+1}k_4}{(2\pi)^{d+1}}(2\pi)^{d+1}\delta(k_1+k_2-k_3-k_4)$$

$$f_{m,n}(k_1,k_2,k_3,k_4)(\bar{\Psi}_{k_1}\tau^m\Psi_{k_3})(\bar{\Psi}_{k_2}\tau^n\Psi_{k_4}) \tag{I.90a}$$

We now show that four of these kernels determine the remaining twelve.

The kernels $f_{m,n}$ are uniquely determined on the support of $\delta(k_1+k_2-k_3-k_4)$ by the condition that

$$\sum_{m,n=0}^{3} f_{m,n}(k_1,k_2,k_3,k_4)(\tau^m)_{\alpha_1,\alpha_3}(\tau^n)_{\alpha_2,\alpha_4} \tag{I.90b}$$

be antisymmetric under $(k_1,\alpha_1)\leftrightarrow(k_2,\alpha_2)$ and $(k_3,\alpha_3)\leftrightarrow(k_4,\alpha_4)$. We make the antisymmetry condition explicit. The kernels $f_{m,n}$ must satisfy:

a) $f_{i,i}(k_1,k_2,k_3,k_4)$ is antisymmetric under $k_1\leftrightarrow k_2$ and $k_3\leftrightarrow k_4$

b) $f_{0,3}(k_1,k_2,k_3,k_4) = -f_{1,2}(k_1,k_2,k_4,k_3)$

$= -f_{2,1}(k_2,k_1,k_3,k_4)$

$= f_{3,0}(k_2,k_1,k_4,k_3)$

c1) $f_{0,1}(k_1,k_2,k_3,k_4) = f_{1,0}(k_2,k_1,k_4,k_3)$

$f_{0,1}(k_1,k_2,k_3,k_4)$ and $f_{1,0}(k_1,k_2,k_3,k_4)$ are antisymmetric under $k_1\leftrightarrow k_2$

c2) $f_{0,2}(k_1,k_2,k_3,k_4) = f_{2,0}(k_2,k_1,k_4,k_3)$

$f_{0,2}(k_1,k_2,k_3,k_4)$ and $f_{2,0}(k_1,k_2,k_3,k_4)$ are antisymmetric under $k_3\leftrightarrow k_4$

c3) $f_{1,3}(k_1,k_2,k_3,k_4) = f_{3,1}(k_2,k_1,k_4,k_3)$

$f_{1,3}(k_1,k_2,k_3,k_4)$ and $f_{3,1}(k_1,k_2,k_3,k_4)$ are antisymmetric under $k_3\leftrightarrow k_4$

c4) $f_{2,3}(k_1,k_2,k_3,k_4) = f_{3,2}(k_2,k_1,k_4,k_3)$

$f_{2,3}(k_1,k_2,k_3,k_4)$ and $f_{3,2}(k_1,k_2,k_3,k_4)$ are antisymmetric under $k_1\leftrightarrow k_2$ (I.90c)

Since $\mathcal{M}$ is invariant under

$$\Psi_k\to i(\bar{\Psi}_k)^t \qquad \bar{\Psi}_k\to i\Psi_k^t \tag{I.91a}$$

$$\Psi_k\to \sigma^2(\bar{\Psi}_{-k})^t \qquad \bar{\Psi}_k\to -(\sigma^2\Psi_{-k})^t \tag{I.91b}$$

$$\Psi_k\to \bar{\Psi}_{Rk} \qquad \bar{\Psi}_k\to \bar{\Psi}_{Rk} \text{ for all } R\in O(d) \tag{I.91c}$$

the kernels obey

$$f_{m,n}(k_1,k_2,k_3,k_4) = f_{\pi(m),\pi(n)}(k_3,k_4,k_1,k_2) \tag{I.92a}$$

where π is the permutation (0) (1,2) (3),

$$f_{m,n}(k_1,k_2,k_3,k_4) = (-1)^{\delta_{m,1}+\delta_{m,2}+\delta_{n,1}+\delta_{n,2}} f_{p(m),p(n)}(-k_3,-k_4,-k_1,-k_2) \tag{I.92b}$$

where p is the permutation (0,3) (1) (2) and

$$f_{m,n}(k_1,k_2,k_3,k_4) = f_{m,n}(Rk_1,Rk_2,Rk_3,Rk_4), \quad R \in O(d). \tag{I.92c}$$

Combining the three parts of (I.92) yeilds

$$f_{m,n}(k_1,k_2,k_3,k_4) = (-1)^{\delta_{m,1}+\delta_{m,2}+\delta_{n,1}+\delta_{n,2}} f_{p\pi(m),p\pi(n)}(Tk_1,Tk_2,Tk_3,Tk_4) \tag{I.92d}$$

where the product $p\pi$ is the permutation (0,3) (1,2) and $T(k_0,\mathbf{k}) = (-k_0,\mathbf{k})$ is time reversal. In particular

$$f_{m,n}\big((0,\mathbf{k}_1),(0,\mathbf{k}_2),(0,\mathbf{k}_3),(0,\mathbf{k}_4)\big)$$
$$= (-1)^{\delta_{m,1}+\delta_{m,2}+\delta_{n,1}+\delta_{n,2}} f_{p\pi(m),p\pi(n)}\big((0,\mathbf{k}_1),(0,\mathbf{k}_2),(0,\mathbf{k}_3),(0,\mathbf{k}_4)\big) \tag{I.92e}$$

Observe that the antisymmetry conditions imply that the nine kernels

$$\begin{matrix} f_{0,0} & f_{0,1} & f_{0,2} & f_{0,3} \\ & f_{1,1} & & f_{1,3} \\ & & f_{2,2} & f_{2,3} \\ & & & f_{3,3} \end{matrix}$$

determine the others. It follows from (I.92a) that the six kernels

$$\begin{matrix} f_{0,0} & f_{0,1} & f_{0,3} \\ & f_{1,1} & f_{1,3} \\ & & \\ & & f_{3,3} \end{matrix}$$

determine the others. Finally (I.92b) implies that only the four kernels $f_{0,0}$ $f_{0,1}$ $f_{0,3}$ $f_{1,1}$ are independent.

Let the group SU(2) act on the fields $\mathbf{\Psi}_k, \bar{\mathbf{\Psi}}_k$ by

$$A = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} \mathbf{\Psi}_k \to a\mathbf{\Psi}_k + ib\sigma^2\bar{\mathbf{\Psi}}^t_{-k} \\ \bar{\mathbf{\Psi}}_{-k} \to \bar{a}\bar{\mathbf{\Psi}}_{-k} - i\bar{b}\mathbf{\Psi}^t_k\sigma^2. \end{matrix} \tag{I.93a}$$

or equivalently on the physical fields $\psi_k, \bar{\psi}_k$ by

$$\begin{matrix} \begin{bmatrix} \psi_{k\uparrow} \\ \psi_{k\downarrow} \end{bmatrix} \to A \begin{bmatrix} \psi_{k\uparrow} \\ \psi_{k\downarrow} \end{bmatrix} \\ \left[\bar{\psi}_{k\uparrow}, \bar{\psi}_{k\downarrow}\right] \to \left[\bar{\psi}_{k\uparrow}, \bar{\psi}_{k\downarrow}\right] A^*. \end{matrix} \tag{I.93b}$$

Observe that the transformation (I.93) sends the term

$$\begin{aligned}&\triangle \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \bar{\Psi}(k)\sigma^1 \Psi(k) g(k)\\ =&\triangle \int \frac{d^{d+1}k}{(2\pi)^{d+1}} g(k) \left[\bar{\psi}_{k\uparrow}\bar{\psi}_{-k\downarrow} + \psi_{-k\downarrow}\psi_{k\uparrow}\right]\end{aligned}$$

to

$$\begin{aligned}&\triangle \int \frac{d^{d+1}k}{(2\pi)^{d+1}} g(k) \left\{(\bar{a}\bar{\psi}_{k\uparrow} + \bar{b}\bar{\psi}_{k\downarrow})(-b\bar{\psi}_{-k\uparrow} + a\bar{\psi}_{-k\downarrow}) + (-\bar{b}\psi_{-k\uparrow} + \bar{a}\psi_{-k\downarrow})(a\psi_{k\uparrow} + b\psi_{k\downarrow})\right\}\\ =&\triangle \int \frac{d^{d+1}k}{(2\pi)^{d+1}} g(k) \left[\bar{\psi}_{k\uparrow}\bar{\psi}_{-k\downarrow} + \psi_{-k\downarrow}\psi_{k\uparrow}\right]\end{aligned}$$

when $g(k) = g(-k)$. Consequently the effective potential $\mathcal{W}^{(h)}$ and the associated connected amputated Green's functions are invariant under the action of SU(2).

Set

$$Q(t', s') := -f_{0,3}(t', s', s', t') \tag{I.94a}$$

where once again $t' = (0, k_F \frac{\mathbf{t}}{|\mathbf{t}|})$. Then by direct calculation SU(2) invariance forces

$$f_{0,0}(t', -t', s', -s') = [Q(t', s') - Q(-t', s')]. \tag{I.95a}$$

Moreover by antisymmetry and (I.93e)

$$\begin{aligned} f_{0,0}(t', -t', s', -s') &= f_{3,3}(t', -t', s', -s')\\ &= [Q(t', s') - Q(-t', s')]\\ f_{0,3}(t', s', s', t') &= f_{3,0}(t', s', s', t')\\ &= -f_{1,2}(t', s', t', s')\\ &= -f_{2,1}(t', s', t', s')\\ &= -Q(t', s') \end{aligned} \tag{I.95b}$$

and

$$Q(t', s') = Q(s', t') = Q(-t', -s') \tag{I.95c}$$

Define an involution * on the Grassman algebra by

$$\begin{aligned} \Psi_k^* &= \Psi_{Tk}\\ \bar{\Psi}_k^* &= \bar{\Psi}_{Tk} \end{aligned} \tag{I.96}$$

and by complex conjugation of scalars. One easily sees that

$$\mathcal{W}(\Psi^e, \bar{\Psi}^e)^* = \mathcal{W}(\Psi^e, \bar{\Psi}^e). \tag{I.97}$$

Consequently

$$f_{m,n}(k_1, k_2, k_3, k_4) = \overline{f_{m,n}(Tk_1, Tk_2, Tk_3, Tk_4)}. \tag{I.98}$$

In particular, $Q(t', s')$ is real and therefore the kernel of a self-adjoint operator on $L^2(k_F S^{d-1})$.

The localization operator $\mathbf{L}^{(h)}, h \le 0$, is now defined by the self-evident analogs of (I.29b-d)

$$\mathbf{L}^{(h)}\text{const} = \text{const} \tag{I.99a}$$

$$\begin{aligned}
&\mathbf{L}^{(h)} \int d\xi_1 d\xi_2 \bar{\Psi}(\xi_1) K(\xi_1 - \xi_2) \Psi(\xi_2) \\
=&\boldsymbol{\ell} \int d\xi_1 d\xi_2 \bar{\Psi}(\xi_1) K(\xi_1 - \xi_2) \Psi(\xi_2) \\
&= \int d\xi \bar{\Psi}(\xi) \tilde{K}(k_0 = 0, |\mathbf{k}| = \sqrt{2m\mu}) \Psi(\xi) \\
&\text{where } K = K_0 \mathbf{1} + K_1 \sigma^1 + K_2 \sigma^2 + K_3 \sigma^3
\end{aligned} \tag{I.99b}$$

$$\mathbf{L}^{(h)} \int d\xi_1 ... d\xi_n K_{i_1, ..., i_n}(\xi_1, ..., \xi_n) \overset{(-)}{\Psi}_{i_1}(\xi_1) ... \overset{(-)}{\Psi}_{i_n}(\xi_n) = 0$$

$$\text{for } n > 4, i_j = 1, 2 \tag{I.99c}$$

and

$$\mathbf{L}^{(h)} \int \prod \frac{d^{d+1}k_i}{(2\pi)^{d+1}} (2\pi)^{d+1} \delta(k_1 + k_2 - k_3 - k_4) f_{m,n}(k_1, k_2, k_3, k_4) \left(\bar{\Psi}^{(i_1)}_{k_1} \tau^m \Psi^{(i_3)}_{k_3} \right) \left(\bar{\Psi}^{(i_2)}_{k_2} \tau^n \Psi^{(i_4)}_{k_4} \right) = 0 \tag{I.99d}$$

for $(m, n) \neq$(0,0), (3,3), (0,3), (3,0), (1,2), (2,1),

$$\begin{aligned}
&\mathbf{L}^{(h)} \int \frac{d^{d+1}t}{(2\pi)^{d+1}} \frac{d^{d+1}s}{(2\pi)^{d+1}} \frac{d^{d+1}q}{(2\pi)^{d+1}} f_{m,n}(t + \frac{q}{2}, -t + \frac{q}{2}, s + \frac{q}{2}, -s + \frac{q}{2}) \\
&\qquad \left(\bar{\Psi}^{(i_1)}_{t+q/2} \tau^m \Psi^{(i_3)}_{s+q/2} \right) \left(\bar{\Psi}^{(i_2)}_{-t+q/2} \tau^n \Psi^{(i_4)}_{-s+q/2} \right) \\
&= \int \frac{d^{d+1}t}{(2\pi)^{d+1}} \frac{d^{d+1}s}{(2\pi)^{d+1}} \frac{d^{d+1}q}{(2\pi)^{d+1}} \rho(|\mathbf{q}| M^{-\frac{1}{2}[i^* + h]}) f_{m,n}(t', -t', s', -s')
\end{aligned}$$

$$\left(\bar{\Psi}^{(i_1)}_{t+q/2}\tau^m\Psi^{(i_3)}_{s+q/2}\right)\left(\bar{\Psi}^{(i_2)}_{-t+q/2}\tau^n\Psi^{(i_4)}_{-s+q/2}\right) \tag{I.99e}$$

for $(m,n)=(0,0),\ (3,3)$ and

$$\mathbf{L}^{(h)}\int\frac{d^{d+1}t}{(2\pi)^{d+1}}\,\frac{d^{d+1}s}{(2\pi)^{d+1}}\,\frac{d^{d+1}q}{(2\pi)^{d+1}}f_{m,n}(t+\frac{q}{2},s-\frac{q}{2},s+\frac{q}{2},t-\frac{q}{2})$$

$$\left(\bar{\Psi}^{(i_1)}_{t+q/2}\tau^m\Psi^{(i_3)}_{s+q/2}\right)\left(\bar{\Psi}^{(i_2)}_{s-q/2}\tau^n\Psi^{(i_4)}_{t-q/2}\right)$$

$$=\int\frac{d^{d+1}t}{(2\pi)^{d+1}}\,\frac{d^{d+1}s}{(2\pi)^{d+1}}\,\frac{d^{d+1}q}{(2\pi)^{d+1}}\rho(|\mathbf{q}|M^{-\frac{1}{2}[i^*+h]})f_{m,n}(t',s',s',t')$$

$$\left(\bar{\Psi}^{(i_1)}_{t+q/2}\tau^m\Psi^{(i_3)}_{s+q/2}\right)\left(\bar{\Psi}^{(i_2)}_{s-q/2}\tau^n\Psi^{(i_4)}_{t-q/2}\right) \tag{I.99f}$$

for $(m,n)=(0,3),\ (3,0)$ and

$$\mathbf{L}^{(h)}\int\frac{d^{d+1}t}{(2\pi)^{d+1}}\,\frac{d^{d+1}s}{(2\pi)^{d+1}}\,\frac{d^{d+1}q}{(2\pi)^{d+1}}f_{m,n}(t+\frac{q}{2},s-\frac{q}{2},t-\frac{q}{2},s+\frac{q}{2})$$

$$\left(\bar{\Psi}^{(i_1)}_{t+q/2}\tau^m\Psi^{(i_3)}_{t-q/2}\right)\left(\bar{\Psi}^{(i_2)}_{s-q/2}\tau^n\Psi^{i_4}_{s+q/2}\right)$$

$$=\int\frac{d^{d+1}t}{(2\pi)^{d+1}}\,\frac{d^{d+1}s}{(2\pi)^{d+1}}\,\frac{d^{d+1}q}{(2\pi)^{d+1}}\rho(|\mathbf{q}|M^{-\frac{1}{2}[i^*+h]})f_{m,n}(t',s',t',s')$$

$$\left(\bar{\Psi}^{(i_1)}_{t+q/2}\tau^m\Psi^{(i_3)}_{t-q/2}\right)\left(\bar{\Psi}^{(i_2)}_{s-q/2}\tau^n\Psi^{(i_4)}_{s+q/2}\right) \tag{I.99g}$$

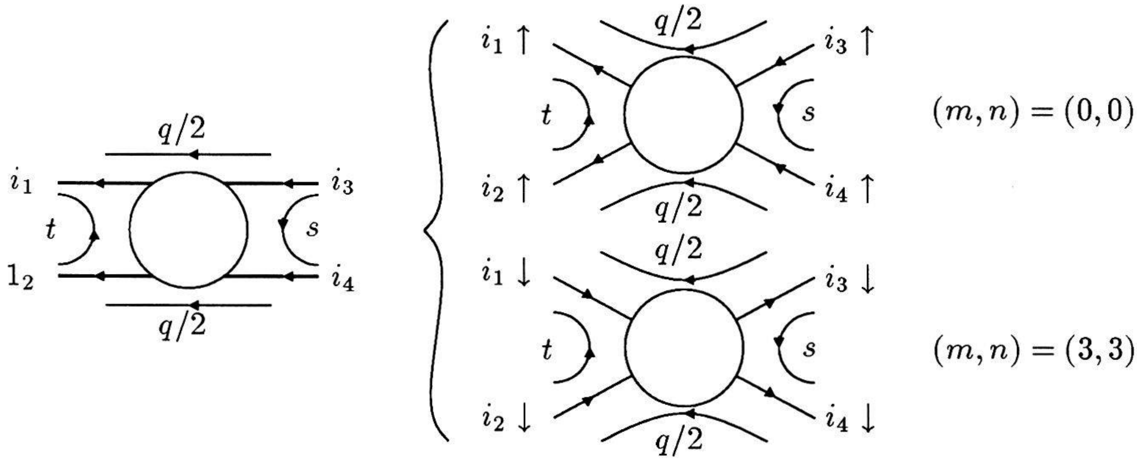
for $(m,n)=(1,2),\ (2,1)$ where, as before,

$$i^*=\max(i_1,i_2,i_3,i_4)$$

$$\rho(r)=1-h(r).$$

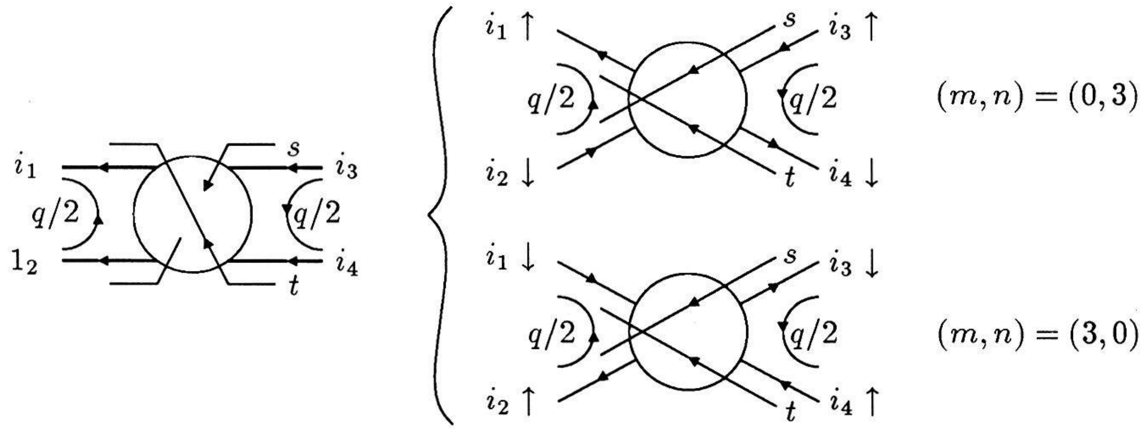
One may visualize the flow of momentum in (I.99) by

Nambu fields Physical fields

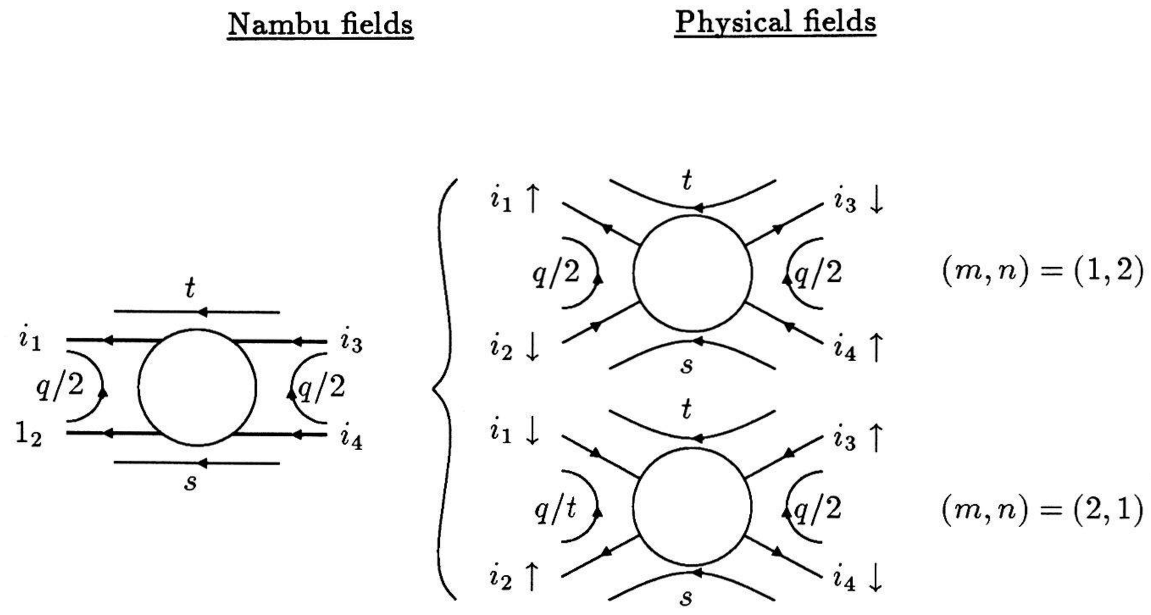


$(I.99e)$

Nambu fields Physical fields



$(I.99f)$



$$(I.99g)$$

Set

$$\mathcal{Q}^{(h)}(\bar{\Psi}^{(i)}, \Psi^{(i)}; i \leq h) := (\mathbf{L}^{(h)} - \ell)\mathcal{W}^{(h)} \left(\sum_{i \leq h} \bar{\Psi}^{(i)}, \sum_{i \leq h} \Psi^{(i)} \right) \tag{I.100}$$

Observe that kernels appearing in the range of $\mathbf{L}^{(h)} - \boldsymbol{\ell}$ can, by (I.95), be expressed in terms of Q. Consequently there exist kernels $Q^{(h)}(t', s'), h \leq 0$, such that

$$\begin{aligned}
\mathcal{Q}^{(h)} = &\sum_{i_1,i_2,i_3,i_4 \leq h} \frac{1}{2} \int \frac{d^{d+1}t}{(2\pi)^{d+1}} \frac{d^{d+1}s}{(2\pi)^{d+1}} \frac{d^{d+1}q}{(2\pi)^{d+1}} \rho(|\mathbf{q}| M^{-\frac{1}{2}[i^*+h]}) \\
&\Bigg\{ \sum_{(m,n)=(0,0),(3,3)} \left[Q^{(h)}(t', s') - Q^{(h)}(-t', s') \right] \left(\bar{\Psi}^{(i_1)}_{t+q/2} \tau^m \Psi^{(i_3)}_{s+q/2} \right) \left(\bar{\Psi}^{(i_2)}_{-t+q/2} \tau^n \Psi^{(i_4)}_{-s+q/2} \right) \\
&+ \sum_{(m,n)=(0,3),(3,0)} (-1) Q^{(h)}(t', s') \left(\bar{\Psi}^{(i_1)}_{t+q/2} \tau^m \Psi^{(i_3)}_{s+q/2} \right) \left(\bar{\Psi}^{(i_2)}_{s-q/2} \tau^n \Psi^{(i_4)}_{t-q/2} \right) \\
&+ \sum_{(m,n)=(1,2),(2,1)} Q^{(h)}(t', s') \left(\bar{\Psi}^{(i_1)}_{t+q/2} \tau^m \Psi^{(i_3)}_{t-q/2} \right) \left(\bar{\Psi}^{(i_2)}_{s-q/2} \tau^n \Psi^{(i_4)}_{s+q/2} \right) \Bigg\}
\end{aligned} \tag{I.101a}$$

$$= \sum_{i_1,i_2,i_3,i_4\le h} 2\int \frac{d^{d+1}t}{(2\pi)^{d+1}} \frac{d^{d+1}s}{(2\pi)^{d+1}} \frac{d^{d+1}q}{(2\pi)^{d+1}} \rho(|\mathbf{q}|M^{-\frac{1}{2}[i^*+h]})$$

$$\frac{1}{2}\left[Q^{(h)}(t',s')+Q^{(h)}(-t',s')\right] \bar\psi^{(i_1)}_{t+q/2\uparrow}\bar\psi^{(i_2)}_{-t+q/2\downarrow}\psi^{(i_4)}_{-s+q/2\downarrow}\psi^{(i_3)}_{s+q/2\uparrow}$$

$$+ \sum_{i_1,i_2,i_3,i_4\le h} \int \frac{d^{d+1}t}{(2\pi)^{d+1}} \frac{d^{d+1}s}{(2\pi)^{d+1}} \frac{d^{d+1}q}{(2\pi)^{d+1}} \rho(|\mathbf{q}|M^{-\frac{1}{2}[i^*+h]})\frac{1}{2}\left[Q^{(h)}(t',s')-Q^{(h)}(-t',s')\right]$$

$$\Big\{\bar\psi^{(i_1)}_{t+q/2\uparrow}\bar\psi^{(i_2)}_{-t+q/2\uparrow}\psi^{(i_4)}_{-s+q/2\uparrow}\psi^{(i_3)}_{s+q/2\uparrow} + \bar\psi^{(i_1)}_{t+q/2\downarrow}\bar\psi^{(i_2)}_{-t+q/2\downarrow}\psi^{(i_4)}_{-s+q/2\downarrow}\psi^{(i_3)}_{s+q/2\downarrow}$$

$$+\frac{1}{2}\left[\bar\psi^{(i_1)}_{t+q/2\uparrow}\bar\psi^{(i_2)}_{-t+q/2\downarrow}+\bar\psi^{(i_1)}_{t+q/2\downarrow}\bar\psi^{(i_2)}_{-t+q/2\uparrow}\right]\left[\psi^{(i_4)}_{-s+q/2\downarrow}\psi^{(i_3)}_{s+q/2\uparrow}\psi^{(i_4)}_{-s+q/2\uparrow}\psi^{(i_3)}_{s+q/2\downarrow}\right]\Big\} \qquad (I.101b)$$

Expression (I.101b) should be compared with the decomposition (I.30) into even orbital angular momentum singlet spin states and odd orbital angular momentum triplet spin states. As explained before these combinations are the Cooper pairs.

The effective potentials $\mathcal{W}^{(h)}$ (I.86) are analyzed using the same strategy. We applied to

$$\mathcal{G}^{(h)}(\psi^{(\le h)},\bar\psi^{(\le h)}) = \log\frac{1}{Z_h}\int \exp[-\lambda\mathcal{V}+\delta\mathcal{V}](\psi^{(\le 0)},\bar\psi^{(\le 0)})\prod_{j>h} d\mu_{C^{(j)}}(\psi^{(j)},\bar\psi^{(j)}). \qquad (I.14)$$

Just as before, $\mathcal{W}^{(h)}$ is the sum of all planar trees (including the trivial trees (I.34c,d)) constructed from R and C forks (I.34a,b) with root scale h and leaves $-\lambda\mathcal{V}$. Now, however, the localization operators ℓ and $L^{(h)}$ are replaced by the operators $\boldsymbol{\ell}$ (I.99) and $\mathbf{L}^{(h)}$ (I.99).

The graphs contributing to the perturbation series for

$$\log\frac{1}{Z}\int \exp[-\lambda\mathcal{V}]d\mu_{\mathbf{C}_\triangle}$$

are, as remarked above, bounded by $(\text{const}_\triangle)^n$ because $[ik_0 - e(\mathbf{k})\sigma^3 - \triangle\sigma^1]^{-1} = O\left(\frac{1}{\triangle}\right)$. As $\triangle$ tends to zero mass subdiagrams diverge and arrays of four-legged subdiagrams produce anomolously large values. The addition of counterterms $\delta\mathcal{V},\mathcal{D}$ in $\mathcal{W}$, (I.69), yields a perturbation series uniform in $\triangle$. Still, as $\triangle$ tends to zero, anomolously large graphs appear. However they are uniformly localized by $\mathbf{L}^{(h)} - \boldsymbol{\ell}$ in $\mathcal{Q}^{(h)}$, (I.100). Precisely, the content of Theorem 1 below is that graphs contributing to trees containing no quartic C forks are exponentially bounded, uniformly in $\triangle$.

The derivations of (I.47) and (I.48) may be applied verbatim to the effective potentials $\mathcal{W}^{(h)}$ and effective interactions $\mathcal{Q}^{(h)}$ to yield the coupled system

$$\begin{aligned}\mathcal{W}^{(h)} &= \mathcal{Q}^{(h)} + \sum_{i>h}(\mathbf{L}^{(i)} - \mathbf{L}^{(i-1)})\mathcal{Q}^{(i)} - \sum_{i\le h} \boldsymbol{\ell}\mathcal{E}^{(i)}(\mathcal{W}^{(i)}) \\ &\quad + \left[-\lambda(\mathbf{1} - \mathbf{L}^{(0)})\mathcal{V} + \sum_{i>h}(\mathbf{1} - \mathbf{L}^{(i-1)})\mathcal{E}^{(i)}(\mathcal{W}^{(i)})\right] + \log\frac{Z_{h+1}}{Z_h} \qquad (I.102a)\end{aligned}$$

$$\mathcal{Q}^{(h-1)} = (\mathbf{L}^{(h-1)} - \boldsymbol{\ell})\mathcal{Q}^{(h)} + (\mathbf{L}^{(h-1)} - \boldsymbol{\ell})\mathcal{E}^{(h)}(\mathcal{W}^{(h)}) \qquad (I.102b)$$

and the boundary conditions

$$(\text{projection onto nonquadratic part of } \mathcal{W}^{(0)}) = -\lambda\mathcal{V} \qquad (I.103a)$$

$$Q^{(0)}(t', s') = -\lambda\langle t', -t'|V|s', -s'\rangle \qquad (I.103b)$$

$$\lim_{h\to-\infty} \boldsymbol{\ell}\mathcal{W}^{(h)} = 0. \qquad (I.103c)$$

Bear in mind that the truncated expectation $\mathcal{E}^{(h)}$ (I.17) is now with respect to the covariance $\mathbf{C}_\triangle$.

Given an arbitrary sequence of quartic monomials $\mathcal{Q}^{(h)}$, iteration of (I.102a) generates $\mathcal{W}^{(h)}$ as the sum of all planar trees constructed from R (I.34a) and c (I.23b) forks with root scale h and leaves $-\lambda(\mathbf{1}-\mathbf{L}^{(0)})\mathcal{V}$ and $\mathcal{Q}^{(j)}+\sum_{i\ge j}(\mathbf{L}^{(i)}-\mathbf{L}^{(i-1)})\mathcal{Q}^{(i)}$. Following the discussion of (I.37) through (I.40) one sees that $\mathcal{W}^{(h)}$is the sum of graphs with generalized vertices $-\lambda(\mathbf{1} - \mathbf{L}^{(0)})\mathcal{V}$ and $\mathcal{Q}^{(j)} + \sum_{i\ge j}(\mathbf{L}^{(i)} - \mathbf{L}^{(i-1)})\mathcal{Q}^{(i)}$. If the renormalization group flow (I.102b) resums the quartic parts of C forks to effective interactions that are uniformly bounded in h then by Theorem I.1, to come, all the graphs contributing to $\mathcal{W}^{(h)}$ are exponentially bounded, uniformly in h. We shall discuss the convergence of (I.102b) in Theorem I.2.

Let $w^{(h)}(\Psi^e, \bar{\Psi}^e)$ be the sum of all planar trees constructed from R (I.34a) and c (I.23b) forks with root scale h, leaves $-\lambda\mathcal{V}$. The scale sums in (I.34a) and (I.23b) are now automatically restricted to the interval (I.85d). Of course, $L^{(h)}, \ell, C^{(h)}$ are replaced by $\mathbf{L}^{(h)}, \boldsymbol{\ell}$ and $\mathbf{C}_\triangle^{(h)}$. We define $w_{2p,n}^{(h)}(\xi_1, \ldots, \xi_{2p})$ by expanding

$$w^{(h)}(\Psi^e, \bar{\Psi}^e) = \sum_{p=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(2p)!}\lambda^n \prod_{i=1}^{p}\left(\int d\xi_{2i-1}d\xi_{2i}\bar{\Psi}^e(\xi_{2i-1})\Psi^e(\xi_{2i})\right) w_{2p,n}^{(h)}(\xi_1, \ldots, \xi_{2p}) \qquad (I.104)$$

Let $\mathcal{T}$ be a tree contributing to $w^{(h)}_{2p,n}$ and $\Gamma_{2p}(\mathcal{T})$ the set of all $2p$-point labelled graphs consistent with $\mathcal{T}$ so that

$$w^{(h)}_{2p,n} = \sum_{\substack{\text{trees } \mathcal{T} \text{ with} \\ n \text{ leaves}}} \sum_{\Gamma_{2p}(\mathcal{T})} \mathrm{Val}_{\Delta}(G^I). \qquad (I.105)$$

We now state the first main result. It immediately implies, that for any fixed graph contributing to (I.105), the sum over scale assignments $i_f, f \in \mathcal{T}$ is exponentially bounded in n, uniformly in Δ provided $|\ell n\Delta| \leq \frac{\text{const}}{\lambda}$. Note that the solution of (I.77) satisfies this inequality. The Theorem is designed so as to accomodate vertices/leaves of the type $\mathcal{Q}^{(j)} + \sum_{i\geq j}(\mathbf{L}^{(i)} - \mathbf{L}^{(i-1)})\mathcal{Q}^i$.

Let us imagine that $\mathcal{Q}^{(j)}, j \leq 0$, has been constructed nonperturbatively by iterating a map of the form (I.102b). Then the first component (I.102a) of the coupled system can be iterated on its own. The solution is a sum over more general trees in which we allow leaves $\mathcal{Q}^{(j)} + \sum_{i\geq j}(\mathbf{L}^{(i)} - \mathbf{L}^{(i-1)}\mathcal{Q}^{(i)}$ and $-\lambda(\mathbf{1} - \mathbf{L}^{(0)})\mathcal{V}$.

Define the norms

$$\|I\|_h = \max\Big\{ M^{(\sum_{j=2}^4 \alpha_j + \sum_{j=1}^4 \beta_j)h} \sup_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4=0} \int d\tau_2 d\tau_3 d\tau_4$$
$$\prod_{j=2}^4 |\tau_j|^{\alpha_j} \left| \prod_{j=1}^4 \nabla^{\beta_j}_{\mathbf{k}_j} I((0,\mathbf{k}_1),(\tau_2,\mathbf{k}_2),(\tau_3,\mathbf{k}_3),(\tau_4,\mathbf{k}_4)) \right| :$$
$$\sum_{j=2}^4 \alpha_j \leq 2, \sum_{j=1}^4 |\beta_j| \leq 1, \alpha_j \geq 0 \Big\} \qquad (I.106)$$

$$\|T\|_h = \max\left\{ M^{(\alpha+\beta)h} \sup_{\mathbf{k}} \int d\tau |\tau|^{\alpha} |\nabla^{\beta}_{\mathbf{k}} S\big((0,-\mathbf{k}),(\tau,\mathbf{k})\big)| : 0 \leq \alpha \leq 2, |\beta| \leq 1 \right\} \qquad (I.107)$$

$$\big|u\big|' = \int d^d\mathbf{k}_1 ... d^d\mathbf{k}_{2n} \delta(\mathbf{k}_1 + ... + \mathbf{k}_{2n}) sup_{\tau_1,...,\tau_{2n}} |u\big((\tau_1,\mathbf{k}_1),...,(\tau_{2n},\mathbf{k}_{2n})\big)| \qquad (I.108)$$

on four legged, two legged and general kernels respectively. Here $(\tau, \mathbf{k})$ refers to mixed (time, vector momentum) coordinates and $|\cdot|$ refers to the tensor norm. Roughly speaking the norms (I.106,107) control two k_0 and one $\mathbf{k}$ derivatives.

We now describe the class of graphs to which Theorem I.1 applies. Let $\mathcal{T}$ be a tree constructed from R forks (I.34a), c forks (I.23b) and n general four legged leaves each of type

$\tau^a \otimes \tau^b$ with $a+b=0 \pmod 3$. These leaves $I_v^{j_{\pi(v)}}$ may depend on the scale $j_{\pi(v)}$ of the fork $\pi(v)$ of $\mathcal{T}$ immediately below v and are assumed to obey

$$\sup_j \|I_v^{(j)}\|_j \leq |\lambda|\omega_v. \tag{I.109}$$

The ultraviolet regime has already been treated in [FT] so we discard $j=0$ and from now on we restrict ourselves to the interval

$$-1 \geq j \geq \text{const} \log \triangle. \tag{I.110}$$

To state the Theorem it is also necessary to describe the effects of Wick ordering (see the discussion following (III.3)). If G^J is a graph contributing to $\mathcal{T}$ then each line of G has a scale label j_f, f a fork $\mathcal{T}$, and is given a hard/soft label. Hard lines carry the covariance (I.85b) while soft lines carry

$$\mathbf{C}_s^{(j)}(\xi) = -\int \frac{d^d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}} e^{-E(\mathbf{k})|\tau|} \frac{\operatorname{sgn}(\tau)E(\mathbf{k})\mathbf{1} + e(\mathbf{k})\sigma^3 + \triangle\sigma^1}{2E(\mathbf{k})} \rho(M^{-2j}E(\mathbf{k})^2) \tag{I.111}$$

Furthermore, defining

$$G_f = \{\ell \in G |\ \text{the fork } f' \text{ of } \ell \text{ has } f' \geq f\}, \tag{I.112}$$

Wick ordering forces each quotient subgraph $G_f/\{G_{f'}|f' > f\}$ to be connected by hard lines and to contain no tadpoles

Theorem I.1 Let $\mathcal{T}$ be a tree constructed from R forks (I.34a), c forks (I.23b) and n general scale dependent four legged leaves $I_v^{(j)}$ each of type $\tau^a \otimes \tau^b$ with $a+b=0 \pmod 3$ and satisfying

$$\sup_j \|I_v^{(j)}\|_j \leq |\lambda|\omega_v$$

Let G^J be a labelled graph contributing to $\mathcal{T}$ as above. Let $|\log\triangle| \leq \frac{\text{const}}{|\lambda|}$. Then

$$\Big|\sum_J \operatorname{Val}(G^J)\Big|' \leq \text{const}^{L(G)} |\lambda|^{3/2} \prod_v [|\lambda|^{1/4}\omega_v].$$

The sum is over

$$\{\mathbf{J}\,|\,j_{\pi(f)} < j_f \le -1 \quad \text{if } f \text{ is an } R \text{ fork and}$$
$$\text{const } \log\triangle \le j_f \le j_{\pi(f)} \quad \text{if } f \text{ is a } c \text{ fork}\}$$

with $j_{\pi(\phi)} = \text{const}\log\triangle$.

Theorem I.1 is an immediate consequence of the more general Theorem III.6.

Verifying Theorem III.6 will amount, in Sections III and IV, to controlling $\mathbf{1}-\mathbf{L}^{(h)}$. When $\mathbf{L}^{(h)}$ acts by (I.99e-g) the discussion of (I.41) applies. A new ingredient is required for (I.99d).

For $(m,n) \neq (0,0),(3,3),(3,0),(0,3),(1,2),(2,1)$, that is $m+n \neq 0(\text{mod } 3)$, the quartic monomial $(\bar{\Psi}\tau^m\Psi)(\bar{\Psi}\tau^n\Psi)$ contains, by (I.89), different numbers of physical ψ's and physical $\bar{\psi}$'s. For example,

$$(\bar{\Psi}_{k_1}\tau^3\Psi_{k_3})(\bar{\Psi}_{k_2}\tau^1\Psi_{k_4}) = \psi_{-k_1\downarrow}\bar{\psi}_{-k_3\downarrow}\bar{\psi}_{k_2\uparrow}\bar{\psi}_{-k_4\downarrow}.$$

Observe that all the interaction leaves (I.60a)

$$\frac{1}{2}\int \langle k_1,k_3|V|k_2,k_4\rangle(\bar{\Psi}_{k_1}[\tau^0-\tau^3]\Psi_{k_3})(\bar{\Psi}_{k_2}[\tau^0-\tau^3]\Psi_{k_4})$$

of any tree $\mathcal{T}$ contain only monomials with $a+b=0(\text{mod } 3)$. (This remains true in Theorem III.6 where $\mathcal{Q}^{(h)}$'s are also admitted as leaves.) Furthermore, the portion $(-1)\frac{ik_0\mathbf{1}+e(\mathbf{k})[\tau^0-\tau^3]}{k_0^2+E(\mathbf{k})^2}$ of $\mathbf{C}_\triangle$ preserves the number of physical ψ's and physical $\bar{\psi}$'s since

$$\tau^i\tau^m = \tau^j \text{ or } \tau^m\tau^i = \tau^j$$

with $i=0,3$ implies $m=j$ and, in particular, $m(\text{mod } 3) = j(\text{mod } 3)$. If G^I is a labelled graph consistent with $\mathcal{T}$ and $a+b \neq 0(\text{mod } 3)$ then it follows from the discussion above that the kernel $f_{a,b}$ of any four-legged subgraph corresponding to an R fork of $\mathcal{T}$ must contain the $\frac{\triangle[\tau^1+\tau^2]}{k_0^2+E(\mathbf{k})^2}$ portion of some covariance.

Recall that

$$\mathbf{C}_\triangle^{(h)}(k) = (-1)\frac{ik_0\mathbf{1}+e(\mathbf{k})[\tau^0-\tau^3]+\triangle[\tau^1+\tau^2]}{k_0^2+E(\mathbf{k})^2}f(M^{-2h}E(\mathbf{k})^2) \tag{I.85b}$$

On the support of $f(M^{-2h}E(\mathbf{k}^2))$

$$\mathbf{C}_1(k) := (-1)\frac{ik_0\mathbf{1}+e(\mathbf{k})[\tau^0-\tau^3]}{k_0^2+E(\mathbf{k})^2} \sim \frac{1}{M^h} \tag{I.113a}$$

$$\mathbf{C}_2(k) := (-1)\frac{\triangle[\tau^1+\tau^2]}{k_0^2+E(\mathbf{k})^2} \sim \frac{\triangle}{M^{2h}}. \tag{I.113b}$$

The portion (I.113a) of $\mathbf{C}_\triangle^{(h)}(k)$ obeys the same power counting bounds as the standard

$$C^{(h)}(k) = \frac{1}{ik_0 - e(\mathbf{k})} f(M^{-2h}e(\mathbf{k})^2).$$

On the other hand the portion (I.113b) is smaller by a factor $\frac{\triangle}{M^h}$.

Suppose $a+b \neq 0 \pmod 3$. Then the kernel $f_{a,b}$ of a four legged subgraph corresponding to an R fork of $\mathcal{T}$ of scale h is bounded by $O\left(\frac{\triangle}{M^h}\right)$, in contrast to the usual power counting bound $O(1)$. We have

$$\sum_{\substack{h \text{ s.t.} \\ M^h \geq \triangle}} \frac{\triangle}{M^h} \leq \text{constant} \tag{I.114}$$

uniformly in $\triangle$. Consequently the subgraph is summable over h, justifying definition (I.99d).

We also exploit

$$\sum_{-1 \geq j \geq \text{const } \log\triangle} 1 \leq \frac{\text{const}}{|\lambda|} \tag{I.115}$$

in the proof of Theorem III.6. See the discussion surrounding (III.21) and (III.23) for more explanation. Care is taken throughout Sections II-IV to ensure that the total number of sums (I.115) does not exceed a fixed fraction (strictly less than one) of the number of leaves.

We now discuss convergence of flows of the form

$$\mathcal{Q}^{(h-1)} = (\mathbf{L}^{(h-1)} - \boldsymbol{\ell})\mathcal{Q}^{(h)} + (\mathbf{L}^{(h-1)} - \boldsymbol{\ell})\hat{\mathcal{E}}_2^{(h)}(\mathcal{Q}^{(0)}, ..., \mathcal{Q}^{(h)}) + (\mathbf{L}^{(h-1)} - \boldsymbol{\ell})\hat{\mathcal{E}}_{\geq 3}^{(h)}(\mathcal{Q}^{(0)}, ..., \mathcal{Q}^{(h)}) \tag{I.116}$$

Here

$\hat{\mathcal{E}}_2^{(h)}(\mathcal{Q}^{(0)}, ..., \mathcal{Q}^{(h)})$ = the part of $\mathcal{E}_\triangle^{(h)}(\mathcal{W}^{(h)})$ that is quartic in the external fields and is homogeneous of degree 2 in $(\mathcal{Q}^{(0)}, ..., \mathcal{Q}^{(h)})$

and $\hat{\mathcal{E}}_{\geq 3}^{(h)}(\mathcal{Q}^{(0)}, ..., \mathcal{Q}^{(h)})$ is to be thought of as the quartic part of $\mathcal{E}_\triangle^{(h)}(\mathcal{W}^{(h)}) - \hat{\mathcal{E}}_2^{(h)}(\mathcal{Q}^{(0)}, ..., \mathcal{Q}^{(h)})$ where $\mathcal{W}^{(h)}$ is expressed nonperturbatively in terms of $(\mathcal{Q}^{(0)}, ..., \mathcal{Q}^{(h)})$. Once again, Wick ordering implies that $\mathcal{E}_1^{(h)}(\mathcal{W}^{(h)}) = 0$. By (I.101) we may equivalently discuss the flow

$$Q^{(h-1)} = Q^{(h)} + \hat{E}_2^{(h)}(Q^{(0)}, ..., Q^{(h)}; \triangle) + \hat{E}_3^{(h)}(Q^{(0)}, ..., Q^{(h)}; \lambda, \triangle) \tag{I.117a}$$

of the kernel $Q^{(h)}(t', s')$ of $\mathcal{Q}^{(h)}$. Here

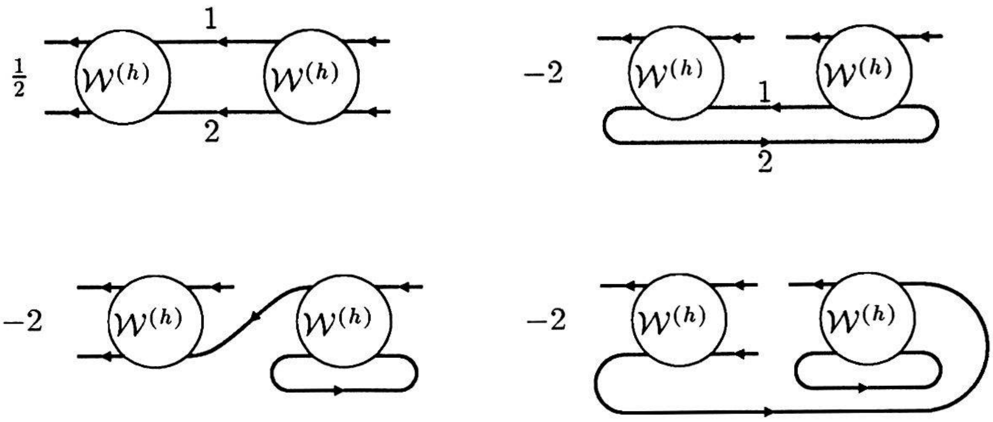
$$\hat{E}_2^{(h)}(t', s') = \hat{E}_2^{(h)}(Q^{(0)}, ..., Q^{(h)}; \triangle)(t', s'), \qquad (I.117b)$$

$$\hat{E}_3^{(h)}(t', s') = \hat{E}_3^{(h)}(Q^{(0)}, ..., Q^{(h)}; \lambda, \triangle)(t', s') \qquad (I.117c)$$

are the kernels of $\hat{\mathcal{E}}_2^{(h)}$ and $\hat{\mathcal{E}}_{\geq 3}^{(h)}$ respectively.

Observe that (equation (I.118))

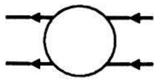
[the quartic part of $\frac{1}{2!}\mathcal{E}_2^{(h)}$ (quartic part of $\mathcal{W}^{(h)}$, quartic part of $\mathcal{W}^{(h)}$)]=



where $\frac{1}{2}$  is the antisymmetric kernel (see (I.90)) of the quartic part of $\mathcal{W}^{(h)}$. We easily show in Section IV that the last two diagrams are irrelevant. The first two are more subtle. We proceed to analyze these diagrams.

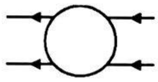
The first step is to separate off the $(-1)\frac{\triangle[\tau^1+\tau^2]}{k_0^2+E(\mathbf{k})^2}$ portion of the covariances on lines 1 and 2. As in (I.113b) and (I.114) they generate summable contributions. Next we remove the $\sum_{i>h}(\mathbf{1}-\mathbf{L}^{(i-1)})\mathcal{E}^{(i)}(\mathcal{W}^{(i)})$ contributions to $\mathcal{W}^{(h)}$. They are part of $\hat{\mathcal{E}}_{\geq 3}^{(h)}$. The third step is to identify electron-hole ladders. We will show in Section IV that they are also irrelevant contributions. This is most easily done in the physical field formalism.

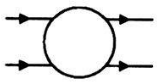
Let the Nambu graph

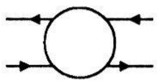


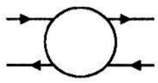
represent the monomial $\frac{1}{2}\int f_{m,n}(\bar{\Psi}\tau^m\Psi)(\bar{\Psi}\tau^n\Psi)$with $0 \leq m, n \leq 3, \;\; m+n = 0(\text{mod } 3)$.

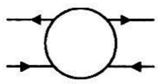
All contributions with $m+n \neq 0 (\mathrm{mod}\ 3)$ have already been placed in $\hat{\mathcal{E}}^{(h)}_{\geq 3}$. In physical fields this graph corresponds, by (I.89), to

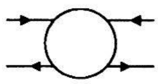
 for $(m,n)=(0,0)$

 for $(m,n)=(3,3)$

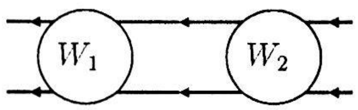
 for $(m,n)=(0,3)$

 for $(m,n)=(3,0)$

 for $(m,n)=(1,2)$

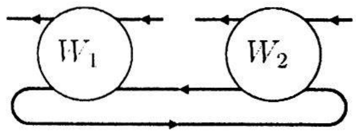
 for $(m,n)=(2,1)$

Consequently, for

$\frac{1}{2}$ 

$(I.119)$

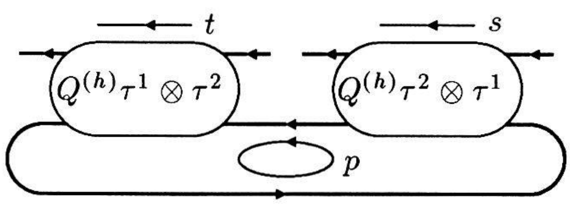
we get zero or an electron-hole ladder except when W_1 and W_2 are both of type $\tau^0 \otimes \tau^0$ or both of type $\tau^3 \otimes \tau^3$. Then the whole graph is of type $\tau^0 \otimes \tau^0$ and $\tau^3 \otimes \tau^3$ respectively. For

-2 

$(I.120)$

we get zero or an electron hole ladder except when one of W_1 and W_2 is of type $\tau^1 \otimes \tau^2$ and the other is of type $\tau^2 \otimes \tau^1$.

The last step is to apply $(\mathbf{L}^{(h-1)} - \boldsymbol{\ell})$ to the remaining parts of (I.119), (I.120) and identify $Q^{(h-1)}$ from the result. By (I.101a) it suffices to antisymmetrize (I.119), (I.120) and select the coefficient of $\tau^1 \otimes \tau^2$. Graph (I.119) does not contribute to this coefficient. By (I.99g), setting $q = 0$ for the whole graph (I.120) forces $q = 0$ in the kernels W_1 and W_2 with the result that the $(\mathbf{L}^{(i)} - \mathbf{L}^{(i-1)})\mathcal{Q}^{(i)}$ contributions to W_i are also zero. The term $(\mathbf{1} - \mathbf{L}^0)\mathcal{V}$ of (I.102a) does not contain $\tau^1 \otimes \tau^2$ or $\tau^2 \otimes \tau^1$. This leaves

$\mathbf{L}^{(h-1)}\ (-1)$ 

$(I.121)$

Antisymmetrization has replaced the coefficient (-2) by (-1). The kernel of (I.121) is

$$-4\int \frac{d^4p}{(2\pi)^4}\frac{1}{2}Q^{(h)}(t',p')\frac{1}{2}Q^{(h)}(s',p') \sum_{\substack{i,j\le h \\ i \text{ or } j=h}} tr\tau^2\mathbf{C}_1^{(i)}(p)\tau^1\mathbf{C}_1^{(j)}(p) \qquad (I.122)$$

where $\mathbf{C}_n^{(h)}(p) = \mathbf{C}_n(p)f(M^{-2h}E(\mathbf{p})^2)$ and $\mathbf{C}_n(p)$ is given in (I.113). The sum $\sum_{\substack{i,j\le h \\ i \text{ or } j=h}}$ results, as usual, from Wick ordering. The only nonzeros traces are

$$\begin{aligned} tr\tau^2\mathbf{1}\tau^1\mathbf{1} &= tr\tau^2\mathbf{1}\tau^1\tau^3 \\ &= tr\tau^2\tau^0\tau^1\mathbf{1} = tr\tau^2\tau^0\tau^1\tau^3 = 1 \end{aligned}$$

so that

$$tr\tau^2\mathbf{C}_1(p)\tau^1\mathbf{C}_1(p) = -\frac{p_0^2+e(\mathbf{p})^2}{[p_0^2+E(\mathbf{p})]^2}.$$

Hence the integral (I.122) becomes

$$\beta_\triangle^{(h)}\int dp' Q^{(h)}(t',p')Q^{(h)}(p',s')$$

where

$$\beta_\triangle^{(h)} = \int \frac{dp_0\, d|\mathbf{p}|}{(2\pi)^{d+1}}\left[\frac{|\mathbf{p}|}{k_F}\right]^{d-1}\frac{p_0^2+e(\mathbf{p})^2}{[p_0^2+E(\mathbf{p})^2]^2}\left\{\rho^2(M^{-2(h+1)}E(\mathbf{p})^2)-\rho^2(M^{-2h}E(\mathbf{p})^2)\right\} \qquad (I.123)$$

The conclusion of the discussion above is that we may rewrite the flow equation (I.117) as

$$\begin{aligned} Q^{(h-1)}(t',s') = Q^{(h)}(t',s') + \beta_\triangle^{(h)}\int dp' Q^{(h)}(t',p')Q^{(h)}(p',s') + S^{(h)}(Q^{(0)},\dots,Q^{(h)};\triangle)(t',s') \\ +H^{(h)}(Q^{(0)},\dots,Q^{(h)};\lambda,\triangle)(t',s'). \qquad (I.124a) \end{aligned}$$

Here all the contributions of third and higher order have been placed in $H^{(h)}$. The "summable" second order contributions have been placed in $S^{(h)}$. Precisely

$S^{(h)} =$ the second order parts of the last two diagrams of (I.118)

+ those second order parts of the first two diagrams of (I.118) for which either propagator 1 or 2 is a C_2 (see (I.113b))

+ all second order parts of the first diagram of (I.118) except those for which both kernels are of type $\tau^0 \otimes \tau^0$ or of type $\tau^3 \otimes \tau^3$

+ all second order parts of the second diagram of (I.118) except those for which one kernel is of type $\tau^1 \otimes \tau^2$ and the other is of type $\tau^2 \otimes \tau^1$. (I.124b)

The only other second order contribution is that from the physical field electron-electron ladder and this has produced $\beta_\triangle^{(h)} Q^{(h)2}$.

To see what is going on we once again start by considering the ladder approximation

$$Q^{(h-1)}(t',s') = Q^{(h)}(t',s') + \beta_\triangle^{(h)} \int dp' Q^{(h)}(t',p') Q^{(h)}(p',s') \tag{I.125}$$

to the flow and compare its behaviour to that of (I.53). By (I.92c), (I.95c) and (I.98) $Q^{(h)}(t',s')$ defines a real, self-adjoint, rotation invariant operator on $L^2(k_F S^{d-1})$ and hence may be decomposed into spherical harmonics

$$Q^{(h)}(t',s') = \sum_{n\geq 0} \lambda_n^{(h)}(\triangle) \pi_n(t',s'). \tag{I.126}$$

The analogue of (I.55) is

$$\lambda_n^{(h-1)}(\triangle) = \lambda_n^{(h)}(\triangle) + \beta_\triangle^{(h)} (\lambda_n^{(h)}(\triangle))^2, \; n \geq 0 \tag{I.127}$$

with initial data determined by

$$\sum_{n\geq 0} \lambda_n^{(0)} \pi_n(t',s') = -\lambda \langle t', -t' | V | s', -s' \rangle.$$

The solution of (I.127) behaves very differently from that of (I.55) because

$$\begin{aligned} \sum_{h=-\infty}^{0} \beta_\triangle^{(h)} &= \int \frac{dp_0 d|\mathbf{p}|}{(2\pi)^{d+1}} \left[\frac{|\mathbf{p}|}{k_F}\right]^{d-1} \frac{p_0^2 + e(\mathbf{p})^2}{[p_0^2 + E(\mathbf{p})^2]^2} \rho^2(M^{-2}E(\mathbf{p})^2) \\ &= \frac{m}{(2\pi)^d k_F} |\log \triangle| + O(1) \end{aligned} \tag{I.128}$$

in contrast to $\lim_{h\to-\infty} \beta^{(h)} = \beta$.

We show that if

$$|\lambda_n^{(0)}| \sum_{h=-\infty}^{0} \beta_\triangle^{(h)} \leq \gamma < 1, \; |\lambda_n^{(0)}| << 1 \tag{I.129}$$

then

$$\lambda_n^{(h)} = (1 - \lambda_n^{(0)} \sum_{j>h} \beta_\triangle^{(j)} - b_n^{(h)})^{-1} \lambda_n^{(0)} \tag{I.130a}$$

with

$$|b_n^{(h)}| \le \frac{1-\gamma}{2\gamma}|\lambda_n^{(0)}|\sum_{j>h}\beta_\triangle^{(j)} \le \frac{1-\gamma}{2}. \tag{I.130b}$$

First observe that (I.130b) implies

$$\begin{aligned}|\lambda_n^{(h)}| &\le (1-\gamma-\frac{1-\gamma}{2})^{-1}|\lambda_n^{(0)}| \\ &= \frac{2}{1-\gamma}|\lambda_n^{(0)}|\end{aligned} \tag{I.131}$$

The flow of the tail is given by

$$b_n^{(h-1)} = b_n^{(h)} - \lambda_n^{(0)}\beta_\triangle^{(h)^2}\lambda_n^{(h)}(1+\lambda_n^{(h)}\beta_\triangle^{(h)})^{-1}.$$

The inductive bound (I.130b) follows from

$$\begin{aligned}&|\lambda_n^{(0)}\beta_\triangle^{(h)^2}\lambda_n^{(h)}(1+\lambda_n^{(h)}\beta_\triangle^{(h)})^{-1}| \\ &\le |\lambda_n^{(0)}|\beta_\triangle^{(h)}\left\{\left(\sup_j \beta_\triangle^{(j)}\right)\frac{2}{1-\gamma}|\lambda_n^{(0)}|\left(1-\frac{2}{1-\gamma}|\lambda_n^{(0)}|\sup_j\beta_\triangle^{(j)}\right)^{-1}\right\} \\ &\le |\lambda_n^{(0)}|\beta_\triangle^{(h)}\frac{1-\gamma}{2\gamma}\end{aligned}$$

provided $|\lambda_n^{(0)}|$ is small enough. Consequently the sequence generated by (I.127) converges (though not to zero) for all sufficiently small initial data <u>irrespective of sign</u> when (I.129) is satisfied. Recall that, in contrast, the solution of

$$\lambda_n^{(h-1)} = \lambda_n^{(h)} + \beta^{(h)}(\lambda_n^{(h)})^2 \tag{I.55}$$

$$\lim_{h\to\infty}\beta^{(h)} = \beta > 0$$

converges to zero like

$$\lambda_n^{(h)} \sim \frac{\lambda_n^{(0)}}{1+\lambda_n^{(0)}\beta h},\ h\to-\infty$$

when $\lambda_n^{(0)} < 0$ and diverges when $\lambda_n^{(0)} > 0$.

Condition (I.129) is anticipated in the discussion following (I.81). After all, solving (I.125) amounts to summing ladders as in (I.84). In fact γ is, up to sign, the largest eigenvalue of $\Lambda^\pm$ with corresponding eigenvector (I.82) $b_\mp = \begin{bmatrix}1\\i\end{bmatrix}\otimes\begin{bmatrix}1\\-i\end{bmatrix}\mp\begin{bmatrix}1\\-i\end{bmatrix}\otimes\begin{bmatrix}1\\i\end{bmatrix}$.

In evaluating the first two graphs of (I.118) we painstakingly distinguished between physical field particle-particle and particle-hole ladders. The former determine the leading behaviour of the flow while the latter were placed in $S^{(h)}$ (I.124b). This essential difference is evident in $\Lambda^{\pm}$. As $\lambda, \triangle \to 0$ with

$$\gamma = -\lambda \int_{|e(\mathbf{k})|\leq\omega} \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{2E(\mathbf{k})}$$

held fixed, $\Lambda^{\pm}$ becomes

$$\gamma\frac{1}{2}\{\mp\sigma^3 \otimes \sigma^3 + \mathbf{1} \otimes \mathbf{1}\}.$$

The limit has two eigenvalues γ (for the particle-particle ladder) and two eigenvalues 0 (for the particle-hole ladder). Thus, particle-hole interactions produce a relatively small effect.

As we have observed, (I.84) is marginally divergent when $\gamma = 1$, that is when $\triangle$ is determined by the BCS gap equation (I.75). But, it is convergent when $\triangle$ is determined by (I.80) that is $\gamma \approx \frac{\triangle - J}{\triangle} < 1$. We introduced J in (I.79) to eliminate Goldstone bosons. Our discussion of the ladder approximation is finished and we now return to the full flow (I.117).

Four legged diagrams are, by natural power counting (see Lemma V.3 [FT,p. 209] and the discussion preceeding it), dimensionless. That is, a four legged diagram whose lowest particle lines are at scale h tends to obey a bound proportional to M^{0h}. However, as we have pointed out (see (I.85d)) there are only $O\left(\frac{1}{|\lambda|}\right)$ scales. The net effect is to improve M^{0h} to $\frac{M^h}{M^h+\triangle}$. We shall see in Lemma IV.4 that for "summable" second order diagrams M^{0h} is improved even further. Thus one would expect, in the course of a nonperturbative construction, using determinant bounds and Theorem III.6 to sum perturbation theory, that $S^{(h)}$ and $H^{(h)}$ satisfy

$$\|\sum_{h'\geq j\geq h} S^{(j)}(Q^{(0)}, ..., Q^{(h)}; \triangle)\|_h \leq \nu \|Q^{(0)}\|_0^2 \tag{I.132a}$$

and

$$\|H^{(h)}(Q^{(0)}, ..., Q^{(h)}; \lambda, \triangle)\|_h \leq \eta \frac{M^h}{M^h + \triangle} \|Q^{(0)}\|_0^{9/4} \tag{I.132b}$$

for $\|Q^{(1)}\|_1, ..., \|Q^{(h)}\|_h \leq \zeta \|Q^{(0)}\|_0$. The constants ν and η depend on ζ of course, but not on $\triangle$. This leads us to formulate

<u>Theorem I.2</u>

Let

$$Q^{(0)}(t',s') = -\lambda\langle t',-t'|V|s',-s'\rangle = \sum_{n\geq 0}\lambda_n^{(0)}\pi_n(t',s'),$$

where $t' = (0, \frac{\mathbf{t}}{|\mathbf{t}|}k_F)$ and π_n is the projection in $L^2(k_F S^{d-1})$ onto the space of homogeneous harmonic polynomials of degree n, satisfy

$$\sup_{n\geq 0}|\lambda_n^{(0)}|\sum_{h=-\infty}^{0}\beta_\triangle^{(h)} \leq \gamma < 1$$

where $\beta_\triangle^{(h)}$ is defined in (I.123) and $\|Q^{(0)}\|_0 < \infty$. Let $S^{(h)}$ and $H^{(h)}$ be maps from the h-fold Cartesian product of $\mathcal{B}(L^2(k_F S^{d-1}))$ to $\mathcal{B}(L^2(k_F S^{d-1}))$ obeying (I.132). Then there exists a constant $\in$, independent of $\triangle$, such that for $|\lambda| <\in$ the sequence $Q^{(h)}$ generated by the flow (I.124) satisfies

$$\|Q^{(h)}\|_h \leq \text{const}\|Q^{(0)}\|_0. \qquad (I.133)$$

The const depends on γ, but not on $\triangle$.

Theorem I.2 is an immediate consequence of the more general Theorem V.1. The convergence of the sequence $Q^{(h)}$ is discussed following Theorem V.1. The convergence is in a norm weaker than all the $\|\cdot\|_h$'s. For constructive purposes the boundedness (I.133) is much more important than the nature of the convergence.

The coupling constant λ and gap width $\triangle$ appear as independent parameters in the coupled system (I.102). In Theorem I.1 we required

$$|\lambda\log\triangle| \leq \text{const}. \qquad (I.134)$$

In Theorem I.2 we further required

$$\sup_{n\geq 0}|\lambda_n^{(0)}|\sum_{h=-\infty}^{0}\beta_\triangle^{(h)} \leq \gamma < 1 \qquad (I.135)$$

Here,$\lambda_n^{(0)}$ is proportional to λ and $\sum_{h=-\infty}^{0}\beta_\triangle^{(h)}$, given in (I.128), is proportional to $|\log\triangle|$. Inequality (I.135) is more stringent than (I.134) and, indeed, implies it.

It is now easy to summarize our main conclusions so far. Assume (I.135). Then the flow (I.102b) converges when (I.132) is satisfied (and, by Theorem III.6, this is the case if (I.102b) is truncated at any finite order of λ) and every graph contributing to (I.102a) is exponentially bounded. The last step is to impose (I.80) so that the coupled system (I.102) gives the physical model (I.60). See (I.75).

Recall that $D(\lambda,\mu,\triangle)$ is the coefficient of $\int d\xi\bar{\Psi}(\xi)\sigma^1\Psi(\xi)$ in the action $-\lambda\mathcal{V}+\delta\mathcal{V}+\mathcal{D}$. See (I.69). The order λ contribution to D is given in (I.76). The n^{th} order contribution, $n\geq 2$, is the coefficient of σ^1 in the sum of all trees with n leaves, root scale 0 and lowest fork a c fork. By Lemma III.7, each graph contributing to such a tree is bounded by $\text{const}^n|\lambda|^{5/4}\triangle|\log\triangle|$. Therefore, to any finite order the constraint

$$\triangle - J = -D(\lambda,\mu,\triangle) \tag{I.80}$$

becomes (following the discussion after (I.77))

$$\begin{aligned}\triangle - J &= -\lambda\int\frac{d^{d+1}p}{(2\pi)^{d+1}}\langle k',-k'|V|p,-p\rangle\frac{\triangle}{p_0^2+E(\mathbf{p})^2}\rho(E^2(\mathbf{p}))\\ &\qquad + O(\triangle|\log\triangle||\lambda|^{5/4})\\ &= \int\frac{d^{d+1}p}{(2\pi)^{d+1}}\lambda_0(p_0,|\mathbf{p}|)\frac{\triangle}{p_0^2+E(\mathbf{p})^2}\rho(E^2(\mathbf{p}))\\ &\qquad + O(\triangle|\log\triangle||\lambda|^{5/4})\\ &= \int\frac{d^{d+1}p}{(2\pi)^{d+1}}\lambda_0(0,k_F)\rho^2(M^{-2}E^2(\mathbf{p}))\frac{\triangle}{p_0^2+E(\mathbf{p})^2}\\ &\qquad + O(\triangle|\lambda|)+O(\triangle|\log\triangle||\lambda|^{5/4})\end{aligned} \tag{I.136}$$

Just as before, (I.136) has for any $J\geq 0$ a solution $\triangle > \max\left\{J,\exp\left[-\frac{\text{const}}{|\lambda|}\right]\right\}$ provided $\lambda_0(0,k_F) = \lambda_0^{(0)} > 0$, and $|\lambda|$ is sufficiently small. Suppose, $\lambda_0^{(0)} > 0, \lambda_0^{(0)} \geq |\lambda_n^{(0)}|$ for all $n\geq 1$, and $J>0$. Then (I.136) implies that

$$\begin{aligned}\sup_{n\geq 0}|\lambda_n^{(0)}|\sum_{h=-\infty}^{0}\beta_\triangle^{(h)} &= \lambda_0^{(0)}\int\frac{dp_0 d|\mathbf{p}|}{(2\pi)^{d+1}}\left[\frac{|\mathbf{p}|}{k_F}\right]^{d-1}\frac{p_0^2+e^2(\mathbf{p})}{[p_0^2+E^2(\mathbf{p})]^2}\rho^2(M^{-2}E^2(\mathbf{p}))\\ &= \lambda_0^{(0)}\left(\int\frac{dp_0 d|\mathbf{p}|}{(2\pi)^{d+1}}\frac{\rho^2(M^{-2}E^2(\mathbf{p}))}{p_0^2+E^2(\mathbf{p})}+O(1)\right)\\ &= \frac{\triangle - J}{\triangle}+O(|\lambda|^{1/4})\\ &< 1\end{aligned} \tag{I.137}$$

when $|\lambda|$ is sufficiently small, depending on J. The inequality $\triangle > J$ tells us that the symmetry breaking term (I.79) does not produce a gap larger than the physical $\triangle$.

Theorem I.3. Let $\langle k_1, k_2|V|k_3, k_4\rangle$ be a general two-body interaction such that the symmetries (I.6) are satisfied on the support of $\delta(k_1 + k_2 - k_3 - k_4)$ and $\|V\|_0 < \infty$. Suppose $\lambda_0^{(0)} > 0$ and $\lambda_0^{(0)} \geq |\lambda_n^{(0)}|, n \geq 1$, where (see Theorem I.2)

$$-\lambda\langle t', -t'|V|s', -s'\rangle = \sum_{n\geq 0} \lambda_n^{(0)} \pi_n(t', s').$$

Then, for $|\lambda|$ sufficiently small, the constraint (I.80) has a solution $\triangle$ such that the hypotheses of Theorem I.1 and Theorem I.2I.32 are satisfied for the pair $(\lambda, \triangle)$. Consequently, the coupled system (I.102) gives the physical model (I.60), the flow (I.102b) converges when (I.132) is satisfied (and this is the case when (I.102b) is truncated at any finite order of λ) and finally, every graph contributing to (I.102a) is exponentially bounded.

Recall the interaction

$$\langle k_1, k_2|V|k_3, k_4\rangle = \hat{U}(\mathbf{k}_3 - \mathbf{k}_1) - \gamma^2\theta(\omega_D - \omega(\mathbf{k}_3 - \mathbf{k}_1))^2 \frac{\omega(\mathbf{k}_3 - \mathbf{k}_1)^2}{(k_3 - k_1)_0^2 + \omega(\mathbf{k}_3 - \mathbf{k}_1)^2} \qquad (I.5c)$$

In this case

$$-\lambda\langle t', -t'|V|s', -s'\rangle = -\lambda\hat{U}(s' - t') + \lambda\gamma^2\theta(\omega_D - \omega(s' - t'))^2 \qquad (I.138)$$

If the two-body potential $\hat{U}$ and phonon frequency ω are smooth and ω is bounded away from zero, we have $\|V\|_0 < \infty$. It is easy to see from (I.138) that for any $\hat{U}, \lambda_0^{(0)} > 0$ and $\lambda_0^{(0)} \geq |\lambda_n^{(0)}|, n \geq 1$ when γ^2 is big enough. Thus the hypotheses of Theorem I.3 are satisfied. This completes our small-field analysis of the effective potential (I.5).

We now summarize the rest of the paper. Section II begins with simple estimates (Lemma II.1) of the covariance $\mathbf{C}_\triangle^{(j)}$ that are formulated in terms of the "dual" mixed time, d-momentum norms (II.5). They tell us that the power counting dimension (in the sense of (III.5)) of each particle line is 1. The rest of Section II is devoted to strings (II.6) of renormalized mass subdiagrams that we regard as generalized covariances. The appropriate estimates are given in Lemmas (II.2) and (II.2′). They are more involved than the analogous [FT Lemma VII.3] because (II.4) does not localize k_0 near zero in contrast to [FT§2].

Section III culminates in an inductive proof of the graph estimates Theorem III.6 and Lemma III.7. The argument relies on the Abstract Power Counting Lemma III.1 that reduces the problem of estimating graphs to those for which renormalization cancellations must be exploited. It also relies on the fact that $\sum_{\substack{h\ s.t.\\ M^h\geq\triangle}} \frac{\triangle}{M^h}, \lambda \sum_{\substack{h\ s.t.\\ M^h\geq\triangle}} 1 = \mathrm{const}|\lambda||\log\triangle|$ are uniformly bounded in $\triangle$. Such a priori bounds allow us via the yes/no Lemma III.5 (and general string estimates of Section II) to further reduce the proof to controlling the special second order, two and four legged graphs of Section IV. Lemma III.5 counts the fraction of coupling constants eaten up by $\lambda \sum_{M^h\geq\triangle} 1$.

Section IV is the most technical. Second order four legged graphs (IV.1) and two legged graphs (IV.48) are treated by hand. The main tool is the k-volume bound Lemma IV.2. It is used to obtain improved estimates of four legged electron-hole graphs with general kernels (Lemma IV.4) and four legged electron-electron graphs with at least one renormalized kernel (Lemma IV.5). Volume restrictions are also used to estimate two-legged graphs in Lemma III.6.

Section V is more straight forward. It treats an abstract flow of the type (I.124).

We hope in the future to combine the ideas and estimates of this paper with the exclusion principle (determinant bounds) to produce a convergent expansion for the effective potential (I.5) with $J > 0$. We also hope to treat the Coulomb interaction with the Anderson-Higgs mechanism (to eliminate the Goldstone boson) and produce a convergent expansion for $J = 0$.

II. Properties of the covariance

Fix a number $M > 1$. Let h be a smooth monotone function obeying

$$h(x) = \begin{cases} 0 & \text{for } x \leq 1 \\ 1 & \text{for } x \geq M^2 \end{cases} \tag{II.1a}$$

and let

$$f(x) = h(x)[1 - h(x/M^2)] = \begin{cases} h(x) & \text{for } x \leq M^2 \\ 1 - h(x/M^2) & \text{for } x \geq M^2 \end{cases} \tag{II.1b}$$

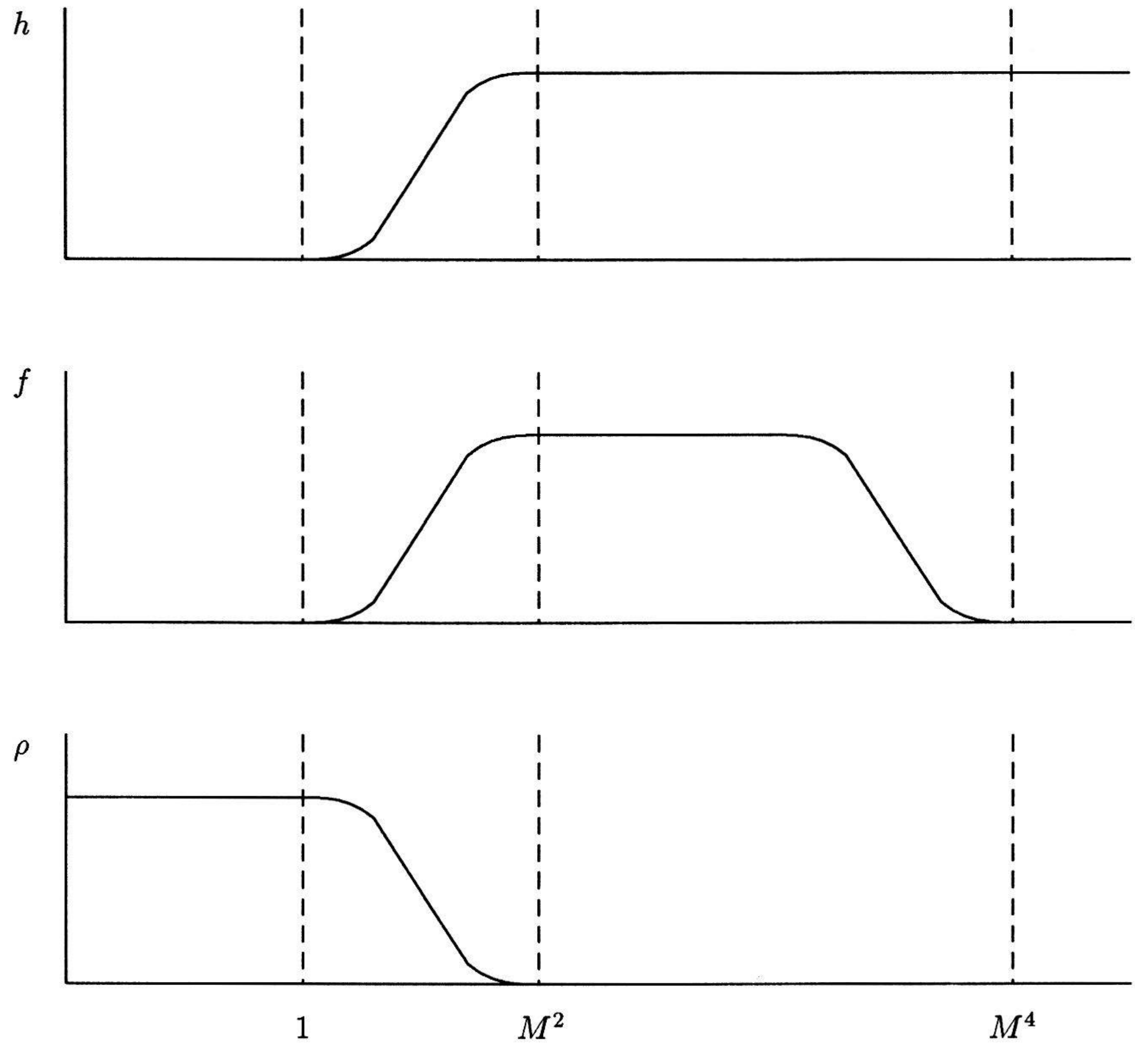
and

$$\rho(x) = 1 - h(x). \tag{II.1c}$$

We have

$$1 = h(x) + \sum_{i=-\infty}^{-1} f(M^{-2i}x). \tag{II.2}$$

These basic functions are illustrated below.



Recall that, for any $\triangle \geq 0$, the covariance

$$\begin{aligned} \mathbf{C} = \mathbf{C}_\triangle(k) &= \left[ik_0 \mathbf{1} - e(\mathbf{k})\sigma^3 - \triangle\sigma^1 \right]^{-1} \\ &= -\frac{ik_0 \mathbf{1} + e(\mathbf{k})\sigma^3 + \triangle\sigma^1}{k_0^2 + E(\mathbf{k})^2} \end{aligned} \tag{II.3}$$

where

$$E(\mathbf{k}) = (e(\mathbf{k})^2 + \triangle^2)^{1/2} \geq 0,$$
$$e(\mathbf{k}) = \frac{1}{2m}\mathbf{k}^2 - \mu$$

and

$$\sigma^0 = \mathbf{1} \qquad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

As before, we decompose

$$\mathbf{C} = \mathbf{C}_\triangle = \sum_{j=-\infty}^{0} \mathbf{C}_\triangle^{(j)} \tag{II.4a}$$

where

$$C_\triangle^{(j)}(\xi) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{i\langle k,\xi\rangle_-} [ik_0\mathbf{1} - e(\mathbf{k})\sigma^3 - \triangle\sigma^1]^{-1} \begin{cases} f(M^{-2j}E(\mathbf{k})^2), & j \leq -1 \\ h(E(\mathbf{k})^2), & j = 0 \end{cases} \tag{II.4b}$$

Evaluating the k_0 integral, for $\tau \neq 0$,

$$\mathbf{C}_\triangle^{(j)}(\xi) = -\int \frac{d^d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}} e^{-E(\mathbf{k})|\tau|} \frac{(\mathrm{sgn}\tau)E(\mathbf{k})\mathbf{1} + e(\mathbf{k})\sigma^3 + \triangle\sigma^1}{2E(\mathbf{k})} \begin{cases} f(M^{-2j}E(\mathbf{k})^2) \\ h(E(\mathbf{k})^2) \end{cases}. \tag{II.4c}$$

When $\tau = 0$, the $\mathbf{C}_{1,1}$ matrix element is defined by the limit $\tau \to 0-$ while the $\mathbf{C}_{2,2}$ matrix element is defined by the limit $\tau \to 0+$.

The basic estimates on $\mathbf{C}_\triangle^{(j)}$ are stated in terms of the norms

$$|u| := \sup_{\mathbf{k}} \int d\tau |u(\tau, \mathbf{k})| \tag{II.5a}$$

$$|u|' := \int \frac{d^d\mathbf{k}}{(2\pi)^d} \sup_\tau |u(\tau, \mathbf{k})|. \tag{II.5b}$$

When u is matrix valued, $|u(\tau, \mathbf{k})|$ refers to the matrix norm. Observe that

$$|uv| \leq |u||v| \tag{II.5c}$$

$$|uv|' \leq \min\{|u||v|', |u|'|v|\} \tag{II.5d}$$

where uv is a product in $\mathbf{k}$ and convolution in τ.

<u>Lemma II.1</u> For $j < 0$

$$|\mathbf{C}_\triangle^{(j)}| \leq \text{const } M^{-j}$$

$$|\mathbf{C}_\triangle^{(j)}|' \leq \text{const } M^{j}$$

uniformly for $\triangle \geq 0$.

Proof. The first estimate is an immediate consequence of

$$|\mathbf{C}_{\triangle}^{(j)}(\tau,\mathbf{k})| \leq \text{const } e^{-M^j|\tau|}.$$

The second follows from

$$\sup_{\tau} |\mathbf{C}_{\triangle}^{(j)}(\tau,\mathbf{k})| \leq f(M^{-2j}E(\mathbf{k})^2).$$

■

The next lemma, an analog of [FT, Lemma VII.3], is essential for controlling strings

$$S(\xi) = \overset{0}{\longrightarrow}_{j_1} rT_1 \longrightarrow_{j_2} rT_2 \longrightarrow \cdots \longrightarrow rT_s \longrightarrow_{j_{s+1}} \ell T_{s+1} \longrightarrow_{j_{s+2}} \ell T_{s+2} \longrightarrow \cdots \longrightarrow \ell T_{s+t} \overset{\xi}{\longrightarrow}_{j_{s+t+1}} \qquad (II.6)$$

of renormalized mass subdiagrams. It will be applied in Section III. Recall that the localization operator (see [I.99b])

$$(\boldsymbol{\ell}T)(k) = T(k') \qquad \left(k' = \left(0, \frac{\mathbf{k}}{|\mathbf{k}|}k_F\right)\right)$$

and that

$$\mathbf{r} = \mathbf{1} - \boldsymbol{\ell}.$$

Lemma II.2 Suppose that there is a j with each $j_\alpha = j$ or $j+1$. Let

$$\|\nabla_k^n T_\alpha(k)\|_{L^\infty} \leq M^{-nh_\alpha}\omega_\alpha, \qquad n \leq 2$$

with $h_\alpha \geq j$ for $1 \leq \alpha \leq s$. Then

$$\left|S\right| \leq \text{const}^{s+t} M^{-j} \prod_{\alpha=1}^{s} M^{-h_\alpha}\omega_\alpha \prod_{\beta=s+1}^{s+t} [|\boldsymbol{\ell}T_\beta| M^{-j}]$$

$$\left|S\right|' \leq \text{const}^{s+t} M^{j} \prod_{\alpha=1}^{s} M^{-h_\alpha}\omega_\alpha \prod_{\beta=s+1}^{s+t} [|\boldsymbol{\ell}T_\beta| M^{-j}]$$

uniformly for $\triangle \geq 0$.

Proof. It suffices to prove the intermediate estimate

$$|S(\tau,\mathbf{k})| \leq \text{const}^{s+t}[1+M^j|\tau|]^{-2} \prod_{\alpha=1}^{s} M^{-h_\alpha}\omega_\alpha \prod_{\beta=s+1}^{s+t} [|\boldsymbol{\ell}T_\beta| M^{-j}]\chi(E(\mathbf{k}) \leq \text{const}M^j). \quad (II.7)$$

Define $\eta = \min\{h_\alpha | 1 \le \alpha \le s\}$. We decompose the k_0 integral in

$$S(\tau,\mathbf{k}) = \int \frac{dk_0}{2\pi} e^{-ik_0\tau}[ik_0\mathbf{1} - e(\mathbf{k})\sigma^3 - \triangle\sigma^1]^{-1} \Big\{ \prod_{\alpha=1}^{s} (\mathbf{r}T_\alpha)(k)[ik_0 - e(\mathbf{k})\sigma^3 - \triangle\sigma^1]^{-1}$$

$$\prod_{\beta=s+1}^{s+t} (\boldsymbol{\ell}T_\beta)[ik_0\mathbf{1} - e(\mathbf{k})\sigma^3 - \triangle\sigma^1]^{-1} \prod_{i=1}^{s+t+1} f(M^{-2j_i}E(\mathbf{k})^2) \Big\} \qquad (II.8)$$

using

$$1 = \rho(M^{-2\eta}k_0^2) + \sum_{\nu=\eta}^{\infty} f(M^{-2\nu}k_0^2).$$

First, fix any k_0 scale $\nu \ge \eta$. Let $0 \le n \le 2$. The volume of integration in

$$(M^j\tau)^n \int \frac{dk_0}{2\pi} e^{-ik_0\tau} f(M^{-2\nu}k_0^2)[ik_0\mathbf{1} - e(\mathbf{k})\sigma^3 - \triangle\sigma^1]^{-1} \prod_{\alpha=1}^{s} (\mathbf{r}T_\alpha)(k)[ik_0 - e(\mathbf{k})\sigma^3 - \triangle\sigma^1]^{-1}$$

$$\prod_{\beta=s+1}^{s+t} (\boldsymbol{\ell}T_\beta)[ik_0\mathbf{1} - e(\mathbf{k})\sigma^3 - \triangle\sigma^1]^{-1}$$

$$= \int \frac{dk_0}{2\pi} e^{-ik_0\tau} \left(-iM^j \frac{d}{dk_0}\right)^n \Big\{ f(M^{-2\nu}k_0^2)[ik_0\mathbf{1} - e(\mathbf{k})\sigma^3 - \triangle\sigma^1]^{-1}$$

$$\prod_{\alpha=1}^{s} (\mathbf{r}T_\alpha)(k)[ik_0 - e(\mathbf{k})\sigma^3 - \triangle\sigma^1]^{-1} \prod_{\beta=s+1}^{s+t} (\boldsymbol{\ell}T_\beta)[ik_0\mathbf{1} - e(\mathbf{k})\sigma^3 - \triangle\sigma^1]^{-1} \Big\} \qquad (II.9)_\nu$$

is M^ν. We suppress unimportant constants. The matrix norm of each $[ik_0\mathbf{1} - e(\mathbf{k})\sigma^3 - \triangle\sigma^1]^{-1}$ is bounded by $M^{-\nu}$. For α with $h_\alpha > \nu$ we apply Taylor's Theorem to obtain

$$\begin{aligned} |\mathbf{r}T_\alpha(k)| &\le [|k_0| + |k - k'|] \sup |\nabla_k T_\alpha| \\ &\le [M^\nu + M^j] M^{-h_\alpha}\omega_\alpha \\ &\le M^{-(h_\alpha - \nu)}\omega_\alpha \end{aligned}$$

on the support of $\prod_{i=1}^{s+t+1} f(M^{-2j_i}E(\mathbf{k})^2) \le \chi(E(\mathbf{k}) \le \text{const}M^j)$. On the other hand, for α with $h_\alpha \le \nu$, and when a $\frac{d}{dk_0}$ acts on T_α, we simply ignore the effect of renormalization and bound

$$|\mathbf{r}T_\alpha(k)| \le |T_\alpha(k)| + |\boldsymbol{\ell}T_\alpha| \le 2\omega_\alpha.$$

Thus, no more than two derivatives ever act on T_α. Each derivative $M^j\frac{d}{dk_0}$ produces an $M^{-(\nu-j)}\le 1$ when it acts on $f(M^{-2\nu}k_0^2)$ or on a covariance, or $M^{-(h_\alpha-j)}\ge M^{-(h_\alpha-\nu)}$ when it acts on T_α. Thus, for $\nu\ge\eta$

$$\begin{aligned}|(II.9)_\nu| &\le M^\nu (M^{-\nu})^{s+t+1}\prod_{h_\alpha>\nu}M^\nu\prod_{h_\alpha\le\nu}M^{h_\alpha}\prod_\alpha M^{-h_\alpha}\omega_\alpha\prod_\beta(\boldsymbol{\ell}T_\beta)\\ &\le M^{-t(\nu-j)}\prod_{h_\alpha\le\nu}M^{-(\nu-h_\alpha)}\prod_\alpha M^{-h_\alpha}\omega_\alpha\prod_\beta(|\boldsymbol{\ell}T_\beta|M^{-j}).\end{aligned}$$

The sum

$$\sum_{\nu\ge\eta}\prod_{h_\alpha\le\nu}M^{-(\nu-h_\alpha)}\le\sum_{\nu\ge\eta}M^{-(\nu-\eta)}\le \text{const}$$

so

$$\sum_{\nu\ge\eta}|(II.9)_\nu|\le\prod_\alpha M^{-h_\alpha}\omega_\alpha\prod_\beta(|\boldsymbol{\ell}T_\beta|M^{-j}) \tag{II.10}$$

on the support of $\chi(E(\mathbf{k})\le \text{const}M^j)$.

This leaves the $\rho(M^{-2\eta}k_0^2)$ contribution. To handle it, write

$$\begin{aligned}&\prod_{\alpha=1}^{s}[(\mathbf{r}T_\alpha)(k)\mathbf{C}_\triangle(k)]\\ &=\prod_{\alpha=1}^{s}[T_\alpha(k_0,\mathbf{k})-T_\alpha(0,k')]\mathbf{C}_\triangle(k)\\ &=\prod_{\alpha=1}^{s}\{[T_\alpha(k_0,k')-T_\alpha(0,k')]+[T_\alpha(k_0,\mathbf{k})-T_\alpha(k_0,k')]\}\,\mathbf{C}_\triangle(k)\\ &=\sum_{A\subset\{1,\dots,s\}}\mathcal{O}\prod_{\alpha\in A}[ik_0\tau_\alpha(k_0)\mathbf{C}_\triangle(k)]\prod_{\alpha\notin A}[T_\alpha(k_0,\mathbf{k})-T_\alpha(k_0,k')]\mathbf{C}_\triangle(k)\end{aligned}$$

where the symbol $\mathcal{O}$ places the product over α in the correct order and

$$\tau_\alpha(k_0)=\frac{1}{ik_0}[T_\alpha(k_0,k')-T_\alpha(0,k')].$$

Anytime an $M^j\frac{d}{dk_0}$ acts on an $(\mathbf{r}T_\alpha)$ the corresponding α is also put in A^C. Further, write the term with $A=\{1,\dots,s\}$

$$\begin{aligned}&\prod_{\alpha=1}^{s}[ik_0\tau_\alpha(k_0)\mathbf{C}_\triangle(k)]\\ &=\prod_{\alpha=1}^{s}[\tau_\alpha(k_0)+\tau_\alpha(k_0)(e(\mathbf{k})\sigma^3+\triangle\sigma^1)\mathbf{C}_\triangle(k)]\\ &=\sum_{B\subset\{1,\dots,s\}}\mathcal{O}\prod_{\alpha\in B}\tau_\alpha(k_0)\prod_{\alpha\notin B}[\tau_\alpha(k_0)(e(\mathbf{k})\sigma^3+\triangle\sigma^1)\mathbf{C}_\triangle(k)]\end{aligned}$$

All contributions with $t \neq 0$ or $A \neq \{1,...,s\}$ or $B \neq \{1,...,s\}$ are treated as follows. The k_0 domain of integration, which is now determined by $\rho(M^{-2\eta}k_0^2)$, is refined using

$$\rho(M^{-2\eta}k_0^2) = \rho(M^{-2j}k_0^2) + \sum_{\nu=j}^{\eta-1} f(M^{-2\nu}k_0^2).$$

We shall refer to the $\rho(M^{-2j}k_0^2)$ term as having $\nu = j-1$. Fix any k_0 scale ν with $j-1 \leq \nu \leq \eta-1$. This contribution to $(II.9)_\nu$ (for $\nu = j-1$, replace $f(M^{-2\nu}k_0^2)$ by $\rho(M^{-2j}k_0^2)$ in $(II.9)_\nu$) has volume of integration M^ν. On the support of $\prod_{i=1}^{s+t+1} f(M^{-2j_i}E(\mathbf{k})^2)$ we have

$$\begin{aligned}
|\mathbf{C}_\triangle(k)| &\leq M^{-\nu} \\
|[T_\alpha(k_0,\mathbf{k}) - T_\alpha(k_0,k')]\mathbf{C}_\triangle(k)| &\leq |\mathbf{k}-k'| \sup|\nabla_k T_\alpha| M^{-\nu} \\
&\leq M^{-(\nu-j)}M^{-h_\alpha}\omega_\alpha \\
|\mathbf{C}_\triangle(k)\left(M^j\frac{d}{dk_0}\right)^m (\mathbf{r}T_\alpha)(k)| &\leq M^{-(h_\alpha-j)m}\omega_\alpha M^{-\nu}, \quad m=1,2 \\
&\leq M^{-(\nu-j)}M^{-h_\alpha}\omega_\alpha \\
|\tau_\alpha(k_0)| &\leq M^{-h_\alpha}\omega_\alpha \\
|\tau_\alpha(k_0)(e(\mathbf{k})\sigma^3 + \triangle\sigma^1)\mathbf{C}_\triangle(k)| &\leq M^{-(\nu-j)}M^{-h_\alpha}\omega_\alpha \\
|(\boldsymbol{\ell}T_\beta)\mathbf{C}_\triangle(k)| &\leq M^{-\nu}|\boldsymbol{\ell}T_\beta|.
\end{aligned}$$

Hence, the contribution to $(II.9)_\nu$, $j-1 \leq \nu \leq \eta-1$, corresponding to any given A, B is at most

$$M^{-(\nu-j)t}\prod_{\alpha\notin A} M^{-(\nu-j)}\prod_{\alpha\notin B} M^{-(\nu-j)}\prod_{\alpha=1}^{s} M^{-h_\alpha}\omega_\alpha \prod_{\beta=s+1}^{s+t} M^{-j}|\boldsymbol{\ell}T_\beta|.$$

By assumption either $t > 0$ or A^C is nonempty or B^C is nonempty so that, on the support of $\chi(E(\mathbf{k}) \leq \text{const}M^j)$,

$$\begin{aligned}
&\sum_{j-1\leq\nu\leq\eta-1} |(II.9)_\nu \text{ excluding the } A=\{1,...,s\}, B=\{1,...,s\} \text{ contribution when } t=0| \\
&\leq \prod_{\alpha=1}^{s} M^{-h_\alpha}\omega_\alpha \prod_{\beta=s+1}^{s+t} M^{-j}|\boldsymbol{\ell}T_\beta| \sum_{j-1\leq\nu\leq\eta-1}\sum_{A,B} M^{-(\nu-j)t}\prod_{\alpha\notin A} M^{-(\nu-j)t}\prod_{\alpha\notin B} M^{-(\nu-j)t} \\
&\leq \prod_{\alpha=1}^{s} M^{-h_\alpha}\omega_\alpha \prod_{\beta=s+1}^{s+t} M^{-j}|\boldsymbol{\ell}T_\beta| \sum_{j-1\leq\nu} M^{-(\nu-j)} \\
&\leq \prod_{\alpha=1}^{s} M^{-h_\alpha}\omega_\alpha \prod_{\beta=s+1}^{s+t} M^{-j}|\boldsymbol{\ell}T_\beta|.
\end{aligned} \tag{II.11}$$

Recall that we suppress unimportant constants.

One term remains, namely

$$\int \frac{dk_0}{(2\pi)} e^{-ik_0\tau}\left(-iM^j\frac{d}{dk_0}\right)^n \left\{\rho(M^{-2\eta}k_0^2)[ik_0\mathbf{1}-e(\mathbf{k})\sigma^3-\triangle\sigma^1]^{-1}\prod_{\alpha=1}^{s}\tau_\alpha(k_0)\right\}.$$

Let

$$\tau(k_0)=\prod_{\alpha=1}^{s}\tau_\alpha(k_0).$$

Then, by hypothesis

$$|\tau(k_0)|\le\prod_\alpha M^{-h_\alpha}\omega_\alpha \qquad (II.12a)$$

$$|\tau(k_0)-\tau(0)|\le s|k_0|M^{-\eta}\prod_\alpha M^{-h_\alpha}\omega_\alpha. \qquad (II.12b)$$

Consequently,

$$\left|\int\frac{dk_0}{2\pi}e^{-ik_0\tau}\left(-iM^j\frac{d}{dk_0}\right)^n\{\rho(M^{-2\eta}k_0^2)[ik_0-e(\mathbf{k})\sigma^3-\triangle\sigma^1]^{-1}[\tau(k_0)-\tau(0)]\}\right|$$

$$\le M^\eta M^{-\eta}\prod_\alpha M^{-h_\alpha}\omega_\alpha \qquad (II.13)$$

as desired. We are left with

$$\tau(0)(M^j\tau)^n\int\frac{dk_0}{2\pi}e^{-ik_0\tau}[ik_0-e(\mathbf{k})\sigma^3-\triangle\sigma^1]^{-1}\rho(M^{-2\eta}k_0^2) \qquad (II.14)$$

Combining part a) ($m=n=0$) of the technical lemma below with (II.12a) bounds (II.14).

The proof of the intermediate estimate (II.7) is concluded by adding (II.10), (II.11), (II.13) and (II.14) with $n=0,2$.

■

The following lemma is used to estimate the effect of renormalization for two legged subdiagrams as above and for four legged subdiagrams in section IV.

Technical Lemma II.3 Let x be a 2×2 hermitian matrix with eigenvalues obeying const$M^j\le|x_i|\le$ constM^j with $j\le\eta$. Then, for any integers $n,m,N\ge 0$

a) $\left|\int\frac{dk_0}{2\pi}e^{-ik_0\tau}[ik_0-x]^{-1-n}k_0^m\rho\left(M^{-2\eta}k_0^2\right)\right|$

$$\le \mathrm{const}^{n+m}[1+|M^j\tau|^N]^{-1}\begin{cases}M^{-j(n-m)} & \text{if } n\ge m\\ M^{\eta(m-n)} & \text{if } m\ge n\end{cases}$$

b) $\left|\int \frac{dk_0}{2\pi} e^{-ik_0\tau}[ik_0 - x]^{-1-n} k_0^m h(M^{-2\eta}k_0^2)\right|$

$$\leq \text{const}^n [1 + |M^\eta \tau|^N]^{-1} M^{-j(n-m)} \quad \text{for} \quad n \geq m$$

<u>Proof</u> a) We choose a basis for $\mathbf{C}^2$ in which x is diagonal and first consider $m = 0, n \geq 0$. Setting

$$B(x,\tau) = \int \frac{dk_0}{2\pi} e^{-ik_0\tau}[ik_0 - x]^{-1-n} \rho(M^{-2\eta}k_0^2)$$
$$b(k_0) = [ik_0 - x]^{-1-n}$$
$$\sigma(k_0) = \rho(k_0^2)$$

we have

$$B(x,\tau) = \hat{b} * [\sigma(M^{-\eta}\cdot)]\hat{}.$$

Since

$$\int \frac{dk_0}{2\pi} e^{-ik_0\tau} \frac{1}{(ik_0 - x_i)^{-1-n}} = \begin{cases} \frac{(-\tau)^n}{n!} e^{-x_i\tau} & x_i\tau \geq 0 \\ 0 & x_i\tau < 0 \end{cases} \tag{II.15}$$

it follows that

$$|\hat{b}(\tau)| \leq \text{const}^n M^{-jn} e^{-\text{const} M^j |\tau|}$$

On the other hand

$$[\sigma(M^{-\eta}\cdot)]\hat{}(\tau) = M^\eta \hat{\sigma}(M^\eta \tau).$$

Consequently, applying

$$\begin{aligned} |\tau^{N'} B(x,\tau)| &= \left| \int d\tau' \tau^{N'} \hat{b}(\tau - \tau') M^\eta \hat{\sigma}(M^\eta \tau') \right| \\ &\leq \text{const} \int d\tau' [|\tau - \tau'|^{N'} + |\tau'|^{N}] |\hat{b}(\tau - \tau')| M^\eta |\hat{\sigma}(M^\eta \tau')| \\ &\leq \text{const}^n M^{-jn} \int d\tau' [M^{-jN'} + |\tau'|^{N'}] M^\eta |\hat{\sigma}(M^\eta \tau')| \\ &\leq \text{const}^n M^{-jn} [M^{-jN'} + M^{-\eta N'}] \\ &\leq \text{const}^n M^{-jn} M^{-jN'} \end{aligned}$$

with $N' = 0, N$ yields the desired bound.

We now consider $m \geq 0$. Since

$$(ik_0)^m = \sum_{\ell=0}^{m} \binom{m}{\ell} (ik_0 - x_i)^\ell x_i^{m-\ell}$$

it suffices to bound

$$B_\ell = \int \frac{dk_0}{(2\pi)} e^{-ik_0\tau}[ik_0 - x_i]^{-1-n+\ell} x_i^{m-\ell} \rho(M^{-2\eta}k_0^2).$$

When $\ell \leq n$ then, by the above bound,

$$\begin{aligned} |B_\ell| &\leq \mathrm{const}^n [1 + |M^j\tau|^N]^{-1} M^{-j(n-\ell)} M^{j(m-\ell)} \\ &= \mathrm{const}^n [1 + |M^j\tau|^N]^{-1} M^{-j(n-m)} \end{aligned} \tag{II.16}$$

When $\ell > n$ we apply

$$\begin{aligned} &\int \frac{dk_0}{(2\pi)} e^{-ik_0\tau} (ik_0)^{\ell-n-1} \rho(M^{-2\eta}k_0^2) \\ &= \left(-\frac{d}{d\tau}\right)^{\ell-n-1} [M^\eta \hat{\sigma}(M^\eta\tau)] \end{aligned}$$

and get

$$\begin{aligned} |B_\ell| &\leq \mathrm{const}^\ell M^\eta [1 + |M^\eta\tau|^N]^{-1} M^{\eta(\ell-n-1)} M^{j(m-\ell)} \\ &\leq \mathrm{const}^\ell [1 + |M^j\tau|^N]^{-1} M^{\eta(m-n)} \end{aligned} \tag{II.17}$$

Part a) is now a consequence of (II.16) and (II.17)

b) As before we may assume that $x = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}$. We first treat $|M^\eta\tau| \leq 1$. Then

$$\begin{aligned} &\int \frac{dk_0}{2\pi} e^{-ik_0\tau}[ik_0 - x_i]^{-1-n} k_0^m h(M^{-2\eta}k_0^2) \\ &= \int \frac{dk_0}{2\pi} e^{-ik_0\tau} \frac{k_0^m}{(ik_0 - x_i)^{1+n}} - \int \frac{dk_0}{2\pi} e^{-ik_0\tau}[ik_0 - x_i]^{-1-n} k_0^m \rho(M^{-2\eta}k_0^2). \end{aligned}$$

Differentiating (II.15), the first integral

$$\begin{aligned} \left|\int \frac{dk_0}{2\pi} e^{-ik_0\tau} \frac{k_0^m}{(ik_0 - x_i)^{1+n}}\right| &\leq \left|\frac{1}{n!}\frac{d^m}{d\tau^m}\left[\tau^n \begin{cases} e^{-x_i\tau} & x_i\tau \geq 0 \\ 0 & x_i\tau < 0 \end{cases}\right]\right| \\ &\leq \mathrm{const}^n \max_{0\leq\ell\leq m} M^{-\eta(n-\ell)} M^{j(m-\ell)} \\ &\leq \mathrm{const}^n M^{-j(n-m)} \end{aligned}$$

satisfies b) in this regime. The second integral is accounted for by a).

For $|M^\eta\tau| \geq 1$

$$\begin{aligned} &|M^\eta\tau|^N \left|\int \frac{dk_0}{2\pi} e^{-ik_0\tau}[ik_0 - x_i]^{-1-n} k_0^m h(M^{-2\eta}k_0^2)\right| \\ &= \left|\int \frac{dk_0}{2\pi} e^{-ik_0\tau} \left(M^\eta \frac{d}{dk_0}\right)^N \{[ik_0 - x_i]^{-1-n} k_0^m h(M^{-2\eta}k_0^2)\}\right| \end{aligned}$$

$$\leq \text{const}^n \max_{\substack{\alpha,\beta,\gamma\geq 0\\ \alpha+\beta+\gamma=N\\ \beta\leq m}} \left| \int \frac{dk_0}{2\pi} e^{-ik_0\tau} M^{\eta N}[ik_0 - x_i]^{-1-n-\alpha} k_0^{m-\beta} \begin{cases} h(M^{-2\eta}k_0^2) & \text{if } \gamma = 0 \\ M^{-\gamma\eta}\sigma^{(\gamma)}(M^{-\eta}k_0) & \text{if } \gamma > 0 \end{cases} \right|$$

$$\leq \text{const}^n \max_{\alpha,\beta,\gamma} \left| \int \frac{dk_0}{2\pi} M^{\eta N} |k_0|^{-1-(n-m)-(\alpha+\beta)} \begin{cases} h(M^{-2\eta}k_0^2) & \text{if } \gamma = 0 \\ M^{-\gamma\eta}\sigma^{(\gamma)}(M^{-\eta}k_0) & \text{if } \gamma > 0 \end{cases} \right|$$

$$\leq \text{const}^n \max_{\alpha,\beta,\gamma} \begin{cases} M^{\eta N} M^{-\eta[(n-m)+N]} & \text{if } \gamma = 0 \\ M^{\eta N} M^{\eta[-1-(n-m)-(\alpha+\beta)]} M^{-\gamma\eta} M^{\eta} & \text{if } \gamma > 0 \end{cases}$$

$$= \text{const}^n M^{-\eta(n-m)}.$$

The proof of the technical lemma is now complete.

■

In the next section a small generalization of Lemma II.2 is required. Let

$$\mathbf{C}^{(j)} = \mathbf{C}_1^{(j)} + \mathbf{C}_2^{(j)} \tag{II.18a}$$

$$\mathbf{C}_1^{(j)}(k) = -\frac{ik_0\mathbf{1} + e(\mathbf{k})\sigma^3}{k_0^2 + E(\mathbf{k})^2} f(M^{-2j}E(\mathbf{k})^2) \tag{II.18b}$$

$$\mathbf{C}_2^{(j)}(k) = -\frac{\Delta\sigma^1}{k_0^2 + E(\mathbf{k})^2} f(M^{-2j}E(\mathbf{k})^2) \tag{II.18c}$$

Lemma II.2′ Let $S(\xi)$ be a generalization of the string (II.6) in the sense that each particle line may be assigned any of the covariances (II.18a,b,c) and the ℓT_α, $s+1 \leq \alpha \leq s+t$, are replaced by general kernels $T_\alpha(k)$. Suppose that there is a j with each $j_\alpha = j$ or $j+1$. Let

$$\|T_\alpha(k)\|_{h_\alpha} := \max\left\{ M^{(\gamma+\delta)h_\alpha} \sup_{\mathbf{k}} \int d\tau |\tau|^\gamma |\nabla_{\mathbf{k}}^\delta T_\alpha(\tau,\mathbf{k})| : 0 \leq \gamma \leq 2, |\delta| \leq 1 \right\} \leq \omega_\alpha$$

with $h_\alpha \geq j$ for $1 \leq \alpha \leq s$ and $h_\alpha = j$ for $s+1 \leq \alpha \leq s+t$. Then

$$\sup_{\substack{0\leq\gamma\leq 2\\ |\delta|\leq 1}} \left| M^{j(\gamma+|\delta|)} |\tau|^\gamma \nabla_{\mathbf{k}}^\delta S \right| \leq \text{const}^{s+t} M^{-j} \prod_{\alpha=1}^{s+t} M^{-h_\alpha}\omega_\alpha \left(\frac{\Delta}{M^j}\right)^{\#(\mathbf{C}_2 \text{ covariances})}$$

$$\sup_{\substack{0\leq\gamma\leq 2\\ |\delta|\leq 1}} \left| M^{j(\gamma+|\delta|)} |\tau|^\gamma \nabla_{\mathbf{k}}^\delta S \right|' \leq \text{const}^{s+t} M^{j} \prod_{\alpha=1}^{s+t} M^{-h_\alpha}\omega_\alpha \left(\frac{\Delta}{M^j}\right)^{\#(\mathbf{C}_2 \text{ covariances})}$$

uniformly for $\Delta \geq 0$.

Proof We first consider the case $\gamma = \delta = 0$. Suppose all the covariances are of type (II.18a). Then, writing,

$$T_\alpha = \mathbf{r}T_\alpha + \ell T_\alpha \qquad s+1 \leq \alpha \leq s+t$$

we may apply Lemma II.2. Since $\mathbf{C}_1^{(j)} = \mathbf{C}^{(j)} - \mathbf{C}_2^{(j)}$ it suffices to consider strings whose covariances are of type (II.18a) and (II.18c). Now, one simply repeats the proof of Lemma II.2 using

$$|(M^j\nabla_k)^n \mathbf{C}_2^{(j)}(k)| \leq M^{-(\nu-j)}\frac{\triangle}{M^j}M^{-\nu}$$

$$|(M^j\nabla_k)^n k_0\tau_\alpha(k_0)\mathbf{C}_2^{(j)}(k)| \leq M^{-h_\alpha}\omega_\alpha M^{-(\nu-j)}\frac{\triangle}{M^j}$$

when k_0 is of scale $\nu \geq j-1$.

We now consider general γ, δ with $0 \leq \gamma \leq 2, |\delta| \leq 1$. Apply the "derivatives" $(M^j|\tau|)^\gamma$ and $(M^j\nabla_{\mathbf{k}})^\delta$ using the "product rule". Consider any term in the resulting sum. Use (II.5c,d) to separate those T_α's with derivatives acting on them from the rest of the string. Each factor

$$\left|(M^j|\tau|)^{\gamma'}(M^j\nabla_{\mathbf{k}})^{\delta'}T_\alpha\right| \leq M^{-(h_\alpha-j)}\omega_\alpha$$

The rest of the string is estimated as in the last paragraph to produce the required bound.

■

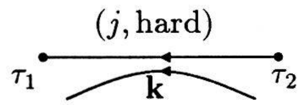
III. Graph Estimates

The purpose of this section is to extend the estimates of [FT] to superconducting systems and, more importantly, to show that the renormalization of four legged subdiagrams by $\mathbf{1} - \mathbf{L}^{(h)}$ eliminates factorials to give exponential bounds. For graphs without four legged subdiagrams we give a much simplified derivation, independent of [FT].

Our estimates will be made in the mixed (τ-time, $\mathbf{k}$=d-momemtum) representation. However, it is often notationally convenient to write expressions in the pure $(k_0, \mathbf{k})$ momentum representation. For this reason we will pass between them without further comment.

We begin by defining the pertinent class of labelled graphs. These graphs are assembled from two kinds of, scale dependent, particle lines and four kinds of vertices.

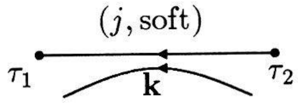
The particle lines are



$$= \mathbf{C}^{(j)}(\tau_1 - \tau_2, \mathbf{k})$$

$$= -e^{-E(\mathbf{k})|\tau_1-\tau_2|}\frac{(\mathrm{sgn}(\tau_1-\tau_2))E(\mathbf{k})\mathbf{1}+e(\mathbf{k})\sigma^3+\triangle\sigma^1}{2E(\mathbf{k})} \times \begin{cases} f(M^{-2j}E(\mathbf{k})^2), & j\le -1 \\ h(E(\mathbf{k})^2), & j=0 \end{cases} \qquad (III.1a)$$

and



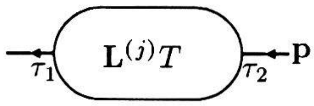
$$= \mathbf{C}_s^{(j)}(\tau_1-\tau_2,\mathbf{k})$$

$$:= \sum_{j'<j} \mathbf{C}^{(j')}(\tau_1-\tau_2,\mathbf{k})$$

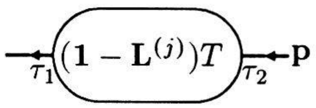
$$= -e^{-E(\mathbf{k})|\tau_1-\tau_2|}\frac{(\mathrm{sgn}(\tau_1-\tau_2))E(\mathbf{k})\mathbf{1}+e(\mathbf{k})\sigma^3+\triangle\sigma^1}{2E(\mathbf{k})} [1-h(M^{-2j}E(\mathbf{k})^2)] \qquad (III.1b)$$

We will explain below that soft particle lines of scale j implement Wick ordering at that scale. From now on we discard $j=0$, thus introducing an ultraviolet cutoff. The ultraviolet end is relatively simple and is treated in [FT§3].

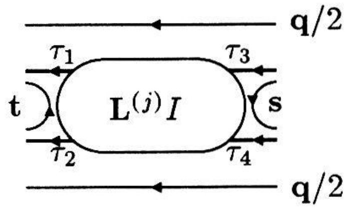
The vertices are



$$\begin{aligned} &= (\boldsymbol{\ell}T)(\tau_1-\tau_2,\mathbf{p}) \\ &= \int d\tau T\left(\tau,\frac{\mathbf{p}}{|\mathbf{p}|}k_F\right)\delta(\tau_1-\tau_2) \\ &= T(p')\delta(\tau_1-\tau_2) \end{aligned} \qquad (III.2a)$$

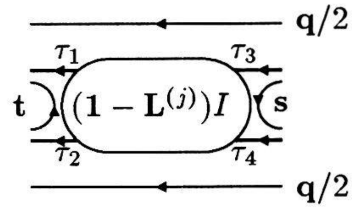


$$\begin{aligned} &= ((\mathbf{1}-\boldsymbol{\ell})T)(\tau_1-\tau_2,\mathbf{p}) \\ &= T(\tau_1-\tau_2,\mathbf{p})-T(\mathbf{p}')\delta(\tau_1-\tau_2) \end{aligned} \qquad (III.2b)$$



$(III.2c)$

and



$(III.2d)$

Here $I = \sum_{m,n=0}^{3} f_{m,n}\tau^m \otimes \tau^n$ satisfies conditions (I.90)-(I.92) and the localization operator, $\mathbf{L}^{(j)}$, is defined, in momentum space, by (I.99). (Note that the momenta $\mathbf{t}, \mathbf{s}$ and $\mathbf{q}$ of (III.2c,d) must not be confused with those of (I.99f,g).)

Let $L\Gamma_{2n}(m)$ be the set of labelled connected graphs constructed from m vertices (III.2) by joining all but $2n$ vertex legs with properly oriented particle lines (II.1). We impose the additional requirement that these graphs remain connected when all soft lines are cut. The value, in momentum space, of such a graph is

$$\widetilde{Val}(G)(2\pi)^{d+1}\delta(W_G) = \mathrm{sgn}(G)\int \prod_{\substack{\text{hard}\\ \text{lines}}} \frac{d^{d+1}k_\ell}{(2\pi)^{d+1}} \mathbf{C}^{j_\ell}(k_\ell) \prod_{\substack{\text{soft}\\ \text{lines}}} \frac{d^{d+1}k_\ell}{(2\pi)^{d+1}} \mathbf{C}_s^{(j_\ell)}(k_\ell)$$
$$\prod_{\substack{2-\text{legged}\\ \mathbf{L}-\text{vertices}}} \mathbf{L}^{(j_v)} T(p_v) \prod_{\substack{2-\text{legged}\\ (\mathbf{1}-\mathbf{L})-\text{vertices}}} (\mathbf{1} - \mathbf{L}^{(j_v)}) T(p_v)$$
$$\prod_{\substack{4-\text{legged}\\ \mathbf{L}-\text{vertices}}} \mathbf{L}^{(j_v)} I(t_v, s_v, q_v) \prod_{\substack{4-\text{legged}\\ (\mathbf{1}-\mathbf{L})-\text{vertices}}} (\mathbf{1} - \mathbf{L}^{(j_v)}) I(t_v, s_v, q_v)$$
$$\prod_{\text{vertices}} (2\pi)^{d+1}\delta(w_v). \qquad (III.3)$$

The momenta flowing in the $2n$ external legs of G have been suppressed. They are the arguments of the kernel $\widetilde{Val}(G)$. Here t_v, s_v, q_v and p_v are respectively the momenta flowing through the four and two-legged vertices as indicated in the diagrams of (III.2). The total momentum flowing into the vertex v is denoted w_v. The overall energy-momentum conserving delta function $(2\pi)^{d+1}\delta(w_G)$ has been explicitly extracted. The signature $\mathrm{sgn}(G)$ is the usual fermion factor and is computed by the recipe given on [FT page 164].

In Section I, (I.37)-(I.40) we formulated a set of rules defining the value of a labelled graph in momentum space. Definition (III.3) is a slight modification of those rules that incorporates Wick ordering. Specifically, all monomials $\prod \Psi^{(\le h)}(\xi_i)\prod \bar{\Psi}^{(\le h)}(\zeta_j)$ appearing in effective potentials at scale h (e.g. (I.24a)), as well as in the definitions of localization operators (e.g. (I.99)), are replaced by monomials $:\prod \Psi^{(\le h)}(\xi_i)\prod \bar{\Psi}^{(\le h)}(\zeta_j):$ that are Wick ordered with respect to $\mathbf{C}^{(\le h)}$. In particular, the value of every truncated expectation $\mathcal{E}^{(h)}$, see (I.17), is expressed as a sum of Wick ordered monomials.

We must, of course, Wick order the initial

$$
\begin{aligned}
\mathcal{W}^{(0)} = &-\frac{\lambda}{2}\int \frac{dt}{(2\pi)^{d+1}}\frac{ds}{(2\pi)^{d+1}}\frac{dq}{(2\pi)^{d+1}}\left(\bar{\Psi}_{t+\frac{q}{2}}\sigma^3\Psi_{s+\frac{q}{2}}\right)\\
&\langle t+\tfrac{q}{2},-t+\tfrac{q}{2}|V|s+\tfrac{q}{2},-s+\tfrac{q}{2}\rangle\left(\bar{\Psi}_{-t+\frac{q}{2}}\sigma^3\Psi_{-s+\frac{q}{2}}\right)\\
&+\delta\mu\int\frac{dk}{(2\pi)^{d+1}}\bar{\Psi}_k\sigma^3\Psi_k + D\int\frac{dk}{(2\pi)^{d+1}}\bar{\Psi}_k\sigma^1\Psi_k
\end{aligned}
$$

by hand. The result is

$$
\begin{aligned}
\mathcal{W}^{(0)} = &-\frac{\lambda}{2}\int \frac{dt}{(2\pi)^{d+1}}\frac{ds}{(2\pi)^{d+1}}\frac{dq}{(2\pi)^{d+1}}:\left(\bar{\Psi}_{t+\frac{q}{2}}\sigma^3\Psi_{s+\frac{q}{2}}\right)\\
&\langle t+\tfrac{q}{2},-t+\tfrac{q}{2}|V|s+\tfrac{q}{2},-s+\tfrac{q}{2}\rangle\left(\bar{\Psi}_{-t+\frac{q}{2}}\sigma^3\Psi_{-s+\frac{q}{2}}\right):\\
&+\lambda\int\frac{dk}{(2\pi)^{d+1}}\left\{\int\frac{dp}{(2\pi)^{d+1}}\langle p,k|V|p,k\rangle\mathbf{C}^{(0)}_{\triangle,s}(p)\right\}:\bar{\Psi}_k\sigma^3\Psi_k:\\
&-\lambda\int\frac{dk}{(2\pi)^{d+1}}:\bar{\Psi}_k\left\{\int\frac{dp}{(2\pi)^{d+1}}\langle k,p|V|p,k\rangle\sigma^3\mathbf{C}^{(0)}_{\triangle,s}(p)\sigma^3\right\}\Psi_k:\\
&+\delta\mu\int\frac{dk}{(2\pi)^{d+1}}:\bar{\Psi}_k\sigma^3\Psi_k: + D\int\frac{dk}{(2\pi)^{d+1}}:\bar{\Psi}_k\sigma^1\Psi_k:\\
&+\text{const}
\end{aligned}
$$

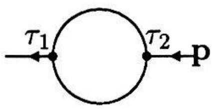
where the Wick dots are with respect to the ultraviolet cutoff covariance $\mathbf{C}^{(0)}_{\triangle,s}(p) = \rho(E(\mathbf{p})^2)\mathbf{C}_\triangle(p)$.

The introduction of Wick ordering has five important effects. First, localization operators become orthogonal projections (as indicated following (I.29)), since Wick monomials of different degree are, by construction, orthogonal with respect to $d\mu_{\mathbf{C}^{(\le h)}}$. Second, at the level of trees, there are no longer any two forks. Third, at the level of graphs, there are hard and soft lines. Fourth, the condition, in (I.39), that each subgraph

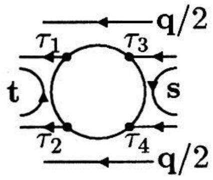
$$G_f^I = \{\text{lines } \ell \in G \text{ and connecting vertices} \mid I(\ell) = i_{f'}, f' \geq f\}$$

be connected is transformed into the condition that it be connected by hard lines. Finally the quotient graphs $G_f^I / \{G_{f'}^I | f' > f\}$ do not contain tadpoles. For a more extended discussion see [FT 199-201].

As a warm up exercise we derive bounds on unrenormalized labelled graphs. Consider a general connected labelled graph G^J, not necessarily arising from our model. It contains two and four legged vertices



$$= T(\tau_2 - \tau_1, \mathbf{p})$$



$$= I(\tau_1, \tau_2, \tau_3, \tau_4, \mathbf{t}, \mathbf{s}, \mathbf{q})$$

with kernels that are translation invariant in the time components and obey the bounds

$$|T|, |I| < \infty$$

where

$$|T| := \sup_{\mathbf{p}} \int d\tau |T(\tau, \mathbf{p})| \tag{III.4a}$$

as in (II.5a) and

$$|I| := \sup_{\mathbf{t},\mathbf{s},\mathbf{q}} \int d\tau_2 d\tau_3 d\tau_4 |I(\tau_1, \tau_2, \tau_3, \tau_4, \mathbf{t}, \mathbf{s}, \mathbf{q})| \tag{III.4b}$$

External lines of G^J are amputated, that is, discarded. Each internal line ℓ carries a hard/soft label, a dimension δ_ℓ, a scale $j_\ell \leq -1$ and a covariance $C_\ell(k)$, which when hard satisfies

$$\begin{aligned} \left|C_\ell\right| &\leq a_\ell M^{(\delta_\ell-2)j_\ell} \\ \left|C_\ell\right|' &\leq a_\ell M^{\delta_\ell j_\ell} \end{aligned} \qquad (III.5\ Hard)$$

(see (II.5b)) and when soft satisfies

$$\left|C_\ell\right|' \leq a_\ell M^{\delta_\ell j_\ell}. \qquad (III.5\ Soft)$$

Let us define, in the mixed representation

$$\begin{aligned} \mathrm{Val}(G^J)(2\pi)^d\delta(\mathbf{w}_{G^J}) = \mathrm{sgn}(G^J)\int & \prod_{\substack{\text{internal}\\ \text{times}}} d\tau_i \prod_\ell \frac{d^d\mathbf{k}_\ell}{(2\pi)^d} C_\ell(\tau_{1,\ell},\tau_{2,\ell},\mathbf{k}_\ell) \\ & \prod_{\substack{2-\text{legged}\\ \text{vertices}}} T(\tau_{1,v},\tau_{2,v},\mathbf{p}_v) \\ & \prod_{\substack{4-\text{legged}\\ \text{vertices}}} I(\tau_{1,v},\tau_{2,v},\tau_{3,v},\tau_{4,v},\mathbf{t}_v,\mathbf{s}_v,\mathbf{q}_v) \\ & \prod_{\text{vertices}} (2\pi)^d\delta(\mathbf{w}_v) \end{aligned} \qquad (III.6)$$

where $\tau_{1,\ell}$, $\tau_{2,\ell}$ and $\mathbf{k}_\ell$ are the temporal and d-momentum arguments of the line ℓ, $\tau_{i,v},\mathbf{p}_v,\mathbf{t}_v,\mathbf{s}_v$ and $\mathbf{q}_v$ are the temporal and d-momentum arguments of the vertex v and $\mathbf{w}_v$ is the total d-momentum entering the vertex v.

We associate to each general graph G^J a tree $t(G^J)$. The forks f of this tree are the connected components G^J_f of all the subgraphs

$$\{\ell \in G^J | j_\ell \geq h\}, \quad h \leq -1.$$

We only consider labelled graphs G^J for which each G^J_f is connected by hard lines.

The subgraphs are partially ordered by inclusion to form $t(G^J)$. As usual $\pi(f)$ denotes the predecessor fork of f. The scale of a fork is defined by

$$j_f = \min\{j_\ell | \ell \in G^J_f\} \qquad (III.7)$$

and obeys $j_f > j_{\pi(f)}$, that is the scale of forks strictly increases as you move up the tree.

Let $u = u((\tau_1, \mathbf{k}_1), ..., (\tau_{2n}, \mathbf{k}_{2n}))$ be a general $(\mathbf{C}^2)^{\otimes 2p}$ tensor valued kernel. Define the norms

$$|u| := \sup_{\mathbf{k}_1,...,\mathbf{k}_{2n}} \int d\tau_2...d\tau_{2n} |u((0,\mathbf{k}_1),...,(\tau_{2n},\mathbf{k}_{2n}))| \tag{III.8a}$$

$$|u|' := \int d\mathbf{k}_2,...,d\mathbf{k}_{2n} \sup_{\tau_1,...,\tau_{2n}} |u((\tau_1, -\mathbf{k}_2 - ... - \cdots - \mathbf{k}_{2n}),(\tau_2,\mathbf{k}_2),\cdots,(\tau_{2n},\mathbf{k}_{2n}))| \tag{III.8b}$$

Here $|u|$ is the tensor norm. The first norm will be estimated in terms of degrees

$$D_f = \sum_{\ell \in G_f^J} \delta_\ell - 2(\#\{\text{vertices of } G_f^J\} - 1) \tag{III.9}$$

of the subgraphs G_f^J. The second norm will be estimated in terms of the degrees

$$\begin{aligned} \triangle_f &= -\frac{1}{2} \sum_{\substack{\text{external lines} \\ \ell \text{ of } G_f^J}} \delta_\ell \\ \triangle_v &= -\frac{1}{2} \sum_{\substack{\text{external lines} \\ \text{of } v}} \delta_\ell \end{aligned} \tag{III.10}$$

of the subgraphs G_f^J and vertices v of G^J. A line ℓ of G^J is an external line of G_f^J or v if it is hooked to G_f^J or v but, in the former case, is not a line of G_f^J. (See the example on [FT p. 202].)

Abstract Power Counting Lemma III.1

Let $G = G^J$ be a general labelled graph such that each fork G_f of the associated tree $t(G)$ is connected by hard lines. Recalling definition (III.6)

a)

$$|\mathrm{Val}(G)| \le \prod_\ell a_\ell \prod_v |T_v| \prod_v |I_v| M^{D_\phi j_\phi} \prod_{\substack{f \in t(G) \\ f > \phi}} M^{D_f (j_f - j_{\pi(f)})}$$

Here ϕ is the lowest fork of $t(G)$, j_f is given by (III.7) and D_f by (III.9).

b) Assume, in addition, that each internal vertex is dimensionless in the sense that

$$\frac{1}{2} \sum_{\ell \text{ hooked to } v} \delta_\ell = 2. \tag{III.11}$$

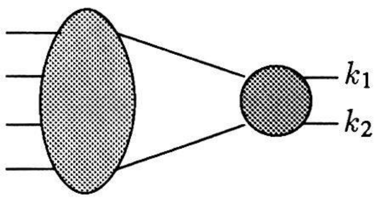
By convention, there is one external momentum for each external vertex (rather than line) of G. Then

$$|\mathrm{Val}(G)|' \le \prod_v |T_v| \prod_v |I_v| \prod_\ell a_\ell \prod_{\substack{f > \phi \\ G_f \text{ contains no external} \\ \text{verticesof } G}} M^{D_f (j_f - j_{\pi(f)})}$$

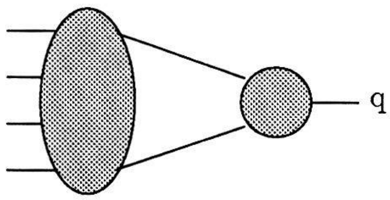
$$\prod_{\substack{f>\phi \\ G_f \text{ contains an external} \\ \text{vertex of } G}} M^{\triangle_f (j_f - j_{\pi(f)})}$$
$$\prod_{\substack{\text{external} \\ \text{vertices of } G}} M^{\triangle_v (0 - j_{\pi(v)})}$$

Here $\pi(v)$ is highest fork f of $t(G)$ for which v is a vertex of G_f and $\triangle_f, \triangle_v$ are defined in (III.10).

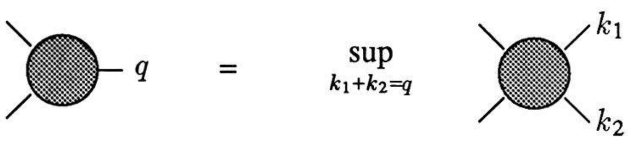
The convention that there is one external momentum per external vertex means, for example, that



is viewed as



with



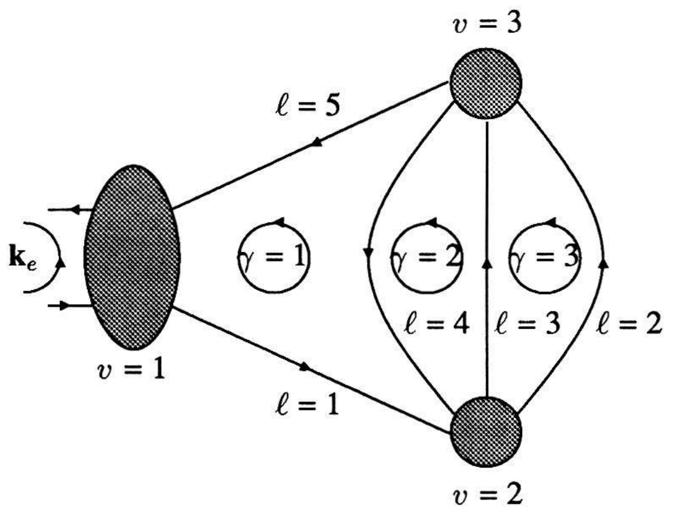
We remark that the power counting dimension 2 (independent of the physical dimension d) in (III.9) and (III.11) is motivated by the covariance estimates of Lemma II.1. (It appears, for example, as $\frac{1}{j}$ (the difference between the exponents appearing the estimates of $|C^{(j)}|'$ and $|C^{(j)}|$).

As usual, the momentum conserving delta functions in (II.26) are eliminated by selecting a basis of d-momentum loops for G. We have (suppressing the temporal arguments)

$$\mathrm{Val}(G^J) = \mathrm{sgn}(G^J) \int \prod_{\substack{\text{internal}\\ \text{times}}} d\tau_i \prod_{\substack{\text{momentum}\\ \text{loops}\gamma}} \frac{d^d \mathbf{r}_\gamma}{(2\pi)^d}$$

$$\prod_{\substack{\text{lines}\\ \ell}} C_\ell(\mathbf{k}_\ell) \prod_{\substack{2-\text{legged}\\ \text{vertices}}} T_v(\mathbf{p}_v) \prod_{\substack{4-\text{legged}\\ \text{vertices}}} I_v\left(\mathbf{t}_v, \mathbf{s}_v, \mathbf{q}_v\right) \tag{III.12}$$

where $\mathbf{k}_\ell$ is the signed sum of all loop and external d-momenta flowing through ℓ. The momenta $\mathbf{p}_v, \mathbf{t}_v, \mathbf{s}_v$ and $\mathbf{q}_v$ are similarly expressed in terms of loop and external momenta. For example



$\mathbf{k}_e =$ external momentum

$\mathbf{k}_1 = \mathbf{k}_5 = \mathbf{r}_1$

$\mathbf{k}_2 = \mathbf{r}_3$

$\mathbf{k}_3 = \mathbf{r}_2 - \mathbf{r}_3$

$\mathbf{k}_4 = \mathbf{r}_2 - \mathbf{r}_1$

$\mathbf{t}_1 = -\mathbf{s}_1 = \frac{1}{2}(\mathbf{k}_e - \mathbf{r}_1)\ , \mathbf{q}_1 = \mathbf{k}_e + \mathbf{r}_1$

$\mathbf{t}_2 = \frac{1}{2}\mathbf{r}_2 - \mathbf{r}_3\ ,\ \mathbf{s}_2 = \frac{1}{2}\mathbf{r}_2 - \mathbf{r}_1\ ,\ \mathbf{q}_2 = \mathbf{r}_2$

$\mathbf{t}_3 = \mathbf{r}_1 - \frac{1}{2}\mathbf{r}_2\ ,\ \mathbf{s}_3 = \frac{1}{2}\mathbf{r}_2 - \mathbf{r}_3\ ,\ \mathbf{q}_2 = \mathbf{r}_2$

To construct a basis of d-momentum loops of G^J it is helpful to introduce the notion of a spanning tree of lines of G^J. This tree is

<u>*not to be confused with the tree*</u> $t(G^J)$ <u>*ofsubgraphsof*</u> G^J

. A spanning tree T is a connected subgraph of G^J without loops that contains all of the vertices of G^J. To each line $\ell \in G^J \backslash T$ corresponds a unique loop γ_ℓ consisting of ℓ and the linear subtree of T joining the vertices at the ends of ℓ. Obviously, the loops $\gamma_\ell, \ell \in G^J \backslash T$, are independent since ℓ belongs to γ_ℓ and none of the other loops. They are in fact a basis since, for any connected graph

$$\begin{aligned}\#\text{ independent internal loops} &= \#\text{ lines} - \#\text{ vertices} + 1\\ &= L(G^J) - L(T).\end{aligned} \tag{III.13}$$

To prepare for the proof of Lemma III.1 we first construct a special spanning tree in the

Technical Lemma III.2 There exists a spanning tree T for G^J such that

(1) Every line $\ell \in T$ is hard.

(2) $T \cap G^J_f$ is connected for all $f \in t(G^J)$.

(3) $\sum_{f\in t(G^J)}(p_f - 1) = V(G^J) - 1$, where p_f is the number of upward branches leaving f.

(4) $L\left(T \cap \left[G^J_f \backslash \cup_{f'>f} G^J_{f'}\right]\right) = p_f - 1$.

Here $L(G)$ and $V(G)$ are the number of lines and vertices of G respectively .

Proof Construct T inductively working through the forks $f \in t(G^J)$ from high to low scale and using the hypothesis that each G^J_f is connected by hard lines. Properties (3) and (4) are general properties of trees.

■

Proof of Lemma III.1 a) Let T be the spanning tree of Lemma III.2 and, for $\ell \in G\backslash T, \gamma_\ell$ the d-momentum loop associated to ℓ. Then

$$\left|\mathrm{Val}G\right| \leq \sup_{\substack{\text{external}\\ \text{momenta}}} \int \prod_{\substack{\text{all internal and}\\ \text{external times}\\ \text{save one}}} d\tau_i \left| \int \prod_{\ell\in G\backslash T} \left(\frac{d^d\mathbf{r}_{\gamma_\ell}}{(2\pi)^d} C_\ell(\tau_{\ell,2} - \tau_{\ell,1}, \mathbf{k}_\ell)\right) \right.$$

$$\left. \prod_{\ell\in T} C_\ell(\tau_{\ell,2} - \tau_{\ell,1}, \mathbf{k}_\ell) \prod_v T_v \prod_v I_v \right|$$

$$\leq \sup_{\substack{\text{external}\\ \text{momenta}}} \int \prod_{\ell\in G\backslash T} \frac{d^d\mathbf{r}_{\gamma_\ell}}{(2\pi)^d} \prod_{\ell\in G\backslash T} \sup_\tau |C_\ell(\tau, \mathbf{k}_\ell)|$$

$$\sup_{\mathbf{k}_\ell} \int \prod d\tau_i \prod_{\ell\in T} |C_\ell(\tau_{\ell,2} - \tau_{\ell,1}, \mathbf{k}_\ell)| \prod_v |T_v| \prod_v |I_v|$$

$$\leq \prod_{\ell\in T} |C_\ell| \prod_v |T_v| \prod_v |I_v| \sup_{\substack{\text{external}\\ \text{momenta}}} \int \prod_{\ell\in G\backslash T} \left(\frac{d^d\mathbf{r}_{\gamma_\ell}}{(2\pi)^d} \sup_\tau |C_\ell(\tau, \mathbf{k}_\ell)|\right)$$

$$= \prod_{\ell\in T} |C_\ell| \prod_v |T_v| \prod_v |I_v| \prod_{\ell\in G\backslash T} |C_\ell|'$$

$$\leq \prod_v |T_v| \prod_v |I_v| \prod_{\ell\in G\backslash T} a_\ell M^{\delta_\ell j_\ell} \prod_{\ell\in T} M^{(\delta_\ell - 2)j_\ell}$$

by (III.5). In the step between the second and third lines we perform the time integrals starting at the extremities farthest from the single nonintegrated time. The step between the second and third lines uses the change of variables

$$\mathbf{r}_{\gamma_\ell} = \mathbf{k}_\ell + \text{ external momenta.}$$

It is justifed by the fact that γ_ℓ is the only internal d-momentum loop containing ℓ.

Since,

$$M^{\alpha j_\ell} = M^{\alpha j_\phi} \prod_{\substack{f \in t(G) \\ f > \phi \\ \ell \in G_f}} M^{\alpha(j_f - j_{\pi(f)})}$$

we have

$$\begin{aligned} \prod_{\ell \in G} M^{\delta_\ell j_\ell} \prod_{\ell \in T} M^{-2 j_\ell} &= M^{j_\phi [\sum_{\ell \in G} \delta_\ell - 2L(T)]} \\ &\quad \prod_{\substack{f \in t(G) \\ f > \phi}} M^{(j_f - j_{\pi(f)}) \left[\sum_{\ell \in G_f} \delta_\ell - 2L(G_f \cap T)\right]} \\ &= M^{D_\phi j_\phi} \prod_{f > \phi} M^{D_f (j_f - j_{\pi(f)})} \end{aligned}$$

because $L(T) = \#\text{vertices}(T) - 1$ for any tree.

■

<u>Proof of Lemma III.1b</u>

We prove the bound on the dual norm $|\text{Val}(G)|'$ by applying the estimate of part a) to an extension G^* of G. The dual norm is morally an L^1-norm in $(d+1)$-momentum space. Thus, the rough idea is to perform the integral over external $(d+1)$-momenta $k_1, \ldots, k_r$ subject to the constraint $\sum_{i=1}^{r} k_i = 0$ by adjoining one extra vertex that is hooked to the r lines carrying these momenta. This observation suggests the following construction of G^*.

The vertices of G^* are the vertices of G plus one extra vertex v^*. All vertices of G^*, with the exception of v^*, are internal. The lines of G^* are the lines of G plus one extra line for each external vertex of G. Each new line, denoted $\ell_i^*, 1 \le i \le r$, joins v^* to a different external vertex of G. These new lines are assigned scale zero and covariance $C_i^* = \delta(\tau_i^* - \tau_i)$ where τ_i is the temporal component of the vertex of G to which ℓ_i^* is attached and τ_i^* is an arbitrary constant. (Recall that we wish to sup over the temporal components of the external vertices of G.) We have

$$\left|C_i^*\right| \le M^{(\delta_{\ell_i^*}-2)j_{\ell_i^*}}, \qquad j_{\ell_i^*}=0,$$

where

$$\delta_{\ell_i^*} = \left[2-\frac{1}{2}\sum_{\substack{\ell \text{ hooked to external} \\ \text{vertex number } i}} \delta_\ell\right]$$

so that all vertices of G^*, with the possible exception of v^*, are, in the sense of the Lemma dimensionless.

All other lines ℓ of G^* inherit scales and dimensions δ_ℓ from the corresponding lines of G. By definition their covariances are the absolute values of the covariances of the lines of G. It follows that

$$\begin{aligned}\left|\mathrm{Val}(G)\right|' &\le \sup_{\tau_i^*, 1\le i\le r} \mathrm{Val}(G^*)\\ &= \sup_{t_i^*, 1\le i\le r}\left|\mathrm{Val}(G^*)\right|\end{aligned}$$

Now we apply part a) to obtain

$$\left|\mathrm{Val}(G)\right|' \le \prod_{\ell\in G} a_\ell \prod_v \left|T_v\right| \prod_v \left|I_v\right| \quad M^{D_{G^*} j_\phi} \prod_{\substack{f\in t(G^*)\\ f>\phi}} M^{D_f(j_f - j_{\pi(f)})}$$

The fact that $\left|C_i^*\right|' = \infty$ is harmless. We can place all these lines in the tree of Lemma III.2.

Observe that

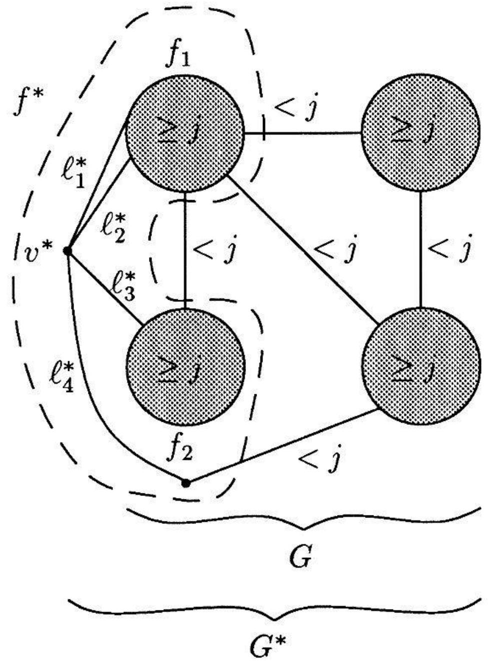
$$\begin{aligned} D_{G^*} &= \sum_{\ell\in G^*} \delta_\ell - 2\left(\#\{\text{vertices of } G\}+1-1\right)\\ &= \sum_{\ell\in G}\delta_\ell + \sum_{\substack{\text{external vertices}\\ v_i \text{ of } G}} \delta_{\ell_i^*} - 2\,\#\{\text{vertices of } G\}\\ &= \sum_{\ell\in G}\delta_\ell + \sum_{\substack{\text{external vertices}\\ v_i \text{ of } G}} \left[2-\frac{1}{2}\sum_{\substack{\ell \text{ hooked}\\ \text{to } v_i}}\delta_\ell\right] - 2\,\#\{\text{vertices of } G\}\\ &= 0\end{aligned}$$

since, by hypothesis, the internal vertices of G are dimensionless. To analyse the last factor it is necessary to express the tree and forest structure of G^* in terms of G. We have

$$\prod_{\substack{f\in t(G^*)\\ f>\phi}} M^{D_f(j_f-j_{\pi(f)})} = \prod_{j=j_\phi+1}^{0} \prod_{f\in C_j^*} M^{D_f} \tag{III.14}$$

where C_j^* is the set of connected components of $\{\ell \in G^* | j_\ell \geq j\}$.

We wish to rewrite (III.14) in terms of C_j, the set of connected components of $\{\ell \in G | j_\ell \geq j\}$. To do so notice that C_j^* consists precisely of those elements of C_j that do not contain external vertices of G together with a single element f^* combining the lines $\ell_1^*, ..., \ell_r^*$ with the elements $f_1, ..., f_{r'}$ of C_j that do contain external vertices. (See the diagram below.)



The degree of f^* is

$$
\begin{aligned}
D_{f^*} &= \sum_{\ell \in G_{f^*}} \delta_\ell - 2\left(\#\{\text{vertices of } G_{f^*}\} - 1\right) \\
&= \sum_{i=1}^{r'} \sum_{\ell \in G_{f_i}} \delta_\ell + \sum_{i=1}^{r} \left[2 - \frac{1}{2} \sum_{\substack{\ell \text{ hooked} \\ \text{to } v_i}} \delta_\ell \right] \\
&\quad - 2\left\{ \sum_{i=1}^{r'} (\#\text{vertices of } G_{f_i}) + \#\text{external vertices of } G \text{ not in any } G_{f_i} + 1 - 1 \right\}
\end{aligned}
$$

Further manipulation yields

$$D_{f^*} = \sum_{i=1}^{r'} \left\{ \sum_{\substack{\text{vertices} \\ v \text{ of } G_{f_i}}} \left[-2 + \sum_{\substack{\text{lines of } G_{f_i} \\ \text{hooked to} v}} \frac{1}{2}\delta_\ell \right] \right\} + \sum_{\substack{\text{external vertices} \\ v \text{ of } G}} \left[2 - \frac{1}{2} \sum_{\substack{\text{lines of } G \\ \text{hooked to} v}} \delta_\ell \right]$$
$$- 2\, \#\{\text{external vertices of } G \text{ not contained in any } G_{f_i}\}$$
$$= \sum_{i=1}^{r'} \triangle_{f_i} + \sum_{\substack{\text{external vertices } v \\ \text{of } G \text{ not in any } G_{f_i}}} \triangle_v$$

Thus,

$$|\mathrm{Val}(G)|' \le \prod_{\ell \in G} a_\ell \prod_v |T_v| \prod_v |I_v| \prod_{j=j_\phi+1}^{0} \left[\prod_{\substack{f \in \mathcal{C}_j \\ f \text{ internal}}} M^{D_f} \prod_{\substack{f \in \mathcal{C}_j \\ f \text{ external}}} M^{\triangle_f} \prod_{\substack{\text{external vertices} \\ v \text{ of } G \text{ not in} \\ \text{any } G_f, f \in \mathcal{C}_j}} M^{\triangle_v} \right]$$

$$= \prod_{\ell \in G} a_\ell \prod_v |T_v| \prod_v |I_v| \prod_{\substack{f \in t(G)\ f > \phi \\ G_f \text{ contains no} \\ \text{external vertices of } G}} M^{D_f (j_f - j_{\pi(f)})} \prod_{\substack{f \in t(G)\ f > \phi \\ G_f \text{ contains an} \\ \text{external vertex of } G}} M^{\triangle_f (j_f - j_{\pi(f)})}$$
$$\prod_{\substack{\text{external vertices} \\ \text{of } G}} M^{\triangle_v (0 - j_{\pi(v)})}$$

■

To implement renormalization cancellations it is necessary to control derivatives. For this reason we formulate a self-evident supplement to the abstract power counting lemma.

Supplement to Lemma III.1

Let $G = G^J$ be general 2p-point labelled graph as in Lemma III.1. Let δ_p be the $\binom{2p}{2}$ vector of differences between its external temporal arguments and let $\mathbf{k}_p$ be the $(2p-1)d$ vector of external momenta. Suppose the covariances corresponding to hard lines obey

$$\sup_{0 \le |m| \le |n|} \left| [M^{j_\ell}(\tau, \nabla_{\mathbf{k}})]^m C_\ell \right| \le a_\ell M^{(\delta_\ell - 2) j_\ell} \qquad (III.15a)$$

$$\sup_{0 \le |m| \le |n|} \left| [M^{j_\ell}(\tau, \nabla_{\mathbf{k}})]^m C_\ell \right|' \le a_\ell M^{\delta_\ell j_\ell} \qquad (III.15b)$$

Then

$$\left|[M^{j_\phi}(\delta_p, \nabla_{\mathbf{k}_p})]^n \mathrm{Val}(G)\right|$$

and

$$\left|[M^{j_\phi}(\delta_p, \nabla_{\mathbf{k}_p})]^n \mathrm{Val}(G)\right|'$$

obey the bounds of Lemma III.1 when $\prod_v |T_v| \prod_v |I_v|$ is replaced by

$$\sup_{\sum |n_v| \leq |n|} \prod_v \left|[M^{j_\phi}(\tau, \nabla_{\mathbf{k}})]^{n_v} T_v\right| \prod_v \left|[M^{j_\phi}(\delta_2, \nabla_{\mathbf{k}_2})]^{n_v} I_v\right|.$$

By Lemma II.1 the covariances (III.1) obey (III.5) with $\delta_\ell = 1$. Consequently, all four legged vertices are dimensionless in the sense of (III.11). A two legged vertex is not. However, the artifice of writing $T_v = [T_v M^{-j_\pi(v)}] M^{j_\pi(v)}$ and assigning the factor $M^{j_\pi(v)}$ to the vertex makes it so. These remarks imply that our model obeys Lemma III.1 with $|T_v|$ replaced by $|T_v| M^{-j_\pi(v)}$ and degrees

$$\begin{aligned} D_f &= \sum_{\ell \in G_f^J} 1 - 2(V_2 + V_4 - 1) + V_2 \\ &= \frac{1}{2}[2V_2 + 4V_4 - E_f] - V_2 - 2V_4 + 2 \\ &= \frac{1}{2}(4 - E_f) \end{aligned} \tag{III.16a}$$

and

$$\triangle_f \leq -\frac{1}{2}, \quad \triangle_v \leq -\frac{1}{2}. \tag{III.16b}$$

Here, V_2 and V_4 are the number of two and four legged vertices in G_f and the last term in D_f arises from the extra $M^{j_\pi(v)}$ that we assigned to the two legged vertices.

Lemma III.1 reduces the problem of estimating a general graph to that of controlling two and four legged subgraphs. This is done by several techniques. The most subtle are applied in Section IV where special low order subgraphs are renormalized by hand. In this section we exploit the fact that $\sum_{\substack{h \text{ s.t.} \\ M^h \geq \triangle}} \frac{\triangle}{M^h}$ and $\lambda \sum_{\substack{h \text{ s.t.} \\ M^h \geq \triangle}} 1$ are uniformly bounded in $\triangle$ for the superconducting model.

It is necessary to introduce the decomposition (see (I.69))

$$\mathbf{C}^{(j)} = \mathbf{C}_1^{(j)} + \mathbf{C}_2^{(j)} \tag{III.17a}$$

$$\mathbf{C}_1^{(j)}(\tau,\mathbf{k}) = -e^{-E(\mathbf{k})|\tau|}\frac{(\mathrm{sgn}\tau)E(\mathbf{k})\mathbf{1} + e(\mathbf{k})\sigma^3}{2E(\mathbf{k})} f(M^{-2j}E(\mathbf{k})^2) \tag{III.17b}$$

$$\mathbf{C}_2^{(j)}(\tau,\mathbf{k}) = -e^{-E(\mathbf{k})|\tau|}\frac{\triangle\sigma^1}{2E(\mathbf{k})} f(M^{-2j}E(\mathbf{k})^2) \tag{III.17c}$$

and its soft analogue

$$\mathbf{C}_s^{(j)} = \mathbf{C}_{1,s}^{(j)} + \mathbf{C}_{2,s}^{(j)} \tag{III.18a}$$

$$\mathbf{C}_{1,s}^{(j)}(\tau,\mathbf{k}) = -e^{-E(\mathbf{k})|\tau|}\frac{\mathrm{sgn}(\tau)E(\mathbf{k})\mathbf{1} + e(\mathbf{k})\sigma^3}{2E(\mathbf{k})} \rho(M^{-2j}E(\mathbf{k})^2) \tag{III.18b}$$

$$\mathbf{C}_{2,s}^{(j)}(\tau,\mathbf{k}) = -e^{-E(\mathbf{k})|\tau|}\frac{\triangle\sigma^1}{2E(\mathbf{k})} \rho(M^{-2j}E(\mathbf{k})^2) \tag{III.18c}$$

where $\triangle > 0$ and $E(\mathbf{k})^2 = e(\mathbf{k})^2 + \triangle^2$.

Lemma III.3 Let $d \geq 1$. For $j < 0$ and $m \in \mathbf{N}^{d+1}$, with $|m| = \sum m_i$,

a) $\mathbf{C}_1^{(j)} = \mathbf{C}_2^{(j)} = \mathbf{C}_{1,s}^{(j)} = \mathbf{C}_{2,s}^{(j)} = 0$ for $M^{j+2} < \triangle$

b) $\left|(\tau,\nabla_{\mathbf{k}})^m \mathbf{C}_1^{(j)}(\tau,\mathbf{k})\right| \leq \mathrm{const}_m M^{-(1+|m|)j}$

$\left|(\tau,\nabla_{\mathbf{k}})^m \mathbf{C}_1^{(j)}(\tau,\mathbf{k})\right|' \leq \mathrm{const}_m M^{(1-|m|)j}$

$\left|\mathbf{C}_{1,s}^{(j)}(\tau,\mathbf{k})\right|' \leq \mathrm{const} M^j$

c) $\left|(\tau,\nabla_{\mathbf{k}})^m \mathbf{C}_2^{(j)}(\tau,\mathbf{k})\right| \leq \mathrm{const}_m \frac{\triangle}{M^j} M^{-(1+|m|)j}$

$\left|(\tau,\nabla_{\mathbf{k}})^m \mathbf{C}_2^{(j)}(\tau,\mathbf{k})\right|' \leq \mathrm{const}_m \frac{\triangle}{M^j} M^{(1-|m|)j}$

$\left|\mathbf{C}_{2,s}^{(j)}(\tau,\mathbf{k})\right|' \leq \mathrm{const} \frac{\triangle}{M^j} \ell n\left(\frac{M^j}{\triangle}\right) M^j$

Proof The bounds on $\mathbf{C}_1^{(j)}$ and $\mathbf{C}_2^{(j)}$ follow from

$$|\nabla_{\mathbf{k}}^n \mathbf{C}_1^{(j)}(\tau,\mathbf{k})| \leq \mathrm{const}_n (M^{-j} + |\tau|)^{|n|} e^{-M^j|\tau|}\chi(E(\mathbf{k}) \leq \mathrm{const} M^j)$$
$$|\nabla_{\mathbf{k}}^n \mathbf{C}_2^{(j)}(\tau,\mathbf{k})| \leq \mathrm{const}_n \frac{\triangle}{M^j}(M^{-j} + |\tau|)^{|n|} e^{-M^j|\tau|}\chi(E(\mathbf{k}) \leq \mathrm{const} M^j)$$

and the observation that the volume of the support of $\chi(E(\mathbf{k}) \leq \mathrm{const} M^j)$ is M^j. The estimates on $\mathbf{C}_{1,s}^{(j)}, \mathbf{C}_{2,s}^{(j)}$ are obtained by setting $m = 0$ and summing over $j' < j$ such that $M^{j+2} < \triangle$.

■

Lemma III.4 Let $G = G^J$ be a labelled graph with two and four legged vertices and $L(G)$ particle lines each of which has covariance $\mathbf{C}^{(j)}, \mathbf{C}_s^{(j)}, \mathbf{C}_1^{(j)}, \mathbf{C}_{1,s}^{(j)}, \mathbf{C}_2^{(j)}$ or $\mathbf{C}_{2,s}^{(j)}$. See (III.1,17,18).

Denote by $L_2(G)$ the number of $\mathbf{C}_2^{(j)}, \mathbf{C}_{2,s}^{(j)}$ lines of G. Let each fork of the associated tree $t(G)$ be connected by hard lines and denote by E_f the number of external lines of G_f. Let each vertex obey

$$\sup_{|n_v|\le|n|} \left|[M^{j_\phi}(\tau, \nabla_{\mathbf{k}})^{n_v} T_v\right| M^{-j_{\pi(v)}} \le \omega_v$$

$$\sup_{|n_v|\le|n|} \left|M^{j_\phi}(\delta_2, \nabla_{\mathbf{k}_2})^{n_v} I_v\right| \le \omega_v.$$

Recall that the notation δ_p and $\mathbf{k}_p$ was introduced in Supplementary Lemma III.1. Then the value Val(G) given in (III.6) obeys

a) $\left|[M^{j_\phi}(\delta_p, \nabla_{\mathbf{k}_p})]^n \mathrm{Val}(G^J)\right|$

$$\le \mathrm{const}^{L(G)} \left[\prod_v \omega_v\right] M^{\frac{1}{2}(4-E_\phi)j_\phi} \left[\frac{\triangle}{M^{j_\phi}} \log\left(\frac{M^{j_\phi}}{\triangle}\right)\right]^{L_2(G)} \prod_{\substack{f\in t(G)\\ f>\phi}} M^{\frac{1}{2}(4-E_f-L_2(G_f))(j_f-j_{\pi(f)})}$$

b) $\left|[M^{j_\phi}(\delta_p, \nabla_{\mathbf{k}_p})]^n \mathrm{Val}(G^J)\right|'$

$$\le \mathrm{const}^{L(G)} \left[\prod_v \omega_v\right] \prod_{\substack{G_f \text{ contains no external}\\ \text{vertices of } G}} M^{\frac{1}{2}(4-E_f-L_2(G_f))(j_f-j_{\pi(f)})}$$

$$\prod_{\substack{f>\phi, G_f \text{ contains an}\\ \text{external vertex of } G}} M^{-\frac{1}{2}(j_f-j_{\pi(f)})} \prod_{\substack{\text{external vertices}\\ \text{of } G}} M^{-\frac{1}{2}(0-j_{\pi(v)})}$$

c) Assume that every $G_f, f > \phi$ has at least six external legs. Then

$$\sum_{\substack{\{j_f|f>\phi\}\\ 0>j_f>j_{\pi(f)}}} \left|[M^{j_\phi}(\delta_p, \nabla_{\mathbf{k}_p})]^n \mathrm{Val}(G^J)\right|$$

$$\le \mathrm{const}^{L(G)} M^{\frac{1}{2}(4-E_\phi)j_\phi} \left[\frac{\triangle}{M^{j_\phi}} \log\left(\frac{M^{j_\phi}}{\triangle}\right)\right]^{L_2(G)} \prod_v \omega_v$$

and

$$\sum_{\substack{\{j_f|f\ge\phi\}\\ 0>j_f>j_{\pi(f)}}} \left|[M^{j_\phi}(\delta_p, \nabla_{\mathbf{k}_p})]^n \mathrm{Val}(G^J)\right|' \le \mathrm{const}^{L(G)} \prod_v \omega_v.$$

All constants are independent of $\triangle$.

Proof Lemma III.3 implies that the hard covariances obey (III.15). We apply

$$\begin{aligned}
\frac{\triangle}{M^{j_f}} &\le \frac{\triangle}{M^{j_f}} \log\left(\frac{M^{j_f}}{\triangle}\right) \\
&\le \frac{\triangle}{M^{j_\phi}} \log\left(\frac{M^{j_\phi}}{\triangle}\right) \prod_{\phi<f'\le f} \frac{M^{j_{\pi(f')}}}{M^{j_{f'}}} \frac{\log(M^{j_{f'}}/\triangle)}{\log(M^{j_{\pi(f')}}/\triangle)} \\
&= \frac{\triangle}{M^{j_\phi}} \log\left(\frac{M^{j_\phi}}{\triangle}\right) \prod_{\phi<f'\le f} \mathrm{const}\, M^{-\frac{1}{2}(j_{f'}-j_{\pi(f')})}
\end{aligned}$$

to the $\triangle$ factors arising from Lemma III.3.c. Lemma III.4 now follows from the supplement to Lemma III.1.

■

We now prepare for the proof of Theorem I.1. Let $\mathcal{T}$ be a tree constructed from R forks (I.34a), c forks (I.23b) and n general four legged leaves. Let G^J be a graph contributing to $\mathcal{T}$. That is $t(G^J) = \mathcal{T}$. It is among the graphs having

- particle lines $\mathbf{C}^{(j)}, \mathbf{C}_s^{(j)}, \mathbf{C}_1^{(j)}, \mathbf{C}_{1,s}^{(j)}, \mathbf{C}_2^{(j)}, \mathbf{C}_{2,s}^{(j)}$ (III.1,17,18) (III.19a)
- local and renormalized four legged vertices (III.2) (III.19b)
- for each R fork f of $\mathcal{T}$, a renormalization operator $(\mathbf{1} - \mathbf{L}^{(j_f-1)})$ (I.99) acting on G_f (III.19c)
- for each c fork f of $\mathcal{T}$, a localization operator $\boldsymbol{\ell}$ (I.99) acting on G_f (III.19d)
- each G_f connected by hard lines (III.19e)
- for each R fork f of $\mathcal{T}$, $j_{\pi(f)} < j_f < 0$ (III.19f)
- for each c fork f of $\mathcal{T}$, $\log\triangle \le j_f \le j_{\pi(f)}$. (III.19g)

We first ignore any potential gain from the renormalization of four legged subgraphs to motivate an estimate of the sum $\sum_j G^J$. First, suppose that G_f is a two legged c fork of scale j_f. The factor $M^{\frac{1}{2}(4-E_\phi)j_\phi}$ of Lemma III.4.a, when applied to G_f becomes M^{j_f}. Since

$$\sum_{\log\triangle\le j_f\le j_{\pi(f)}} M^{j_f} \le \mathrm{const}\, M^{j_{\pi(f)}}$$

G_f acts, after summing over j_f, as a generalized dimensionless vertex. We may therefore assume that there are no c forks.

If (III.19c) is not implemented Lemma III.4b gives

$$|\mathrm{Val}(G^J)|' \le \mathrm{const}^{L(G)} \left[\prod_v \omega_v\right] \prod_{\text{internal}} M^{\frac{1}{2}(4-E_f-L_2(G_f))(j_f-j_{\pi(f)})} \prod_{\text{external}} M^{-\frac{1}{2}(j_f-j_{\pi(f)})} \prod_{\text{external}} M^{-\frac{1}{2}(0-j_{\pi(v)})} \qquad (III.20)$$

The string Lemma II.2′ implements renormalization of two legged subgraphs and improves (III.20) to

$$\begin{aligned} |\mathrm{Val}(G^J)|' \le \mathrm{const}^{L(G)} \left[\prod_v \omega_v\right] & \prod_{\substack{\text{internal,}\\ E_f\ge 4}} M^{\frac{1}{2}(4-E_f-L_2(G_f))(j_f-j_{\pi(f)})} \\ & \prod_{\substack{\text{internal}\\ E_f=2}} M^{-\frac{1}{2}L_2(G_f)(j_f-j_{\pi(f)})} \prod_{\text{external}} M^{-\frac{1}{2}(j_f-j_{\pi(f)})} \\ & \prod_{\text{external}} M^{-\frac{1}{2}(0-j_{\pi(v)})} \end{aligned} \qquad (III.21)$$

It follows that in the sum of (III.21) over J there are two kinds of factors. If f has $E_f \ge 6$ or $L_2(G_f) \ge 1$ or if G_f contains on external vertex of G, the sum over j_f is uniformly bounded. On the other hand, when f has $E_f = 2, 4$ and $L_2(G_f) = 0$ the sum produces a factor

$$\begin{aligned} \sum_{\max(j_{\pi(f)}, \ln \Delta) < j_f < 0} 1 &\le \log \Delta \\ &= \mathrm{const} \frac{1}{|\lambda|} \end{aligned} \qquad (III.22)$$

for relevant λ, Δ. Each ω_v is proportional to λ and compensates a sum (III.22). We must ensure that the total number of sums (III. 22) does not exceed a fixed fraction (strictly less than one) of the number of leaves. The following lemma provides a sufficient condition for this to be the case.

<u>Yes/No Lemma III.5</u> Let $\mathcal{T}$ be an abstract rooted tree. Each fork has a branching number (the number of branches leaving f upward) $b_f \ge 2$ and is assigned a scale h_f and a variable $s_f \in \{\text{yes}, \text{no}\}$. Assume that the functions $a_f(h_f, h_{\pi(f)})$ of the scales $h_f, h_{\pi(f)}$ satisfy

(1) $0 \le a_f(h_f, h_{\pi(f)}) \le 1$

(2) $\sum_{h_f = h_{\pi(f)}+1}^{-1} a_f(h_f, h_{\pi(f)}) \le 1$ if $s_f = \text{yes}$.

Here $h_{\pi(\phi)} = -1/|\lambda|$. If, for each $f > \phi$, either

(a) s_f =yes

(b) $b_{\pi(f)} \geq 3$

(c) $b_{\pi(f)} = 2$ and at least one f' with $\pi(f') = \pi(f)$ has $s_{f'}$ =yes

or

(d) $s_{\pi(f)}$ =yes

then

$$\sum_{h_f} a_f(h_f, h_{\pi(f)}) \leq |\lambda|^{\frac{1}{2}-\frac{3}{4}\#\text{leaves}} \begin{cases} |\lambda|, & s_\phi = \text{yes} \\ 1, & s_\phi = \text{no} \end{cases}$$

Here the sum runs over $\{\{h_f\}|0 > h_f > h_{\pi(f)}, 0 > h_\phi > -1/|\lambda|\}$.

<u>Remark</u> A minor modification of the proof below actually gives a slightly stronger result. The $|\lambda|^{-\frac{3}{4}\#\text{leaves}}$ is replaced by

$$\prod_{\substack{\text{leaves}\, v \\ \text{with } b_{\pi(v)}=2}} |\lambda|^{-3/4} \prod_{\substack{\text{leaves}\, v \\ \text{with } b_{\pi(v)}\geq 3}} |\lambda|^{-1/2} \tag{III.23}$$

<u>Proof</u> Clearly,

$$\sum_{h_f} a_f(h_f, h_{\pi(f)}) \leq |\lambda|^{-\#\{f|s_f=\text{no}\}}. \tag{III.24}$$

Suppose that, for every $f > \phi, s_f$ =no and that, for every nonmaximal $f, b_f \geq 3$. In this case, we prove, by induction on the size of $\mathcal{T}$, that

$$\frac{3}{4}\#(\text{leaves}) - \#\{f|s_f = \text{no}, f > \phi\} \geq 3/2 \tag{III.25}$$

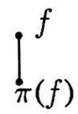
Since $\frac{3}{4}2 - 0 = \frac{3}{2}$, (III.25) is satisfied for the smallest possible tree

Let the successor forks of ϕ be denoted $f_1, ..., f_m$ and the successor leaves $v_1, ..., v_n$. Then $b_f = m + n$ and we have, by the inductive hypothesis,

$$\begin{aligned}
&\frac{3}{4}\#(\text{leaves of } \mathcal{T}) - \#\{f|s_f = \text{no}, f > \phi\} \\
&\geq \sum_{i=1}^{m} [\frac{3}{4}\#(\text{leaves above } f_i) - \#\{f|s_f = \text{no}, f > f_i\}] - m + \frac{3}{4}n \\
&\geq \frac{3}{2}m - m + \frac{3}{4}n \\
&\geq 3/2
\end{aligned}$$

since $m+n \geq 3$.

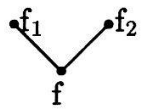
Let $\mathcal{T}$ be a general tree. Construct from it a new tree $\mathcal{T}'$ by collapsing every branch



with s_f =yes to a point. The resulting fork f' inherits $s_{f'}$, from $s_{\pi(f)}$. The new tree has the same number of leaves and the same number of no forks as $\mathcal{T}$. All its forks, except possibly the first, ϕ', are no forks. Finally, every nonmaximal f' has $b_{f'} \geq 3$ by (a), (b), (c), (d). We have already verified (III.25) for $\mathcal{T}'$.

■

Lemma III.5 implies that the sum over J of the right hand side of (III.21) will be bounded by $\text{const}^{L(G)} \prod_v [\omega_v/\lambda^{3/4}]$ provided $t(G^J)$ does not have a two fork



such that all of G_f, G_{f_1}, G_{f_2} have at most four external legs, no C_2 or $C_{2,s}$ lines and no external vertices of G. In the discussion above we have reduced the problem of controlling $\sum\limits_J \text{Val}(G_J)$ to consideration of a small number of second order graphs. They are treated, by hand, in Section IV, where the improvement due to renormalization is carefully extracted.

We now formalize the remarks made above. Introduce the norms

$$\begin{aligned}\|T\|_j &= \max\{M^{(\alpha+|\beta|)j} \sup_{\mathbf{k}} \int d\tau |\tau|^\alpha \nabla_{\mathbf{k}}^\beta T(\tau,\mathbf{k})| : 0 \leq \alpha \leq 2, |\beta| \leq 1\} \\ &= \max\{M^{(\alpha+|\beta|)j} \left|\delta_1^\alpha \nabla_{\mathbf{k}_1}^\beta T\right| : 0 \leq \ a \leq 2, |\beta| \leq 1\}\end{aligned} \tag{III.26a}$$

$$\|I\|_j = \max\{M^{(|\alpha|+|\beta|)j} \sup_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4=0} \int d\tau_2 d\tau_3 d\tau_4 \prod_{j=2}^{4} |\tau_j|^{\alpha_j}$$

$$\left| \prod_{j=1}^{4} \nabla_{\mathbf{k}_j}^{\beta_j} I\big((0,\mathbf{k}_1),(\tau_2,\mathbf{k}_2),(\tau_3,\mathbf{k}_3),(\tau_4,k_4)\big) \right| : \qquad (III.26b)$$

$$\alpha_j \geq 0, \beta_j \geq 0, |\alpha| \leq 2, |\beta| \leq 1\}$$

on two and four point kernels.

Theorem III.6 Let $\mathcal{T}$ be a tree constructed from R forks (I. 34a), c forks (I.23b) and n general four legged leaves each of type $\tau^a \otimes \tau^b$ with $a+b=0$ (mod 3). These leaves $I_v^{(j_{\pi(v)})}$ may depend on the scale $j_{\pi(v)}$ of the fork $\pi(v)$ of $\mathcal{T}$ immediately below v and are assumed to obey

$$\sup_j \|I_v^{(j)}\|_j \leq |\lambda|\omega_v.$$

Let G^J be a labelled graph, with $t(G^J)=\mathcal{T}$, satisfying (III.19). Let $|\log\triangle| \leq \frac{\text{const}}{|\lambda|}$. Then

$$|\sum_J \text{Val}(G^J)|' \leq \text{const}^{L(G)}|\lambda|^{3/2}\prod_{v=1}^{n}[|\lambda^{1/4}|\omega_v]$$

Recall that, by (III.19f,g), the sum is over

$$\{J | j_{\pi(f)} < j_f \leq -1 \text{ if } f \text{ is an } R \text{ fork and}$$

$$\log\triangle \leq j_f \leq j_{\pi(f)} \text{ if } f \text{ is a } c \text{ fork}\}$$

with $j_{\pi(\phi)} = \log\triangle$.

If G^J is two-legged

$$\sum_{J \text{ s.t. } j_\phi \leq r} |\ell\text{Val}(G^J)| \leq \text{const}^{L(G)} M^r |\lambda|^{3/2} \prod_v [|\lambda|^{1/4}\omega_v].$$

If G^J is four-legged

$$\sum_{J \text{ s.t. } j_\phi = r} \|\text{Val}(G^J)\|_r \leq \text{const}^{L(G)}|\lambda|^{3/2}\prod_v[|\lambda|^{1/4}\omega_v].$$

All the const's above are uniform in $\triangle$.

Proof The proof will proceed by repeatedly trimming off portions of J above forks with $E_f=2$ and above "dangerous" forks with $b_f=2$. First assign $s_f=$yes to each fork f of $\mathcal{T}$ such that $E_f \geq 6$ or $L_2(G_f) \geq 1$ or G_f contains an external vertex of G. Assign $s_f=$no to the remainings forks. As we proceed some of these no's will be changed to yes's. The final s_f's will obey (a)-(d) of Lemma III.5.

In each step we consider all remaining forks that are maximal in the class of all forks that are (i) two legged forks or (ii) two forks f_1 whose successor forks/leaves f_2, f_3 are not both leaves and have $s_{f_1} = s_{f_2} = s_{f_3} =$ no. We trim just below the two legged forks and f_2, f_3 forks. The portions trimmed off will be treated as leaves v in the next step. In the case of a c-fork the leaf will be viewed as a local two-legged vertex of scale j_v. The sum over scales for the c-fork itself is, by definition, included in the trimmed off portion, that is, the new leaf. The remaining new leaves are viewed as two or four legged renormalized vertices. Their scales run from $j_{\pi(v)} + 1$ to -1. The sum over these scales and the renormalization operator are included in $\mathcal{T}\backslash\{\text{new leaves}\}$. The quotient graph $G_{f_1}/\{G_{f_2}, G_{f_3}\}$ is one of those considered in Lemmas IV.4, IV.5 and IV.6. The bounds there are of three kinds. One kind controls the sum over the scale of G_{f_2} or G_{f_3} with that of G_{f_1} held fixed. An example is the factor $M^{-(j_1-h)}$ in Lemma IV.6b). In this case change s_{f_2} or s_{f_3} to yes. The second kind controls the scale of G_{f_1} with those of G_{f_2} or G_{f_3} held fixed as in Lemma IV.6 a). In this case change s_{f_1} to yes. Finally there is the term j_β in Lemma IV.5. This term will provide decay $M^{-(j_{f_1}-j_{f_4})}$ between the scale j_{f_1} of G_{f_1} and the scale j_{f_4} of the line hooked to τ_2 so change s_f to yes for $f = f_1$ as well as for all

$$f_1 > f > f_4. \tag{III.28}$$

All the forks $f_1 \geq f > f_4$ are by construction in $\mathcal{T}\backslash\{\text{new leaves}\}$ and so affect the trimming rules at the next step.

We now bound the new leaf $\text{Val}(G_f)$ that is obtained by trimming $\{f' > f\}$ off the tree. The graph G_f has general two and four legged scale dependent leaves v, generated by previous trimmings. We assume, by induction, that the local two and four legged leaves have scale $j_{\pi(v)}$ and kernels satisfying

$$\|T_v^{(j_{\pi(v)})}\|_{j_{\pi(v)}} M^{-j_{\pi(v)}}, \|I_v^{(j_{\pi(v)})}\|_{j_{\pi(v)}} \leq \left[\prod_{v'>v} |\lambda||\omega_{v'}|\right] |\lambda|^{-\#\{f\geq v | s_f = \text{no}\}}. \tag{III.29a}$$

The renormalized two and four legged leaves come equipped with $(\mathbf{1} - \mathbf{L})$'s, have scales summed from $j_{\pi(v)} + 1$ to -1 and have kernels obeying (before $\mathbf{1} - \mathbf{L}$ is applied and the scale is summed over)

$$\|T_v^{(j)}\|_j M^{-j}, \|I_v^{(j)}\|_j \leq \left[\prod_{v'>v} |\lambda|\omega_{v'}\right] |\lambda|^{-\#\{f>v|s_f=\text{no}\}} \qquad (III.29b)$$

The f's in (III.29) refer to the forks of the original tree $\mathcal{T}$, s_f is the final assignment of yes/no to f, and the i's in (III.29) refer to the original leaves of $\mathcal{T}$. We shall suppress all constants that can be absorbed in $\text{const}^{L(G)}$.

We must verify that, if G_f is two legged

$$\sum_{\substack{\{j_{f'}|f'\geq f\} \\ j_f\leq r}} |\ell\text{Val}(G_f^J)|M^{-r} \leq \left[\prod_{v'>f} |\lambda|\omega_{v'}\right] |\lambda|^{-\#\{f'\geq f|s_{f'}=\text{no}\}} \qquad (III.30a)$$

and

$$\sum_{\{j_{f'}|f'>f\}} \|\text{Val}(G_f^J)\|_{j_f} M^{-j_f} \leq \left[\prod_{v'>f} |\lambda|\omega_{v'}\right] |\lambda|^{-\#\{f'>f|s_{f'}=\text{no}\}} \qquad (III.30b)$$

that, if G_f is four legged

$$\sum_{\{j_{f'}|f'>f\}} \|\text{Val}(G_f^J)\|_{j_f} \leq \left[\prod_{v'>f} |\lambda|\omega_{v'}\right] |\lambda|^{-\#\{f'>f|s_{f'}=\text{no}\}} \qquad (III.30c)$$

and that, when $f = \phi$

$$\sum_J |\text{Val}(G_\phi^J)|' \leq \left[\prod_v |\lambda|\omega_v\right] |\lambda|^{-\#\{f|s_f=\text{no}\}} \qquad (III.30d)$$

The Theorem will then follow from Lemma III.5, or more precisely (III.25).

By construction, G_f contains no $f' > f$ with $E_{f'} = 2$. Further more if f_1 is a two fork with s_{f_i} =no or f_i a leaf for $i = 1,2,3$ then f_2 and f_3 must both be leaves. The first step in bounding $\text{Val}(G_f)$ is to apply Lemma's IV.4, 5, 6 to $\text{Val}(G_{f_1})$ and treat the latter as a generalized vertex in G_f. We have defined $s_{f_i}, i = 2,3$, so that the sum over j_{f_i} yields const if s_{f_i} =yes and $1/\lambda$ if s_{f_i} =no. This generalized vertex may of course be a $j_{\beta,en}$. The next step is to apply Lemma II.2$'$ to convert all strings of two legged diagrams into new covariances. The last step is to apply the power counting Lemma III.4.

One technical nuisance remains. We need to prepare $j_{\beta,en}$ so that Lemma III.4 automatically generates the desired renormalization decay factor. If $\mathbf{C}^{(j=j_{f_4})}$ is the covariance of G_f hooked to τ_2 and $j_{\beta,en}$ is of scale $h = j_{f_1}$ write (suppressing irrelevant arguments)

$$\int d\tau_2 \mathbf{C}^{(j)}(\tau_2)[j_{\beta,en}(\tau_1,\tau_2) - \delta(\tau_1-\tau_2)\int d\alpha_2 j_{\beta,en}(\tau_1,\alpha_2)]$$

$$\begin{aligned}
&= \int d\tau_2[\mathbf{C}^{(j)}(\tau_2) - \mathbf{C}^{(j)}(\tau_1)]j_{\beta,en}(\tau_1,\tau_2)\\
&= \int d\tau_2[\mathbf{C}^{(j)}(\tau_2)h(M^{\frac{1}{2}(j+h)}|\tau_2|) - \mathbf{C}^{(j)}(\tau_1)h(M^{\frac{1}{2}(j+h)}|\tau_1|)]j_{\beta,en}(\tau_1,\tau_2)\\
&\qquad + \int d\tau_2 \mathbf{C}^{(j)}(\tau_2)\rho(M^{\frac{1}{2}(j+h)}|\tau_2|)j_{\beta,en}(\tau_1,\tau_2)\\
&\qquad - \int d\tau_2 \mathbf{C}^{(j)}(\tau_1)\rho(M^{\frac{1}{2}(j+h)}|\tau_1|)j_{\beta,en}(\tau_1,\tau_2)
\end{aligned}$$

$$= \int_0^1 d\alpha \int d\tau_2 \frac{d}{d\tau}[\mathbf{C}^{(j)}(\tau)h(M^{\frac{1}{2}(j+h)}|\tau|)](\tau_1+\alpha(\tau_2-\tau_1))j_{\beta,en}(\tau_1,\tau_2)(\tau_2-\tau_1) \quad (III.31a)$$

$$+ \int d\tau_2 \mathbf{C}^{(j)}(\tau_2)\rho(M^{\frac{1}{2}(j+h)}|\tau_2|)j_{\beta,en}(\tau_1,\tau_2) \quad (III.31b)$$

$$- \int d\tau_2 \mathbf{C}^{(j)}(\tau_1)\rho(M^{\frac{1}{2}(j+h)}|\tau_1|)j_{\beta,en}(\tau_1,\tau_2) \quad (III.31c)$$

For (III.31a) we simply treat $\frac{d}{d\tau}[\mathbf{C}^{(j)}(\tau)h(M^{\frac{1}{2}(j+h)}|\tau|)]$ as a new covariance, obeying

$$\Big|\frac{d}{d\tau}\mathbf{C}^{(j)}(\tau)h(M^{\frac{1}{2}(j+h)}|\tau|)\Big|, \Big|\frac{d}{d\tau}\mathbf{C}^{(j)}(\tau)h(M^{\frac{1}{2}(j+h)}|\tau|)\Big|'$$

$$\leq M^{\frac{1}{2}(j+h)} \times \text{ old bound on } \mathbf{C}^{(j)}$$

and treat $j_{\beta,en}(\tau_1,\tau_2)(\tau_2-\tau_1)$ as a new vertex obeying

$$\big|j_{\beta,en}(\tau_1,\tau_2)(\tau_2-\tau_1)\big| \leq M^{-h}\|j_{\beta,en}\|_h.$$

Thus we gain $M^{-\frac{1}{2}(h-j)}$ from (III.31a) as desired.

Contributions (III.31b,c) are easy to treat when $\mathbf{C}^{(j)}$ is in the spanning tree of technical Lemma III.2. In this case we merely replace

$$\int dt|\mathbf{C}^{(j)}(\tau,\mathbf{k})| \leq \text{const} M^{-j}\chi(E(\mathbf{k}) \approx M^j)$$

by

$$\begin{aligned}\int d\tau |\mathbf{C}^{(j)}(\tau,\mathbf{k})|\rho(M^{\frac{1}{2}(j+h)}|\tau|) \\ \leq \mathrm{const}\chi(E(\mathbf{k})\approx M^j)\int d\tau\rho(M^{\frac{1}{2}(j+h)}|\tau|) \\ = \mathrm{const}M^{-\frac{1}{2}(j+h)}\chi(E(\mathbf{k})\approx M^j).\end{aligned} \tag{III.32}$$

Once again renormalization generates an extra factor of $M^{-\frac{1}{2}(h-j)}$.

When $\mathbf{C}^{(j)}$ is not in the spanning tree we take its $\big|\cdot\big|'$ norm (II.5b). Then, since $\sup\rho = 1$, there is no gain.

To circumvent this difficulty we modify the abstract power counting lemma to allow separate spanning trees for τ and $\mathbf{k}$. The norms (III.5) on the covariances are generalized to

$$\sup_{\mathbf{k}}\sup_{\tau}|C_\ell(\tau,\mathbf{k})| \leq a_\ell 1\times 1 \tag{III.33a}$$

$$\big|C_\ell\big| = \sup_{\mathbf{k}}\int d\tau|C_\ell(\tau,\mathbf{k})| \leq a_\ell 1\times M^{-j_\ell} \tag{III.33b}$$

$$\big|C_\ell\big|' = \int\frac{d\mathbf{k}}{(2\pi)^d}\sup_{\tau}|C_\ell(\tau,\mathbf{k})| \leq a_\ell M^{i_\ell}\times 1 \tag{III.33c}$$

$$\int\frac{d\mathbf{k}}{(2\pi)^d}\int d\tau|C_\ell(\tau,\mathbf{k})| \leq a_\ell M^{i_\ell}\times M^{-j_\ell}. \tag{III.33d}$$

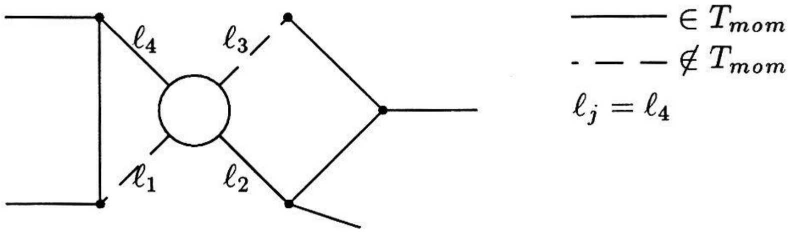
The hard covariances $\mathbf{C}^{(j)},\mathbf{C}_1^{(j)},\mathbf{C}_2^{(j)}$ (III.17) and the string (II.6) all obey (III.32a-d) with $i_\ell = j_\ell = j$. The soft covariances $\mathbf{C}_s^{(j)},\mathbf{C}_{1,s}^{(j)},\mathbf{C}_{2,s}^{(j)}$ (III.18) obey (III.32a,c) with $i_\ell = j_\ell = j$. The product of any of $\mathbf{C}^{(j)},\mathbf{C}_s^{(j)},\mathbf{C}_1^{(j)},\mathbf{C}_{1,s}^{(j)},\mathbf{C}_2^{(j)},\mathbf{C}_{2,s}^{(j)}$ or the string with $\rho(M^{\frac{1}{2}(j+h)}\tau)$ obeys (III.32a-d) with $i_\ell = j, j_\ell = \frac{1}{2}(j+h) > j$.

We select any two spanning trees T_{mom}, T_{time} for the graph G. When the line $\ell\in T_{\mathrm{mom}}\cap T_{\mathrm{time}}$ (resp. $T_{\mathrm{mom}}\cap T_{\mathrm{time}}^c$, $T_{\mathrm{mom}}^c\cap T_{\mathrm{time}}$, $T_{\mathrm{mom}}^c\cap T_{\mathrm{time}}^c$) we require that C_ℓ obey (III.33b) (resp. (III.33a), (III.33d), (III.33c)). Then mimicing the proof of Lemma III.1.a one obtains

$$\big|\mathrm{Val}(G)\big| \leq \prod_\ell a_\ell\prod_v|T_v|\prod_v|I_v|\prod_{\ell\in T_{\mathrm{mom}}^c}M^{i_\ell}\prod_{\ell\in T_{\mathrm{time}}}M^{-j_\ell} \tag{III.34}$$

We now apply (III.34) to extract the required decay factor for (III.31b,c). Call the four lines hooked to $j_{\beta,en},\ell_1,\ell_2,\ell_3$ and ℓ_4 with $C^{(j)}$ being the covariance for ℓ_1. Let T_{mom} be original spanning tree of Technical Lemma III.2. By hypotheses $\ell_1\notin T_{\mathrm{mom}}$. But at least one ℓ_2,ℓ_3,ℓ_4 is. The tree $T_{\mathrm{mom}}\backslash\{\ell_2,\ell_3,\ell_4\}$ is a union of at most three connected components.

one ℓ_2, ℓ_3, ℓ_4 is. The tree $T_{\mathrm{mom}} \backslash \{\ell_2, \ell_3, \ell_4\}$ is a union of at most three connected components. The vertex of G at the far end of ℓ_1 is in one of those components, say the component ending at ℓ_α. Use $M^{\frac{1}{2}(j_\alpha+h)}$ rather than $M^{\frac{1}{2}(j+h)}$ in (III.31). (The improvement for (III.31a) is now $M^{\frac{1}{2}(h-j_\alpha)}$.)



We choose $T_{\mathrm{time}} = T_{\mathrm{mom}} \backslash \{\ell_2\} \cup \{\ell_1\}$. Thus the only consequence of changing the time tree is that the time integral is done using ℓ_1 (which now gives, by (III.32), $M^{-\frac{1}{2}(j_\alpha+h)}$ instead of 1) rather than ℓ_a (which now gives 1 instead of M^{-j_α}). The net improvement is $M^{-\frac{1}{2}(h-j_\alpha)}$.

■ sGw.

<u>Lemma III.7</u> Let G^J be a labelled two legged graph with $t(G^J) = \mathcal{T}$ with G^J and $\mathcal{T}$ satisfying the hypotheses of Theorem III.6. If G^J is of type τ^1 or τ^2 then

$$\sum_J |\boldsymbol{\ell}\mathrm{Val}(G^J)| \le \mathrm{const}^{L(G)} \triangle |\ell n \triangle| |\lambda|^{1/2} \prod_v [|\lambda|^{1/4} \omega_v] \begin{cases} \lambda & \text{if } \#\{v\} = 2 \\ 1 & \text{otherwise} \end{cases} \qquad (III.35)$$

<u>Proof</u> The proof is almost identical to that of Theorem III.6. However now, since G^J is of type τ^1 or τ^2, it must contain at least one line of type $\mathbf{C}_2$ or $\mathbf{C}_{2,s}$. Hence Lemma III.3 provides an extra factor of $\frac{\triangle}{M^j} \log\left(\frac{M^j}{\triangle}\right) \le \frac{\triangle}{M^j} |\log \triangle|$ for this line. We are only interested in gaining one extra $\triangle$ (and $|\log \triangle|$'s) so we may as well assume that there is only one $\mathbf{C}_2/\mathbf{C}_{2,s}$ line.

Denote by $f_\triangle$ the highest f for which the line is in G^J_f. First suppose that there is no c fork between $f_\triangle$ and ϕ (including $f_\triangle$ but excluding ϕ). Then since

$$\frac{\triangle}{M^{j_{f_\triangle}}} \log \triangle = \left(\frac{\triangle}{M^{j_\phi}} \log \triangle\right) \prod_{\phi < f \le f_\triangle} M^{-(j_f - j_{\pi(f)})} \qquad (III.36)$$

we get, as in Theorem III.6,

$$\sum_{\{j_f, |f'>\phi\}} |\boldsymbol{\ell}\mathrm{Val}(G^J)| \le \mathrm{const}^{L(G)} M^{j_\phi} \left(\frac{\triangle}{M^{j_\phi}} |\log \triangle|\right) \lambda^{3/2} \prod_v [|\lambda|^{1/4}\omega_v] \qquad (III.37)$$

The sum over j_ϕ from $\log \triangle$ to 0 gives another $|\log \triangle| \le \frac{\mathrm{const}}{|\lambda|}$ and hence (III .35) when G is at least third order. When G is of second order Lemma IV.6 insures that the sum over j_ϕ is bounded uniformly in $\triangle$ so that we do not loose the $\frac{\mathrm{const}}{|\lambda|}$ in this case.

The proof continues by induction with previously estimated ℓG_f^J's fed in as 2-legged vertices. Each such vertex comes with a coefficient

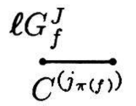
$$M^{j_{\pi(f)}} \left(\frac{\triangle}{M^{j_{\pi(f)}}} |\log \triangle|\right) |\lambda|^{1/2} \omega_f \qquad (III.38a)$$

where

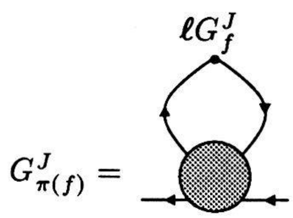
$$\omega_f = \mathrm{const}^{L(G_f)} \prod_{v>f} [|\lambda|^{1/4}\omega_v]. \qquad (III.38b)$$

The factor $M^{j_{\pi(f)}}$ renders the 2-legged vertex dimensionless as usual and the factor $\left(\frac{\triangle}{M^{j_{\pi(f)}}} |\log \triangle|\right)$ mimics a $\mathbf{C}_2/\mathbf{C}_{2,s}$ line in $G^J_{\pi(f)}$. At first sight $|\lambda|^{1/2}\omega_f$ seems to have too small a power of $|\lambda|$, since the original vertices had $|\lambda|^{3/4}[|\lambda|^{1/4}\omega_v]$. However, by the remark following the statement of Lemma III.5, $|\lambda|^{1/2}$ is sufficient provided $b_{\pi(f)} \ge 3$.

It remains only to consider $b_{\pi(f)} = 2$. If $\boldsymbol{\ell} G_f^J$ is an external vertex, then when $\boldsymbol{\ell}$ is applied to G_ϕ^J.



the hard covariance $\mathbf{C}^{(j_{\pi(f)})}$, connecting $\boldsymbol{\ell} G_f^J$ to the rest of G_ϕ^J, is evaluated on the fermi surface and vanishes. If $\boldsymbol{\ell} G_f^J$ is internal and $b_{\pi(f)} = 2$ then



Once again Lemma IV.6 saves us a factor of λ.

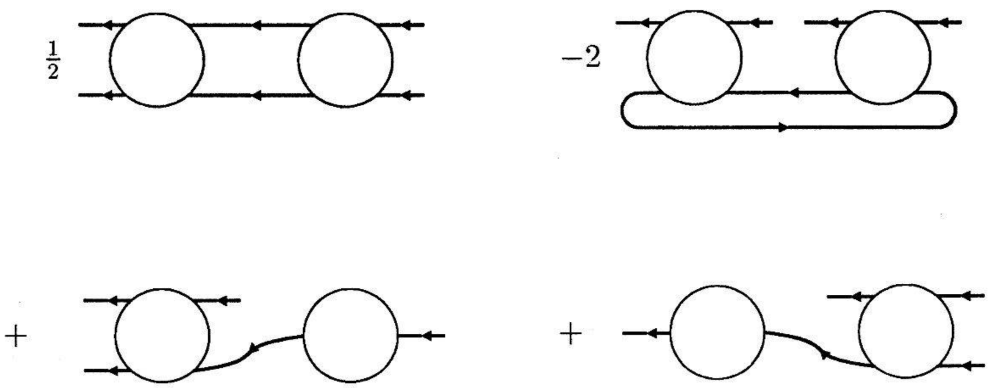
■

IV. Second Order Graphs

The purpose of this section is to control the quadratic and quartic parts of $\frac{1}{2!}\mathcal{E}_2^{(h)}(\mathcal{U},\mathcal{U})$ where $\mathcal{U}$ is the part of $\mathcal{W}^{(h)}$ of degree at most four (see [I.102a]). The truncated expectation $\mathcal{E}_2^{(h)}$ (1.17) is with respect to the superconducting covariance $\mathbf{C}_\triangle^{(h)}$. Here, as in Section III, all monomials are Wick ordered and degree is interpreted accordingly.

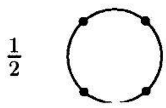
We begin with the quartic part. The quadratic part is treated following (IV.48). In more detail

$$\text{quartic part of } \frac{1}{2!}\mathcal{E}_2^{(h)}(\mathcal{U},\mathcal{U}) =$$

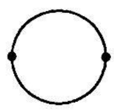


$(IV.1)$

The kernel



of the quartic part of $\mathcal{U}$ is a sum of renormalized $(\mathbf{1}-\mathbf{L}^{(i)})I$ and local (i.e. in the range of $\mathbf{L}^{(j)}$, see (I.99)) kernels I. Also, the kernel



of the quadratic part of $\mathcal{U}$ consists of renormalized and local contributions $(\mathbf{1}-\mathbf{L}^{(s)})S$ and (see (I.99a)) $\boldsymbol{\ell}S$. By [I.102] the renormalized contributions are respectively of scales $i, s > h$, while the (resummed [I.102b]) local quartic part has scale $i = h$ and the local quadratic part has scale $s \leq h$.

As in the last section (III.26) we use the norms

$$\|I\|_i = \max\Bigg\{ M^{\left(\sum_{j=2}^4 \alpha_j + \sum_{j=1}^4 \beta_j\right) i} \sup_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4=0} \int d\tau_2 d\tau_3 d\tau_4$$

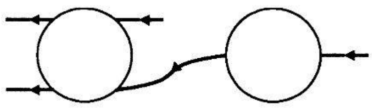
$$\prod_{j=2}^4 |\tau_j|^{\alpha_j} \left| \prod_{j=1}^4 \nabla_{\mathbf{k}_j}^{\beta_j} I((0,\mathbf{k}_1),(\tau_2,\mathbf{k}_2),(\tau_3,\mathbf{k}_3),(\tau_4,\mathbf{k}_4)) \right| :$$

$$\alpha_j \geq 0, \quad \sum_{j=2}^4 \alpha_j \leq 2, \quad \sum_{j=1}^4 |\beta_j| \leq 1 \Bigg\} \tag{IV.2a}$$

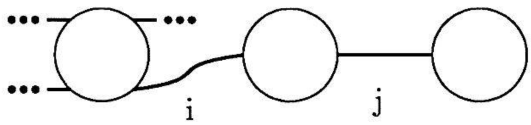
$$\|S\|_s = \max\left\{ M^{(\alpha+\beta)s} \sup_{\mathbf{k}} \int d\tau |\tau|^\alpha |\nabla_{\mathbf{k}}^\beta S((0,-\mathbf{k}),(\tau,\mathbf{k}))| : 0 \leq \alpha \leq 2, \quad |\beta| \leq 1 \right\} \tag{IV.2b}$$

to measure the size of four and two legged kernels.

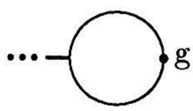
The bottom two diagrams are easy to treat. If



ends up as part of a larger diagram



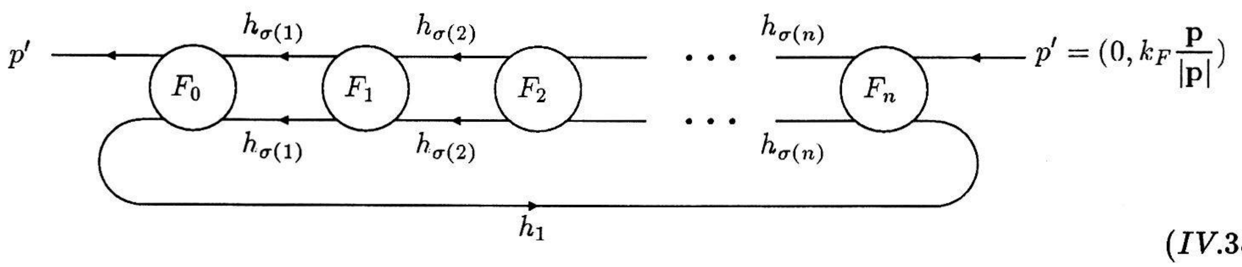
then, by conservation of momentum, the scales i and j differ by at most one and the second order diagram may be absorbed into the larger one. The latter, being of at least third order, was treated in the last section. If



ends up integrated against an external test function g there is a summable factor $M^{\triangle(h-h_\pi)}$ (see Lemma III.1) associated with the diagram rendering it harmless. This was also treated in the last section.

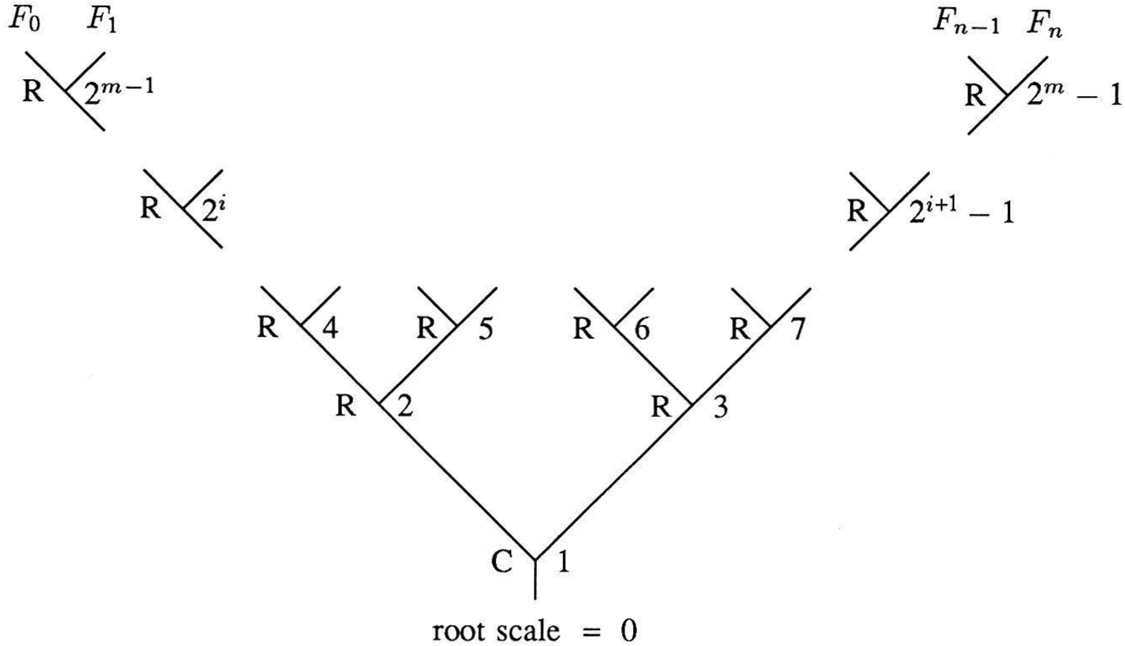
We now consider the top two diagrams of (IV.1) when at least one of the generalized vertices is renormalized. Our goal will be to extract exponential decay between scales hidden in the renormalization. In order to control the action of $(\mathbf{1} - \mathbf{L})$ the technique of [FT] is supplemented by a detailed analysis of the volume of momentum space that is actually integrated over.

To illustrate the mechanism consider, as an example, the labelled graph



$(IV.3a)$

contributing to the binary tree



$(IV.3b)$

Here, $n = 2^m - 1$ and the indices

$$\sigma(j) = \frac{1}{2}[(2^m + j)/2^{e_j} - 1]$$

where 2^{e_j} is the largest power of 2 dividing $2^m + j$. For example, in the case of the $2^3 - 1$ ladder

$$\sigma(1) = 4$$
$$\sigma(2) = 2$$
$$\sigma(3) = 5$$
$$\sigma(4) = 1$$
$$\sigma(5) = 6$$
$$\sigma(6) = 3$$
$$\sigma(7) = 7.$$

Note that we have written h_j as j at the forks of figure (IV.3b).

The tree rules (I.34) require that we sum over all h_j obeying $0 \geq h_j > h_{\pi(j)}$ for $j = 2, ..., 2^m - 1$ and $h_1 \leq 0$, where $\pi(j)$ is the index of the predecessor fork of j, that is the integer part of $j/2$.

In discussing (IV.3) we restrict ourselves to the most subtle term, (I.41b), of the operator $(\mathbf{1} - \mathbf{L}^{(h_i)})$ at R forks. See the discussion following (I.41). To make the example especially simple, we assume that all lines are hard, that the kernels $F_i = \tau^0 \otimes \tau^0$ and that $\triangle = 0$.

Then the value of (IV.3a)

$$\sum_{h_i} \int \frac{dk}{(2\pi)^{d+1}} \mathbf{C}_{\triangle}^{(h_1)}(k)_{1,1} \prod_{i=1}^{n} \left[\int \frac{dt_i}{(2\pi)^{d+1}} \mathbf{C}_{\triangle}^{(h_i)}(t_i + p')_{1,1} \mathbf{C}_{\triangle}^{(h_i)}(-t_i + k)_{1,1} \right]$$

$$\prod_{j=2}^{n} \left[1 - \rho(|\mathbf{p}' + \mathbf{k}| M^{-\frac{1}{2}[h_{\pi(j)} + h_j]}) \right] \tau^0 \tag{IV.3c}$$

$$= \sum_{h_i} \int \frac{dk}{(2\pi)^{d+1}} \frac{1}{ik_0 - e(\mathbf{k})} \prod_{i=1}^{n} \int_{e(\mathbf{t}_i + \mathbf{p}')e(-\mathbf{t}_i + \mathbf{k}) > 0} \frac{d\mathbf{t}_i}{(2\pi)^d}$$

$$[-ik_0 \operatorname{sgn} e(\mathbf{t}_i + \mathbf{p}') + |e(\mathbf{t}_i + \mathbf{p}')| + |e(-\mathbf{t}_i + \mathbf{k})|]^{-1}$$

$$f(M^{-2h_1} e(\mathbf{k})^2) \prod_{i=1}^{n} [f(M^{-2h_i} e(\mathbf{t}_i + \mathbf{p}')^2) f(M^{-2h_i} e(-\mathbf{t}_i + \mathbf{k})^2)]$$

$$\prod_{j=2}^{n} \left[1 - \rho\left(|\mathbf{p}' + \mathbf{k}| M^{-\frac{1}{2}[h_{\pi(j)} + h_j]} \right) \right] \tau^0$$

Evaluating the k_0-integral yields a sum of terms one of which is

$$\sum_{h_i} -\mathrm{sgn}(e(\mathbf{k})) \int \frac{d\mathbf{k}}{(2\pi)^d} f(M^{-2h_1}e(\mathbf{k})^2) \prod_{i=1}^{n} \int_{e(\mathbf{t}_i+\mathbf{p}')e(-\mathbf{t}_i+\mathbf{k})>0} \frac{d\mathbf{t}_i}{(2\pi)^d}$$

$$\frac{f(M^{-2h_i}e(\mathbf{t}_i+\mathbf{p}')^2)f(M^{-2h_i}e(-\mathbf{t}_i+\mathbf{k})^2)}{|e(\mathbf{t}_i+\mathbf{p}')|+|e(-\mathbf{t}_i+\mathbf{k})|-e(\mathbf{k})\mathrm{sgn}e(\mathbf{t}_i+\mathbf{p}')}$$

$$\prod_{j=2}^{n} \left[1-\rho(|\mathbf{p}'+\mathbf{k}|M^{-\frac{1}{2}[h_{\pi(j)}+h_j]})\right] \tau^0 \qquad (IV.4)$$

The analysis of the remaining terms is similar to that of (IV.4) so we concentrate on the latter. It is bounded in magnitude by

$$\sum_{h_i}(\mathrm{const})^n \prod_{i=1}^{n} M^{-h_i} \int d\mathbf{k} f(M^{-2h_1}e(\mathbf{k})^2) \prod_{i=1}^{n} \int d\mathbf{t}_i f(M^{-2h_i}e(\mathbf{t}_i+\mathbf{p}')^2) f(M^{-2h_i}e(-\mathbf{t}_i+\mathbf{k})^2)$$

$$\prod_{j=2}^{n} \left[1-\rho(|\mathbf{p}'+\mathbf{k}|M^{-\frac{1}{2}[h_{\pi(j)}+h_j]})\right]$$

Ignoring the last product, that is the effect of $\prod_{i=2}^{n}(\mathbf{1}-\mathbf{L}^{(h_i)})$, the volume of integration over $\mathbf{k}$ and each $\mathbf{t}_i, 1 \le i \le n$, may be estimated as usual by M^{h_1} and M^{h_i} respectively. This yields the "unrenormalized" bound

$$(\mathrm{const})^n \sum_{\substack{h_{\pi(j)}<h_j\le 0 \\ h_1 \le 0}} M^{h_1} \prod_{i=1}^{n} M^{-h_i}M^{h_i}$$

$$= (\mathrm{const})^n \sum_{h_1=-\infty}^{0} M^{h_1} \sum_{h_{\pi(j)}<h_j\le 0} 1$$

One sees by induction that

$$\sum_{h_{\pi(j)}<h_j\le 0} 1 \sim (\mathrm{const})^n |h_1|^{n-1}.$$

Therefore, the unrenormalized bound is

$$(\mathrm{const})^n \sum_{h_1=-\infty}^{0} |h_1|^{n-1} M^{h_1} \sim (\mathrm{const})^n n! \qquad (IV.5a)$$

To eliminate the $n!$ in (IV.5a) we exploit the last product in (IV.4) by applying Lemma IV.2 below. For each $2 \le j \le n$ the jth factor of this product forces $q = p' + k$ to be large compared to the scale of the predecessor fork $i = \pi(j)$. To each fork with $1 \le i < 2^{m-1} = \frac{1}{2}(n+1)$ there correspond two such factors, namely $j = 2i, 2i+1$. Selecting the one that gives the larger gap between scales and hence better decay leads us, via Lemma IV.2, to

$$\left\{1 - \rho(|\mathbf{p}' + \mathbf{k}|M^{-\frac{1}{2}[h_i + \max(h_{2i}, h_{2i+1})]})\right\} \int d\mathbf{t}_i f(M^{-2h_i} e(\mathbf{t}_i + \mathbf{p}')^2) f(M^{-2h_i} e(-\mathbf{t}_i + \mathbf{k})^2)$$

$$\le \text{const} M^{-\frac{1}{2}\{\frac{1}{2}[h_i + \max(h_{2i}, h_{2i+1})] - h_i\}} M^{h_i}$$

$$= \text{const} M^{-\{\frac{1}{4}\max(h_{2i}, h_{2i+1}) - \frac{1}{4}h_i\}} M^{h_i}$$

$$\le \text{const} M^{-\frac{1}{8}(h_{2i} - h_i)} M^{-\frac{1}{8}(h_{2i+1} - h_i)} M^{h_i}$$

for each $1 \le i < 2^{m-1}$ (Lemma IV.2 is used in the second line) yielding the renormalized bound

$$(\text{const})^n \sum_{h_i} M^{h_1} \prod_{i=1}^{n} M^{-h_i} M^{h_i} \prod_{j=2}^{n} M^{-\frac{1}{8}(h_j - h_{\pi(j)})}$$

$$\le (\text{const})^n \sum_{h_1 = -\infty}^{0} M^{h_1} \prod_{j=2}^{n} \left(\sum_{0 \ge h_j > h_{\pi(j)}} M^{-\frac{1}{8}(h_j - h_{\pi(j)})} \right)$$

$$\le (\text{const})^n \qquad (IV.5b)$$

Consider a pair of particles or holes with momenta $\pm t + q/2$. Pairs that bind have $||\pm \mathbf{t} + \mathbf{q}/2| - k_F|$ small. Our immediate goal is to show that the volume of $\{\mathbf{t} : ||\pm \mathbf{t} + \mathbf{q}/2| - k_F| \le 0(M^i)\}$ for component particles of scale i is small when the scale of the composite momentum q is comparatively high. This effect was exactly what was required in the derivation of the renormalized bound (IV.5b). To do this we first prove the Technical Lemma IV.1 from which we shall derive Lemma IV.2 that estimates the amount of momentum space available for condensation.

Technical Lemma IV.1

a) If $\nu \ge 0$ or $|\mathbf{q}| \ge 2k_F$, then

$$\int_{\substack{|\mathbf{t}+\mathbf{q}/2| \le k_F \\ |-\mathbf{t}+\mathbf{q}/2| \le k_F}} d^d\mathbf{t} \quad \delta(e(\mathbf{t} + \mathbf{q}/2) + e(-\mathbf{t} + \mathbf{q}/2) - \nu) = 0$$

If $\nu < 0$ and $|\mathbf{q}| < 2k_F$, then

$$\int_{\substack{|\mathbf{t}+\mathbf{q}/2|\le k_F\\ |-\mathbf{t}+\mathbf{q}/2|\le k_F}} d^d\mathbf{t} \quad \delta(e(\mathbf{t}+\mathbf{q}/2)+e(-\mathbf{t}+\mathbf{q}/2)-\nu)$$

$$\le \mathrm{const}_d \begin{cases} \frac{|\nu|}{|\mathbf{q}|}\left[m\nu + k_F^2 - \frac{|\mathbf{q}|^2}{4}\right]^{\frac{d-3}{2}}, & \text{if } m|\nu| \le \frac{|\mathbf{q}|}{2}(2k_F - |\mathbf{q}|) \\ \left[m\nu + k_F^2 - \frac{|\mathbf{q}|^2}{4}\right]^{\frac{d-2}{2}}, & \text{if } \frac{|\mathbf{q}|}{2}(2k_F-|\mathbf{q}|) \le m|\nu| \le k_F^2 - \frac{|\mathbf{q}|^2}{4} \\ 0, & \text{if } k_F^2 - \frac{|\mathbf{q}|^2}{4} \le m|\nu| \end{cases} \tag{IV.6}$$

b) If $\nu \le 0$, then

$$\int_{\substack{|\mathbf{t}+\mathbf{q}/2|\ge k_F\\ |-\mathbf{t}+\mathbf{q}/2|\le k_F}} d^d\mathbf{t} \quad \delta(e(\mathbf{t}+\mathbf{q}/2)-e(-\mathbf{t}+\mathbf{q}/2)-\nu) = 0$$

If $\nu > 0$ and $|\mathbf{q}| \ge 2k_F$, then

$$\int_{\substack{|\mathbf{t}+\mathbf{q}/2|\ge k_F\\ |-\mathbf{t}+\mathbf{q}/2|\le k_F}} d^d\mathbf{t} \quad \delta(e(\mathbf{t}+\mathbf{q}/2)-e(-\mathbf{t}+\mathbf{q}/2)-\nu)$$

$$\le \mathrm{const}_d \begin{cases} 0, & \text{if } m\nu \le \frac{|\mathbf{q}|}{2}(|\mathbf{q}|-2k_F) \\ \left[k_F^2 - \left(\frac{|\mathbf{q}|}{2} - \frac{m\nu}{|\mathbf{q}|}\right)^2\right]^{\frac{d-1}{2}}, & \text{if } \frac{|\mathbf{q}|}{2}(|\mathbf{q}|-2k_F) \le m\nu \le \frac{|\mathbf{q}|}{2}(|\mathbf{q}|+2k_F) \\ 0, & \text{if } \frac{|\mathbf{q}|}{2}(|\mathbf{q}|+2k_F) \le m\nu \end{cases} \tag{IV.7}$$

If $\nu > 0$ and $|\mathbf{q}| \le 2k_F$, then

$$\int_{\substack{|\mathbf{t}+\mathbf{q}/2|\ge k_F\\ |-\mathbf{t}+\mathbf{q}/2|\le k_F}} d^d\mathbf{t} \quad \delta(e(\mathbf{t}+\mathbf{q}/2)-e(-\mathbf{t}+\mathbf{q}/2)-\nu)$$

$$\le \mathrm{const}_d \begin{cases} \frac{\nu}{|\mathbf{q}|}\left[k_F^2 - \left(\frac{|\mathbf{q}|}{2} - \frac{m\nu}{|\mathbf{q}|}\right)^2\right]^{\frac{d-3}{2}}, & \text{if } m\nu \le \frac{|\mathbf{q}|}{2}(2k_F-|\mathbf{q}|) \\ \left[k_F^2 - \left(\frac{|\mathbf{q}|}{2} - \frac{m\nu}{|\mathbf{q}|}\right)^2\right]^{\frac{d-1}{2}}, & \text{if } \frac{|\mathbf{q}|}{2}(2k_F-|\mathbf{q}|) \le m\nu \le \frac{|\mathbf{q}|}{2}(2k_F+|\mathbf{q}|) \\ 0, & \text{if } \frac{|\mathbf{q}|}{2}(2k_F+|\mathbf{q}|) \le m\nu \end{cases} \tag{IV.8}$$

c) If $\nu \le 0$, then

$$\int_{\substack{|\mathbf{t}+\mathbf{q}/2|\ge k_F\\ |-\mathbf{t}+\mathbf{q}/2|\ge k_F}} d^d\mathbf{t} \quad \delta(e(\mathbf{t}+\mathbf{q}/2)+e(-\mathbf{t}+\mathbf{q}/2)-\nu) = 0$$

If $\nu \ge 0$ and $|\mathbf{q}| \ge 2k_F$ then

$$\int_{\substack{|\mathbf{t}+\mathbf{q}/2|\ge k_F\\ |-\mathbf{t}+\mathbf{q}/2|\ge k_F}} d^d\mathbf{t} \quad \delta(e(\mathbf{t}+\mathbf{q}/2)+e(-\mathbf{t}+\mathbf{q}/2)-\nu)$$

$$\leq \text{const}_d \begin{cases} 0, & \text{if } m\nu \leq \frac{|\mathbf{q}|^2}{4} - k_F^2 \\ \left[m\nu + k_F^2 - \frac{|\mathbf{q}|^2}{4}\right]^{\frac{d-2}{2}}, & \text{if } \frac{|\mathbf{q}|^2}{4} - k_F^2 \leq m\nu \leq \frac{|\mathbf{q}|}{2}\left(|\mathbf{q}| - 2k_F\right) \\ \frac{\nu}{|\mathbf{q}|}\left[m\nu + k_F^2 - \frac{|\mathbf{q}|^2}{4}\right]^{\frac{d-3}{2}}, & \text{if } \frac{|\mathbf{q}|}{2}\left(|\mathbf{q}| - 2k_F\right) \leq m\nu \leq \frac{|\mathbf{q}|}{2}\left(|\mathbf{q}| + 2k_F\right) \\ \left[m\nu + k_F^2 - \frac{|\mathbf{q}|^2}{4}\right]^{\frac{d-2}{2}}, & \text{if } \frac{|\mathbf{q}|}{2}\left(|\mathbf{q}| + 2k_F\right) \leq m\nu \end{cases} \tag{IV.9}$$

If $\nu \geq 0$ and $|\mathbf{q}| \leq 2k_F$, then

$$\int_{\substack{|\mathbf{t}+\mathbf{q}/2| \geq k_F \\ |-\mathbf{t}+\mathbf{q}/2| \geq k_F}} d^d\mathbf{t} \quad \delta(e(\mathbf{t} + \mathbf{q}/2) + e(-\mathbf{t} + \mathbf{q}/2) - \nu)$$

$$\leq \text{const}_d \begin{cases} \frac{\nu}{|\mathbf{q}|}\left[m\nu + k_F^2 - \frac{|\mathbf{q}|^2}{4}\right]^{\frac{d-3}{2}}, & \text{if } 0 \leq m\nu \leq \frac{|\mathbf{q}|}{2}\left(|\mathbf{q}| + 2k_F\right) \\ \left[m\nu + k_F^2 - \frac{|\mathbf{q}|^2}{4}\right]^{\frac{d-2}{2}}, & \text{if } \frac{|\mathbf{q}|}{2}\left(|\mathbf{q}| + 2k_F\right) \leq m\nu \end{cases} \tag{IV.10}$$

Proof a) Observe that

$$e(\mathbf{t} + \mathbf{q}/2) + e(-\mathbf{t} + \mathbf{q}/2) - \nu = \frac{1}{m}(\mathbf{t}^2 - \alpha)$$

with

$$\alpha = m\nu + 2m\mu - \mathbf{q}^2/4.$$

Let $d\sigma_n(\phi_1, ..., \phi_{n-1}, \theta)$ be the surface measure on S^n expressed in polar coordinates. Then, for $n \geq 2$,

$$d\sigma_n(\phi_1, ..., \phi_{n-1}, \theta) = \sin^{n-1}\phi_1 d\phi_1 d\sigma_{n-1}(\phi_2, ..., \phi_{n-1}, \theta)$$

so that, for $d \geq 3, \alpha \geq 0$

$$\begin{aligned} \int_{\frac{\pi}{2}-\epsilon \leq \phi_1 \leq \frac{\pi}{2}+\epsilon} d^d\mathbf{t} \quad \delta\left(\frac{1}{m}(\mathbf{t}^2 - \alpha)\right) &= \int dr d\phi_1 d\sigma_{d-2} r^{d-1} \sin^{d-2}\phi_1 \delta\left(\frac{1}{m}(r^2 - \alpha)\right) \\ &= m\alpha^{\frac{d-2}{2}} \Phi(\epsilon) \end{aligned} \tag{IV.11}$$

where

$$\Phi(\epsilon) := \frac{1}{2}\omega_{d-2} \int_{-\epsilon}^{\epsilon} d\phi_1 \cos^{d-2}\phi_1 \leq \omega_{d-2}\epsilon$$

Formula (IV.11) also applies for $d = 2$ with $\omega_0 = 2$.

There are five cases:

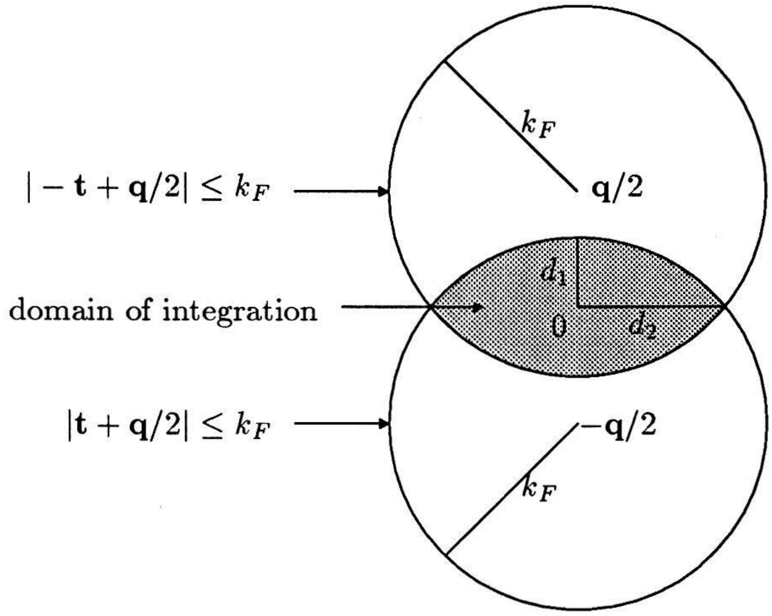
case 1 $\nu \geq 0$: The integrand is zero, since on the domain of integration $e(\mathbf{t} + \mathbf{q}/2), e(-\mathbf{t} + \mathbf{q}/2) < 0$.

case 2 $|\mathbf{q}| \geq 2k_F$: The domain of integration is empty since if $|\mathbf{t}+\mathbf{q}/2| < k_F$ and $|-\mathbf{t}+\mathbf{q}/2| < k_F$, then $|\mathbf{q}| = |\mathbf{q}/2 + \mathbf{t} + \mathbf{q}/2 - \mathbf{t}| < 2k_F$.

case 3 $-k_F^2 + \frac{|\mathbf{q}|^2}{4} \leq m\nu \leq -\frac{|\mathbf{q}|}{2}(2k_F - |\mathbf{q}|)$: Let $d_1 := \left(k_F - \frac{|\mathbf{q}|}{2}\right)$ and $d_2 := \left[k_F^2 - \left(\frac{|\mathbf{q}|}{2}\right)^2\right]^{1/2}$. Of course, $d_1 \leq d_2$. If, $0 \leq \alpha \leq d_1^2$, or equivalently,

$$0 \leq m\nu + k_F^2 - \frac{|\mathbf{q}|^2}{4} \leq k_F^2 - k_F|q| + \frac{|\mathbf{q}|^2}{4},$$

the support of $\delta\left(\frac{1}{m}(\mathbf{t}^2 - \alpha)\right)$ is contained in the domain of integration and we may therefore apply (IV.11) with $\epsilon = \frac{\pi}{2}$.



case 4 $-\frac{|\mathbf{q}|}{2}(2k_F - |\mathbf{q}|) \leq m\nu \leq 0$: If $d_1^2 \leq \alpha \leq d_2^2$, or equivalently,

$$k_F^2 - k_F|q| + \frac{|\mathbf{q}|^2}{4} \leq m\nu + k_F^2 - \frac{|\mathbf{q}|^2}{4} \leq k_F^2 - \frac{|\mathbf{q}|^2}{4},$$

we may apply (IV.11) with ϵ determined by

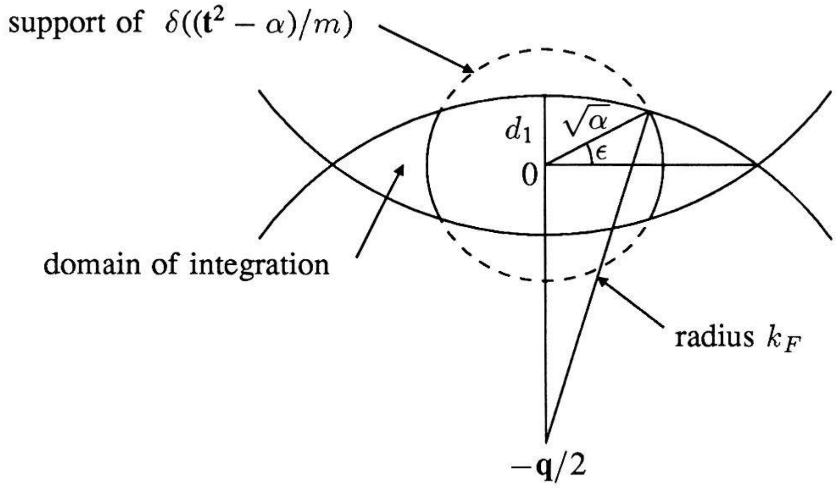
$$k_F^2 = \alpha + \frac{|\mathbf{q}|^2}{4} + \sqrt{\alpha}|\mathbf{q}|\sin\epsilon.$$

Consequently,

$$\int_{\substack{|\mathbf{t}+\mathbf{q}/2|<k_F \\ |-\mathbf{t}+\mathbf{q}/2|<k_F}} d^d\mathbf{t} \quad \delta(e(\mathbf{t}+\mathbf{q}/2) + e(-\mathbf{t}+\mathbf{q}/2) - \nu)$$

$$= m\left[m\nu + k_F^2 - \frac{|\mathbf{q}|^2}{4}\right]^{\frac{d-2}{2}} \Phi\left(\sin^{-1}\frac{-m\nu}{|\mathbf{q}|[m\nu + k_F^2 - \frac{|\mathbf{q}|}{4}]}\right)$$

$$= \mathrm{const}_d\left(-\frac{\nu}{|\mathbf{q}|}\left[m\nu + k_F^2 - \frac{|\mathbf{q}|^2}{4}\right]^{\frac{d-3}{2}}\right)$$



case 5 $m\nu \le -k_F^2 + \frac{|\mathbf{q}|^2}{4}$: If $\alpha \le 0$ the support of the delta function does not intersect the domain of integration.

b) Now

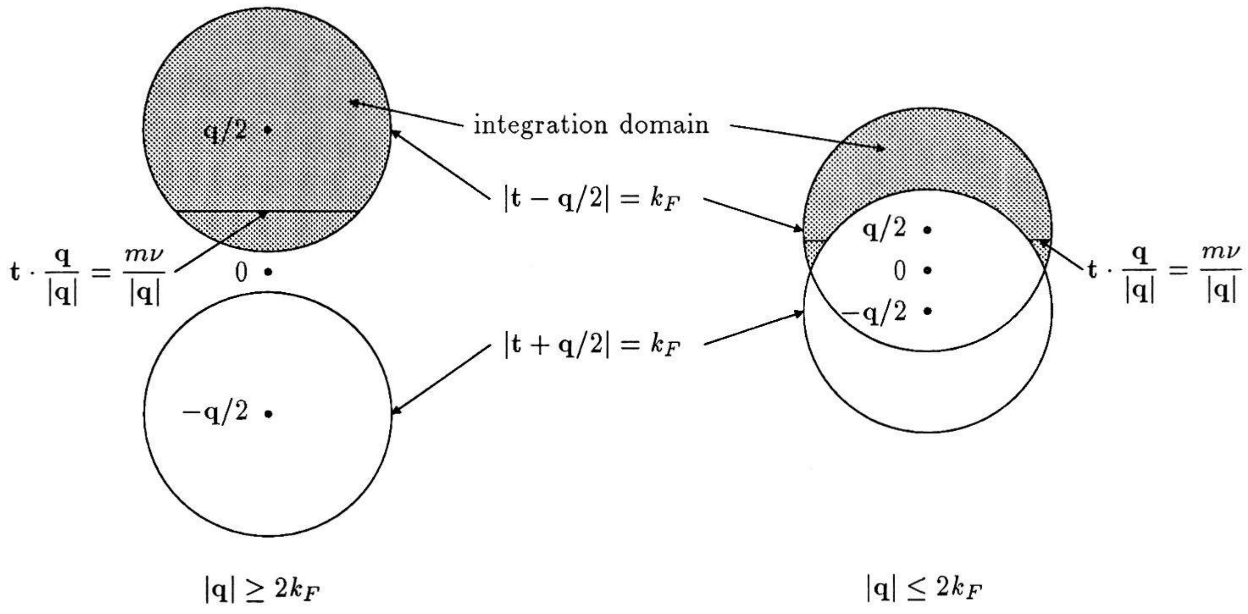
$$e(\mathbf{t} + \mathbf{q}/2) - e(-\mathbf{t} + \mathbf{q}/2) - \nu = \frac{1}{m}\mathbf{t}\cdot\mathbf{q} - \nu$$

so that

$$\int_\Omega d^n\mathbf{t} \quad \delta(e(\mathbf{t} + \mathbf{q}/2) - e(-\mathbf{t} + \mathbf{q}/2) - \nu) = \frac{m}{|\mathbf{q}|}\int_\Omega d^n\mathbf{t} \quad \delta\left(\mathbf{t}\cdot\frac{\mathbf{q}}{|\mathbf{q}|} - \frac{m\nu}{|\mathbf{q}|}\right)$$

$$= \frac{m}{|\mathbf{q}|}\mathrm{Vol}\Omega \cap \left\{\mathbf{t} : \mathbf{t}\cdot\frac{\mathbf{q}}{|\mathbf{q}|} = \frac{m\nu}{|\mathbf{q}|}\right\} \qquad (IV.12)$$

There are three cases.



case 1: If $\nu \leq 0$ or, $|\mathbf{q}| \geq 2k_F, \frac{m\nu}{|\mathbf{q}|} \leq \frac{|\mathbf{q}|}{2} - k_F$ or, $|\mathbf{q}| \geq 2k_F, \frac{m\nu}{|\mathbf{q}|} \geq \frac{|\mathbf{q}|}{2} + k_F$ or, $|\mathbf{q}| \leq 2k_F, \frac{m\nu}{|\mathbf{q}|} \geq \frac{|\mathbf{q}|}{2} + k_F$ then the hyperplane $\mathbf{t} \cdot \frac{\mathbf{q}}{|\mathbf{q}|} = \frac{m\nu}{|\mathbf{q}|}$ fails to intersect the domain of integration and the integral is zero.

case 2: If $|\mathbf{q}| \geq 2k_F, \frac{|\mathbf{q}|}{2} - k_F \leq \frac{m\nu}{|\mathbf{q}|} \leq \frac{|\mathbf{q}|}{2} + k_F$ or, $|\mathbf{q}| \leq 2k_F, k_F - \frac{|\mathbf{q}|}{2} \leq \frac{m\nu}{|\mathbf{q}|} \leq \frac{|\mathbf{q}|}{2} + k_F$ then the hyperplane $\mathbf{t} \cdot \frac{\mathbf{q}}{|\mathbf{q}|} = \frac{m\nu}{|\mathbf{q}|}$ intersects the domain of integration in a $(d-1)$ dimensional ball of radius $\left[k_F^2 - \left(\frac{|\mathbf{q}|}{2} - \frac{m\nu}{|\mathbf{q}|}\right)^2\right]^{1/2}$. Now apply (IV.12).

case 3: If $|\mathbf{q}| \leq 2k_F$ and $0 \leq \frac{m\nu}{|\mathbf{q}|} \leq k_F - \frac{|\mathbf{q}|}{2}$ then the hyperplane intersects the domain of integration in a $(d-1)$ dimensional ball of radius $R = \left[k_F^2 - \left(\frac{|\mathbf{q}|}{2} - \frac{m\nu}{|\mathbf{q}|}\right)^2\right]^{1/2}$ with a $(d-1)$ dimensional ball of radius $r = \left[k_F^2 - \left(\frac{|\mathbf{q}|}{2} + \frac{m\nu}{|\mathbf{q}|}\right)^2\right]^{1/2}$ excluded. By (IV.12) the desired integral is

$$
\begin{aligned}
&\frac{m}{|\mathbf{q}|}\mathrm{const}_d[R^{d-1} - r^{d-1}]\\
=&\frac{m}{|\mathbf{q}|}\mathrm{const}_d\frac{R^2 - r^2}{R+r}[R^{d-2} + rR^{d-3} + ... + Rr^{d-3} + r^{d-2}]\\
\leq&\frac{m}{|\mathbf{q}|}\mathrm{const}_d\frac{R^2 - r^2}{R}[(d-1)R^{d-2}]\\
=&\mathrm{const}_d\frac{\nu}{|\mathbf{q}|}\left[k_F^2 - \left(\frac{|\mathbf{q}|}{2} - \frac{m\nu}{|\mathbf{q}|}\right)^2\right]^{\frac{d-3}{2}}.
\end{aligned}
$$

c) The proof is similar to that of part a)

■

The estimates of Technical Lemma IV.1 are in the spirit of, for example, [FW pages 160-162].

We are ready to estimate the amount of momentum space available for condensation. Consider particle (holes) of momenta $\pm\mathbf{t} + \mathbf{q}/2$ constrained by $f(M^{-2i}e(\pm\mathbf{t} + \mathbf{q}/2)^2) > 0$. That is, particles with momenta lying in a shell of thickness $0(M^i)$ about the Fermi surface. The momentum of the composite is further constrained by $1 - \rho(M^{-j}|\mathbf{q}|) = h(M^{-j}|\mathbf{q}|) > 0$. In other words $|\mathbf{q}| \geq 0(M^j)$. See (I.29a) and (I.41b).

Lemma IV.2 Let $d \geq 2$ and $q = u + v \in \mathbf{R}^{d+1}$. Let $h(M^{-j}|\mathbf{q}|) > 0$ with $j \geq i$. Then,

$$\mathrm{vol}\{\mathbf{t} \in \mathbf{R}^d : f(M^{-2i}e(\mathbf{t} + \mathbf{u})^2) > 0, \quad \mathbf{f}(\mathbf{M}^{-2i}\mathbf{e}(-\mathbf{t} + \mathbf{v})^2) > 0\}$$

$$\leq \mathrm{vol}\{\mathbf{t} \in \mathbf{R}^d : \rho(M^{-2(i+1)}e(\mathbf{t} + \mathbf{u})^2) > 0, \quad \rho(\mathbf{M}^{-2(i+1)}\mathbf{e}(-\mathbf{t} + \mathbf{v})^2) > 0\}$$

$$\leq \mathrm{const}M^{-\frac{1}{2}(j-i)}\mathrm{vol}\{\mathbf{t} \in \mathbf{R}^d : f(M^{-2i}e(\mathbf{t})^2) > 0\}$$

Proof By shifting t it suffices to consider $u = v = q/2$. Since $\rho(x) = 1 - h(x) = \sum_{i=-\infty}^{-1} f(xM^{-2i}) \geq f(xM^2)$ the first inequality is self evident.

The right hand side is bounded below by

$$\mathrm{const}(k_F, d)M^{-\frac{1}{2}(j-i)}M^i.$$

The left hand side is the volume of three disjoint sets. They are determined by the positions of $\pm\mathbf{t} + \mathbf{q}/2$ relative to the Fermi surface.

Inside-inside. The set in which $\pm\mathbf{t} + \mathbf{q}/2$ are both inside the Fermi surface has volume at most.

$$\int_{\substack{-0(M^i)\le e(\mathbf{t}+\mathbf{q}/2)\le 0\\ -0(M^i)\le e(-\mathbf{t}+\mathbf{q}/2)\le 0}} d^d\mathbf{t}$$

$$\begin{aligned}
&= \int d^d\mathbf{t} \int_{|\nu|\le 0(M^i)} d\nu \quad \delta(e(\mathbf{t}+\mathbf{q}/2) + e(-\mathbf{t}+\mathbf{q}/2) - \nu) \\
&= \int_{|\nu|\le 0(M^i)} d\nu \int_{\substack{e(\mathbf{t}+\mathbf{q}/2)\le 0\\ e(-\mathbf{t}+\mathbf{q}/2)\le 0}} d^d\mathbf{t} \quad \delta(e(\mathbf{t}+\mathbf{q}/2) + e(-\mathbf{t}+\mathbf{q}/2) - \nu)
\end{aligned} \tag{IV.13}$$

The contribution with $|\nu| \le \frac{1}{2m}|\mathbf{q}|(2k_F - |\mathbf{q}|) := \nu(\mathbf{q})$ is bounded by

$$\int_{-\min(\nu(\mathbf{q}),0(M^i))}^{0} d\nu \quad \frac{|\nu|}{|q|}[m\nu + k_F^2 - \frac{|\mathbf{q}|^2}{4}]^{\frac{d-3}{2}}. \tag{IV.14}$$

If $d \ge 3$ or if $|\mathbf{q}| \le k_F$, then $[m\nu + k_F^2 - \frac{|\mathbf{q}|^2}{4}]^{\frac{d-3}{2}} \le$ const and (IV.14) is bounded by M^{2i}/M^j as desired. If $d = 2$ and $|\mathbf{q}| \ge k_F$, then

$$\begin{aligned}
(II.16) &\le \text{const} M^i \int_{-\min(\nu(\mathbf{q}),0(M^i))}^{0} d\nu[m\nu + k_F^2 - \frac{|\mathbf{q}|^2}{4}]^{-1/2} \\
&\le \text{const} M^i \int_{\sigma_0}^{\sigma_0+0(M^i)} d\sigma\ \sigma^{-1/2} \\
&\le \text{const} M^{\frac{3}{2}i} \le \text{const} M^{-\frac{1}{2}(j-i)} M^i.
\end{aligned} \tag{IV.15}$$

where $\sigma_0 = -m\min(\nu(\mathbf{q}), 0(M^i)) + k_F^2 - \frac{|\mathbf{q}|^2}{4} \ge \frac{1}{4}(2k_F - |\mathbf{q}|)^2 \ge 0$.

The contribution with $\frac{1}{2}|\mathbf{q}|(2k_F - |\mathbf{q}|) \le m|\nu| \le k_F^2 - \frac{|\mathbf{q}|^2}{4}$ is, by (IV.6), at most

$$\int_{\substack{-(4k_F^2-|\mathbf{q}|^2)\le m\nu\le -2|\mathbf{q}|(2k_F-|\mathbf{q}|)\\ |\nu|\le 0(M^i)}} d\nu \quad \left[m\nu + k_F^2 - \frac{|\mathbf{q}|^2}{4}\right]^{\frac{d-2}{2}} \le \text{const} \int d\nu \tag{IV.16}$$

In order for (IV.16) to be nonzero it is necessary that

$$|\mathbf{q}|(2k_F - |\mathbf{q}|) \le 0(M^i)$$

so that either $|\mathbf{q}| \le 0(M^i)$, or $(2k_F - |\mathbf{q}|) \le 0(M^i)$. In the former case

$$(IV.16) \le 0(M^i) = M^{j-i}0(M^i)$$

while in the latter case

$$(IV.16) \le \text{const}\left[\left(k_F^2 - \frac{|\mathbf{q}|^2}{4}\right) - \frac{1}{2}|\mathbf{q}|\,(2k_F - |\mathbf{q}|)\right]$$

$$= \text{const}\,(2k_F - |\mathbf{q}|)^2 = 0(M^{2i}) \le M^{-(j-i)} 0(M^i)$$

Outside-inside. The set in which $\pm\mathbf{t} + \mathbf{q}/2$ are on opposite sides of the Fermi surface has volume at most

$$\int_{\substack{0 \le e(\mathbf{t}+\mathbf{q}/2) \le 0(M^i) \\ -0(M^i) \le e(-\mathbf{t}+\mathbf{q}/2) \le 0}} d^d\mathbf{t}$$

$$\le \int_{|\nu| \le 0(M^i)} d\nu \int_{\substack{e(\mathbf{t}+\mathbf{q}/2) \ge 0 \\ e(-\mathbf{t}+\mathbf{q}/2) \le 0}} d^d\mathbf{t} \quad \delta(e(\mathbf{t}+\mathbf{q}/2) - e(-t+\mathbf{q}/2) - \nu) \qquad (IV.17)$$

The part of (IV.17) with $\nu > 0, |q| \le 2k_F$ and $m\nu \le \frac{1}{2}|\mathbf{q}|(2k_F - |\mathbf{q}|)$ is bounded by

$$\int_{\substack{0 \le m\nu \le \frac{1}{2}|\mathbf{q}|(2k_F - |\mathbf{q}|) \\ \nu \le 0(M^i)}} d\nu \quad \frac{\nu}{|\mathbf{q}|} \left[k_F^2 - \left(\frac{|\mathbf{q}|}{2} - \frac{m\nu}{|\mathbf{q}|} \right)^2 \right]^{\frac{d-3}{2}} . \qquad (IV.18)$$

For $d \ge 3$, this at most const$\frac{\nu}{|\mathbf{q}|}$ and $(IV.18) \le \text{const} M^{-(j-i)} M^i$. For $d = 2$, the integrand factors

$$\frac{\nu}{|\mathbf{q}|} \left[k_F - \frac{|\mathbf{q}|}{2} + \frac{\nu m}{|\mathbf{q}|} \right]^{-1/2} \left[k_F + \frac{|\mathbf{q}|}{2} - \frac{m\nu}{|\mathbf{q}|} \right]^{-1/2} .$$

When $|\mathbf{q}| \ge k_F$, the second radical is $0(1)$ and $(IV.18) \le \text{const} M^{\frac{3}{2}i}$ as in (IV.15). When $|\mathbf{q}| \le k_F$, the first radical is $0(1)$ and

$$\begin{aligned} (IV.18) &\le \text{const} \int d\nu \quad \frac{\nu}{|\mathbf{q}|} \left[k_F + \frac{|q|}{2} - \frac{m\nu}{|\mathbf{q}|} \right]^{-1/2} \\ &= \text{const} \int d\nu \quad \frac{\nu}{|\mathbf{q}|^{1/2}} \left[k_F|\mathbf{q}| + \frac{|\mathbf{q}|^2}{4} - m\nu \right]^{-1/2} \\ &\le \text{const} M^{(3/2)i} M^{-j/2}. \end{aligned}$$

The part of (IV.17) with $\frac{1}{2}|\mathbf{q}|\,|2k_F - |\mathbf{q}|| \le m\nu \le \frac{1}{2}|\mathbf{q}|(2k_F + |\mathbf{q}|)$ is bounded by

$$\int d\nu \left[k_F^2 - \left(\frac{|q|}{2} - \frac{m\nu}{|\mathbf{q}|} \right)^2 \right]^{\frac{d-1}{2}} \le \int d\nu \left[\frac{1}{|q|} \left(k_F|\mathbf{q}| - \frac{|\mathbf{q}|^2}{2} + m\nu \right) \left(k_F + \frac{|\mathbf{q}|}{2} - m\frac{\nu}{|\mathbf{q}|} \right) \right]^{\frac{d-1}{2}}$$

$$\le \text{const} \int d\nu \left[\frac{\nu}{q} \right]^{\frac{d-1}{2}} = \text{const} M^{-(j-i)\frac{d-1}{2}} M^i$$

Outside-outside. Finally, the set in which $\pm\mathbf{t} + \mathbf{q}/2$ are both outside the Fermi surface has volume at most

$$\int_{\substack{0 \le e(\mathbf{t}+\mathbf{q}/2) \le 0(M^i) \\ 0 \le e(-\mathbf{t}+\mathbf{q}/2) \le 0(M^i)}} d^d\mathbf{t} \le \int_{|\nu| \le 0(M^i)} d\nu \int_{e(\pm\mathbf{t}+\mathbf{q}/2) \ge 0} d^d\mathbf{t} \quad \delta(e(\mathbf{t}+\mathbf{q}/2) + e(-t+\mathbf{q}/2) - \nu).$$

If $|\mathbf{q}| \geq 2k_F$, $\frac{|\mathbf{q}|^2}{4} - k_F^2 \leq m\nu \leq \frac{|\mathbf{q}|}{2}[|\mathbf{q}| - 2k_F]$ then

$$\left[m\nu + k_F^2 - \frac{|\mathbf{q}|^2}{4}\right]^{\frac{d-2}{2}} \leq \text{const}$$

and ν is integrated over an interval of length

$$\frac{1}{2}|\mathbf{q}|[|\mathbf{q}| - 2k_F] - \frac{1}{4}|\mathbf{q}|^2 - k_F^2 = \frac{1}{4}[|\mathbf{q}| - 2k_F]^2$$
$$\leq \text{const}M^{2i}.$$

If

$$\max\left(0, \frac{1}{2}|\mathbf{q}|\,[|\mathbf{q}| - 2k_F]\right) \leq m\nu \leq \frac{1}{2}|\mathbf{q}|\,[|\mathbf{q}| + 2k_F]$$

we must bound

$$\int d\nu \quad \frac{\nu}{|\mathbf{q}|}\left[m\nu + k_F^2 - \frac{|\mathbf{q}|^2}{4}\right]^{\frac{d-3}{2}} \tag{IV.19}$$

It is handled precisely as (IV.14). Finally, $\frac{1}{2}|\mathbf{q}|[|\mathbf{q}| + 2k_F] \leq m\nu$ forces $|\mathbf{q}| \leq \text{const}\nu$, that is $j \leq i + \text{const}$. So we have

$$\int_{0 \leq e(\pm\mathbf{t}+\mathbf{q}/2) \leq 0(M^i)} d^d\mathbf{t} \leq 0(M^{-\frac{1}{2}(j-i)}M^i)$$

and the proof is complete.

■

We remark that, as a brief inspection of the above proof shows, for $d \geq 3$, the $M^{-\frac{1}{2}(j-i)}$ in the statement of Lemma IV.2 may be replaced by $M^{-(j-i)}$.

<u>Lemma IV.3</u> Let $|\mathbf{q}| \leq \text{const}$. Then,

$$\int_{e(\mathbf{t}+\mathbf{q}/2)e(-\mathbf{t}+\mathbf{q}/2)<0} d^d\mathbf{t} \quad \delta(\nu - |e(\mathbf{t}+\mathbf{q}/2)| - |e(-\mathbf{t}+\mathbf{q}/2)|)$$

$$\leq \text{const}\begin{cases} \nu^{1/2} + \frac{\nu}{|\mathbf{q}|^{1/2}}(2k_F|\mathbf{q}| - |\mathbf{q}|^2 - 2m\nu)^{-1/2}\chi(0 \leq \nu \leq \frac{|\mathbf{q}|}{2m}(2k_F - |\mathbf{q}|)) & d = 2 \\ \nu + \frac{\nu}{|\mathbf{q}|}\chi(|\mathbf{q}| \geq \text{const}\nu) & d \geq 3 \end{cases}$$

<u>Proof</u> By Technical Lemma IV.1b), there are two cases. Namely,

$$|2k_F - |\mathbf{q}|| \leq \frac{2m\nu}{|\mathbf{q}|} \leq 2k_F + |\mathbf{q}|$$

and

$$0 \leq \frac{2m\nu}{|\mathbf{q}|} \leq 2k_F - |\mathbf{q}|, \quad |\mathbf{q}| \leq 2k_F.$$

In the first case, the integral is bounded by

$$\text{const}\left|k_F - \frac{|\mathbf{q}|}{2} + \frac{m\nu}{|\mathbf{q}|}\right|^{\frac{d-1}{2}} \left|k_F + \frac{|\mathbf{q}|}{2} - \frac{m\nu}{|\mathbf{q}|}\right|^{\frac{d-1}{2}}$$

$$\leq \begin{cases} \text{const}\left|\frac{2m\nu}{|\mathbf{q}|}\right|^{\frac{d-1}{2}} |\text{const}|^{\frac{d-1}{2}}, & \text{if } |\mathbf{q}| \geq k_F \\ \text{const}\left|k_F - \frac{|\mathbf{q}|}{2} + k_F + \frac{|\mathbf{q}|}{2}\right|^{\frac{d-2}{2}} \left|\frac{|\mathbf{q}|}{2} + \frac{|\mathbf{q}|}{2}\right|^{\frac{d-1}{2}}, & \text{if } |\mathbf{q}| \leq k_F \end{cases}$$

$$\leq \begin{cases} \text{const}\nu^{\frac{d-1}{2}}, & \text{if } |\mathbf{q}| \geq k_F \\ \text{const}\left|\frac{2m\nu}{2k_F - |\mathbf{q}|}\right|^{\frac{d-1}{2}}, & \text{if } |\mathbf{q}| \leq k_F \end{cases}$$

$$\leq \text{const}\nu^{\frac{d-1}{2}}.$$

In the second case, the integral is bounded by

$$\text{const}\frac{\nu}{|\mathbf{q}|}\left|k_F - \frac{|\mathbf{q}|}{2} + \frac{m\nu}{|\mathbf{q}|}\right|^{\frac{d-3}{2}} \left|k_F + \frac{|\mathbf{q}|}{2} - \frac{m\nu}{|\mathbf{q}|}\right|^{\frac{d-3}{2}}$$

$$\leq \begin{cases} \text{const}\frac{\nu}{|\mathbf{q}|}|\text{const}|^{\frac{d-3}{2}}|\text{const}|^{\frac{d-3}{2}} & d \geq 3 \\ \text{const}\frac{\nu}{|\mathbf{q}|}\left|\frac{2m\nu}{|\mathbf{q}|}\right|^{-1/2} |\mathbf{q}|^{-1/2} & d = 2, |\mathbf{q}| \geq k_F \\ \text{const}\frac{\nu}{|\mathbf{q}|}\left(\frac{k_F}{2}\right)^{-1/2}\left(k_F + \frac{|\mathbf{q}|}{2} - \frac{m\nu}{|\mathbf{q}|}\right)^{-1/2} & d = 2, |\mathbf{q}| \leq k_F \end{cases}$$

$$\leq \begin{cases} \text{const}\frac{\nu}{|\mathbf{q}|} & d \geq 3 \\ \text{const}\nu^{1/2} & d = 2, |\mathbf{q}| \geq k_F \\ \text{const}\frac{\nu}{|\mathbf{q}|^{1/2}}(k_F|\mathbf{q}| + \frac{1}{2}|\mathbf{q}|^2 - m\nu)^{-1/2} & d = 2, |\mathbf{q}| \leq k_F \end{cases}$$

and is zero unless $|\mathbf{q}| \geq \frac{2m\nu}{2k_F - |\mathbf{q}|}$.

■

We now return to our analysis of the top two diagrams of (IV.1).

As before, we decompose

$$\mathbf{C}_\triangle = \mathbf{C}_1 + \mathbf{C}_2$$

$$\mathbf{C}_1 = (-1)\frac{ik_0\mathbf{1} + e(\mathbf{k})\sigma^3}{k_0^2 + E(\mathbf{k})^2}$$

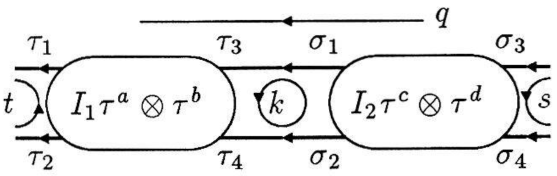
$$\mathbf{C}_2 = (-1)\frac{\triangle\sigma^1}{k_0^2 + E(\mathbf{k})^2}.$$

Each of the two diagrams become a sum of four terms. Three of these terms contain at least one $\mathbf{C}_2$. As explained in the introduction, and implemented in Section III, with every occurence of a $\mathbf{C}_2$ there is an accompanying uniformly summable factor of $\frac{\Delta}{M^h}$. See (I.114). Hence, we only consider $\mathbf{C}_1$ in this section.

To obtain the necessary information about the top two diagrams that is required for the scheme of Section III we must consider the following problems, that naturally divide into the case of two local kernels and the case of at least one renormalized kernel. For the former we need only consider, in the language of physical fields, the electron-hole ladder, since the electron-electron and hole-hole ladders have been put in the flow. The latter is more involved. We must extract summable factors from three distinct sources, namely, momentum space constraints, cancellations of the type $f(p) - f(p')$ when $|p - p'|$ is small and effective infrared cutoffs when p_0 is large.

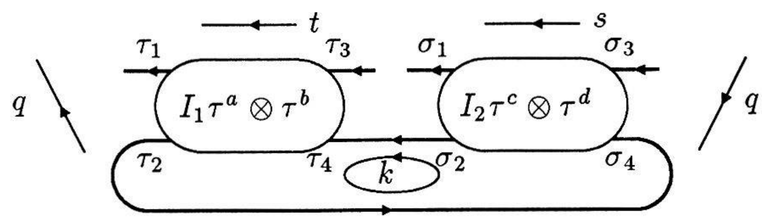
For the rest of this section I_1 and I_2 denote arbitrary kernels proportional to pure tensor products $\tau^i \otimes \tau^j$ and

$J_1 :=$



$(IV.20a)$

$J_2 :=$



$(IV.20b)$

The internal lines of J_i are of type

$$\mathbf{C}_1^{(h)}(\tau, \mathbf{k}) = (-1)e^{-E(\mathbf{k})|\tau|} \frac{(\operatorname{sgn}\tau)E(\mathbf{k})\mathbf{1} + e(\mathbf{k})\sigma^3}{2E(\mathbf{k})} f(M^{-2h}E(\mathbf{k})^2)$$

or its soft analogue (f replaced by ρ). At least one internal line is hard. We shall, without loss of generality, assume that it is the upper one.

The case of an electron-hole ladder is covered by the lemma below. In the expansion of Theorem I.1 each kernel I_1, I_2 is either local and of scale h or renormalized and of scale $j_i > h$. We do not need to exploit the effect of renormalization in the electron-hole ladder so we replace renormalized kernels by general ones.

Lemma IV.4 Let $r \leq -1$. Let $I_1 \tau^a \otimes \tau^b$ (resp. $I_2 \tau^c \otimes \tau^d$) be a local or general kernel with $a + b = 0 \pmod 3$ (resp. $c + d = 0 \pmod 3$). In the former case it may depend on h. Let

$$r' \leq \begin{cases} -1 & \text{if } I_1, I_2 \text{ are both local} \\ j_1 & \text{if } I_1 \text{ is general and } I_2 \text{ is local} \\ j_2, & \text{if } I_1 \text{ is local and } I_2 \text{ is general} \\ \min(j_1, j_2), & \text{if } I_1, I_2 \text{ are both general} \end{cases} \tag{IV.21}$$

and define

$$B_i = \begin{cases} \|I_i\|_{j_i}, & \text{if } I_i \text{ is general} \\ \max_{r'<h<r} \|I_i^{(h)}\|_r, & \text{if } I_i \text{ is local} \end{cases} \tag{IV.22}$$

a) Suppose that I_1 and I_2 are not both of type $\tau^0 \otimes \tau^0$ and not both of type $\tau^3 \otimes \tau^3$. Then

$$\| \sum_{r'>h>r} J_1^{(h)} \|_r \leq \text{const} B_1 B_2.$$

b) Suppose that I_1, I_2 are not of type $\tau^1 \otimes \tau^2, \tau^2 \otimes \tau^1$ or vice versa. Then

$$\| \sum_{r'>h>r} J_2^{(h)} \| \leq \text{const} B_1 B_2$$

Proof a) The proof is made by combining estimates on $J_1(\triangle = 0)$ and $\frac{d}{d\triangle} J_1$. When $\triangle = 0$

$$J_1^{(h)} = \int \frac{d^d k}{(2\pi)^d} d\tau_3 d\tau_4 d\sigma_1 d\sigma_2 \quad F(M^{-2h} e(\mathbf{k}+\mathbf{q})^2, M^{-2h} e(-\mathbf{k})^2)$$

$$I_1(\tau_1, \tau_2, \tau_3, \tau_4, \mathbf{t}, \mathbf{k}, \mathbf{q}) I_2(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \mathbf{k}, \mathbf{s}, \mathbf{q})$$

$$e^{-|e(\mathbf{k}+\mathbf{q})||\sigma_1 - \tau_3|} e^{-|e(-\mathbf{k})||\sigma_2 - \tau_4|}$$

$$\tau^a [\text{sgn}(\sigma_1 - \tau_3)\mathbf{1} + \text{sgn}(e(\mathbf{k}+\mathbf{q}))\sigma^3] \tau^c$$

$$\otimes \tau^b [\text{sgn}(\sigma_2 - \tau_4)\mathbf{1} + \text{sgn}(e(-\mathbf{k}))\sigma^3] \tau^d \tag{IV.23}$$

with

$$F(M^{-2h} e(\mathbf{k}+\mathbf{q}), M^{-2h} e(-\mathbf{k})^2) = f(M^{-2h} e(\mathbf{k}+\mathbf{q})^2)(\rho \text{ or } f)(M^{-2h} e(-\mathbf{k})^2) \tag{IV.24}$$

supplying the scale cutoff.

First consider the case in which $(\sigma_1 - \tau_3)(\sigma_2 - \tau_4) > 0$. Denote this part of $J_1^{(h)}$ by $J_+^{(h)}$ and the other part by $J_-^{(h)}$. We claim that unless $e(\mathbf{k}+\mathbf{q})e(-\mathbf{k}) < 0$ the integrand of $J_+^{(h)}$ vanishes. To see this observe that when $(\sigma_1 - \tau_3)(\sigma_2 - \tau_4) > 0$ and $e(\mathbf{k}+\mathbf{q})e(-\mathbf{k}) > 0$,

$$\begin{aligned}&(\mathrm{sgn}(\sigma_1-\tau_3)\mathbf{1} + \mathrm{sgn}(e(\mathbf{k}+\mathbf{q}))\sigma^3, \mathrm{sgn}(\sigma_2-\tau_4)\mathbf{1} + \mathrm{sgn}(e(-\mathbf{k})\sigma^3))\\ &\in \quad 2\mathrm{sgn}(\sigma_1-\tau_3)\{(\tau^0,\tau^0),(\tau^3,\tau^3)\}. \qquad (IV.25)\end{aligned}$$

Since

$$\begin{aligned}\tau^i\tau^0\tau^j &= 0 \quad \text{unless } (i,j) \in \{(0,0),(0,1),(2,1),(2,0)\}\\ \tau^i\tau^3\tau^j &= 0 \quad \text{unless } (i,j) \in \{(1,2),(1,3),(3,2),(3,3)\}\end{aligned} \qquad (IV.26)$$

and $a+b, c+d = 0(\mathrm{mod}\ 3)$ and furthermore, by assumption $(a,b,c,d) \neq (0,0,0,0),(3,3,3,3)$ we have

$$\begin{aligned}\left(\tau^a\otimes\tau^b\right)\left(\tau^0\otimes\tau^0\right)\left(\tau^c\otimes\tau^d\right) &= 0\\ \left(\tau^a\otimes\tau^b\right)\left(\tau^3\otimes\tau^3\right)\left(\tau^c\otimes\tau^d\right) &= 0.\end{aligned} \qquad (IV.27)$$

Therefore, when $\triangle = 0$,

$$\begin{aligned}&\int d\tau_2 d\sigma_3 d\sigma_4 |J_+^{(h)}(\tau_1,\tau_2,\sigma_3,\sigma_4,\mathbf{t},\mathbf{s},\mathbf{q})|\\ &\leq \mathrm{const} M^{-h}\left|I_1\right|\,\left|I_2\right| \int_{e(\mathbf{k}+\mathbf{q})e(-\mathbf{k})<0} d^d\mathbf{k} \quad F(M^{-2h}e(\mathbf{k}+\mathbf{q})^2, M^{-2h}e(-\mathbf{k})^2) \qquad (IV.28)\end{aligned}$$

So

$$\begin{aligned}&\int d\tau_2 d\sigma_3 d\sigma_4 \left|J_+^{(h)}\right|\\ &\leq \mathrm{const}\left|I_1\right|\left|I_2\right| M^{(-h)} \int_{\mathrm{const} M^h}^{\mathrm{const}\prime M^h} d\nu \int_{e(\mathbf{k}+\mathbf{q})e(-\mathbf{k})<0} \frac{d^d\mathbf{k}}{(2\pi)^d} \quad \delta(\nu - |e(\mathbf{k}+\mathbf{q})| - |e(-\mathbf{k})|)\\ &\leq \mathrm{const}\left|I_1\right|\left|I_2\right| \begin{cases} M^h + \frac{M^h}{|\mathbf{q}|}\chi(|\mathbf{q}| \geq \mathrm{const} M^h), & \text{if } d \geq 3\\ M^{h/2} & \\ +\frac{1}{|\mathbf{q}|^{1/2}} \int_{\mathrm{const} M^h}^{\mathrm{const}' M^h} d\nu |2m\nu - 2k_F|\mathbf{q}| + |\mathbf{q}|^2|^{-\frac{1}{2}} \chi(|\mathbf{q}| \geq \mathrm{const} M^h), & \text{if } d = 2\end{cases}\\ &\leq \mathrm{const}\left|I_1\right|\left|I_2\right| \begin{cases} M^h + \frac{M^h}{|\mathbf{q}|}\chi(|\mathbf{q}| \geq \mathrm{const} M^h) & d \geq 3\\ M^{h/2} + \frac{M^{h/2}}{|\mathbf{q}|^{1/2}}\chi(|\mathbf{q}| \geq \mathrm{const} M^h) & d = 2\end{cases} \qquad (IV.29)\end{aligned}$$

where Lemma IV.3 is used in the third line.

We next consider $\int d\tau_2 d\sigma_3 d\sigma_4 |J_-^{(h)}|$. Now, the domain of integration in (IV.21a) is restricted by $(\sigma_1 - \tau_3)(\sigma_2 - \tau_4) < 0$. Hence,

$$\begin{aligned}|\tau_3 - \sigma_1| &\le |\tau_3 - \sigma_1 + \sigma_2 - \tau_4| \\ &\le |\tau_3 - \tau_4| + |\sigma_1 - \sigma_2|\end{aligned}$$

Suppose, I_1, I_2 are both general. Then, recalling $r' = \min(j_1, j_2)$,

$$\begin{aligned}\int d\tau_2 d\sigma_3 d\sigma_4 |J_-^{(h)}| \le &\int_{(\sigma_1-\tau_3)(\sigma_2-\tau_4)<0} \frac{d^d\mathbf{k}}{(2\pi)^d} \prod_{i=2}^{4} d\tau_i \prod_{i=1}^{4} d\sigma_i |I_1^{(j_1)}||I_2^{(j_2)}| F \\ &[1 + M^{r'}|\tau_3 - \sigma_1|]^{-2}[1 + M^{r'}|\tau_3 - \sigma_1|]^2 \\ \le &\int \frac{d^d\mathbf{k}}{(2\pi)^d} \prod_{i=2}^{4} d\tau_i \prod_{i=1}^{4} d\sigma_i |I_1^{(j_1)}||I_2^{(j_2)}| F \\ &[1 + M^{r'}|\tau_3 - \sigma_1|]^{-2}[1 + M^{j_1}|\tau_3 - \tau_4| + M^{j_2}|\sigma_1 - \sigma_2|]^2 \\ \le &\,\mathrm{const} M^h \|I_1^{(j_1)}\|_{j_1} \|I_2^{(j_2)}\|_{j_2} M^{-r'}.\end{aligned} \tag{IV.30}$$

When only one kernel, say I_1, is general the bound (IV.30) still applies since then I_2 forces $\sigma_1 = \sigma_2$. When both are local $J_-^{(h)}$ vanishes.

The next step is to bound s, t and q derivatives of J_1, while retaining $\triangle = 0$. By a q_0-derivative we mean multiplication by τ. Observe that t derivatives must act on I_1 and s derivatives must act on I_2, while q derivatives may act on $I_1, I_2, e^{-|e(\mathbf{k}+\mathbf{q})||\sigma_1-\tau_3|}$, the cutoff $F(M^{-2h}e(\mathbf{k}+\mathbf{q})^2, M^{-2h}e(-\mathbf{k})^2)$ or on the $\mathrm{sgn}(e(\mathbf{k}+\mathbf{q}))$. If the derivative acts on $\mathrm{sgn}(e(\mathbf{k}+\mathbf{q}))$, we get zero because the hardness of the upper line implies that the supports of $\delta(e(\mathbf{k}+\mathbf{q}))$ and F are disjoint.

The result of applying any derivative to I_1, I_2 is estimated immediately. Only slightly more involved is the action of a q derivative on $e^{-|e(\mathbf{k}+\mathbf{q})||\sigma_1-\tau_3|}$ or $F(M^{-2h}e(\mathbf{k}+\mathbf{q})^2, M^{-2h}e(-\mathbf{k})^2)$. In both cases each derivative produces an extra M^{-h} (possibly via $\int d\tau |\tau|^n e^{-M^h|\tau|} \le \mathrm{const}_n M^{-(n+1)h}$).

Recalling (IV.29) and (IV.30) it follows that when $\triangle = 0, \alpha_j \ge 0, \beta_j \ge 0$,

$$\sum_{j=1}^{3} \alpha_j \le 2, \quad \sum_{j=1}^{3} |\beta_j| \le 1$$

$$\int d\tau_2 d\sigma_3 d\sigma_4 ||\tau_2 - \tau_1|^{\alpha_1} |\tau_1 - \sigma_3|^{\alpha_2} |\sigma_4 - \sigma_3|^{\alpha_3} \partial_{\mathbf{t}}^{\beta_1} \partial_{\mathbf{s}}^{\beta_2} \partial_{\mathbf{q}}^{\beta_3} J_1^{(h)}| M^{h(\sum \alpha_j + \sum |\beta_j|)}$$

$$\le \text{const}B_1B_2\left[M^{-(r'-h)}+\begin{cases}M^h+\frac{M^h}{|\mathbf{q}|}\chi(|\mathbf{q}|\ge\text{const}M^h) & d\ge 3\\ M^{h/2}+\frac{M^{h/2}}{|\mathbf{q}|^{1/2}}\chi(|\mathbf{q}|\ge\text{const}M^h) & d=2\end{cases}\right] \tag{IV.31}$$

or, summing over h,

$$\begin{aligned}\|\sum_{r'>h>r}J_1^{(h)}\|_r &\le \text{const}B_1B_2\left\{1+\sup_{\mathbf{q}}\sum_{h<0}\left[M^{h/2}+\frac{M^{h/2}}{|\mathbf{q}|^{1/2}}\chi(|\mathbf{q}|\ge\text{const}M^h)\right]\right\}\\ &\le \text{const}B_1B_2\left\{1+\sup_{\mathbf{q}}\sum_{\substack{h\ s.t.\\ M^h\le\text{const}|\mathbf{q}|}}\frac{M^{h/2}}{|\mathbf{q}|^{1/2}}\right\}\\ &\le \text{const}B_1B_2\end{aligned} \tag{IV.32}$$

The next step is to consider

$$\frac{d}{d\triangle}J_1^{(h)}=\sum_{h_2\le h}\frac{d}{d\triangle}\int\frac{d^d\mathbf{k}}{(2\pi)^d}d\tau_3d\tau_4d\sigma_1d\sigma_2I_1\tau^a\otimes\tau^b\mathbf{C}_1(\sigma_1-\tau_3,\mathbf{k}+\mathbf{q})\otimes\mathbf{C}_1(\sigma_2-\tau_4,-\mathbf{k})$$
$$\tau^c\otimes\tau^dI_2f(M^{-2h}E(\mathbf{k}+\mathbf{q})^2)f(M^{-2h}E(-\mathbf{k})^2) \tag{IV.33}$$

We apply

$$|\mathbf{C}_1(\tau,\mathbf{p})|f(M^{-2h}E(\mathbf{p})^2)\le e^{-M^j|\tau|}\chi(E(\mathbf{p})\le M^j) \tag{IV.34a}$$

$$\left|\frac{d}{d\triangle}\mathbf{C}_1(\tau,\mathbf{p})f(M^{-2j}E(\mathbf{p})^2)\right|\le M^{-j}e^{-M^j|\tau|}\chi(E(\mathbf{p})\le M^j) \tag{IV.34b}$$

to get

$$\begin{aligned}\int d\tau_2d\sigma_3d\sigma_4\left|\frac{d}{d\triangle}J_1^{(h)}\right| &\le \sum_{h_2=\ell n\triangle}^{h}|I_1||I_2|M^{-h_2}M^{-h}M^{h_2}\\ &=|I_1||I_2|M^{-h}\log\left(\frac{M^h}{\triangle}\right).\end{aligned} \tag{IV.35}$$

Just as for $\triangle=0$, the result of applying any t,s,q "derivative" to I_1,I_2 is immediately bounded. As expected each "$(\tau,\nabla_{\mathbf{q}})$ derivative" acting on

$$\mathbf{C}_1(\sigma_1-\tau_3,\mathbf{k}+\mathbf{q})\otimes\mathbf{C}_1(\sigma_2-\tau_4,-\mathbf{k})F(M^{-2h}E(\mathbf{k}+\mathbf{q})^2,M^{-2h}E(-\mathbf{k})^2)$$

produces an M^{-h}. Thus

$$\|\frac{d}{d\triangle}J_1^{(h)}\|_h\le\text{const}B_1B_2M^{-h}\log(M^h/\triangle) \tag{IV.36}$$

and

$$
\begin{aligned}
\Big\| \sum_{r'>h>r} J_1^{(h)} \Big\|_r &\le \Big\| \sum_{r'>h>r} J_1^{(h)}\big|_{\Delta=0} \Big\|_r + \sum_{r'>h>r} \int_0^{\Delta} d\Delta' \Big\| \frac{dJ_1}{d\Delta} \Big\|_r \\
&\le B_2 B_2 \left[1 + \sum_{0>h>r} \int_0^{\Delta} d\Delta' M^{-h} \log(M^h/\Delta') \right] \\
&\le B_1 B_2 \left[1 + \sum_{0>h>r} \frac{\Delta}{M^h} \log(M^h/\Delta) \right] \\
&\le B_1 B_2
\end{aligned}
\tag{IV.37}
$$

ignoring irrelevant constants.

b) From (IV.20b), when $\Delta = 0$,

$$
\begin{aligned}
J_2^{(h)} = \int \frac{d^d\mathbf{k}}{(2\pi)^d} & d\tau_2 d\tau_4 d\sigma_2 d\sigma_4 F(M^{-2h} e(\mathbf{k}+\mathbf{q})^2, M^{-2h} e(\mathbf{k})^2) I_1 I_2 \tau_a \otimes \tau_c \\
& e^{-|e(\mathbf{k}+\mathbf{q})||\sigma_2-\tau_4|} e^{-|e(\mathbf{k})||\tau_2-\sigma_4|} \\
& tr\tau^b[\mathrm{sgn}(\tau_2-\sigma_4)\mathbf{1} + \mathrm{sgn}(e(\mathbf{k})\sigma^3]\tau^d[\mathrm{sgn}(\sigma_2-\tau_4)\mathbf{1} + \mathrm{sgn}(e(\mathbf{k}+\mathbf{q}))\sigma^3].
\end{aligned}
\tag{IV.38}
$$

As in part (a), we first verify that when $(\tau_2 - \sigma_4)(\sigma_2 - \tau_4) < 0$ and $e(\mathbf{k})e(\mathbf{k}+\mathbf{q}) > 0$ the integrand vanishes. This is a consequence of

$$
tr\tau^b\tau^0\tau^d\tau^3 = tr\tau^b\tau^3\tau^d\tau^0 = 0
$$

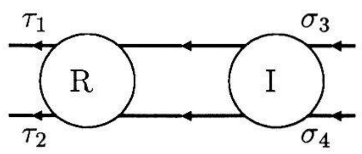
when $(b,d) \neq (1,2)$ or $(2,1)$. The hypothesis on $\tau^a \otimes \tau^b, \tau^c \otimes \tau^d$ forces $(b,d) \neq (1,2)$ or $(2,1)$. The argument continues as in part (a).

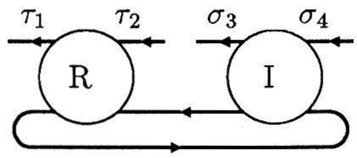
■

At this point we have finished our analysis of the "wrong way ladder" contributions to the first two diagrams of (IV.1). (Recall that the second two diagrams have already been completely treated.) We now treat "right way ladders" and assume that at least one of the kernels appearing in each of the first two diagrams is renormalized.

Lemma IV.5 Let I and K be kernels of scale i and k respectively. Let $R = (\mathbf{1} - \mathbf{L}^{(k)})K$ be the renormalized kernel. Define

$J_1 :=$



$J_2 :=$ 

For J_1, assume that I and K are both of type $\tau^0 \otimes \tau^0$ or are both of type $\tau^3 \otimes \tau^3$. For J_2, assume that I, K are of types $\tau^1 \otimes \tau^2, \tau^2 \otimes \tau^1$ or vice versa. Then

$$J_\beta(\tau_1,\tau_2,\sigma_3,\sigma_4,\mathbf{t},\mathbf{s},\mathbf{q}) = j_{\beta,\mathrm{int}}(\tau_1,\tau_2,\sigma_3,\sigma_4,\mathbf{t},\mathbf{s},\mathbf{q})$$

$$+j_{\beta,\mathrm{mom}}(\tau_1,\tau_2,\sigma_3,\sigma_4,\mathbf{t},\mathbf{s},\mathbf{q})$$

$$+\{j_{\beta,\mathrm{en}}(\tau_1,\tau_2,\sigma_3,\sigma_4,\mathbf{t},\mathbf{s},\mathbf{q}) - \int d\alpha_2 j_{\beta,\mathrm{en}}(\tau_1,\alpha_2,\sigma_3,\sigma_4,\mathbf{t},\mathbf{s},\mathbf{q})\delta(\tau_1-\tau_2)\}$$

for $\beta = 1,2$ where

$$\|j_{\beta,\mathrm{int}}\|_h \le \mathrm{const}\|K\|_k\|I\|_i\left[M^{-\frac{1}{2}(k-h)} + \frac{\triangle}{M^h}\log\left(\frac{M^h}{\triangle}\right)\right]$$

$$\|j_{\beta,\mathrm{mom}}\rho(M^{-2h}E^2(\mathbf{t}\pm\mathbf{q}/2))\|_h \le \mathrm{const}\|K\|_k\|I\|_i M^{-\frac{1}{2}(k-h)}$$

$$\|j_{\beta,\mathrm{en}}\|_h \le \mathrm{const}\|K\|_k\|I\|_i.$$

Here, $h \le i,k$ is the scale of the internal lines. The external momenta $\mathbf{t},\mathbf{s}$ and $\mathbf{q}$ are defined as in (I.99) and $\mathbf{t}' = k_F\mathbf{t}/|\mathbf{t}|$.

Remark The subscript int indicates that the renormalization operator $(\mathbf{1}-\mathbf{L}^{(k)})$ acts on the internal lines of J_β. The subscripts mom and en indicate that it acts on the external momentum and energy arguments of J_β.

Proof. To avoid the tedium of repeated conversion from Nambu to physical fields we shall assume that K and I are both proportional to $\tau^0 \otimes \tau^0$. The remaining eight cases are treated in exactly the same way. Evaluating (IV.20a), using the notation (IV.21b),

$$J_1(t,s,q) = \int \frac{d^4p}{(2\pi)^4}(\mathbf{1}-\mathbf{L}^{(k)})K(t,p,q)[\tau^0\mathbf{C}_1(p+q/2)\tau^0] \otimes [\tau^0\mathbf{C}_1(-p+q/2)\tau^0]I(p,s,q)$$

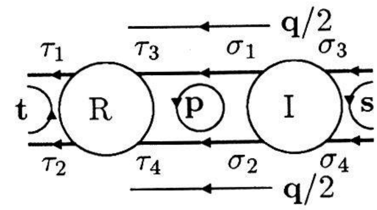
$$F(M^{-2h}E(\mathbf{p}+\mathbf{q}/2)^2, M^{-2h}E(-\mathbf{p}+\mathbf{q}/2)^2)$$

$$= \int \frac{d^4p}{(2\pi)^4}(\mathbf{1}-\mathbf{L}^{(k)})K(t,p,q)\frac{i(p_0+q_0/2)+e(\mathbf{p}+\mathbf{q}/2)}{(p_0+q_0/2)^2+E(\mathbf{p}+\mathbf{q}/2)^2}\frac{i(-p_0+q_0/2)+e(-\mathbf{p}+\mathbf{q}/2)}{(-p_0+q_0/2)^2+E(-\mathbf{p}+\mathbf{q}/2)^2}$$

$$I(p,s,q)F(M^{-2h}E(\mathbf{p}+\mathbf{q}/2)^2, M^{-2h}(-\mathbf{p}+\mathbf{q}/2)^2)$$
$$= J_1(\triangle = 0) + \int_0^{\triangle} d\triangle' \frac{dJ}{d\triangle}(\triangle'). \qquad (IV.39)$$

To facilitate the estimates we work in the mixed $(\tau, \mathbf{k})$ representation with the variable names displayed in

$$J_1((\tau_1, \mathbf{t}+\mathbf{q}/2), (\tau_2, -\mathbf{t}+\mathbf{q}/2), (\sigma_3, \mathbf{s}+\mathbf{q}/2), (\sigma_4, -\mathbf{s}+\mathbf{q}/2))$$

=


For $\triangle = 0$, recalling the definition (I.99) of $\mathbf{L}^{(h)}$,

$$J_1(\tau_1, \tau_2, \sigma_3, \sigma_4, \mathbf{t}, \mathbf{s}, \mathbf{q})$$
$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3} d\tau_3 d\tau_4 d\sigma_1 d\sigma_2 F(M^{-2h}e(\mathbf{p}+\mathbf{q}/2)^2, M^{-2h}e(-\mathbf{p}+\mathbf{q}/2)^2)$$
$$R(\tau_1, \tau_2, \tau_3, \tau_4, \mathbf{t}, \mathbf{p}, \mathbf{q}) I(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \mathbf{p}, \mathbf{s}, \mathbf{q})$$
$$e^{-|e(\mathbf{p}+\mathbf{q}/2)||\sigma_1-\tau_3|}[\chi(-e(-\mathbf{p}+\mathbf{q}/2))\chi(\sigma_1-\tau_3) - \chi(e(\mathbf{p}+\mathbf{q}/2))\chi(\tau_3-\sigma_1)]$$
$$e^{-|e(-\mathbf{p}+\mathbf{q}/2)||\sigma_2-\tau_4|}[\chi(-e(\mathbf{p}+\mathbf{q}/2))\chi(\sigma_2-\tau_4) - \chi(e(-\mathbf{p}+\mathbf{q}/2))\chi(\tau_4-\sigma_2)] \qquad (IV.40a)$$

where

$$R = K(\tau_1, \tau_2, \tau_3, \tau_4, \mathbf{t}, \mathbf{p}, \mathbf{q}) -$$
$$- \rho\left(|\mathbf{q}|M^{-\frac{1}{2}(h+k)}\right) \prod_{j=2}^{4} \delta(\tau_1 - \tau_j) \int \prod_{j=2}^{4} d\alpha_j K(\tau_1, \alpha_2, \alpha_3, \alpha_4, \mathbf{t}', \mathbf{p}', 0) \qquad (IV.40b)$$

We decompose the renormalized factor R (IV.40b) into the sum of nine pieces:

$$R_a = \left[1 - \rho\left(|\mathbf{q}|M^{-\frac{1}{2}(h+k)}\right)\right] K(\tau_1, \tau_2, \tau_3, \tau_4, \mathbf{t}, \mathbf{p}, \mathbf{q}) \qquad (IV.41a)$$

$$R_b = \rho\left(|\mathbf{q}|M^{-\frac{1}{2}(h+k)}\right) [K(\tau_1, \tau_2, \tau_3, \tau_4, \mathbf{t}, \mathbf{p}, \mathbf{q}) - K(\tau_1, \tau_2, \tau_3, \tau_4, \mathbf{t}, \mathbf{p}, 0)] \qquad (IV.41b)$$

$$R_c = \rho\left(|\mathbf{q}|M^{-\frac{1}{2}(h+k)}\right) [K(\tau_1, \tau_2, \tau_3, \tau_4, \mathbf{t}, \mathbf{p}, 0) - K(\tau_1, \tau_2, \tau_3, \tau_4, \mathbf{t}, \mathbf{p}', 0)] \qquad (IV.41c)$$

$$R_d = \rho\left(|\mathbf{q}|M^{-\frac{1}{2}(h+k)}\right)[K(\tau_1,\tau_2,\tau_3,\tau_4,\mathbf{t},\mathbf{p}',0) - K(\tau_1,\tau_2,\tau_3,\tau_4,\mathbf{t}',\mathbf{p}',0)] \qquad (IV.41d)$$

$$R_e = A_1\int d\alpha_4[K(\tau_1,\tau_2,\tau_3,\tau_4,\mathbf{t}',\mathbf{p}',0)\delta(\tau_1-\alpha_4) - K(\tau_1,\tau_2,\tau_3,\alpha_4,\mathbf{t}',\mathbf{p}',0)\delta(\tau_1-\tau_4)] \qquad (IV.41e)$$

$$R_f = A_1\delta(\tau_1-\tau_4)\int d\alpha_3 d\alpha_4$$

$$[K(\tau_1,\tau_2,\tau_3,\alpha_4,\mathbf{t}',\mathbf{p}',0)\delta(\tau_1-\alpha_3) - K(\tau_1,\tau_2,\alpha_3,\alpha_4,\mathbf{t}',\mathbf{p}',0)\delta(\tau_1-\tau_3)] \qquad (IV.41f)$$

$$R_g = A_1\delta(\tau_1-\tau_3)\delta(\tau_1-\tau_4)\int d\alpha_2 d\alpha_3 d\alpha_4$$

$$[K(\tau_1,\tau_2,\alpha_3,\alpha_4,\mathbf{t}',\mathbf{p}',0)\delta(\tau_1-\alpha_2) - K(\tau_1,\alpha_2,\alpha_3,\alpha_4,\mathbf{t}',\mathbf{p}',0)\delta(\tau_1-\tau_2)] \qquad (IV.41g)$$

$$R_h = A_2 K(\tau_1,\tau_2,\tau_3,\tau_4,\mathbf{t}',\mathbf{p}',0) \qquad (IV.41h)$$

$$R_i = A_2(-1)\prod_{j=2}^{4}\delta(\tau_1-\tau_j)\int\prod_{j=2}^{4} d\alpha_j K(\tau_1,\alpha_2,\alpha_3,\alpha_4,\mathbf{t}',\mathbf{p}',0) \qquad (IV.41i)$$

where

$$A_1 = \rho\left(|\mathbf{q}|M^{-\frac{1}{2}(h+k)}\right) h\left(M^{\frac{1}{2}(h+k)}(\tau_3-\sigma_1)\right) h\left(M^{\frac{1}{2}(h+k)}(\tau_4-\sigma_2)\right)$$

and

$$A_2 = \rho\left(|\mathbf{q}|)M^{-\frac{1}{2}(h+k)}\right)\left[1 - h\left(M^{\frac{1}{2}(h+k)}(\tau_3-\sigma_1)\right) h\left(M^{\frac{1}{2}(h+k)}(\tau_4-\sigma_2)\right)\right]$$

We have

$$R = R_a + R_b + R_c + R_d + R_e + R_f + R_g + R_h + R_i.$$

As a preliminary exercise, we first, as in example (IV.3), bound J_1 (still with $\triangle = 0$) without exploiting the renormalization cancellations (IV.41) in R. Recalling the norm (IV.2) and our convention that the upper line, joining τ_3 and σ_1, is hard

$$\begin{aligned}
&\int d\tau_2 d\sigma_3 d\sigma_4 |J_1(\tau_1,\tau_2,\sigma_3,\sigma_4,\mathbf{t},\mathbf{s},\mathbf{q})| \\
&\quad\le \int \frac{d^3\mathbf{p}}{(2\pi)^3}\prod_{j=2}^{4} d\tau_j \prod_{j=1}^{4} d\sigma_j F(M^{-2h}e(\mathbf{p}+\mathbf{q}/2)^2, M^{-2h}e(-\mathbf{p}+\mathbf{q}/2)^2)e^{-M^h|\sigma_1-\tau_3|} \\
&\qquad |R(\tau_1,\tau_2,\tau_3,\tau_4,\mathbf{t},\mathbf{p},\mathbf{q})||I(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\mathbf{p},\mathbf{s},\mathbf{q})| \qquad (IV.42a) \\
&\quad\le \sup_{\mathbf{p},\mathbf{s},\mathbf{q}}\int d\sigma_2 d\sigma_3 d\sigma_4 |I(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\mathbf{p},\mathbf{s},\mathbf{q})|
\end{aligned}$$

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \prod_{j=2}^{4} d\tau_j d\sigma_1 F |R(\tau_1,\tau_2,\tau_3,\tau_4,\mathbf{t},\mathbf{p},\mathbf{q})| e^{-M^h|\sigma_1-\tau_3|} \tag{IV.42b}$$

$$\leq M^{-h}\|I\|_i \int \frac{d^3\mathbf{p}}{(2\pi)^3} \prod_{j=2}^{4} d\tau_j F |R(\tau_1,\tau_2,\tau_3,\tau_4,\mathbf{t},\mathbf{p},\mathbf{q})| \tag{IV.42c}$$

$$\leq M^{-h}\|I\|_i \sup_{\mathbf{t},\mathbf{p},\mathbf{q}} \int d\tau_1 d\tau_2 d\tau_3 |R(\tau_1,\tau_2,\tau_3,\tau_4,\mathbf{t},\mathbf{p},\mathbf{q})|$$
$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} F(M^{-2h}e(\mathbf{p}+\mathbf{q}/2)^2, M^{-2h}e(-\mathbf{p}+\mathbf{q}/2)^2) \tag{IV.42d}$$
$$\leq M^{-h}\|I\|_i\|R\|_k M^h$$
$$= \|I\|_i\|R\|_k. \tag{IV.42e}$$

We have suppressed constants.

We now explain how each term in (IV.41) improves (IV.42e). For R_a, we improve (IV.42d) by using the immediate consequence

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} F(M^{-2h}e(\mathbf{p}+\mathbf{q}/2)^2, M^{-2h}e(-\mathbf{p}+\mathbf{q}/2)^2)\left[1-\rho\left(|\mathbf{q}|M^{-\frac{1}{2}(h+k)}\right)\right]$$
$$\leq M^{-\frac{1}{2}(k-h)}M^h \tag{IV.43a}$$

of Lemma IV.2. This replaces (IV.42e) by $M^{-\frac{1}{2}(k-h)}\|I\|_i\|K\|_k$.

For R_b we improve (IV.42c) by

$$\begin{aligned}
&\rho\left(|\mathbf{q}|M^{-\frac{1}{2}(h+k)}\right)\int \prod d\tau_j |K(\{\tau_i\},\mathbf{t},\mathbf{p},q) - K(\{\tau_i\},\mathbf{t},\mathbf{p},0)| \\
&= \rho \int \prod d\tau_j \left|\int_0^1 d\epsilon \frac{d}{d\epsilon} K(\{\tau_i\},\mathbf{t},\mathbf{p},\epsilon\mathbf{q})\right| \\
&\leq \rho \int_0^1 d\epsilon \int \prod d\tau_j |\mathbf{q}\cdot\nabla_{\mathbf{q}} K(\{\tau_i\},\mathbf{t},\mathbf{p},\epsilon\mathbf{q})| \\
&\leq |\mathbf{q}|\rho \int_0^1 d\epsilon \sup_{\mathbf{t},\mathbf{p},\mathbf{q}} \int \prod d\tau_j |\nabla_{\mathbf{q}} K(\{\tau_i\},\mathbf{t},\mathbf{p},\epsilon\mathbf{q})| \\
&= |\mathbf{q}|\rho \sup_{\mathbf{t},\mathbf{p},\mathbf{q}} \int \prod d\tau_j |\nabla_{\mathbf{q}} K(\{\tau_i\},\mathbf{t},\mathbf{p},\mathbf{q})| \\
&\leq M^{\frac{1}{2}(h+k)} \sup_{\mathbf{t},\mathbf{p},\mathbf{q}} \int \prod d\tau_j |\nabla_{\mathbf{q}} K(\{\tau_i\},\mathbf{t},\mathbf{p},\mathbf{q})|
\end{aligned} \tag{IV.43b}$$

The norm (IV.2a) "converts" the $\nabla_{\mathbf{q}}$ into M^{-k} yielding an extra $M^{\frac{1}{2}(h-k)} = M^{-\frac{1}{2}(k-h)}$ in (IV.42e). For R_c, we improve (IV.42c) as above but with $\mathbf{q}\cdot\nabla_{\mathbf{q}}$ replaced by $(\mathbf{p}-\mathbf{p}')\cdot\nabla_{\mathbf{p}}$.

As observed following (I.41)

$$
\begin{aligned}
F\rho|\mathbf{q}-\mathbf{p}'| &= F\rho||\mathbf{p}|-k_F| \\
&\le F\rho[||\mathbf{p}+\mathbf{q}/2|-k_F|+|\mathbf{q}|/2] \qquad (IV.43c)\\
&\le F\rho[M^h+M^{\frac{1}{2}(h+k)}]
\end{aligned}
$$

Once again the norm generates from (IV.43c) an extra $M^{-\frac{1}{2}(k-h)}$ in (IV.42e). For R_d, the same argument contributes $\rho|\mathbf{t}-\mathbf{t}'|M^{-k}$.

For R_e, we first manipulate (IV.40a)

$$
\begin{aligned}
&\int \frac{d^3\mathbf{p}}{(2\pi)^3} d\tau_3 d\tau_4 d\sigma_1 d\sigma_2 d\alpha_4 F\rho I h\left(M^{\frac{1}{2}(h+k)}(\tau_3-\sigma_1)\right) h\left(M^{\frac{1}{2}(h+k)}(\tau_4-\sigma_2)\right)\\
&\qquad [K(...,\tau_4,...)\delta(\tau_1-\alpha_4)-K(...,\alpha_4,...)\delta(\tau_1-\tau_4)]\\
&\qquad e^{-|e(\mathbf{p}+\mathbf{q}/2)||\sigma_1-\tau_3|}[\chi(-e(\mathbf{p}+\mathbf{q}/2))\chi(\sigma_1-\tau_3)-\chi(e(\mathbf{p}+\mathbf{q}/2))\chi(\tau_3-\sigma_1)]\\
&\qquad e^{-|e(-\mathbf{p}+\mathbf{q}/2)||\sigma_2-\tau_4|}[\chi(-e(-\mathbf{p}+\mathbf{q}/2))\chi(\sigma_2-\tau_4)-\chi(e(-\mathbf{p}+\mathbf{q}/2))\chi(\tau_4-\sigma_2)]\\
&=\int \frac{d^3\mathbf{p}}{(2\pi)^3} d\tau_3 d\tau_4 d\sigma_1 d\sigma_2 F\rho I h\left(M^{\frac{1}{2}(h+k)}(\tau_3-\sigma_1)\right) K(\tau_1,\tau_2,\tau_3,\tau_4,\mathbf{t}',\mathbf{p}',0)\\
&\qquad e^{-|e(\mathbf{p}+\mathbf{q}/2)||\sigma_1-\tau_3|}[\chi(-e(\mathbf{p}+\mathbf{q}/2))\chi(\sigma_1-\tau_3)-\chi(e(\mathbf{p}+\mathbf{q}/2))\chi(\tau_3-\sigma_1)]\\
&\qquad \Big\{e^{-|e(-\mathbf{p}+\mathbf{q}/2)||\sigma_2-\tau_4|}[\chi(-e)\chi(\sigma_2-\tau_4)-\chi(e)\chi(\tau_4-\sigma_2)]h\left(M^{\frac{1}{2}(h+k)}|\tau_4-\sigma_2|\right)\\
&\qquad -e^{-|e(-\mathbf{p}+\mathbf{q}/2)||\sigma_2-\tau_1|}[\chi(-e)\chi(\sigma_2-\tau_1)-\chi(e)\chi(\tau_1-\sigma_2)]h\left(M^{\frac{1}{2}(h+k)}|\tau_1-\sigma_2|\right)\Big\}. \qquad (IV.44)
\end{aligned}
$$

For the $K(...,\alpha_4,...)\delta(\tau_1-\tau_4)$ we evaluated the τ_4 integral using $\delta(\tau_1-\tau_4)$ and then made the change of variables $\tau_4=\alpha_4$. We next apply Taylor's theorem to $\{...\}$ in (IV.44). This yields

$$
(\tau_4-\tau_1)\int_0^1 d\alpha \frac{d}{d\tau}\left\{e^{-|e||\sigma_2-\tau|}[\chi(-e)\chi(\sigma_2-\tau)-\chi(e)\chi(\tau-\sigma_2)]h\left(M^{\frac{1}{2}(h+k)}(\tau-\sigma_2)\right)\right\}
$$

evaluated at

$$
\tau=\tau_1+\alpha(\tau_4-\tau_1)
$$

The $\frac{d}{d\tau}$ may act on $e^{-|e||\sigma_2-\tau|}$, producing $|e(-\mathbf{p}+\mathbf{q}/2)|\le M^h$, or on $h\left(M^{\frac{1}{2}(h+k)}(\tau-\sigma_2)\right)$, producing $M^{\frac{1}{2}(h+k)}$. It may not act on $\chi(\pm(\tau-\sigma_2))$ since $h\left(M^{\frac{1}{2}(h+k)}(\tau-\sigma_2)\right)$ is supported away from $\tau = \sigma_2$. The $|\tau_1 - \tau_4|$ produces an M^{-k} via the norm (IV.2a). Thus, altogether, Taylor's theorem yields an $M^{-\frac{1}{2}(k-h)}$. The analysis of R_f is similar.

Substituting R_g for R in (IV.40a) yields

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} d\tau_3 d\tau_4 d\sigma_1 d\sigma_2 d\alpha_2 F\rho h\left(M^{\frac{1}{2}(h+k)}(\tau_1-\sigma_1)\right) h\left(M^{\frac{1}{2}(h+k)}(\tau_1-\sigma_2)\right) I$$

$$e^{-|e(\mathbf{p}+\mathbf{q}/2)||\sigma_1-\tau_1|}[\chi(-e(\mathbf{p}+\mathbf{q}/2))\chi(\sigma_1-\tau_1) - \chi(e(\mathbf{p}+\mathbf{q}/2))\chi(\tau_1-\sigma_1)]$$

$$e^{-|e(-\mathbf{p}+\mathbf{q}/2)||\sigma_2-\tau_1|}[\chi(-e(-\mathbf{p}+\mathbf{q}/2))\chi(\sigma_2-\tau_1) - \chi(e(-\mathbf{p}+\mathbf{q}/2))\chi(\tau_1-\sigma_2)]$$

$$[K(\tau_1,\tau_2,\tau_3,\tau_4,\mathbf{t}',\mathbf{p}',0)\delta(\tau_1-\alpha_2) - K(\tau_1,\alpha_2,\tau_3,\tau_4,\mathbf{t}',\mathbf{p}',0)\delta(\tau_1-\tau_2)]$$

$$:= j_{1,\mathrm{en}}(\tau_1,\tau_2,\sigma_3,\sigma_4,\mathbf{t}',\mathbf{s},\mathbf{q}) - \int d\alpha_2 j_{1,\mathrm{en}}(\tau_1,\alpha_2,\sigma_3,\sigma_4,\mathbf{t}',\mathbf{s},\mathbf{q})\delta(\tau_1-\tau_2) \qquad (IV.45a)$$

where

$$\sup_{\mathbf{t},\mathbf{s},\mathbf{q}} \int d\tau_2 d\sigma_3 d\sigma_4 |j_{i,\mathrm{en}}(\tau_1,\tau_2,\sigma_3,\sigma_4,\mathbf{t}',\mathbf{s},\mathbf{q})| \le \mathrm{const}\|K\|_k\|I\|_i \qquad (IV.45b)$$

as in (IV.42).

For the remaining contributions R_h and R_i we follow the argument (IV.42), leading to the unrenormalized bound, but exploit the fact that, on the support of $\left[1 - h\left(M^{\frac{1}{2}(h+k)}(\tau_3-\sigma_1)\right) h\left(M^{\frac{1}{2}(h+k)}(\tau_4-\sigma_2)\right)\right]$ at least one of $|\tau_3-\sigma_1|, |\tau_4-\sigma_2|$ is smaller than $M^{-\frac{1}{2}(h+k)}$. In either case the M^{-h} coming from the integration

$$\int d\sigma_1 e^{-M^h|\sigma_1-\tau_3|} \le M^{-h}$$

in (IV.42b) is replaced by

$$\int_{|\sigma_i-\tau_j|\le M^{-\frac{1}{2}(h+k)}} d\sigma_i\, 1 \le M^{-\frac{1}{2}(h+k)}.$$

This is the desired improvement of $M^{-\frac{1}{2}(k-h)}$.

Our "sup" estimate on $J_1(\triangle = 0)$ is now complete. We next derive a "sup" estimate on $\frac{d}{d\triangle}J_1$. After that we include derivatives of J_1.

Just as in (IV.35c) unrenormalized power counting estimates suffice. Recall that, in the mixed $(\tau, \mathbf{k})$ representation,

$$\mathbf{C}_1^{(j)}(\tau,\mathbf{k}) = -e^{-E(\mathbf{k})|\tau|}\frac{(\mathrm{sgn}\tau)E(\mathbf{k})\mathbf{1} + e(\mathbf{k})\sigma^3}{2E(\mathbf{k})} f(M^{-2j}E(\mathbf{k})^2). \qquad (IV.46a)$$

Hence

$$\frac{d}{d\triangle}\mathbf{C}_1^{(j)}(\tau,\mathbf{k}) = e^{-E(\mathbf{k})|\tau|}f(M^{-2j}E(\mathbf{k})^2)\frac{\triangle}{2E(\mathbf{k})^2}\left\{|\tau|[(\mathrm{sgn}\tau)E(\mathbf{k})\mathbf{1}+e(\mathbf{k})\sigma^3]\right.$$

$$\left.+2(\mathrm{sgn}\tau)\mathbf{1}+e(\mathbf{k})/E(\mathbf{k})\sigma^3\right\} - e^{-E(\mathbf{k})|\tau|}f'(M^{-2j}E(\mathbf{k})^2)\frac{\triangle}{M^{2j}}\frac{(\mathrm{sgn}\tau)E(\mathbf{k})\mathbf{1}+e(\mathbf{k})\sigma^3}{E(\mathbf{k})} \qquad (IV.46b)$$

obeys

$$\left|\frac{d}{d\triangle}\mathbf{C}_1^{(j)}(\tau,\mathbf{k})\right| \le \mathrm{const}M^{-h}e^{-E(\mathbf{k})|\tau|}\chi(E(\mathbf{k})\le M^j)[1+|\tau|M^j] \qquad (IV.46c)$$

Decomposing the soft lower line into scales $\ell n\triangle \le h_2 < h$ and then using (IV.46c) in the "unrenormalized" argument (IV.42) gives

$$\int d\tau_2 d\sigma_3 d\sigma_4 \left|\frac{d}{d\triangle}J_1(\tau_1,\tau_2,\sigma_3,\sigma_4,\mathbf{t},\mathbf{s},\mathbf{q})\right| \le \mathrm{const}\|I\|_i\|K\|_k \sum_{h_2=\ell n\triangle}^{h} M^{-h_2}M^{-h}M^{h_2}$$

with the M^{-h_2} coming from the action of $\frac{d}{d\triangle}$ on the lower (soft) propagator (if $\frac{d}{d\triangle}$ acts on the upper line we get $M^{-h} \le M^{-h_2}$), the M^{-h} coming from the σ_1 integral and the M^{h_2} coming from the $d^3\mathbf{p}$ integral. Hence, just as in (IV.35),

$$\int d\tau_2 d\sigma_3 d\sigma_4 \left|\frac{d}{d\triangle}J_1(\tau_1,\tau_2,\sigma_3,\sigma_4,\mathbf{t},\mathbf{s},\mathbf{q})\right| \le \mathrm{const}\|I\|_i\|K\|_k M^{-h}\ell n\left(\frac{M^h}{\triangle}\right). \qquad (IV.47)$$

Denoting by $j_{1,\mathrm{int}}$ the contributions to J_1 from $R_a, R_b, R_c, R_e, R_f, R_h, R_i$ and $\frac{d}{d\triangle}J_1$ we have

$$J_1 = j_{1,\mathrm{int}} + j_{1,\mathrm{mom}} + \left\{j_{1,\mathrm{en}}(\tau_1,\tau_2,\ldots) - \int d\alpha_2 j_{1,\mathrm{en}}(\tau_1,\alpha_2,\ldots)\delta(\tau_1-\tau_2)\right\} \qquad (IV.47a)$$

where the second (resp. third) term is the contribution from R_d (resp. R_g). So far we have shown

$$\sup_{\mathbf{t},\mathbf{s},\mathbf{q}}\int d\tau_2 d\sigma_3 d\sigma_4 |j_{1,\mathrm{int}}| \le \mathrm{const}\|K\|_k\|I\|_i\left[M^{-\frac{1}{2}(k-h)} + \frac{\triangle}{M^h}\ell n\left(\frac{M^h}{\triangle}\right)\right] \qquad (IV.47b)$$

$$\sup_{\mathbf{t},\mathbf{s},\mathbf{q}}\int d\tau_2 d\sigma_3 d\sigma_4 |j_{1,\mathrm{mom}}|\rho\left(M^{-2h}E^2(\mathbf{t}+\mathbf{q}/2)\right) \le \mathrm{const}\|K\|_k\|I\|_i M^{-\frac{1}{2}(k-h)} \qquad (IV.47c)$$

(since $j_{1,\mathrm{mom}}$ contains a factor $|\mathbf{t}-\mathbf{t}'|$ which, as in (IV.43c), is bounded by $M^{\frac{1}{2}(h+k)}$)

$$\sup_{\mathbf{t},\mathbf{s},\mathbf{q}}\int d\tau_2 d\sigma_3 d\sigma_4 |j_{1,\mathrm{en}}| \le \mathrm{const}\|K\|_k\|I\|_i. \qquad (IV.47d)$$

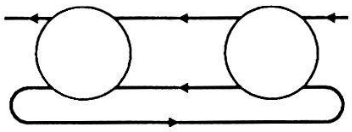
To complete the bound on J_1 it is necessary to show that each derivative with respect to the external momenta $\mathbf{t}, \mathbf{s}, \mathbf{q}$ and each multiplication by the external time differences $|\tau_2 - \tau_1|, |\sigma_3 - \tau_1|, |\sigma_4 - \tau_1|$ produces at worst an M^{-h}. This is reasonable, since K has scale k, I has scale i, the internal lines have scale h, ρ has scale $\frac{1}{2}(h+k)$ and $k, i, h, \frac{1}{2}(h+k) \geq h$. The proof is a straightforward variation of the above provided one first makes the change of variables $\mathbf{p} \to \mathbf{p} + \mathbf{q}/2$ (so that $\mathbf{q}$ derivatives cannot act on the soft line) and one applies the momentum derivatives before setting $\triangle = 0$ (so that derivatives cannot act on the characteristic functions $\chi(\pm e)$ in (IV.40)).

The bound on J_2 is proven similarly.

■

We have controlled the quartic part of $\frac{1}{2!}\mathcal{E}_2^{(h)}(\mathcal{U},\mathcal{U})$ where $\mathcal{U}$ is the part of $\mathcal{W}^{(h)}$ (see [I.102a]) of degree at most four. It remains to control the quadratic part. Precisely,

quadratic part of $\frac{1}{2!}\mathcal{E}_2^{(h)}(\mathcal{U},\mathcal{U}) =$



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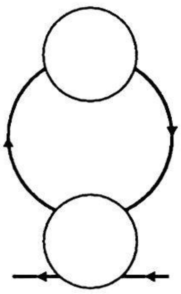

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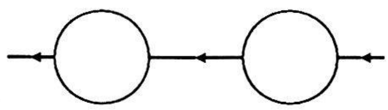
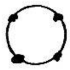


$(IV.48)$

We denote the values of the first, second and third graphs in (IV.48) by T_1, T_2 and T_3 respectively. That is,

$$\text{the quadratic part of } \frac{1}{2!}\mathcal{E}_2^{(h)}(\mathcal{U},\mathcal{U}) = T_1 + T_2 + T_3$$

Recall that $\frac{1}{2}$  is the kernel of the quartic part of $\mathcal{U}$ and is a sum of renormalized $(\mathbf{1}-\mathbf{L}^{(i)})I$ and local (see (I.99) $\mathbf{L}^{(i)}I$ kernels;  is the kernel of the quadratic part of $\mathcal{U}$ and also is a sum of renormalized and local (see(I.99b)) contributions $(\mathbf{1}-\mathbf{L}^{(s)})S$ and $\boldsymbol{\ell}S$; all monomials are Wick ordered. Once again by [I.102] renormalized contributions are of scales $i, s > h$, while the (resummed [I.102b]) local quartic part has scale $i = h$ and the local quadratic part is a number. The last diagram T_3 has already been treated in Lemma II.2′.

Lemma IV.6 Let $I_1^{(j_1)}\tau^a \otimes \tau^b$, $I^{(j_2)}\tau^c \otimes \tau^d$ and $S^{(s)}\sigma^e$ be general (not necessarily in the range of $\mathbf{L}$) kernels, with $j_1, j_2, s \geq h$.

a) Let $T_1(\tau, \mathbf{q})$ be the first diagram of (IV.48). Recall, from (IV.48), that the lines of this diagram are of scale h. Then,

$$\|T_1\|_h \leq \text{const}\|I_1^{(j_1)}\|_h\|I_2^{(j_2)}\|_h M^h\left[M^{\frac{1}{5}h} + \frac{\Delta}{M^h}\log\left(\frac{M^h}{\Delta}\right)\right]$$

b) Suppose, $e \in \{0,3\}$. Let $T_2(\tau, \mathbf{q})$ be the second diagram of (IV.48). Define

$$r = \begin{cases} -1 & I_1, S \text{ both local} \\ j_1 & I_1 \text{ nonlocal, } S \text{ local} \\ s & I_1 \text{ local, } S \text{ nonlocal} \\ \min(j_1, s) & I_1, S \text{ both nonlocal} \end{cases}$$

Then,

$$\|T_2\|_h \leq \text{const}M^h\|I_1^{(j_1)}\|_h[|S^{(s)}|M^{-h}]\left(\frac{\Delta}{M^h}\right) + \text{const}M^h\|I_1^{(j_1)}\|_{j_1}[\|S^{(s)}\|_s M^{-h}]M^{-(r-h)}$$

Proof The strategy is the same as in Lemma IV.4. The volume estimates of Lemma IV.2 are used to control the case $\Delta = 0$. Normal power counting suffices to control $\frac{d}{d\Delta}T_i$.

a) When $\Delta = 0$

$$T_1(\tau, \mathbf{q}) = \frac{1}{8}\int \frac{d^d\mathbf{p}}{(2\pi)^d}\frac{d^d\mathbf{r}}{(2\pi)^d}\prod_{i=1}^{6} d\tau_i I_1(\mathbf{p},\mathbf{q},\mathbf{r},0,\tau_1,\tau_2,\tau_3)I_2(\mathbf{p},\mathbf{q},\mathbf{r},\tau_4,\tau_5,\tau_6,\tau)$$

$$e^{-|e(\mathbf{p}+\mathbf{q})||\tau_4-\tau_2|}e^{-|e(\mathbf{r}-\mathbf{p})||\tau_3-\tau_5|}e^{-|e(\mathbf{r})||\tau_1-\tau_6|}$$

$$\tau^a[\operatorname{sgn}(\tau_4 - \tau_2)\mathbf{1} + \operatorname{sgn}e(\mathbf{p}+\mathbf{q})\sigma^3]\tau^c$$

$$tr\tau^b[\operatorname{sgn}(\tau_5 - \tau_3)\mathbf{1} + \operatorname{sgn}e(\mathbf{r}-\mathbf{p})\sigma^3]\tau^d[\operatorname{sgn}(\tau_1 - \tau_6)\mathbf{1} + \operatorname{sgn}e(\mathbf{r})\sigma^3]$$

$$F(M^{-2h}e(\mathbf{p}+\mathbf{q})^2, M^{-2h}e(\mathbf{r}-\mathbf{p})^2, M^{-2h}e(\mathbf{r})^2) \qquad (IV.49)$$

where F ensures that at least one of the three lines is hard of scale h, while the remaining two are hard or soft of scale h.

Suppose the line carrying $[\operatorname{sgn}(\tau_4 - \tau_2)\mathbf{1} + \operatorname{sgn}e(\mathbf{p}+\mathbf{q})\sigma^3]$ is hard. Then for

$$0 \le \alpha \le 4, \quad |\beta| \le 1$$

we have

$$M^{(\alpha+|\beta|)h} \int d\tau |\tau^\alpha \nabla_{\mathbf{q}}^\beta T_1(\tau, \mathbf{q})| \le \text{const} \|I_1\|_h, \|I_2\|_h \int d\tau_4 e^{-M^h|\tau_4 - \tau_2|}$$

$$\int d^d\mathbf{p} d^d\mathbf{r} \chi(|e(\mathbf{p}+\mathbf{q})| \le M^h)\chi(|e(\mathbf{r}-\mathbf{p})| \le M^h)\chi(|e(\mathbf{r})| \le M^h)$$

Each "derivative" $M^h(\tau, \nabla_1)$ yields a factor $0(1)$ when it acts upon $I_1^{(j_1)}, I_2^{(j_2)}, e^{-|e(\mathbf{p}+\mathbf{q})||\tau_4-\tau_2|}$ or F. If it acts on $\operatorname{sgn}e(\mathbf{p}+\mathbf{q})$ we get zero since the supports of $\delta(e(\mathbf{p}+\mathbf{q}))$ and F are disjoint. Thus,

$$\|T_1(\triangle = 0)\|_h \le \text{const}\|I_1\|_h \|I_2\|_h M^{-h} \int d^d\mathbf{p} d^d\mathbf{r}$$

$$\chi(|e(\mathbf{p}+\mathbf{q})| \le M^h)\chi(|e(\mathbf{r}-\mathbf{p})| \le M^h)\chi(|e(\mathbf{r})| \le M^h) \qquad (IV.50)$$

If the top line is soft we arrive at (IV.50) by, first, changing variables in (IV.49) so that $\mathbf{q}$ flows through a hard line, estimating as above and then changing variables back again.

To estimate the integral on the right hand side of (IV.50) we write $1 = \chi(|p| \le M^{\alpha h}) + \chi(|p| \ge M^{\alpha h})$ with $\alpha = \frac{3}{2d+1}$ and bound

$$\int d^d\mathbf{p} d^d\mathbf{r} \chi(|p| \le M^{\alpha h})\chi(|e(\mathbf{r})| \le M^h) \le M^{d\alpha h} M^h = M^{\frac{3d}{2d+1}h} M^h$$

$$\int d^d\mathbf{p} \chi(|e(\mathbf{p}+\mathbf{q})| \le M^h) \int d^d\mathbf{r} \chi(|p| \ge M^{\alpha h})\chi(|e(\mathbf{r}-\mathbf{p})| \le M^h)\chi(|e(\mathbf{r})| \le M^h)$$

$$\le M^h M^{-\frac{1}{2}(\alpha h - h)} M^h = M^{\frac{3d}{2d+1}h} M^h$$

by Lemma IV.2. This yields

$$\|T_1(\triangle = 0)\|_h \le \text{const}\|I_1\|_h\|I_2\|_h M^h M^{\frac{d-1}{2d+1}h}$$
$$\le \text{const}\|I_1\|_h\|I_2\|_h M^h M^{\frac{1}{5}h}.$$

Finally, by conventional power counting bounds, as in [IV.34, 35, 36]

$$\|\frac{d}{d\triangle}T_1\|_h \le \text{const}\|I_1\|_h\|I_2\|_h \sum_{\ell n\triangle \le h_1 \le h_2 \le h} M^{-h}(M^{-h_1} + M^{-h_2} + M^{-h})M^{h_1}M^{h_2}$$
$$\le \text{const}\|I_1\|_h\|I_2\|_h M^{-h} \sum_{h_2 \le h} M^{h_2} \sum_{h_1 = \ell n\triangle}^{h} 1$$
$$\le \text{const}\|I_1\|_h\|I_2\|_h \ell n\left(\frac{M^h}{\triangle}\right).$$

The argument is completed as in [IV.37].

b) When $\triangle = 0$,

$$T_2(\tau, \mathbf{q}) = \frac{\tau^b}{4}\int \frac{d^d\mathbf{p}}{(2\pi)^d}\prod_{i=1}^{4} d\tau_i I_1(\mathbf{p}, \mathbf{q}, 0, \tau_1, \tau_2, \tau)S(\mathbf{p}, \tau_3, \tau_4)$$
$$e^{-|e(\mathbf{p})|[|\tau_3-\tau_1|+|\tau_4-\tau_2|]}F(M^{-2h}e(\mathbf{p})^2)$$
$$tr\tau^a[\text{sgn}(\tau_3 - \tau_1)\mathbf{1} + \text{sgn}e(\mathbf{p})\sigma^3]\sigma^e[\text{sgn}(\tau_2 - \tau_4)\mathbf{1} + \text{sgn}e(\mathbf{p})\sigma^3]. \qquad (IV.51)$$

Observe that, when $(\tau_3 - \tau_1)(\tau_2 - \tau_4) < 0$,

$$(\text{sgn}(\tau_3 - \tau_1)\mathbf{1} + \text{sgn}e(\mathbf{p})\sigma^3, \text{sgn}(\tau_2 - \tau_4)\mathbf{1} + \text{sgn}e(\mathbf{p})\sigma^3)$$
$$\in 2\text{sgn}e(\mathbf{p})\left\{(\tau^0, -\tau^3), (-\tau^3, \tau^0)\right\}$$

with the result that the trace is zero since $\tau^0\tau^\alpha\tau^3 = \tau^3\tau^\alpha\tau^0 = 0$ for $\alpha = 0, 3$. On the other hand when $(\tau_3 - \tau_1)(\tau_2 - \tau_4) \ge 0$

$$|\tau_3 - \tau_1| \le |\tau_3 - \tau_1 + \tau_2 - \tau_4|$$
$$\le |\tau_1 - \tau_2| + |\tau_4 - \tau_3|.$$

So, recalling that

$$r \le \begin{cases} 0 & \text{if } \tau_1 = \tau_2, \tau_3 = \tau_4 \\ j_1 & \text{if } \tau_3 = \tau_4, \tau_1 \ne \tau_2 \\ s & \text{if } \tau_1 \ne \tau_2, \tau_3 = \tau_4 \\ \min(s, j_1) & \text{if } \tau_1 \ne \tau_2, \tau_3 \ne \tau_4 \end{cases}$$

$$\begin{aligned}\int d\tau |T_2(\tau,\mathbf{q})_{\triangle=0}| &\le \int_{(\tau_3-\tau_1)(\tau_2-\tau_4)>0} \frac{d^d\mathbf{p}}{(2\pi)^d} \prod_{i=1}^{4} d\tau_i d\tau |I_1^{(j_1)}||S^{(s)}|F \\ &\quad [1+M^r|\tau_3-\tau_1|]^{-2}[1+M^r|\tau_3-\tau_1|]^2 \\ &\le \int \frac{d^d\mathbf{p}}{(2\pi)^d} \prod d\tau_i d\tau I_1^{(j_1)} S^{(s)} F [1+M^r|\tau_3-\tau_1|]^{-2} \\ &\quad [1+M^{j_1}|\tau_1-\tau_2|+M^s|\tau_4-\tau_3|]^2 \\ &\le \mathrm{const}\|I_1^{(j_1)}\|_{j_1}\|S^{(s)}\|_s M^h M^{-r}\end{aligned} \tag{IV.52}$$

To estimate $M^{(\alpha+|\beta|)h}\int d\tau|\tau^\alpha\nabla_{\mathbf{q}}^\beta T_2(\tau,\mathbf{q})_{\triangle=0}|$ when $\alpha+|\beta|\ge 1$ it suffices to use straight power counting without exploiting the constraint $(\tau_3-\tau_1)(\tau_2-\tau_4)>0$. The "derivatives" necessarily act on I, and produce $M^{-(\alpha+|\beta|)j_1}$. Consequently,

$$\|T_2(\triangle=0)\|_h \le \mathrm{const}\|I_1^{(j_1)}\|_{j_1}\|S^{(s)}\|_s M^{-(r-h)} \tag{IV.53}$$

As in part a)

$$\|\frac{d}{d\triangle}T_2\|_h \le \mathrm{const}\|I_1\|_h|S|M^{-h}M^{-h}M^h$$

so

$$\int_0^{\triangle} d\triangle'\|\frac{d}{d\triangle}T_2(\triangle')\|_h \le \mathrm{const}\|I_1\|_h[|S|M^{-h}]M^h\frac{\triangle}{M^h}. \tag{IV.54}$$

Adding (IV.53) and (IV.54) finishes the proof.

■

V. The Flow of the Effective Interaction

Let $\mathcal{H}$ be a Banach space and $\mathcal{B}=\mathcal{B}(\mathcal{H})$ the space of bounded operators on $\mathcal{H}$ with operator norm $\|\cdot\|$. Let $\|\cdot\|'_h, h\le 0$, be a sequence of norms (motivated by, but not necessarily equal to (I.106)) on $\mathcal{B}$ obeying

$$\|a\|'_h \le \|a\|'_{h'} \qquad \text{for all } h\le h' \tag{V.1.a}$$

$$\|a\| \le \|a\|'_h \tag{V.1.b}$$

$$\|abc\|'_h \le \|a\|'_h\|b\|\|c\|'_h \tag{V.1.c}$$

for all $a,b,c\in\mathcal{B}$ and $h\le 0$. For example, if $\mathcal{H}=L^2(k_FS^{d-1})$ then

$$\|a\|'_h = (1+k_F^{d-2}\omega_{d-1}) \max_{\substack{|n|\le 2\\|m|\le 2}} \sup_{t',s'\in k_FS^{d-1}} |M^{(|n|+|m|)h}\nabla_{t'}^n\nabla_{s'}^m a(t',s')| \tag{V.1.d}$$

satisfies (V.1)

We consider a flow on $\mathcal{B}$ defined for $h \le 0$ by

$$a_{h-1} = a_h + a_h B_h a_h + S_h(a_0, ..., a_h) + H_n(a_0, ..., a_h) \tag{V.2}$$

where $B_h \in \mathcal{B}$ and S_h, H_h are maps from the h-fold product of $\mathcal{B}$ with itself to $\mathcal{B}$.

Theorem V.1 Let $\gamma < 1, M > 1, \triangle > 0$, α and Γ be constants such that

$$\|a_0 \sum_{h \ge k} B_h\| \le \gamma < 1 \text{ for all } k \le 0 \tag{V.3.a}$$

$$\|B_h\|_h \le \alpha \frac{M^h}{M^h + \triangle} \text{ for all } h \le 0 \tag{V.3.b}$$

and

$$\|a_0\| \sum_{h \le 0} \frac{M^h}{M^h + \triangle} \le \Gamma. \tag{V.3.c}$$

Further, suppose that there are constants ν, η (possibly depending on α, γ, Γ) and $0 < \omega \le 1$ such that

$$\| \sum_{h' \ge j \ge h} S_j(a_0, ..., a_j)\|_h \le \nu \|a_0\|_0^2 \tag{V.4}$$

$$\|H_h(a_0, ..., a_h)\|_h \le \eta \frac{M^h}{M^h + \triangle} \|a_0\|_0^{2+\omega} \tag{V.5}$$

for all $\|a_0\|_0, ..., \|a_h\|_h \le 4\frac{1+\alpha\Gamma}{1-\gamma}\|a_0\|_0$. Then there exists a constant $\in = \in (\gamma, \Gamma, \alpha, \nu, \eta, \omega) > 0$ for which

$$\|a_0\|_0 \le \in \tag{V.6}$$

implies that the sequence $a_0, a_1, a_2, ...$ generated by (V.2) obeys

$$\|a_h\|_h \le 4\frac{1 + \alpha\Gamma}{1 - \gamma}\|a_0\|_0, \qquad h \le 0 \tag{V.7}$$

Proof We first verify, by induction, that the solution of the truncated flow

$$\tilde{a}_{h-1} = \tilde{a}_h + \tilde{a}_h B_h \tilde{a}_h \tag{V.8}$$

is of the form

$$\tilde{a}_h = [\mathbf{1} - a_0 \sum_{0 \ge j > h} B_j - b_h]^{-1} a_0 \tag{V.9a}$$

with $b_0 = 0$ and

$$\|b_h\|_h \le \frac{1-\gamma}{2\Gamma}\|a_0\|_0 \sum_{0\ge j>h} \frac{M^j}{M^j+\triangle} < \frac{1-\gamma}{2}. \qquad (V.9b)$$

Note that (V.3.a) and (V.9b) imply

$$\|a_0 \sum_{0\ge j>h} B_j + b_h\| \le \gamma + \frac{1-\gamma}{2} = \frac{1+\gamma}{2} < 1$$

and hence

$$\begin{aligned}
\|\tilde{a}_h\|_h &= \|[\mathbf{1} - a_0 \sum_{0\ge j>h} B_j - b_h]^{-1} a_0\|_h \\
&\le \sum_{n=0}^{\infty} \|[a_0 \sum B_j + b_h]^h a_0\|_h \\
&\le \left\{1 + \|a_0 \sum B_j + b_h\|_h \sum_{n=1}^{\infty} \|a_0 \sum B_j + b_h\|^{n-1}\right\} \|a_0\|_h \qquad (V.10)\\
&\le \left\{1 + \left(\alpha\Gamma + \frac{1-\gamma}{2}\right)\left(1 - \frac{1+\gamma}{2}\right)^{-1}\right\} \|a_0\|_0 \\
&\le 2\frac{1+\alpha\Gamma}{1-\gamma}\|a_0\|_0
\end{aligned}$$

The bound (V.1.c) is used in the third line.

The flow of the tail b_h is given by

$$b_{h-1} = b_h - a_0 B_h \tilde{a}_h B_h [\mathbf{1} + \tilde{a}_h B_h]^{-1} \qquad (V.11)$$

since substituting (V.9.a) into (V.8) yields

$$\begin{aligned}
\mathbf{1} - a_0 \sum_{j\ge h} B_j - b_{h-1} &= \left\{[\mathbf{1} - a_0 \sum_{j>h} B_j - b_h]^{-1} + \tilde{a}_h B_h [\mathbf{1} - a_0 \sum_{j>h} B_j - b_h]^{-1}\right\}^{-1} \\
&= [\mathbf{1} - a_0 \sum_{j>h} B_j - b_h][\mathbf{1} + \tilde{a}_h B_h]^{-1} \\
&= \mathbf{1} - a_0 \sum_{j>h} B_j - b_h - a_0 B_h + a_0 B_h \tilde{a}_h B_h [\mathbf{1} + \tilde{a}_h B_h]^{-1}
\end{aligned}$$

We have used $[\mathbf{1} - a_0 \sum_{j>h} B_j - b_h]\tilde{a}_h = a_0$ in the last line.

For $\|a_0\|_0 \le \in$,

$$\begin{aligned}
&\|a_0 B_h \tilde{a}_h B_h [\mathbf{1} + \tilde{a}_h B_h]^{-1}\|_{h-1} = \|a_0 B_h [\mathbf{1} + \tilde{a}_h B_h]^{-1} \tilde{a}_h B_h\|_{h-1} \\
&\le \|a_0\|_0 \alpha \frac{M^h}{M^h+\triangle}\left(1 - 2\frac{1+\alpha\Gamma}{1-\gamma}\|a_0\|_0\alpha\right)^{-1} 2\frac{1+\alpha\Gamma}{1-\gamma}\|a_0\|_0\alpha \\
&\le \frac{1-\gamma}{2\Gamma}\|a_0\|_0 \frac{M^h}{M^h+\triangle}
\end{aligned}$$

provided $\in$ is small enough, depending on α, γ, Γ. This completes the verification of (V.9b).

We now show, by induction of course, that the solution of the full flow (V.2) obeys

$$a_h = \tilde{a}_h + \sum_{j>h} S_j(a_0, ..., a_j) + c_h \tag{V.12.a}$$

with $c_0 = 0$ and

$$\|c_h\|_h \le \|a_0\|_0^{1+\omega} \prod_{j>h} \left(1 + \delta \frac{M^j}{M^j + \triangle} \|a_0\|_0\right) \equiv I_h \tag{V.12b}$$

$$\delta = 6\alpha \frac{1 + \alpha\Gamma}{1 - \gamma}(1 + \nu) + \eta$$

The bound (V.12b) implies

$$\begin{aligned}
\|c_h\|_h &\le \|a_0\|_0^{1+\omega} \exp\left(\sum_{j>-\infty} \delta \frac{M^j}{M^j + \triangle} \|a_0\|_0\right) \\
&\le \|a_0\|_0^{1+\omega} e^{\delta\Gamma} \qquad \text{(by (V.3b))} \\
&\le \frac{1 + \alpha\Gamma}{1 - \gamma} \|a_0\|_0
\end{aligned}$$

provided $\in^\omega e^{\delta\Gamma} \le \frac{1+\alpha\Gamma}{1-\gamma}$. Hence, by (V.10), (V.4)

$$\|a_h\|_h \le 4 \frac{1 + \alpha\Gamma}{1 - \gamma} \|a_0\|_0$$

provided $\nu \in \le \frac{1+\alpha\Gamma}{1-\gamma}$.

The flow for the error c_h is

$$c_{h-1} = c_h + c_h B_h a_h + \tilde{a}_h B_h c_h + (\sum_{j>h} S_j) B_h a_h + \tilde{a}_h B_h (\sum_{j>h} S_j) + H_h \tag{V.13}$$

since, substituting (V.12a) into (V.2),

$$\begin{aligned}
a_{h-1} &= a_h + a_h B_h a_h + S_h + H_h \\
&= \tilde{a}_h + \tilde{a}_h B_h \tilde{a}_h + \sum_{j>h} S_j + c_h + (\sum_{j>h} S_j + c_h) B_h a_h \\
&\quad + \tilde{a}_h B_h (\sum_{j>h} S_j + c_h) + S_h + H_h \\
&= \tilde{a}_{h-1} + \sum_{j>h-1} S_j + c_h + (\sum_{j>h} S_j + c_h) B_h a_h \\
&\quad + \tilde{a}_h B_h (\sum_{j>h} S_j + c_h) + H_h
\end{aligned}$$

Assuming the inductive bound (V.12b) for c_h we have

$$\begin{aligned}\|c_{h-1}\|_{h-1} &\leq \|c_h\|_h \left(1+6\alpha\frac{M^h}{M^h+\triangle}\frac{1+\alpha\Gamma}{1-\gamma}\|a_0\|_0\right)\\ &\quad+\left(6\nu\alpha\frac{1+\alpha\Gamma}{1-\gamma}+\eta\right)\frac{M^h}{M^h+\triangle}\|a_0\|_0^{2+\omega}\\ &\leq I_h\left[1+6\alpha\frac{M^h}{M^h+\triangle}\frac{1+\alpha\Gamma}{1-\gamma}\|a_0\|_0+\left(6\nu\alpha\frac{1+\alpha\Gamma}{1-\gamma}+\eta\right)\frac{M^h}{M^h+\triangle}\|a_0\|_0\right]\\ &= I_h\left[1+\delta\frac{M^h}{M^h+\triangle}\|a_0\|_0\right]\end{aligned}$$

where we used $\|c_h\|_h, \|a_0\|_0^{1+\omega} \leq I_h$ in the second line.

It follows from Theorem V.1 that

$$a_h = \sum_{j>h}(A_j+S_j)+a_0 \tag{V.14}$$

where $A_j = a_j B_j a_j + H_j$ obeys

$$\begin{aligned}\sum_j \|A_j\|_j &\leq \sum_j \left[16\left(\frac{1+\alpha\Gamma}{1-\gamma}\right)^2 \alpha\frac{M^j}{M^j+\triangle}\|a_0\|_0^2+\eta\frac{M^j}{M^j+\triangle}\|a_0\|_0^{2+\omega}\right]\\ &\leq \left[16\left(\frac{1+\alpha\Gamma}{1-\gamma}\right)^2\alpha\Gamma+\eta\Gamma\|a_0\|_0^{\omega}\right]\|a_0\|_0\end{aligned} \tag{V.15}$$

Thus the $\sum_{j>h} A_j$ part of a_h converges in any norm $\|\cdot\|'_{-\infty}$ that is smaller than all the $\|\cdot\|'_h$'s. For example, if (V.1.d) is used then the L^∞ norm

$$\|a\|'_{-\infty} = (1+k_F^{d-1}\omega_{d-1}) \sup_{t',s'\in k_F S^{d-1}} |a(t',s')|$$

will do. We see from (IV.29) and (IV.35) that $S_j(t',s')$ is bounded by

$$\text{const}\|a_0\|_0^2\left\{\left[M^{j/2}+\frac{\triangle}{M^j}\log(M^j/\triangle)\right]+\frac{M^{j/2}}{|S'\pm t'|^{1/2}}\chi(|s'\pm t'|\geq \text{const}M^j)\right\} \tag{V.16}$$

The portion in square brackets is summable as in (V.15). The contribution to a_h from the second part converges pointwise and in all L^p norms with $p<\infty$ but not uniformly.

List of Symbols

$\in(\mathbf{k}) = \frac{\mathbf{k}^2}{2m}$

$k = (k_0, \mathbf{k}) \in \mathbf{R}^{d+1}$

$\langle k, (\tau, \mathbf{x})\rangle_- = -k_0\tau + \mathbf{k}\cdot\mathbf{x}$

$\psi_{k,\alpha}$, $\bar\psi_{k,\alpha}$ physical Grassmann fields

ψ^e, $\bar\psi^e$ external Grassmann fields

$C(\xi_1, \xi_2)$ (I.3)

$e(\mathbf{k}) = \frac{\mathbf{k}^2}{2m} - \mu$, μ =chemical potential

$d\mu_C(\psi, \bar\psi)$ fermionic Gaussian measure with covariance C

$\mathcal{G}(\psi^e, \bar\psi^e)$ (I.5a), (I.14)

$\mathcal{V}(\psi, \bar\psi)$ (I.5b)

$\langle k_1, k_2|v|k_3, k_4\rangle$ general two-body interaction satisfying (I.6)

Fermi surface (I.7)

$C^{(j)}$ (I.8)

$h(x)$, $f(x)$ (II.1a) (II.1b)

$\rho(x) = 1 - h(x)$ (II.1c)

$d\mu_{C^{(j)}}(\psi^{(j)}, \bar\psi^{(j)})$ (I.9)

$\mathcal{G}^{(h)}(\phi^{(\leq h)})$ (I.10)

$\phi^{(\leq h)}$ (I.11)

$$\int d\xi := \int d^d\mathbf{x}\,d\tau \sum_{\sigma\in\{\uparrow,\downarrow\}}$$

G_2, S_2, $\sum$ (I.13)

$\mathcal{E}^{(h)}$, $\mathcal{E}_n^{(h)}$ (I.17)

ℓ (I.18)

$\delta\mu$ (I.21)

r, c forks (I.23)

$G_{2p,n}^{(h)}$ (I.24)

$L^{(h)}$ (I.29)

$i^* = \max(i_1, i_2, i_3, i_4)$ (I.29)

$t' = (t_0, \mathbf{t})' = \left(0, \frac{\mathbf{t}}{|\mathbf{t}|} k_F\right), k_F = (2m\mu)^{1/2}$ (I.29)

$q^0 = (q_0, 0)$

H^n homogeneous harmonic polynomials on $\mathbf{R}^d$ of degree n

π_n projection from $L^2(k_F S^{d-1})$ onto H^n

λ_n nth eigenvalue of the rotation invariant kernel $F(t', s')$

R, C forks (I.34), (I.50)

$g^{(h)}$, $g_{2p,n}^{(h)}$ (I.36)

$\mathcal{T}$ rooted planar tree

$s(\mathcal{T})$ (I.38)

G_f^I (I.39)

$\mathrm{Val}(G^I)$ (I.40) (III.3) (III.6) (III.12)

$\mathcal{F}^{(h)}$, $F^{(h)}$ (I.42)

$\beta^{(h)}$, β (I.54)

Ψ_k, $\bar{\Psi}_k$ Nambu fields (I.56)

σ^j, $j = 0, 1, 2, 3$ Pauli matrices (I.57)

$\mathbf{C}_0$ (I.59)

$\mathbf{C} = \mathbf{C}_\triangle$ (I.62a) (II.3)

$\mathbf{C}_\triangle^{(j)}$ (I.85) (II.4a)

$E(\mathbf{k})$ (I.62b)

$\delta\mathcal{V}$, $\delta\mu$ (I.67a)

$\mathcal{D}$, D (I.67b)

$\mathcal{W}$ (I.69)

$\sum(k) = r_0 \mathbf{1} + r_1\sigma^1 + r_2\sigma^2 + r_3\sigma^3$ (I.70)

J (I.79)

$\wedge_n^{\pm}$ (I.81)

$b_{\pm}$ (I.82)

γ (I.83)

$\mathbf{C}^{(j)} = \mathbf{C}_{\Delta}^{(j)}$ (I.85) (III.1a)

$\mathcal{W}^{(h)}$ (I.86)

$\Psi^{(j)}, \bar{\Psi}^{(j)}$ (I.86)

$\tau^0 = \frac{1}{2}(\sigma^0 + \sigma^3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (I.89)

$\tau^1 = \frac{1}{2}(\sigma^1 + i\sigma^2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (I.89)

$\tau^2 = \frac{1}{2}(\sigma^1 - i\sigma^2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ (I.89)

$\tau^3 = \frac{1}{2}(\sigma^0 - \sigma^3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ (I.89)

$f_{m,n}$ (I.90)

$I = \sum_{m,n=0}^{3} f_{mn}\tau^m \otimes \tau^n$ general quartic

$Q(t', s')$ (I.94)

$\mathbf{L}^{(h)}$, $\boldsymbol{\ell}$, $\mathbf{r}$ (I.99) (II.6) (III.19c)

$\mathcal{Q}^{(h)}$, $Q^{(h)}$ (I.100)

$\mathbf{w}^{(h)}$, $w_{2p,n}^{(h)}$ (I.104)

$\|I\|_h$ (I.106) (II.26b)

$\|T\|_h$ (I.107) (III.26a)

$|u|$ (II.5a) (III.4) (III.8a)

$|u|'$ (I.108) (II.5b) (III.8b)

$\pi(f)$ $\pi(f)$ is the fork immediately below the fork f of a tree

$\mathbf{C}_s^{(j)}$ (I.111) (III.1b)

G_f (I.112)

$\hat{\mathcal{E}}_2^{(h)}$, $\hat{\mathcal{E}}_{\geq 3}^{(h)}$ (I.116)

$\hat{E}_2^{(h)}$, $\hat{E}_{\geq 3}^{(h)}$ (I.117)

$\beta_{\triangle}^{(h)}$ (I.123)

$S^{(h)}$, $H^{(h)}$ (I.124)

$\lambda_n(\triangle)$ (I.126)

$(\tau, \mathbf{k}) \in \mathbf{R}^{d+1}$ (time, vector momentum)

G_f^J a connected component of $\{\ell \in G^J | j_\ell \geq h\}$, $h \leq -1$

j_f (III.7)

D_f (III.9)

$\triangle_f$, $\triangle_v$ (III.10)

$L(G)$ number of lines in G

T spanning tree (Technical Lemma III.2)

δ_p, $\mathbf{k}_p$ (supplement to Lemma III.1) (III.15)

$\mathbf{C}_1^{(j)}$, $\mathbf{C}_2^{(j)}$ (II.18) (III.17)

$\mathbf{C}_{1,s}^{(j)}$, $\mathbf{C}_{2,s}^{(j)}$ (III.18)

b_f number of upward branches from f

s_f (Lemma III.5)

E_f number of external legs of G_f

$L_2(G)$ number of $\mathbf{C}_2$ or $\mathbf{C}_{2,s}$ lines in G

J_1, J_2 (IV.20) (Lemma IV.4) (Lemma IV.5)

r' (IV.21)

B_i (IV.22)

$j_{\beta,\mathrm{int}}$, $j_{\beta,\mathrm{mom}}$, $j_{\beta,\mathrm{en}}$ (Lemma IV.5) (IV.45a) (IV.47a)

T_1, T_2, T_3 (IV.48)

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