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Distribution of eigenvalues of 3 × 3 band matrices

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Abstract

The distribution of eigenvalues of 3×3 band matrices when two of the eigenvalues approach each other is found by directly evaluating the δ-function and making use of an expansion of hypergeometric function in terms of log.

1. Introduction

Since the introduction of random matrices [1], there has been extensive mathematical formulation of the distribution of the eigenvalues of such matrices starting from the known distribution of their matrix elements. In most of the studies so far it was assumed that each matrix element has an independent Gaussian distribution. This physical picture applies when one is dealing with the levels of compound nucleus. It so happens that in the area of condensed matter physics many of the off-diagonal elements are zero. This makes the problem of distribution of eigenvalues of such matrices called band matrices extremely complicated. Only recently level repulsion in 3×3 band matrices in which the off-diagonal element $a_{13} = 0$ has been studied [2]. In this study by using the Fourier transform of δ function and showing that a certain integral involving two Bessel functions diverges logarithmically when two eigenvalues approach each other one was able to derive the first term in the distribution of the eigenvalues of 3×3 band matrices. The purpose of the present work is to carry out a different formlation which evaluates δ function directly and makes use of an expansion of the hypergeometric function which not only gives the leading order term but also the next order terms.

2. Formulation

Let us consider a real, symmetric 3×3 matrix A with joint distribution of its diagonal and off-diagonal elements of the Gaussian form $exp(-Tr\ A^2)$. We consider the case in which off-diagonal element $\alpha_{13} = 0$. Using well known technique [3] we can write the distribution of the eigenvalues λ_i $(i = 1, 2, 3)$ as

$$P(\lambda_1, \lambda_2, \lambda_3) = K\ exp(-\sum \lambda_i^2) \prod_{i<j} |\lambda_i - \lambda_j| F, \tag{1}$$

where K is the normalization constant and F is the integral [2] given by

$$F = \int_{\gamma=0}^{2\pi} \int_{\alpha=0}^{2\pi} \int_{\beta=0}^{\pi} \delta[\sin\beta(A\ \sin\alpha\cos\beta + B\cos\alpha] \sin\beta d\beta d\alpha d\gamma. \tag{2}$$

This integral arises because the off-diagonal element a_{13} of 3×3 band matrix is taken to be zero. In expression (2)

$$A = (\lambda_3 - \lambda_1)\sin^2\gamma + (\lambda_3 - \lambda_2)\cos^2\gamma, \tag{3a}$$

$$B = (\lambda_1 - \lambda_2)\cos\gamma\sin\gamma. \tag{3b}$$

By putting $\cos\beta = t$, we can integrate over β, this gives us

$$F = \int\int d\gamma d\alpha [A^2 - (A^2 + B^2)\cos^2\alpha]^{-1/2}, \tag{4}$$

the range of α is now limited to those values of α for which the expression in [ ] bracket remains positive. It must be remarked here that one could cite many examples from theory of random matrices in which the integration over one variable in δ-function restricts the range of the remaining variables. Thus F can be written as

$$F = \frac{4}{\sqrt{A^2 + B^2}} \int d\gamma \int_{\cos^{-1}\sqrt{\frac{A^2}{A^2+B^2}}}^{\pi/2} d\alpha \left[\frac{A^2}{A^2 + B^2} - \cos^2\alpha\right]^{-\frac{1}{2}} \tag{5}$$

By a simple change of variables, expression (5) can be written as complete elliptic integral of first kind [4] and can be written as

$$F = (2\pi) \int_0^{2\pi} d\gamma \frac{1}{\sqrt{A^2 + B^2}} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{A^2}{A^2 + B^2}\right), \tag{6}$$

where ${}_2F_1$ is the hypergeometric function. From expressions (1) and (6) we see that the distribution of the three eigenvalues of the 3×3 band matrix can be expressed as an integral over the single variable γ.

We now use expression (6) to study level repulsion when two eigenvalues say λ_1, λ_2 approach each other. This can be done by using the following expansion of the hypergeometric function [4]

$$\begin{aligned} {}_2F_1(a, b; a + b; z) = & \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} \\ & [2\psi(n + 1) - \psi(a + n) - \psi(b + n) - \ell n(1 + z)](1 - z)^n, \end{aligned} \tag{7}$$

where $\psi(z)$ is the derivative of $log\ \Gamma(z)$.

When $|\lambda_1 - \lambda_2| \to 0$, it suffices to keep the first term ($n = 0$) in the above expansion. Using expressions (6), (7) and carrying out integration over γ we finally arrive at the following expression for F when $|\lambda_1 - \lambda_2| \to 0$

$$F = \frac{8\pi}{|\lambda_3 - \lambda_1|}[-\ell n|\lambda_2 - \lambda_1| + 4\ell n 2 + \ell n|\lambda_3 - \lambda_1| + \cdots]. \tag{7}$$

The first term in square bracket is precisely the logarithmic repulsion obtained earlier [2] using the Fourier transform of δ function and noting the divergence of an integral when two eigenvalues approach each other while the other two terms are the next order terms.

3. Conclusions

By directly evaluating the δ-function we have been able to derive an expression for the distribution of the three eigenvalues of the band matrix in which $a_{13} = 0$. The lowest as well as the next order terms which give the behaviour of level repulsion when two eigenvalues approach each other are obtained by making use of an expansion of hypergeometric function in terms of *log*.

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