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Higher-dimensional homogeneous cosmological models

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Abstract

We study the structure of the field equations for higher-dimensional homogeneous cosmological models and discuss the behavior of their solutions close to the initial singularity. First it is shown that the application of the ADM-formalism to the finite dimensional situation of homogeneous cosmology yields a Hamiltonian system only if the underlying Lie group is unimodular. Otherwise, to obtain the correct field equations, we have to introduce constraint forces perpendicular to the cotangent bundle with respect to the De Witt metric. Using coordinates similar to the Jacobi coordinates of classical mechanics, we generalize the (3+1)-dimensional time-dependent Hamiltonian description to an arbitrary number of spatial dimensions. Subsequently we show how to eliminate the explicit time-dependence and derive an autonomous system for the anisotropy coordinates.

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If the spatial topology is a product of isotropic subspaces, our system becomes two-dimensional and we can use arguments such as in the Poincaré-Bendixon theorem to study the qualitative behavior of the solutions. The existence of a Ljapunov function simplifies the discussion of the general case. Using a geometrical criterion concerning the structure constants we show that due to the Levi-Malcev decomposition for Lie algebras the approach to the initial singularity is regular for all homogeneous cosmological models in an arbitrary number of dimensions, except for the Bianchi type VIII and IX models. Finally, we extend the arguments to the inhomogeneous cases and obtain a chaotic behavior of the generic solutions if the number of spatial dimensions is less than 10.

Contents

Introduction

One of the most interesting features of the general theory of relativity consists in the prediction of a cosmological singularity. The observation of the expansion of the universe by Edwin Hubble in 1929, in agreement with Alexander Friedmann's non-static cosmological solutions of the field equations found in 1922, led to the supposition that the universe was in a state of infinitely high density at a certain time in the past. In fact, Roger Penrose and Steven Hawking [1] finally succeeded in showing that within the framework of (classical) general relativity, a singularity at which the physical laws can no longer hold is inevitable.

This article is concerned with the nature of the initial singularity in cosmological models with an arbitrary number of spatial dimensions. Restricting ouerselves to the framework of classical general relativity, the singularity theorems guarantee the existence of a cosmo- logical singularity if the energy-momentum tensor satisfies reasonable inequalities. These theorems contain, however, little information about the dynamical details of the solutions and the nature of the singularity (for a detailed review on singularity theorems see [2,3]). As a matter of fact, the approach to the singularity may be very complicated and irregular as was first pointed out by Belinskii, Khalatnikov and Lifshitz in 1970 [4,5]. Considering the non-isotropic generalization of the positively curved Friedmann-Robertson-Walker cosmological model (i.e.the Bianchi type IX model), they found an oscillating behavior of the scale factors with both arbitrary large frequencies and magnitudes as $t \to 0$. Introducing the concept of Kasner epochs, the field equations were subsequently replaced by a discrete dynamical system, which later was shown to be ergodic and mixing. Since modern Kaluza-Klein theories are based on higher-dimensional spacetimes, it is interesting to ask, how generic such a chaotic behavior is. In this work we shall study higher- dimensional homogeneous cosmological models, for which the field equations lead to very interesting multidimensional dynamical systens. This structure of the equations also motivates their discussion from a mathematical point of view. We are especially interested in the following questions :

- What is the structure of the field equations considered from the point of view of a Hamiltonian formulation ?
- What is the behavior of the solutions close to the cosmological singularity and to what extent does it depend on the number of spatial dimensions ?

The Hamiltonian formulation of general relativity was given by Arnowitt, Deser and Misner [6]. The application of the ADM-formalism to homogeneous cosmological models does not work, however for all Bianchi types, as was pointed out by Hawking already in 1969 [7]. In the first part of this work, we shall show that the restriction of the ADM-formalism to the finite dimensional case of homogeneous cosmology is not problematic if the underlying Lie group is unimodular. Otherwise, if the adjoint representation of the corresponding Lie algebra is not traceless, some additional forces, which are perpendicular to the co- tangent bundle with respect to the De Witt metric, must be added in order to obtain the correct field equations. We shall also clarify the role of the boundary terms, which have often been held responsible for the failure of the ADM-formalism in homogeneous cosmology.

Using the monotonic behavior of the determinant of the metric and performing a symplectic transformation to coordinates which are adapted to the symmetries of the Ricci scalar, we introduce a new Hamiltonian system in Part II. The kinetic energy term of the new Hamiltonian is positive definite but the potential becomes explicitly time-dependent. The new coordinates are related to the Jacobi coordinates of classical mechanics. The transformation generalizes the well-known explicit construction often used in the discussion of the Bianchi type models in (3+1), and of the Fee models in (4+1) dimensions, respectively [8]. Subsequently, we shall show how to eliminate the explicit time-dependence and how to obtain an autonomous system for the anisotropy coordinates. The new equations form the basis for the subsequent discussion of the scale factors near the initial singularity.

Most of the higher-dimensional space-times discussed in the literature consist in a product topology of two isotropic submanifolds. [9-14]. In order to become familiar with the autonomous system derived in Part II, we first study its application to these cases in Part III. With the help of a Ljapunov function, we are able to find a criterion which must be fulfilled for the scales to approach the singularity in a non-oscillatory manner. Since for all product topologies the logarithmic derivative of the curvature potential with respect to the anisotropy coordinate satisfies this criterion, the solutions have to approach one of the generalized Kasner solutions as $t \to 0$.

In Part IV, we extend the discussion to the full autonomous system derived in Part II. In the literature there exist essentially two different approaches to the general problem. The first method assumes that the generalized Kasner solutions are admissible approximations during successive time intervals and then discusses the discrete dynamical system describing the change of Kasner exponents [15-17]. The second approach is based on the explicitly time-dependent Hamiltonian formulation [18,19]. In order to treat the system within this framework, the equipotential walls are usually assumed to be infinitely steep, and the motion of the "universe point" is approximated by a sequence of free propagations and bounces against the equipotentials. Within the first approach, the criterion for the scales to behave regularly close to the initial singularity, consists in the existence of a set of Kasner indices for which no further transitions can take place. In the Hamiltonian formulation one has to demand the existence of a time t^* such that the equipotential walls move faster than the "universe point" for all $t < t^*$.

The advantage of our method is that it works without the un- controlled approximations just mentioned and that it also takes account of the cases where the original system is not Hamiltonian. Splitting the problem into the discussion of the modulus and the direction of the "velocity" vector corresponding to the autonomous system introduced in Part II, we are able to give a simple geometrical criterion concerning the structure terms for the solutions to behave regularly close to the initial singularity. We shall show that this criterion is satisfied if the number of spatial dimensions increases to $n \geq 10$. However, in homogeneous cosmology, where, due to the Levi-Malcev decomposition for Lie groups [20,21], some structure constants vanish, the condition is already fulfilled for $n > 3$.

We conclude that chaos is a generic feature of cosmological solutions of the field equations in inhomogeneous models with nine or less spatial dimensions, whereas chaotic behavior does not occur in homogeneous cosmological models unless the underlying Lie group is SO(3) or SO(2,1), i.e. for the Bianchi type VIII and IX cosmological models.

I. The Hamiltonian formalism for homogeneous cosmological models

The Bianchi types of the Ellis-MacCallum class B do not form a Hamiltonian system. We show that the restriction of the ADM-formalism to the finite dimensional situation of homogeneous cosmology only produces a Hamiltonian system if the Lie group acting on the spacelike hypersurfaces of spacetime is unimodular. In this case, the structure constants of the corresponding Lie algebra have vanishing traces, and the algebra belongs to the class-A models. We also give a derivation and an interpretation of the modified Hamiltonian equations which are valid for all homogeneous cosmological models.

I.1. Introduction

Arnowitt, Deser and Misner developed the Hamiltonian formalism of general relativity (ADM-formalism) [6]. Fischer and Marsden [22,23] showed that the cotangent bundle of the gravitational manifold carries a symplectic structure in which the evolution equations arise in Hamiltonian form. Homogeneous cosmology provides a pleasant laboratory for applications of this formalism, since in this case the field equations reduce to ordinary differential equations. As was first noted by Hawking [7] in 1969, the ADM-Hamiltonian fails to reproduce all field equations correctly in some cases. Taub and MacCallum showed that Einstein`s equations form a Hamiltonian system only for the Bianchi types [24] belonging to the Ellis-MacCallum class A [25,26]. They pointed out that it may be inconsistent to neglect boundary terms if spatial homogeneity is assumed. In order to obtain the correct constraint equations, Sneddon [27] removed the spatial divergence term from the ADM-action and, introducing a coordinate constraint, he derived the remaining correct field equations.

We shall explain why it is possible to give a correct Hamiltonian description for the class-A models, although the argument of Taub and MacCallum mentioned above holds for all homogeneous models. We shall also show how to modify this description for the class-B types. This modification is due to the fact that the Lie groups of this class do not act as symplectic transformations of the finite dimensional system obtained from reducing the Hamiltonian to homogeneous fields.

In the second section we shall briefly repeat some general features of the homogeneous models and then discuss the structure of the field equations. Using the De Witt metric, we consider the split of the Gauss equation into a kinetic and a potential part (external and internal geometry). We shall see from Bianchi's identity that the variation of the potential part causes additional terms that spoil the Hamiltonian structure for the class-B models.

In the third section we shall explain the different behaviour of the two Ellis-MacCallum classes. In the present context, the question is not only whether the variation of the action and the imposition of spatial homogeneity commute with each other [45]. In order to decide whether the restricted dynamical equations still form a Hamiltonian system, we must examine if a given symmetry is a canonical symmetry of the restricted ADM-system. The answer to this question splits the Bianchi types into the two classes, whereas the problem of boundary terms is a general obstacle arising for the variational principle in homogeneous cosmology.

The correct modified Hamiltonian equations are derived in the fourth section comparing the variation of the Hilbert action in its usual form and in the "3+1 split". We do not consider variations with respect to the lapse and shift functions, but set them equal to one and zero from the beginning, according to the assumption of spatial homogeneity. The advantage of our formulation, which holds for all Bianchi types, is the following : The constraint equations, which must be taken into account as secondary factors, and the extra terms that spoil the Hamiltonian structure, can be treated simultaneously by investigating constraint forces standing perpendicular to the cotangent bundle with respect to the De Witt metric. The system may be described by the flow of a Hamiltonian and an additional non-Hamiltonian vector field.

For the sake of completeness, we extend our formulation in the last section to models with perfect fluids and a cosmological constant. In either case a description in terms of potentials is still possible.

I.2. Homogeneous cosmological models
I.2.1. General features

Before we discuss the field equations of homogeneous cosmology, let us give some definitions, the connection forms and the Einstein tensor.

Like the Friedmann models, the homogeneous models still possess a spatial stratification of equal time, but the subspaces belonging to a fixed time are only assumed to be homogeneous [28]. A space-time (M,g) is a homogeneous cosmological model if a three-dimensional Lie group G acts isometrically and freely on (M,g), such that the orbits Ω are spacelike surfaces. For given points p and q of M, the corresponding orbits $\Omega(p)$ and $\Omega(q)$ are geodetically parallel. Let t be the distance between them. The mapping

$$gp \rightarrow (t,g) \in \mathrm{IR} \times G$$

is a diffeomorphism which is compatible with the group operation and, relative to the corresponding metric, IR is perpendicular to G ;

$$g = dt^2 - \pi^*(h(t)) \quad .$$

For a fixed time t, h(t) is a left-invariant metric of G and has the form

$$h(t) = - g_{ij}(t) \;\; \theta^i \otimes \theta^j \quad , \tag{1}$$

where the θ^i are the basis of left-invariant one-forms on G. The local structure of (M,g) is the following :

$$M = \mathrm{IR} \times G \;\; , \;\; \text{as } G\text{–manifold} \;\; ,$$
$$g = dt \otimes dt \; + \; g_{ij}(t) \;\; \pi^*(\theta^i) \otimes \pi^*(\theta^j) \quad . \tag{2}$$

Solving the structure equations and the Maurer-Cartan equation for the metric (2),

$$d\theta^i + \frac{1}{2} C^i_{jk} \, \theta^j \wedge \theta^k = 0 \;\; , \tag{3}$$

we obtain the connection forms :

$$
\begin{aligned}
\omega^o_o &= 0\\
\omega^o_i &= K_{ij}\,\theta^j \quad ; \quad \omega^i{}_o = -K^i{}_j\,\theta^j\\
\omega_{ij} &= -K_{ij}\,\theta^o + \lambda_{ijk}\,\theta^k\\
\lambda_{ijk} &= -\frac{1}{2}\,(\,C_{ijk} + C_{jki} - C_{kij}\,)\ .
\end{aligned}
\tag{4}
$$

K(X,Y) is the second fundamental form on Ω. In the homogeneous case the components are

$$
K_{ij} = -\frac{1}{2}\,\dot{g}_{ij}\ . \tag{5}
$$

In the chosen basis of one-forms, the $C^i{}_{jk}$ are the structure constants of the Lie group G. Let (e_i) be the dual basis to (θ^i) . Equation (3) then reads

$$
[\,e_i\,,e_j\,] = C^k_{ij}\,e_k\ . \tag{6}
$$

In terms of K_{ij} the components of the Ricci and the Einstein tensor are the following :

$$
R^o_o = \dot{K} - K^2 \qquad (7a) \qquad\qquad G^o_o = \frac{1}{2}\,\bar{R} - \frac{1}{2}\,(\,K^2 - K{\cdot}K\,) \qquad (8a)
$$

$$
R^o_k = G^o_k \qquad (7b) \qquad\qquad G^o_k = -2\,K_{ij}\,(F_k)^{ij} \qquad (8b)
$$

$$
R^i_j = \bar{R}^i{}_j + \frac{1}{\sqrt{g}}(\sqrt{g}\,K^i_j)^{\cdot} \quad (7c) \qquad G^i_j = \bar{G}^i{}_j + (K^i_j - \delta^i_j K)^{\cdot} - K{\cdot}K^i_j + \frac{1}{2}\delta^i_j\,(K^2 + K{\cdot}K) \quad (8c)
$$

where $K := K^i{}_i$, $K^2 := K^i{}_j K^j{}_i$. R_{ij} denotes the Ricci tensor of the three-dimensional Riemannian space (G,h) with negative definite metric h :

$$
\bar{R}_{ij} = -\lambda^k_{il}\,\lambda^l{}_{jk} - C^k_{kl}\,\lambda^l{}_{ij}\ ;\ \bar{R} = -\bar{R}^i{}_i\ . \tag{9}
$$

The F_k are defined as the parts of the G_{ok} which are independent on the derivatives of g_{ij} :

$$
\sqrt{g}\,G_{ok} = \dot{g}_{ij}\,(F_k)^{ij}\ . \tag{10}
$$

The matrix densities $(F_k)^i{}_j$ are traceless expressions in terms of the structure constants of G :

$$(F_k)^i_j := -\frac{\sqrt{g}}{2}\{ C^i_{jk} - \delta^i_k C^l_{lj} \} . \tag{11a}$$

Later on we shall also use the quantity F , which is defined as the linear combination of the F^k with coefficients $C^l{}_{lk}$:

$$(F)^i_j := C^l_{lk}(F^k)^i_j . \tag{11b}$$

I.2.2. Lagrange formulation

In this section we shall discuss the dynamics of the field equations and give a derivation of the modified Euler-Lagrange equations.

The Gauss equation (8a) separates into two terms, of which the first describes the internal geometry of Ω and the second is a function of external properties only. The latter term is a quadratic form in the first derivatives of the metric, while the first part contains no derivatives at all. The split into an internal and an external part thus corresponds to the split into a potential and a kinetic energy term. Together with the fact that the constraint equation for G_{oo} is conserved under the time evolution, this circumstance suggests the interpretation of $\sqrt{g}G_{oo}$ as the Hamiltonian of a "particle" moving in a six-dimensional configuration space Σ with natural coordinates (g_A) (we shall use capital letters for pairs of symmetric indices ($g_I := g_{ij}$, $W^{AC} := W^{abcd}$, etc.).

As is well known [23,29], the variation of the kinetic part generates the equations of the geodesics with respect to the De Witt metric W [30] :

$$W_{ijkl} = \frac{1}{2\sqrt{g}}(g_{i(k}g_{l)j} - g_{ij}g_{kl}) \tag{12a}$$

$$W^{ijkl} = \frac{\sqrt{g}}{2}(g^{i(k}g^{l)j} - 2g^{ij}g^{kl}) \tag{12b}$$

$$W^{ijkl}W_{klab} = \frac{1}{2}\delta^i_{(a}\delta^j_{b)} .$$

These geodesic equations are equivalent to the space-space components of the Einstein equations for the free ($R_{ij}=0$) dynamics. Writing (8a) in the form

$$\sqrt{g}\,G_{oo} = V(\underline{g}) + T(\dot{\underline{g}},\dot{\underline{g}}) = \frac{\sqrt{g}}{2}\bar{R} - \frac{1}{8}<\dot{\underline{g}},\dot{\underline{g}}>_W \tag{13}$$

and applying the Euler-Lagrange derivative

$$D^A := \frac{\partial}{\partial g_A} - \frac{d}{dt}\frac{\partial}{\partial \dot{g}_A} \tag{14}$$

on the kinetic term of (13), the resulting geodesic equation reads

$$0 = D^A\left\{ -\frac{1}{8} W^{IK}\dot{g}_I\,\dot{g}_K \right\} = \frac{1}{4}\{ W^{AC}\ddot{g}_C + \Gamma^{IK,A}\dot{g}_I\,\dot{g}_K \}\ , \tag{15}$$

where the "Christoffel symbols" are defined as usual:

$$\Gamma^{IK,A} := \frac{1}{2}\{ W^{IA,K} + W^{KA,I} - W^{IK,A} \}\ . \tag{16}$$

After a short calculation, (15) is easily seen to be, up to a factor of (-1/2), identical with the "free" part of $\sqrt{g}G^{ab}$ (8c) . We thus can write :

$$D^{ab}\{ \sqrt{g}\,G_{oo} \}_{kin} = -\frac{1}{2}\{ \sqrt{g}\,G^{ab} \}_{free}\ . \tag{17}$$

Now we consider the parts of the equations (8c) and (13) which do not depend on derivatives of the metric. We need an expression for the differentiated internal curvature $\sqrt{g}R$ with respect to g_{ab} . Since R_{ij} consists of products of structure terms, the direct calculation is not very convenient, although it is easy. A more elegant way is to compare the Bianchi identity with the time derivative of the Gauss equation (13) and to eliminate all external quantities. The time derivative of (13) can be written as

$$(\sqrt{g}\,G_{oo})^{\cdot} = \frac{1}{2}(\sqrt{g}\,\bar{R})^{\cdot} - K^j_i\{\sqrt{g}\,K^i_j - \sqrt{g}\,\delta^i_j\,K\}^{\cdot}$$
$$- \frac{1}{2}\sqrt{g}\,K\{K^2 - K{\cdot}K\}\ .$$

Now we use the Gauss equation again in the last term and equation (7c) to replace the derivatives of $\sqrt{g}K^i{}_j$. We obtain :

$$(\sqrt{g}\,G_{oo})^{\cdot} = \frac{1}{2}(\sqrt{g}\,\bar{R})^{\cdot} - \sqrt{g}\,K_{ij}\ \{R^{ij} - g^{ij}R^k_k\}$$
$$+ \sqrt{g}\,K\,G_{oo} + \sqrt{g}\,K_{ij}\,\bar{G}^{ij}\quad . \tag{18}$$

On the other hand we consider the Bianchi identity for $G^{o\mu}$, which reads

$$(-G^{oo})^{\cdot} = G^{oi}{}_{,i} + \omega^o_\mu(e_\nu)\,G^{\mu\nu} + \omega^\nu{}_\mu(e_\nu)\,G^{o\mu}\ .$$

Together with the expressions for the connection forms (4), the identity $G^{ij} + g^{ij}G^o{}_o = R^{ij} - g^{ij}R^k{}_k$ and ($K=-1/2(\ln g)^{\cdot}$ we obtain :

$$(\sqrt{g}\,G_{oo})^{\cdot} = \sqrt{g}K\,G_{oo} - \sqrt{g}K_{ij}\{R^{ij} - g^{ij}R^k_k\} - \lambda^i_{ij}\,\sqrt{g}\,G^{oi}\ . \tag{19}$$

Comparing (18) and (19) we have the following equation

$$\frac{1}{2}\,(\sqrt{g}\bar{R})^{\cdot} = -\sqrt{g}\,\{K_{ij}\,\bar{G}^{ij} + \lambda^l{}_{kl}\,G^{ok}\}\ . \tag{20}$$

Using (10), (11b) and $\lambda^j{}_{ij} = C^j{}_{ji}$, the last term in (20) becomes

$$\sqrt{g}\,\lambda^l{}_{kl}\,G^{ok} = \dot{g}_{ij}\,C^l_{lk}\,(F^k)^{ij} = \dot{g}_{ij}\,(F)^{ij}\ .$$

Since $\sqrt{g}R$ is independent of derivatives of g_{ij} , we obtain the Euler-Lagrange derivative of the potential part of (13) :

$$D^{ab}\left(\frac{1}{2}\sqrt{g}\,\bar{R}\right) = \left(\frac{\sqrt{g}}{2}\bar{G}^{ab} - F^{ab}\right)\ . \tag{21}$$

Let us define the Lagrangian as

$$L := \sqrt{g}\, G^{-}_{oo} := -\frac{1}{2}(\sqrt{g}\,\bar{R}) - \frac{1}{8} <\dot{g}\,,\dot{g}>_W \quad , \tag{22}$$

where the minus sign superscript refers to the changed sign of the potential term compared with G_{oo} (13) . We can now express the Euler-Lagrange derivative in terms of the space-space components of the Einstein tensor and the quantities F_k, which are related to the time-space components by (10) and (11b) . The equations (17) and (21) , which hold for the external and internal geometry respectively, together imply :

$$D^{ab}(\sqrt{g}\, G^{-}_{oo}) = -\frac{1}{2}\{ (\sqrt{g}\, G^{ab})_{kin} + \sqrt{g}\, \bar{G}^{ab} \} + F^{ab}$$

$$= \left\{ F^{ab} - \frac{\sqrt{g}}{2} G^{ab} \right\} \quad . \tag{23}$$

From (23) we conclude that the Euler-Lagrange equations for $\sqrt{g}G^{-}_{oo}$ are only equivalent to the space-space components of the field equations if the force F^{ab} vanishes. This is the case for the class-A models where $C^{l}_{lk}= 0$ holds (11b). If the C^{l}_{lk} do not vanish, the Euler-Lagrange equations must be modified : The G_{ab}=0 equations still hold if

$$D^{ab}L - F^{ab} = 0 \tag{24}$$

is fulfilled. We shall discuss the interpretation of the additional term in (24) in the fourth section. For the moment, we shall only mention that the constraint force F^{ab} is not holonomic. This corresponds to the fact that the Hamiltonian structure can not be saved by additional potentials, as was pointed out by Taub and MacCallum [25]. Introducing the one-form Z, corresponding to F^{ab} , $Z := F^{ab}\, dg_{ab}$, it is easy to verify that the form $Z \wedge dZ$ does in general not vanish and thus F^{ab} is not holonomic.

I.3. Application of the ADM-formalism to homogeneous cosmology

As was shown in the last section and is well known, the application of the ADM-formalism to homogeneous models fails for the Bianchi types of the Ellis MacCallum class B [7,27,29,31-34]. As Taub and MacCallum [25] pointed out, it may not be admissible to impose certain conditions of symmetry to the action before performing its variation. Especially for homogeneous spaces, vanishing variations at the boundary would presuppose vanishing variations in the interior region as well. This argument is independent of the Bianchi type and makes a derivation of the correct field equations from a variational principle for a restricted action impossible (without imposing additional special boundary conditions by hand). In order to understand where the problems arising in the class-B models originate from, we have to examine how the imposition of symmetry conditions on the Hamiltonian affects the canonical equations. Let us therefore briefly review the ADM-formalism :

In the "3+1-split" of general relativity a curve i_λ of space-like imbeddings Ω_λ of a three-dimensional manifold M is considered, i.e.: $i_\lambda(M) = \Omega_\lambda$. The one-parameter family of lapse functions $\alpha_\lambda : M \to \mathbb{R}$ and shift vectors $\beta_\lambda : M \to TM$ are defined as the normal and horizontal projections of the vector field X_λ, $X_\lambda \circ i_\lambda = di_\lambda / d\lambda$. The pulled back metric takes the form

$$g = \alpha_\lambda{}^2 \, d\lambda^2 - (g_\lambda)_{ij} \, (dx^i + \beta_\lambda{}^i \, d\lambda) \, (dx^j + \beta_\lambda{}^j \, d\lambda) \quad . \tag{25}$$

The gravitational configuration manifold Σ is the space of all Riemannian metrics on M . Its tangent and cotangent bundle consist in the two-covariant tensor fields $t_{ij} \in S_2$ and the two-contravariant tensor densities $t^{ij} \in S_d{}^2$, respectively : $T\Sigma = \Sigma \times S_2$, $T^*\Sigma = \Sigma \times S_d{}^2$. Fischer and Marsden [22,23] have shown that $T^*\Sigma$ carries a symplectic structure. The field equations can be written in the following form :

$$\frac{\partial g}{\partial \lambda} = \frac{\delta}{\delta \pi_d} \left(\alpha H_d + \beta_i H_d^i \right) \quad , \tag{26a}$$

$$-\frac{\partial \pi_d}{\partial \lambda} = \frac{\delta}{\delta g} \left(\alpha H_d + \beta_i H_d^i \right) \quad . \tag{26b}$$

$\pi_d \in S_d^2$ denotes the canonical momentum density conjugated to $g \in S_2$ and we have suppressed the indices of $\pi_d{}^{ij}$ and g_{ij} as well as the slice-index λ. Using the tensor part π, the vector part h^i and the scalar part h of the corresponding densities, H_d and $H_d{}^i$ are given as :

$$H_d = G_{\perp\perp}\,\mu(g) = h\,\mu(g) = \frac{1}{2}\left[\bar{R} - 2(2\pi^2 - \pi\pi)\right]\mu(g) \ , \tag{27a}$$

$$H_d^i = G^i_{\perp\|}\,\mu(g) = h^i\mu(g) = \nabla_j\left[\pi^{ij}\,\mu(g)\right] \ . \tag{27b}$$

The ADM-quantities are denoted by capital letters :

$$H := \sqrt{g}\,h \ , \quad H^i := \sqrt{g}\,h^i \ , \quad \Pi := \sqrt{g}\,\pi \ . \tag{28}$$

Let us now consider the case of homogeneous cosmology: Since these models have zero vorticity, the shift vector vanishes and since the slices of equal time are homogeneous, the lapse function may be set equal to one. The space-space components of the field equations read:

$$\left[\frac{dg}{dt} = \frac{\delta H}{\delta \Pi}\right]_r \quad ; \quad \left[-\frac{d\Pi}{dt} = \frac{\delta H}{\delta g}\right]_r \ . \tag{29}$$

The index "r" denotes the restriction of the equations to homogeneous metric fields. On the other hand we can consider the restricted Hamiltonian H_r, which is a function of g and Π only, since the $g_{,i}$ vanish: $H_r = H_r(g,\Pi)$. This is the Hamiltonian of a finite dimensional system with phase-space coordinates g and Π. The corresponding canonical equations are thus :

$$\frac{dg}{dt} = \frac{\partial H_r}{\partial \Pi} \quad ; \quad -\frac{d\Pi}{dt} = \frac{\partial H_r}{\partial g} \ . \tag{30}$$

These equations need not be identical to the field equations (29) above for the following reason: The ADM-Hamiltonian H is obtained from the corresponding Lagrangian $L(g\ ,\ g_{,o}\ ,\ g_{,i}\ )$ by a Legendre transformation in $g_{,o}$ and is thus dependent on g, Π and $g_{,i}$. The restricted functional derivatives on the right hand sides of the equations (29) give the following equations:

$$\frac{dg}{dt} = \frac{\partial H_r}{\partial \Pi} \qquad ; \qquad -\frac{d\Pi}{dt} = \frac{\partial H_r}{\partial g} - \left[\nabla_i \frac{\partial H}{\partial g_{,i}} \right]_r \quad . \tag{31}$$

Replacing the restricted functional derivatives for H by the ordinary partial derivatives for the restricted Hamiltonian H_r is only admissible if the last term in (31) vanishes;

$$\left[\nabla_i \frac{\partial H}{\partial g_{,i}} \right]_r = 0 \quad . \tag{32}$$

Using the connection forms (4), we obtain for the spatial divergence of a vector V^i :

$$[\nabla_i V^i]_r = [(\partial_i + \omega^i_j(e_j)) V^i]_r = C^j_{ji} V^i_r \quad , \tag{33}$$

since ordinary spatial derivatives vanish. As we shall see, the vector $V_r{}^i$ does not vanish and the equations (30), obtained from the restricted Hamiltonian, are not the correct field equations (29) except if the traces of the structure constants vanish. This is the case for the Bianchi types belonging to the Ellis MacCallum class A.

The same conclusion can also be drawn from the following argument: Let us consider the action of the Lie group G on the configuration manifold Σ. The lifted action on the cotangent bundle is a symplectic symmetry if the Hamiltonian transforms like a scalar function under this action. Although the restricted Hamiltonian $H_r(g,\Pi)$ looks like the Hamiltonian function of a finite dimensional system, it is still a density by its definition (28). In order to respect the structure of the equations (30), the action of G must leave H_r invariant, i.e. it must not only conserve the volume on the phase space but also on the configuration manifold. Since the one-forms θ^i transform with the adjoint representation of G, the volume form $\theta^1 \wedge \theta^2 \wedge \theta^3$ is conserved if $\det(AdG) = 1$. Using

$$\det(AdG) = \exp(\operatorname{tr}(ad\mathfrak{g}) \quad , \tag{34}$$

we see that the adjoint representation of the Lie algebra $\mathfrak{g}$ must be trace-less. The components of the adjoint representation are

$$(\mathrm{ad}_X\ )^i{}_j = [\,X,\ \ ]^i{}_j = X^k C^i{}_{kj} \quad ; \tag{35}$$

and the vanishing of its trace is equivalent to the vanishing of the traces of the structure constants. The Lie algebras of the Ellis MacCallum class A correspond to the unimodular Lie groups which leave the Hamiltonian density invariant and thus admit the interpretation of H_r as the scalar Hamiltonian of a finite dimensional system.

I.4. The modified Hamiltonian equations

I.4.1. Derivation of the equations

In order to extend the interpretation of H_r as the Hamiltonian of a finite dimensional system to the class-B Bianchi types, we have to modify the canonical equations (30). Considering a restricted action, the additional boundary terms prevent a derivation of the filed equations from an ordinary variational principle. In this section we shall derive a system of differential identities for the Hilbert Lagrangian in the "3+1-split", which will turn out to be the system of the modified Hamiltonian equations. In order to be consistent, we are not allowed to neglect any boundary terms since two actions differing by such terms are no longer equivalent.
As in the last section, let Π be the ADM-momentum,

$$\Pi^{ij} = -\frac{1}{4} W^{ijkl}\dot{g}_{kl} = \frac{\sqrt{g}}{2}(K^{ij} - g^{ij}K)\ . \tag{36}$$

From now on, the tensor indices of g and Π are suppressed, $\Pi\, g := \Pi^{ij}g_{jk}$. The Hilbert Lagrangian in the 3+1-split reads (ADM-Lagrangian plus divergence terms) :

$$\frac{\sqrt{g}\,R}{2} = \mathrm{tr}(\,\Pi\dot{g}\,) - \alpha H - \beta_i H^i - \mathrm{tr}(\,g\Pi\,)^{\cdot} + \sqrt{g}\left\{\frac{1}{\sqrt{g}}\mathrm{tr}(\,g\Pi\,)\,\beta^j - 2\Pi^{ij}\beta_i + \alpha^{|j}\right\}_{|j} \ . \tag{37}$$

Setting again α=1 and β=0, we obtain for the variation of (37) :

$$\delta\left(\frac{\sqrt{g}\,R}{2}\right) = -\mathrm{tr}\{\,(\dot{\Pi}+H_g)\delta g\,\} \;+\; \mathrm{tr}\{\,(\dot{g}-H_\Pi)\delta\Pi\,\} \;-\; \mathrm{tr}\{\,g\delta\Pi\,\}^{\cdot} \;. \quad (38)$$

On the other hand, we consider the restricted variation of the Hilbert Lagrangian in its usual form :

$$\delta\left(\frac{\sqrt{g}\,R}{2}\right) = \frac{1}{2}\,\delta(\sqrt{g}g_{ij})\,R^{ij} + \sqrt{g}\,\Phi^{\mu}{}_{;\,\mu} \quad , \quad (39)$$

$$\Phi^{\alpha} := g^{\mu[\nu}\,\delta\omega^{\alpha]}{}_{\mu}(e_\nu) \quad .$$

Using the connection forms (4), the components of Φ are (in the homogeneous case) easily found to be :

$$\sqrt{g}\,\Phi^{o} = \frac{\sqrt{g}}{2}(K^{ij}\,\delta g_{ij} + 2\,\delta K^{i}_{i}) = -\mathrm{tr}(\,g\,\delta\Pi\,)\;, \quad (40a)$$

$$\sqrt{g}\Phi^{k} = \frac{\sqrt{g}}{2}(g^{ij}\,\delta\lambda^{k}_{ij} - g^{kj}\,\delta\lambda^{i}_{ji}) = \mathrm{tr}(\,F^{k}\,\delta g\,)\;. \quad (40b)$$

In order to calculate the covariant divergence term in (39), we use

$$\begin{aligned}\sqrt{g}\,\Phi^{\mu}{}_{;\,\mu} &= \sqrt{g}\,\{\,\Phi^{\mu}{}_{,\,\mu} + \omega^{\mu}{}_{\nu}(e_\mu)\,\Phi^{\nu}\,\} = \sqrt{g}\,\{\,\Phi^{\mu}{}_{,\,\mu} + \omega^{\mu}{}_{\mu}(e_\nu)\,\Phi^{\nu} + C^{i}{}_{ij}\,\Phi^{j}\,\} \\ &= (\sqrt{g}\Phi^{\mu})_{,\mu} + \sqrt{g}\,C^{i}{}_{ij}\,\Phi^{j} \quad . \end{aligned} \quad (41)$$

Since ordinary spatial derivatives vanish, only the μ=o term contributes. Together with (40), (41) and $\delta(\sqrt{g}\;g_{ij})R^{ij} = \sqrt{g}\;G^{ij}\,\delta g_{ij}$, equation (39) now reads :

$$\delta\left(\frac{\sqrt{g}\,R}{2}\right) = -\frac{\sqrt{g}}{2}\mathrm{tr}\{\,G\delta g\,\} - \mathrm{tr}\{\,g\,\delta\Pi\,\}^{\cdot} + \mathrm{tr}\{\,F\delta g\,\}\,. \quad (42)$$

Now we can compare the two expressions (38) and (42) for the variation of $\sqrt{g}$ R. The problematic terms containing the time-derivative of $\delta\,\Pi$ cancel each other out, and we have the identity

$$\mathrm{tr}\left\{\left(\dot{\Pi}+H_g+F-\frac{\sqrt{g}}{2}G\right)\delta g\right\} \quad - \quad \mathrm{tr}\left\{\left(\dot{g}-H_\Pi\right)\delta\Pi\right\} = 0 \ , \qquad (43)$$

which immediately shows, that the space-space components of the field equations (G=0) are equivalent to the following modified Hamiltonian equations :

$$\dot{g} \ = \ H_\Pi \quad , \quad -\dot{\Pi} \ = \ H_g \ + \ F \quad , \qquad (44)$$

$$H = \frac{\sqrt{g}}{2}\bar{R} \ - \ 2<\Pi,\Pi>_W \ , \ F \ = \ C^l_{lk}F^k \ . \qquad (45)$$

After a Legendre transformation we obtain the corresponding modified Lagrange equations (24) derived in the second section by explicit calcula- tion:

$$L = \mathrm{tr}\{\ \Pi\dot{g}\ \} \ - \ H \ = \ -\frac{\sqrt{g}}{2} - \frac{1}{8}<\dot{g},\dot{g}>_W \quad , \quad D^{ab}L \ = \ F^{ab} .$$

I.4.2. Interpretation of the modified equations

Summarizing, we can consider the following finite dimensional modified Hamiltonian system:

<u>Proposition 4.1.</u>

Let $x=(g_{uv},\Pi^{uv})$ be the phase space coordinates, X_H the Hamiltonian vector field belonging to H and X_C an additional (constraint) vector field :

$$X_H \ = \ J\,\mathrm{grad}_x H \ ^{*)} \quad , \quad X_C \ = \ (0,-F) \quad , \qquad (46)$$

$$H \ = \ \sqrt{g}\,G_{oo} \ = \ \sqrt{g}\left(\frac{1}{2}\,\bar{R} \ - \ 2<\Pi,\Pi>_W\right) . \qquad (47)$$

Then the field equations are equivalent to the equations for the finite-dimensional dynamical system evolving with zero energy and with

---

*) J is the symplectic standard form :

$$J\left(\frac{\partial H}{\partial g},\frac{\partial H}{\partial \Pi}\right) = \left(-\frac{\partial H}{\partial \Pi},\frac{\partial H}{\partial g}\right)$$

respect to the flow Ψ of the vector field X :

$$X := X_H + X_C \quad . \tag{48}$$

Proof: Let D_X denote the derivative in the direction of X. The total time derivative of a function f(x,t) on the integral trajectories of Ψ is :

$$\dot{f} = \frac{\partial f}{\partial t} + D_{X_H} f + D_{X_C} f \quad .$$

If f does not explicitly depend on t and since X_H is a Hamiltonian field, we can also write, using Poisson bracket notation:

$$\{ f , H \} = \dot{f} - D_{X_C} f \quad . \tag{49}$$

Setting f equal to the component functions of g and Π, we obtain the equations of motion (44):

$$\begin{gathered} H_\Pi = \{ g , H \} = \dot{g} \quad , \quad - H_g = \{ \Pi , H \} = \dot{\Pi} + F \quad ; \\ \Leftrightarrow \\ G_{ij} = 0 \quad . \end{gathered} \tag{50}$$

The zero-energy condition is equivalent to the time-time component of the field equations:

$$H = 0 \qquad \Leftrightarrow \qquad G_{oo} = 0 \quad . \tag{51}$$

Since H must be conserved due to the Bianchi identity, we obtain the following condition, setting f equal to H:

$$O = \{ H , H \} = \dot{H} + tr(F H_\Pi) = tr(F H_\Pi) \quad . \tag{52}$$

This equation is only satisfied if the remaining time-space components of the field equations hold,

$$tr\, (F H_\Pi) = C^l_{lk}\, tr\, (F^k \dot{g}) = C^l_{lk}\, G^{ok} / \sqrt{g} = -4 < \Pi , F >_W \quad . \tag{53}$$

The equations $G_{ok} = 0$ are equivalent to the requirement for the force densities F^k to be perpendicular to the cotangent bundle with respect to the De Witt metric. Δ

In the language of classical mechanics the modified Hamiltonian system describes a particle ("the universe point") evolving under the influence of a potential $\sqrt{g}R$ and three additional nonholonomic d`Alembertian forces F^k (forces that perform no work), which is in some sense the simplest non-Hamiltonian system with energy conservation.

I.5. Perfect fluids and cosmological constant

Let us finally extend our formulation to models with a non-vanishing cosmological constant and matter. For definiteness the matter is assumed to be a perfect fluid with energy-momentum tensor $T^{\mu\nu}$,

$$T^{\mu\nu} = (p+\rho)\, u^{\mu} u^{\nu} - p\, g^{\mu\nu} \quad , \tag{54}$$

pressure p and energy density ρ. u^{μ} is the cosmic 4-velocity, satisfying

$$g_{\mu\nu} u^{\mu} u^{\nu} = 1 \ . \tag{55}$$

Contracting the Bianchi identity for $T^{\mu\nu}$ with u_{μ} and using $u = (1/2\ln g)'$, we obtain

$$p\, \partial_t \sqrt{g} + \partial_t (\rho \sqrt{g}) = 0 \ .$$

Assuming an equation of state of the form

$$\rho = \rho(\sqrt{g}) = c\, \sqrt{g}^{\,-\lambda} , \tag{56}$$

we have the well known relation

$$p = (\lambda - 1)\, \rho \quad , \tag{57}$$

where the cases λ=1 and λ=4/3 correspond to incoherent dust and to threedimensional radiation, respectively. $\lambda=2$ describes "stiff" matter or equivalently homogeneous scalar fields. The components of the energy-momentum tensor can now be written as

$$T^{oo} = c\sqrt{g}^{-\lambda} \quad , \quad T^{ok} = 0 \ ,$$
$$T^{ij} = c\, g^{ij}(1-\lambda)\sqrt{g}^{-\lambda} \ . \tag{58}$$

Defining the matter Hamiltonian as $H^M = \sqrt{g}T_{oo}$, we have the equations

$$H^M_{\Pi^{ij}} = 0 \quad , \quad H^M_{g_{ij}} = \frac{\sqrt{g}}{2} T^{ij} \ , \tag{59}$$

which mean that $\sqrt{g}T_{oo}$ may be considered as a potential and $\sqrt{g}/_2\, T^{ij}$ as the corresponding force. The same statement holds for the cosmological term since

$$(\sqrt{g}\,\Lambda)_{g_{ij}} = \frac{\sqrt{g}}{2} g^{ij}\Lambda \quad , \quad (\sqrt{g}\,\Lambda)_{\Pi^{ij}} = 0 \ . \tag{60}$$

The complete potential now reads (where κ is the coupling constant)

$$V(g) = \frac{\sqrt{g}}{2}\bar{R} - \kappa c\sqrt{g}^{(1-\lambda)} - \sqrt{g}\,\Lambda \ , \tag{61}$$

and the Hamiltonian becomes

$$H = \sqrt{g}\left\{ G_{oo}(g,\Pi) - \kappa T_{oo}(g) - \Lambda \right\} \ . \tag{62}$$

The extension of the Hamiltonian formulation to models with perfect fluids or a cosmological constant does not cause new difficulties since in both cases the additional terms may be described by introducing new potentials.

If we restrict ourselves to the class-A models, which are defined by the condition

$$a_i = 0 \quad \Leftrightarrow \quad C^j_{ji} = 0 \ , \tag{63}$$
where $C^k_{ij} = \varepsilon_{ijl}\, n^{lk} + \delta^k_{[j}\, a_{i]}$, $n^{[l,k]} = 0$,

the additional vector field X_C vanishes and the system becomes Hamiltonian; $X = X_H$. Since in this case the metric can be diagonalized time-independently [35], the dimension of the phase space reduces to six. Moreover, the constraints $0 = \langle \Pi, F^k \rangle_W$ are identically satisfied, since the F^k are traceless. Writing the metric in the coordinates x_i and τ ($x_i := \ln g_{ii}$, $d\tau := g^{-1/2} dt$),

$$g = e^{\Sigma x_i}\, d\tau^2 \; + \; \Sigma\, e^{x_i} (\theta^j)^2 \quad , \tag{64}$$

the Hamiltonian becomes $H(x,p) = g\, H(g,\Pi)$:

$$H(x,p) = (\Sigma p_i)^2 - 2(\Sigma p_i^2) + V(x) \ , \tag{65}$$

$$V(x) = \frac{1}{2} e^{\Sigma x_i} \bar{R}(x) - \kappa c\, e^{(1-\lambda/2)\Sigma x_i} - \kappa \Lambda\, e^{\Sigma x_i} \quad , \tag{66}$$

and the field equations are equivalent to the canonical equations for H and the zero-energy condition :

$$-\frac{\partial H}{\partial x} = \frac{dp}{d\tau} \quad , \quad \frac{\partial H}{\partial p} = \frac{dx}{d\tau} \quad , \quad H = 0 \ . \tag{67}$$

II. Non-Hamiltonian autonomous equations for (n+1)-dimensional homogeneous models

The field equations for the (n+1)-dimensional homogeneous models form a Hamiltonian system with constraint forces . The monotonic behavior of $\sqrt{g}$ renders the interpretation of the evolution as the motion of a particle in an explicitly time-dependent potential. Considering vacuum models, we show that this explicit time-dependence of the potential (and of the extra force terms in class-B models) may be eliminated completely. Using suitable coordinates, we obtain a system with a (2n-2)- dimensional phase space, for which we can also give a Ljapunov function in terms of the n-dimensional Ricci curvature.

II.1. Introduction

The study of higher-dimensional cosmological models is mainly motivated by a possible geometrical unification of the fundamental interactions. In this context, an important question is whether the Einstein equations provide a mechanism which causes a dynamical compactification of the extra dimensions. Another interesting problem concerns the nature of the cosmological singularity, which has been especially treated for highly symmetric cosmological models, such as $\mathbb{R} \times R_c{}^d \times R_c{}^D$ (where $R_c{}^d$ and $R_c{}^D$ are isotropic d- and D-dimensional Riemannian spaces) [9-12,36]. Since even in (3+1) dimensions some anisotropic cosmological models exhibit a very complicated (chaotic) behaviour of the scale factors [5,19,37], it is convenient to consider also anisotropic cases in higher spacetime dimensions [20,38]. Assuming spatial homogeneity, the chaotic behaviour does not occur in models with more than (3+1) dimensions [8,20] whilst it can be present in most inhomogeneous cosmological models up to (9+1) dimensions [15,38].

Discussing the behavior of scale factors is mostly replaced by the examination of the ergodic and mixing properties of the so called mixmaster map (see also section IV.8). This map has been introduced for the first time for the Bianchi type IX [5,19] and subsequently was generalized to the higher dimensional cases [38]. Treating the evolution of scale factors within this framework is based on the assumption that the Kasner solution is as good an approximation during successive time intervals as it represents the only nonchaotic approach to the cosmological singularity. In the following we shall introduce coordinates which are adapted to the symmetries of the internal curvature and to the scaling properties of the Hamiltonian. The assumptions mentioned above will turn out to be provable within this formulation and, additionally, the discussion of the highly symmetric models is simplified to the treatment of the equations in a phase plane.

In the second section the Hamiltonian formalism is generalized to the (n+1)-dimensional cosmological models. If the Lie group acting on the space like hypersurfaces is unimodular then the field equations can be written in Hamiltonian form. Otherwise we must add constraint forces (standing perpendicular to the momenta with respect to the De Witt metric) to the equations of motion as for the class-B Bianchi type models in (3+1) dimensions.

The properties of homogeneity of the Ricci tensor of a n-dimensional Riemannian space (G,γ) are discussed in the third section. Splitting the metric γ into a d- and a D-dimensional part, the Ricci curvature falls into four pieces of which each has a fixed behavior under different transformations in the two subspaces of (G,γ).

In the forth section we give a symplectic transformation to coordinates which are adapted to the scaling properties of the curvature. The new "anisotropy" and "volume" coordinates are related to the Jacobi coordinates [39] of classical mechanics. The potential and the force terms separate into a factor which depends only on the relative (anisotropy) coordinates and a factor which includes the volume-depending part. Since in addition, the kinetic part of the Hamiltonian is indefinite (the signature of the De Witt metric is (n-2)), we can consider a Hamiltonian system for (n-1) degrees of freedom with a time dependent potential instead of an autonomous system with a 2n-dimensional phase space. There also exists a special relative coordinate corresponding to the ratio of the d- and the D-dimensional part of the volume. On the one hand this coordinate will turn out to be suitable in discussing the highly symmetric models (such as $\mathrm{IR} \times R_c{}^d \times R_c{}^D$ in Part III) whilst it yields a necessary criterion for the existence of non-chaotic solutions on the other hand.

Finally, in the last section, we shall show how the explicit time-dependence of the potential and the force terms can be eliminated. Using the scaling properties of the Hamiltonian, one can derive a conserved quantity depending on (n-1) coordinates from the original constraint equation. Instead of the explicitly time-dependent equations of section four, we obtain an autonomeous system for (n-1) degrees of freedom. The treatment of the resulting equations is simplified by the existence of a Ljapunov function which we derive at the end of this section.

II.2. The field equations

Considering (n+1)-dimensional cosmological models we shall repeat some aspects of the Hamiltonian formulation and fix some notations in this section. As in (3+1)-dimensional homogeneous cosmology, the G_{ij}-equations are identical with the equations of motion for the Lagrangian $\sqrt{g}G^{-}_{oo}$ up to some constraint forces F_{ij}

$$D^{ij}L - F^{ij} = 0 \qquad \Leftrightarrow \qquad G^{ij} = 0 \tag{1}$$

$$L := \sqrt{g}\,G^{-}_{oo} := -\frac{1}{2}(\sqrt{g}\,\bar{R}) - \frac{1}{8}<\overset{\circ}{g}, \overset{\circ}{g}>_W \quad , \tag{2}$$

where the minus sign superscript refers to the opposite sign of the first term compared with the expression for $\sqrt{g}G_{oo}$, and $D_{ij}=\partial/\partial g_{ij} - d/dt\,\partial/\partial g_{,ij,t}$ is the Euler-Lagrange derivative with respect to g_{ij}. The n-dimensional De Witt metric and its inverse are

$$W_{ijkl} = \frac{1}{2\sqrt{g}}\left(g_{i(k}\,g_{l)j} - \frac{2}{n-1}\,g_{ij}\,g_{kl}\right) \tag{3a}$$

$$W^{ijkl} = \frac{\sqrt{g}}{2}\,(g^{i(k}g^{l)j} - 2\,g^{ij}g^{kl}) \tag{3b}$$

$$W^{ijkl}\,W_{klab} = \frac{1}{2}\,\delta^{i}_{(a}\,\delta^{j}_{b)} \quad .$$

The forces F_{ab} are expressions in terms of the structure constants of the Lie group G

$$F^a_b := C_k (F^k)^a_b := -\sqrt{g}\, g^{ij}\, C^a_{bij} \tag{4a}$$

$$C^a_{bij} := 1/2\ C_j\ (C^a_{bi} - \delta^a_i\, C_b) \tag{4b}$$

$$C_b := C^i_{ib} \quad . \tag{4c}$$

Using the conjugated momenta Π^{ij}, the Hamiltonian becomes $\sqrt{g}G_{oo}$:

$$H := \Pi^{ij} \mathring{g}_{ij} - L = \sqrt{g}\, G_{oo} = \frac{\sqrt{g}}{2} \bar{R} - 2 <\Pi,\Pi>_W \tag{5}$$

$$\Pi^{ij} := \partial L / \partial \mathring{g}_{ij} = -1/4\ W^{ijkl} \mathring{g}_{kl} \quad . \tag{6}$$

The equations of motion (1) are the modified Hamiltonian equations :

$$\mathring{g}_{ij} = \frac{\partial H}{\partial \Pi^{ij}} \quad , \quad -\mathring{\Pi}^{ij} = \frac{\partial H}{\partial g_{ij}} + F^{ij} \quad . \tag{7}$$

The constraint H=0 implies that the forces must stand perpendicular to the canonical momenta with respect to the De Witt metric, which is again equivalent to the G_{ok} equations :

$$0 = \frac{dH}{dt} = \{ H , H \} - \mathring{g}_{ij}\, F^{ij} = \sqrt{g}\, C_k G^{ok} = -1/4 <F, \Pi>_W \quad . \tag{8}$$

If the metric contains only diagonal elements, $g_{ij}=\delta_{ij}g_i$, we can use the canonical pairs (X_i,P^i), the time coordinate τ and the Hamiltonian h :

$$(X_i , P^i) := (\ln g_i , g_i \Pi^i) \quad ; \quad d\tau := dt / \sqrt{g} \tag{9}$$

$$h(X,P) := \sqrt{g} H = \frac{1}{2} (g\bar{R})(X) + T(P,P) \quad . \tag{10}$$

The modified Hamiltonian equations (7) now read

$$X_{i,\tau} := \frac{\partial h}{\partial P^i} \quad ; \quad -P^i_{,\tau} := \frac{\partial h}{\partial X_i} + F^i \tag{11}$$

where the quadratic form T(P,P) and the forces F^i are defined as

$$T(P,P) := \frac{2}{n-1}(\Sigma P^i)^2 - 2\,\Sigma\,(P^i)^2 \tag{12}$$

$$F^i \quad := \sqrt{g}\,F^i{}_i = -g\,g^{jj}\,C^i{}_{ijj} \quad . \tag{13}$$

Using $\partial/\partial X_i = -P_i\partial/\partial P_i + g_i\partial/\partial g_i$ and H=0 we obtain (11b) from (7) :

$$\frac{\partial h}{\partial X_i} = g_i\frac{\partial(\sqrt{g}H)}{\partial g_i} - \Pi^i\sqrt{g}\,\frac{\partial H}{\partial \Pi^i}$$
$$= -\sqrt{g}\,(\,\Pi^{\circ i}g_i + F^{ii}g_{ii} + \Pi^i\mathring{g}_i\,) = -P^i{}_{,\tau} - F^i\,.$$

The Hamiltonian formulation for (n+1)-dimensional homogeneous cosmological models is thus completely equivalent to the (3+1)-dimensional case. The field equations form a finite dimensional Hamiltonian system if the Lie group acting on the spacelike hypersurfaces is unimodular. Otherwise, if the traces of the structure constants do not vanish, the equations of motion must be modified and the system is no longer Hamiltonian but can be written in the form (7).

II.3. Symmetries of R_{ij}

The discussion of the field equations (7) or (11) is simplified in coordinates which are adapted to the symmetries of the Hamiltonian. We shall thus treat the scaling properties of the Ricci tensor in this section. Let R_{ij} denote the Ricci tensor of the n-dimensional Riemannian space (G,γ) with metric $\gamma = -g_{ij}\theta^i \otimes \theta^j$ and structure constants $C^i{}_{jk}$ of G:

$$d\theta^i + \frac{1}{2}C^i{}_{jk}\,\theta^j \wedge \theta^k = 0 \quad . \tag{14}$$

Defining λ_{kij} and the constants t_{ij} and $s_{ijkl}{}^{ab}$ as

$$\lambda_{kij} := -1/2\ (C_{kij} + C_{ijk} - C_{jki}) \tag{15a}$$

$$t_{ij} := C_iC_j + 1/2\ C^k_{il}\,C^l_{jk} \tag{15b}$$

$$s_{ijkl}{}^{ab} := 1/4\ C^a_{ik}\,C^b_{jl} \quad , \tag{15c}$$

we can write the Ricci tensor and the Ricci curvature as follows :

$$\bar{R}_{ij} = -\lambda^k{}_{il}\,\lambda^l{}_{jk} - C_l\lambda^l{}_{ij} \tag{16}$$

$$\bar{R} = -[\, t_{ij} + s_{ijkl}{}^{ab}\, g^{kl} g_{ab} \,]\, g^{ij} \ . \tag{17}$$

Using the Bianchi identity or by direct calculation we obtain with (4) :

$$\frac{\partial \bar{R}}{\partial g_{ij}} = \bar{R}^{\,ij} - \frac{2}{\sqrt{g}} F^{ij} \ . \tag{18}$$

The homogeneity of R implies the vanishing of the trace of F_{ij} : Since t_{ij} and $s_{ijkl}{}^{ab}$ are constant expressions in terms of the structure constants we have for any $\lambda \in \mathbb{R}$

$$\bar{R}(\lambda g_{ij}) = \lambda^{-1}\bar{R}(g_{ij}) \tag{19}$$

and after a differentiation of (19):

$$g_{ij}\frac{\partial \bar{R}}{\partial g_{ij}} = -\bar{R} \ . \tag{20}$$

Comparing (18) and (20) we obtain $g_{ij}F^{ij} = 0$, which also derives from the algebraic identity $\Sigma_a C^a{}_{aij} = 0$ (4a). In the case of a diagonal metric, $g_{ij} = \delta_{ij}g_i = \delta_{ij}\exp(X_i)$, the expressions (16,17) reduce to

$$\begin{aligned}\bar{R}^{\,a} = {} & \frac{1}{2} C^k{}_{al}\, C^l{}_{ak}\, e^{-X_a} + C_k\, C^a{}_{ak}\, e^{-X_k} \\ & + 2\, s_{ak}{}^{i}\, e^{X_i - X_k - X_a} - s_{ik}{}^{a}\, e^{X_a - X_i - X_k}\end{aligned} \tag{21}$$

$$\bar{R} = -t_i\, e^{-X_i} - s_{ij}{}^{k}\, e^{X_k - X_i - X_j} \ , \tag{22}$$

and instead of (18) we obtain

$$\frac{\partial \bar{R}}{\partial X_i} = \bar{R}^{\,i} - \frac{2}{g} F^{i} \ . \tag{23}$$

The constants t_i and $s_{ij}{}^k$ are defined as

$$t_i \;:=\; t_{ii} \;=\; (C_i)^2 + 1/2 \;\; C^k_{il}\, C^l_{ik} \tag{24a}$$

$$s_{i\,j}{}^k \;:=\; s_{i\,i\,j\,j}{}^{kk} \;=\; 1/4 \;\; (C^k_{ij})^2 \;\geq 0 \quad , \tag{24b}$$

where the semi-definiteness of the $s_{ij}{}^k$ will play an important role in the discussion of the asymptotic behavior of the scale factors. In the following we use little Latin letters for indices running from 1 to n, Greek and capital Latin letters for indices between 1...d and (d+1)...n, respectively. Assuming a metric of the block form

$$(g)_{i\,j} \;=\; \begin{pmatrix} (g)_{\mu\nu} & 0 \\ 0 & (G)_{IJ} \end{pmatrix} \qquad , \tag{25}$$

the curvature and the forces can be written as follows :

$$\bar{R} \;=\; R_d \;+\; R_D \;+\; R_{dD} \;+\; R_{Dd} \tag{26a}$$

$$F^a_{\;b} = \;(F_d)^a_{\;b} \;\;+\;\; (F_D)^a_{\;b} \qquad . \tag{26b}$$

Each of the quantities R_d, F_d etc has a fixed behavior under different rescaling of the d- and the D-dimensional part of the metric (25):

$$g_{\mu\nu} \;\rightarrow\; \lambda\, g_{\mu\nu} \quad , \qquad G_{IJ} \;\rightarrow\; \Lambda\, G_{IJ} \quad . \tag{27}$$

Applying (27), we obtain $(R_d, F_d/\sqrt{g}) \rightarrow \lambda^{-1}(R_d, F_d/\sqrt{g})$, $(R_D, F_D/\sqrt{g}) \rightarrow \Lambda^{-1}(R_D, F_D/\sqrt{g})$, $R_{dD} \rightarrow \lambda\Lambda^{-2} R_{dD}$ and $R_{Dd} \rightarrow \Lambda\lambda^{-2} R_{Dd}$, which is easily seen from the following expressions for R_d, F_d etc:

$$R_d \;:=\; -\,[\,t_{\mu\nu} + s_{\mu\nu\alpha\beta}{}^{\gamma\delta}\, g^{\alpha\beta}\, g_{\gamma\delta} \;+\; 2\; s_{\mu\nu AB}{}^{CD}\; G^{AB}\, G_{CD}\,]\;\; g^{\mu\nu} \tag{28a}$$

$$R_D \;:=\; -\,[\,t_{IJ} + s_{IJAB}{}^{CD}\; G^{AB}\, G_{CD} \;+\; 2 s_{IJ\alpha\beta}{}^{\gamma\delta}\;\; g^{\alpha\beta}\, g_{\gamma\delta}\;\,]\;\; G^{IJ} \tag{28b}$$

$$R_{dD} := -\;\; s_{IJAB}{}^{\gamma\delta}\;\; g_{\gamma\delta}\; G^{IJ}\, G^{AB} \tag{28c}$$

$$R_{Dd} := -\;\; s_{\mu\nu\alpha\beta}{}^{CD}\; G_{CD}\, g^{\mu\nu} g^{\alpha\beta} \tag{28d}$$

$$(F_d/\sqrt{g})^a_{\ b} := -\, C^a_{b\mu\nu}\, g^{\mu\nu} \tag{29a}$$

$$(F_D/\sqrt{g})^a_{\ b} := -\, C^a_{bIJ}\, G^{IJ} \quad . \tag{29b}$$

Now we can set λ and Λ equal to $g^{-1/d}$ and $G^{-1/D}$, respectively, and use

$$R_d(g_{\mu\nu},G_{IJ}) = g^{-1/d}\, R_d(g^{-1/d}g_{\mu\nu}\,,\, G^{-1/D}G_{IJ}) = g^{-1/d} r_d\,(h_{\mu\nu},H_{IJ}) \ ,$$

where the determinants of $h_{\mu\nu}$ and H_{IJ} are equal to one, to write R and $(F/\sqrt{g})^a{}_b$ in the following form :

$$g\bar{R} \;=\; z\,\kappa^{-\alpha}\ [\, r_d + \kappa\, r_D + \kappa^2\, r_{dD} + \kappa^{-1}\, r_{Dd}\,] \tag{30a}$$

$$g(F/\sqrt{g})^a_{\ b} \;=\; z\,\kappa^{-\alpha}\ [\ (f_d)^a_{\ b} + \kappa\,(f_D)^a_{\ b}\ ]\ . \tag{30b}$$

The little r_d, f_d etc are the same expressions in $h_{\mu\nu}$ and H_{IJ} as the capital R_d, F_d etc are in $g_{\mu\nu}$ and G_{IJ}, but they only depend on (n-2) degrees of freedom. The special variables z and κ appearing explicitly in (30) are identified with the "volume" g and the ratio of the two determinants of g and G in (25):

$$z := g^{\gamma^2} \;=\; (g\,G)^{1-1/n} \tag{31a}$$

$$\kappa := (g^{\alpha}\, G^{\alpha-1})^{\omega^2} \;=\; g^{1/d}\, G^{-1/D}\ . \tag{31b}$$

Instead of d, D and n we shall often use the quantities α, ω and γ:

$$\alpha := \frac{D}{n} \ \ ; \ \ \omega := \sqrt{\frac{n}{dD}} \ \ ; \ \ \gamma := \sqrt{\frac{n-1}{n}}\ . \tag{32}$$

Whenever we are able to split the metric such as in (25), there exists, additionally to z, another special coordinate κ which is a measure for the "total" anisotropy. For all homogeneous models (in any space dimension) the potential (30a) is proportional to a polynomial of maximally third degree in κ and is linear in z (and so are the force terms (30b)). The r_d, f_d etc. are independent of z and κ. Since they are only functions of "local" anisotropies, they reduce to constants for all highly symmetric models of the form $\mathrm{IR} \times R_c{}^d \times R_c{}^D$ ($R_c{}^d$ and $R_c{}^D$ are subspaces of constant curvature). In this case the ratio κ and the product z of the determinants are the only variables in the system. Using z as a time coordinate we are finally left with the problem of a one-dimensional motion $\kappa(z)$ as we shall explain in the fifth section. Instead of z and κ we shall often use their logarithms x_n and x_d :

$$x_n := 1/\sqrt{n}\gamma^2 \ln z = 2/\sqrt{n} \ln \sqrt{g} \tag{33a}$$

$$x_d := 1/\omega \ln \kappa = \sqrt{D/nd} \ln g - \sqrt{d/nD} \ln G . \tag{33b}$$

II.4. Volume and anisotropy coordinates

II.4.1. n=3 as an example

The three-dimensional homogeneous models were often treated in the literature (for detailed lists of refs., see [19,33,35]. Discussing the diagonal Bianchi type cosmologies, the introduction of the linear combinations $x_i=A_{ij}X_j$ has turned out to be useful. The X_j are the logarithms of the diagonal elements, and the linear map $A : \underline{X} \in IR^3 \to \underline{x} \in IR^3$ is represented by the matrix

$$(A)_{ij} := \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} . \tag{34}$$

Since $A^{-1} = A^T$, the momenta also transform with A: $\underline{P} \in IR^3 \to \underline{p} \in IR^3$ and the kinetic part in (10) becomes together with (12)

$$T(P,P) \to T_A(p,p) = T(A^Tp,A^Tp) = p_3^2 - 2(p_1^2+p_2^2) . \tag{35}$$

The first two components of $\underline{x}$ are anisotropy coordinates and the third is the volume coordinate introduced in (33a): $x_n= x_3= 1/\sqrt{3}\ \Sigma X_i = 2/\sqrt{n} \ln\sqrt{g} = 1/(\sqrt{n}\gamma^2) \ln(z)$. Since gR contains only linear terms in z, it must separate in all cases into an x_3-dependent part and a part which is independent on the volume coordinate. Taking the Bianchi types VIII (lower sign) or IX as an example, we obtain from

$$\bar{R} = \pm e^{-X_1} \pm e^{-X_2} + e^{-X_3} - \frac{1}{2}\,[\, e^{X_1-X_2-X_3} + e^{X_2-X_1-X_3} + e^{X_3-X_1-X_2}] \qquad (36)$$

after using the transformation $\underline{X} = A^T\underline{x}$ the following expression for the potential:

$$g\bar{R} = e^{2/\sqrt{3}x_3}\ [\ e^{\sqrt{2/3}x_2}\ \{1-\mathrm{ch}(\sqrt{2}x_1)\}\ -\ 1/2\ e^{-2\sqrt{2/3}x_2}$$

$$\pm\ 2\,e^{-1/\sqrt{6}x_2}\,\mathrm{ch}(x_1/\sqrt{2})\ ]\ =\ z\,\Psi(x_1,x_2)\quad . \qquad (37)$$

The Hamiltonian now reads

$$h\ =\ p_3^2 - 2\,(\,p_1^2\ +\ p_2^2\,)\ +\ \frac{1}{2}\,z(x_3)\,\Psi(x_1,x_2) \qquad (38)$$

where only $\Psi(x_1,x_2)$ contains the information about the Lie group. Introducing the new time $T{:=}\sqrt{2}x_3$ and using the constraint $h=0$, we can consider the explicitely time-dependent Hamiltonian $h_T = -p_3(x_{1,2},p_{1,2},T)$ whenever the solution $x_3(\tau)$ is a monotonic function of τ. This is the case for $\tau \in\ ]-\infty,\tau_o[$ where τ_o is the solution of $\sqrt{g(\tau_o)} \geq \sqrt{g(\tau)}\ \forall\tau$ for the Bianchi type IX and $\tau_o = \infty$ for all other three-dimensional models :

$$h_T\ :=\ -p_3\,(x_i,p_i,T)\,/\sqrt{2}\ =\left[\ (\,p_1^2\ +\ p_2^2\,)\ -\ \frac{1}{4}\,e^{\gamma T}\,\Psi(x_1,x_2)\ \right]^{1/2} . \qquad (39)$$

The equations for the evolution of $x_{1,2}$ and $p_{1,2}$ are the Hamiltonian equations for the explicitely time-dependent Hamiltonian h_T:

$$\frac{dx_i}{dT} = \frac{x_{i,\tau}}{\sqrt{2}x_{3,\tau}} = \frac{1}{\sqrt{2}}\,\frac{\partial h/\partial p_i}{dh/\partial p_3} = \frac{\partial h_T}{\partial p_i} \qquad (40a)$$

$$\frac{dp_i}{dT} = \frac{p_{i\cdot\tau}}{\sqrt{2}x_{3,\tau}} = -\frac{1}{\sqrt{2}}\,\frac{\partial h/\partial x_i}{\partial h/\partial p_3} = -\frac{\partial h_T}{\partial x_i}\quad . \qquad (40b)$$

The configuration space is now two-dimensional and the solutions of the field equations may be considered as the trajectories of the "universe point" moving in a time-dependent potential of the form $\exp(\gamma T)\Psi(\underline{x})$. We shall next generalize this concept to an arbitrary number of dimensions.

II.4.2. The general case n≥3

In this section we shall give the generalization of the transformation (34) for the higher dimensional cases. Additionally to the volume coordinate z (or x_n) we shall also introduce the special anisotropy coordinate κ (or x_d) (31,33). In the new coordinates we can write the po- tential in the form $z\Psi(\kappa\ ,\ x_1 \ldots x_{d-1}\ ,\ x_{d+1} \ldots x_{n-1})$ where $\kappa^{1+\alpha}\Psi$ is a polynom of maximally third degree in κ (30):

$$g\bar{R} = z\,\kappa^{-\alpha}\,[\,r_d + \kappa\, r_D + \kappa^2\, r_{dD} + \kappa^{-1}\, r_{Dd}\,] \quad . \tag{41}$$

The functions r_d and r_D etc depend only on the (n-2) coordinates $x_1 \ldots x_{d-1}$, $x_{d+1} \ldots x_{n-1}$.

<u>Proposition 4.1.</u>
Let A and B_k (k=d,D) denote the linear transformations $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $B_k : \mathbb{R}^k \rightarrow \mathbb{R}^{k-1}$ represented by the matrices

$$A := \begin{pmatrix} & B_d & & & 0 & \\ \sqrt{D/nd} & \ldots\ldots & \sqrt{D/nd} & -\sqrt{d/nD} & \ldots\ldots & -\sqrt{d/nD} \\ & 0 & & & B_D & \\ 1/\sqrt{n} & \ldots\ldots & 1/\sqrt{n} & 1/\sqrt{n} & \ldots\ldots & 1/\sqrt{n} \end{pmatrix} \tag{42}$$

$$B_k := \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 & \ldots & \ldots & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \ldots & 0 \\ -1/\sqrt{k(k-1)} & \ldots & \ldots & \ldots & -1/\sqrt{k(k-1)} & (k-1)/\sqrt{k(k-1)} \end{pmatrix}$$

Then the mapping $X \rightarrow x = AX$, $P \rightarrow p = AP$ is a symplectic transformation of the 2n-dimensional phase space which especially introduces the coordinates $x_n = \ln z/\sqrt{n}\gamma^2$ and $x_d = \ln\kappa/\omega$, diagonalizes the quadratic form $T(P;P)$ (12):

$$T_A(p,p) = \frac{2}{n-1} p_n^2 - 2\sum_{i=1}^{n-1} p_i^2 \tag{43}$$

and brings the function $gR(X)$ (17) into the form (41).

Proof: Since $A^{-1} = A^T$, the transformation is symplectic. The lines of A are orthogonal to each other and the sum of all elements in each line vanishes (except in the n´th):

$$\sum_{j=1}^{n} A_{ij} = \sqrt{n}\,\delta_{in} \quad , \quad \sum_{j=1}^{n} A_{ji} A_{jk} = (A^T A)_{ik} = \delta_{ik} \; .$$

For the n´th and the d´th component of x we have

$$x_n = A_{nj} X_j = \frac{1}{\sqrt{n}} \sum X_j = \frac{1}{\sqrt{n}} \ln g = \frac{1}{\sqrt{n}\gamma^2} \ln z \, .$$

$$x_d = A_{dj} X_j = \sqrt{\frac{D}{nd}} \sum_{1}^{d} X_j - \sqrt{\frac{d}{nD}} \sum_{d+1}^{n} X_j = \frac{1}{\omega} \ln \kappa \; .$$

Using (12) we obtain for the kinetic part of the Hamiltonian the diagonal expression (43): $T(P,P) \to T_A(p,p) = T(A^T p, A^T p) = 2/(n-1)\{\Sigma_i (A^T p)_i\}^2 - 2(p, AA^T p) = 2/(n-1)(\sqrt{n}\,\delta_{jn}\,p_j)^2 - 2p^2 = 2/(n-1)\,p_n^2 - 2[p_1^2 + p_2^2 + \ldots + p_{n-1}^2]$. Δ

Instead of the pair (x_n, p_n) we shall use $(x_o, p_o) := \{x_n\sqrt{(n-1)}\,,\ p_n/\sqrt{(n-1)}\}$ and the notation $x = (x_o, \underline{x})$, $p = (p_o, \underline{p})$. The kinetic and the potential part of the Hamiltonian now read (with $\eta_{ij} :=$ diag $(1,-1,\ldots,-1)$:

$$T_A(p,p) = 2\,\eta_{ij}\, p_i p_j = 2\,\eta\,(p,p) \tag{44a}$$

$$(g\bar{R})(x) = e^{\gamma x_o}\,\Psi(\underline{x}) \; . \tag{44b}$$

In the new coordinates we thus have the following

Proposition 4.2.

The modified Hamiltonian equations in the canonical conjugated coordinates (x,p) read :

$$h = 4\left[\frac{1}{2}\eta(p,p) + \frac{1}{8}e^{\gamma x_o}\,\Psi(\underline{x})\right] \tag{45}$$

$$x_{o\,,\tau} := \frac{\partial h}{dp_o} \quad ; \quad -p_{o\,,\tau} := \frac{\partial h}{\partial x_o} \tag{46a}$$

$$\underline{x}_{\,,\tau} := \frac{\partial h}{\partial \underline{p}} \quad ; \quad -\underline{p}_{\,,\tau} := \frac{\partial h}{\partial \underline{x}} + \frac{1}{2}e^{\gamma x_o}\,\underline{\varphi}(\underline{x}) \tag{46b}$$

where the n´th (o´th) component of the extra force $\underline{\varphi}(\underline{x})$ vanishes. The potential Ψ and the force are given as

$$\Psi(\underline{x}) = \kappa^{-\alpha}\,[\,r_d + \kappa\, r_D + \kappa^2\, r_{dD} + \kappa^{-1}\, r_{Dd}\,] \tag{47a}$$

$$\underline{\varphi}(\underline{x}) = 2\,\kappa^{-\alpha}\,[\,A\,\underline{f}_d + \kappa\,A\,\underline{f}_D\,]\;, \tag{47b}$$

where r_d, f_d etc. depend only on $\underline{x}$ without x_d and $\underline{\varphi}(\underline{x})$ stands perpendicular to the momentum with respect to the Eukliedian metric of IR^{n-1}.

Proof: From Proposition 4.1. we know that the modified Hamiltonian equations written in the coordinates (x,p) must still be of the form (11). Using equations (13) and (30b) the transformation of the force terms yields

$$F^i \rightarrow A_{ij}\,F^j = A_{ij}\sqrt{g}\,F^j_{\;j} = A_{ij}\,z\,\kappa^{-\alpha}\,[\,(f_d)^j_{\,j} + \kappa\,(f_D)^j_{\,j}\,] = \frac{1}{2}e^{\gamma x_o}\,\varphi_i \tag{48a}$$

where the n´th component of the force vanishes since the traces $\Sigma_i C^i{}_{ijk}$ are zero for all j,k (46):

$$\varphi_n = 2A_{nj}\,\kappa^{-\alpha}\,[\,\underline{f}_d^{\,j} + \kappa\,\underline{f}_D^{\,j}\,] = -\frac{2}{\sqrt{n}}\kappa^{-\alpha}(\,C^j_{\;j\mu\mu}\,h^{\mu\mu} + C^j_{\;jKK}\,H^{KK}) = 0.$$

The orthogonality is finally obtained from (8) with $A^{-1} = A^T$ since the n´th component vanishes :

$$0 = \langle F, \Pi \rangle_W = -\frac{1}{4g} X_{i,\tau} F^i \propto 2(A^T x_{,\tau})_i F^i$$
$$= 2x_{j,\tau} (AF)_j = z\, x_{j,\tau} \varphi_j \propto (\underline{p}, \underline{\varphi}) \; . \qquad \Delta \qquad (48b)$$

The Hamiltonian (45) represents the generalization of (38) for all higher dimensional diagonal models. The information on the Lie group is contained in the force vector $\underline{\varphi}(x)$ and in the potential $\Psi(\underline{x})$. The equations of motion for (x_o, p_o) are Hamiltonian equations without modification, no matter whether the Lie group is unimodular or not.

As in the (3+1)-dimensional case, we can introduce the time T $= \sqrt{(n-1)}\, x_n = x_o$ and use the constraint h=0 to obtain a time-dependent Hamiltonian $h_T = -p_o(\underline{x}, \underline{p}, T)$:

$$h_T := -p_o(\underline{x}, \underline{p}, T) = \left[\; |p|^2 - \frac{1}{4} e^{\gamma T} \Psi(\underline{x}) \right]^{1/2} . \qquad (49a)$$

The evolution equations for h_T can be written in Hamiltonian form whenever the equations for h are Hamiltonian :

$$\frac{d\underline{x}}{dT} = \frac{\underline{x}_{,\tau}}{x_{o,\tau}} = \frac{\partial h/\partial \underline{p}}{4p_o} = \frac{\partial h_T}{\partial \underline{p}} \qquad (49b)$$
$$\frac{d\underline{p}}{dT} = \frac{\underline{p}_{,\tau}}{x_{o,\tau}} = -\frac{\partial h/\partial \underline{x}}{4p_o} = -\frac{\partial h_T}{\partial \underline{x}} \; . \qquad (49c)$$

If we also have to take the extra force terms into account, the equations (49c) for $\underline{p}$ must be modified and have the unpleasant form

$$\frac{d\underline{p}}{dT} = -\frac{\partial h_T}{\partial \underline{x}} + \frac{1}{8} h_T^{-1} e^{\gamma x_o} \underline{\varphi} \; .$$

The discussion of the explicitly time-dependent equations (49) turns out to be rather difficult even in (3+1) dimensions where the system has only two degrees of freedom, x_1 and x_2. The potentials have exponentially steep walls which expand as $T \to -\infty$ $(t \to 0)$. The question of whether a solution behaves regular near the cosmological singularity is equivalent to the question of whether there is an infinite sequence of reflections from the potential walls [19]. If the equipotential hypersurfaces (lines) are not closed (as is the case in (3+1)-dimensional cosmology for the Bianchi types I...VII), the universe point can move in directions where no collisions take place and the solution behaves regular as $T \to -\infty$. If, on the other hand, the equipotentials are closed or if there is only a set of directions of vanishing measure where they are open (as in the Bianchi type VIII and IX models), the problem is more difficult and one has to find out if the universe point is (after each collision) fast enough to catch up again with one of the expanding walls. In order to make the system treatable in this form, the exponentially steep walls are usually replaced by infinitely steep ones and the universe point is assumed to propagate freely between two reflections. Using these assumptions, the following properties of the system are known:

In (3+1) dimensions the solutions for type VIII and IX behave ergodic [40] and chaotic [19] near the cosmological singularity, whereas the solutions for the other Bianchi types show a regular approach to the singularity since their equipotentials are not closed.

The (4+1)-dimensional homogeneous models have been classified by Fee [41]. Since none of the Fee types has closed equipotential walls, chaos is not expected to occur in this dimension [8] (this need not be true for inhomogeneous models as we shall explain in part IV).

Within the framework of moving equipotentials, models in more than (4+1) dimensions have mainly been discussed for cases with higher symmetries [11] and were found to behave regular as $t \to 0$ (see Part IV).

The method of explicitly time-dependent potentials is (especially in higher dimensions) rather unpleasant since it requires assumptions of the mentioned kind. Another approach, which has turned out to be successful, starts with the question whether there is a time t_0 and a constant vector $\sigma \in \mathbb{R}^n$ (with $\Sigma\sigma_i = \Sigma\sigma_i^2 = 1$) such that a solution of the form $g_i(t) = t^{2\sigma_i}$ may be a good approximation for all $t < t_0$. Using the Levi-Malcev decomposition for Lie algebras [21], such a stable generalized Kasner solution may be found for all homogeneous models in more than (3+1) dimensions [20]. In this context one has to assume that the generalized Kasner solution is the only regular approach to the cosmological singularity and that the transitions from one Kasner epoch to the next one takes place instantaneously.

In the next section we shall show that in all homogeneous vacuum models the explicit time-dependence of the modified Hamiltonian equations can be eliminated. Within the new set of autonomous equations we shall later (Part IV) find a necessary condition for the existence of regular solutions and we shall also show that the generalized Kasner solution is in fact the only regular general approach to the singularity in homogeneous cosmology.

II.5. Reduction to an autonomous system

The invariance of the Hamiltonian constraint under simultaneous rescaling of the volume coordinate and the momenta and the behavior of $\sqrt{g}$ as a function of τ make it possible to eliminate the explicite time dependence in the equations of motion (49). In order to show this, we first consider the function $x_0(\tau)$:

Proposition 5.1.

Let $\xi(\tau) = (x(\tau) , p(\tau))$ be a non-stationary solution of the modified Hamil- tonian system (46)

$$\xi_{,\tau} = \{ \xi , h \} - f \ ; \qquad (f, J\, \xi_{,\tau}) = 0 \tag{50}$$

with Hamiltonian $h(\xi) = 0$ and additional constraint force f of the form

$$f = \frac{1}{2} e^{\gamma x_0} \,(0, \underline{0}, 0, \underline{\varphi}(\underline{x})) \tag{51}$$

$$h = 2\eta\,(p,p) + \frac{1}{2} e^{\gamma x_0}\, \Psi(\underline{x}) = T_A + V . \tag{52}$$

Then $x_0(\tau)$ is either monotonically increasing for all times or it has exactly one local extremum which is also the global maximum.

Proof: Let $x_0'(\tau_0) = 0$. We first show that $x_0''(\tau_0) \le 0$: Using $x_0' = 4p_0$ and $p_0' = -\gamma V(x)$ we obtain from $h=0$ at $\tau=\tau_0$: $p_0' = -2\gamma\, |\underline{p}|^2$ and thus $x_0''(\tau_0) \le 0$. If $x_0''(\tau) \neq 0$ then any critical point of x_0 is a local maximum

and the statement follows. Let us thus consider the case where $x_0''(\tau_0)$ also vanishes:
Let $x_0'(\tau_0) = x_0''(\tau_0) = 0$ .We show that $x_0'''(\tau_0) = 0$, $x_0^{(4)}(\tau_0) \leq 0$: From $(f, J\xi') = 0$ and (51) we know that $(\underline{p}, \underline{\varphi}) = 0$. Using this and $\underline{p} = -1/2 \exp(\gamma x_0) (\underline{\nabla}\Psi + \underline{\varphi})$ and $\Psi' = -4(\underline{\nabla}\Psi, \underline{p})$, we obtain

$$\Psi' = 8\, e^{-\gamma x_0} (\underline{p}, \underline{p}')$$
$$\Psi'' = 8\, e^{-\gamma x_0} [(\underline{p}', \underline{p}') + (\underline{p}, \underline{p}'') - 4\gamma p_0 (\underline{p}, \underline{p}')] .$$

Together with $p_0(\tau_0) = p_0'(\tau_0) = 0$ and $p_0'(\tau_0) = -2\gamma |\underline{p}|^2(\tau_0) = 0$ we have $\Psi'(\tau_0) = 0$ and $\Psi''(\tau_0) = 8 \exp(-\gamma x_0) |\underline{p}'|^2 \geq 0$. Differentiating the equation $x_0'' = -2\gamma \exp(\gamma x_0) \Psi(\underline{x})$ twice with respect to τ yields at $\tau = \tau_0$: $x_0'''(\tau_0) = 0$ and $x_0^{(4)}(\tau_0) = -2\gamma \exp(\gamma x_0) \Psi''(\tau_0) = -16\gamma |\underline{p}'|^2 \leq 0$ (and $= 0$ only if $\underline{p}' = 0$ at τ_0). If $x_0^{(4)}(\tau_0) \neq 0$ the statement follows again since the first non-vanishing even derivative of x_0 is strictly negative.
If, finally, $x_0^{(4)}(\tau_0)$ also vanishes, we have $p_0(\tau_0) = \underline{p}(\tau_0) = 0$ and $p_0'(\tau_0) = \underline{p}'(\tau_0) = 0$ which imply $p_0^{(k)}(\tau_0) = \underline{p}^{(k)}(\tau_0) = 0$ for all $k \geq 1$. This is immediately seen from the coupled equations of motion for $p_0(\tau)$ and $\underline{p}(\tau)$ at τ_0 by induction. Assuming that $p_0(\tau)$ and $\underline{p}(\tau)$ are analytic functions for finite τ, we thus obtain the stationary solution $p(\tau) = 0$. Δ

Using x_0 as a new time coordinate we shall now eliminate the explicite time dependence of the modified Hamiltonian equations (49) by considering the following

Proposition 5.2.

Let the modified Hamiltonian system (50-52) be given as in Proposition 5.1 and let I be the τ-interval where x_0 is increasing $I :=]-\infty, \tau_0[$. Then (for $p_0 \neq 0, \infty$) there exists a diffeomorphism $\eta : \xi \in \mathbb{R}^{2n} \to \eta(\xi) \in \mathbb{R}^{2n}$, a projector $P : \eta \in \mathbb{R}^{2n} \to P\eta \in \mathbb{R}^{2n-2}$ and a vector field $W : P\eta \in \mathbb{R}^{2n-2} \to W(P\eta) \in \mathbb{R}^{2n-2}$ such that on I the 2n modified Hamiltonian equations (50) together with the constraint $h=0$ correspond to the (2n-2)-dimensional autonomous system

$$\frac{d}{dT} P\eta(\xi) = W(P\eta(\xi)) \tag{53}$$

and the equation $\tau = \int^{T} F(P\eta(T'))\,dT'$, where $T := -x_o(\tau)$.

Proof: For $p_o \neq 0,\infty$ let $\eta(\xi)$ be the regular transformation

$$\eta(\xi) := (q_0, \underline{x}, y_o, \underline{y}) = \left(x_o - \frac{1}{\gamma}\ln p_o^2 , \underline{x} , \ln p_o^2 , \frac{\underline{p}}{p_o} \right)$$

and the projector $(P\eta)_i := (1 - \delta_{oi} \otimes \delta_{oi})$ i.e. $P\eta = (\underline{x}, \underline{y})$. We shall show that it is possible to find an autonomous system (53) for $(\underline{x}, \underline{y})$. Having solved these equations, τ (T) is obtained by an integration :

$$\tau(T) = -\int^{T} 2e^{\gamma T'/2} \left| \frac{|\underline{y}(T')|^2 - 1}{\Psi(\underline{x}(T'))} \right|^{1/2} dT' \quad , \tag{54}$$

and one has the solution in parametric form $\underline{x} = \underline{x}(T)$, $x_o = -T$, $\tau = \tau(T)$ for $T \in \tau^{-1}(I)$, which is equivalent to the solution $\underline{x} = \underline{x}(\tau)$, $x_o = x_o(\tau)$ of (50-52) after having solved $\tau = \tau(T)$ for T on the interval I. The Hamiltonian in the η-coordinates separates in the y_o-dependent part

$$h(\eta) = e^{y_o}\, \varepsilon(\underline{x}, \underline{y}, q_o) = e^{y_o}\left[2\,(1 - |\underline{y}|^2) + \frac{1}{2} e^{\gamma q_o}\, \Psi(\underline{x}) \right] \tag{55}$$

and so do the equations of motion for $\underline{x}$ and $\underline{y}$:

$$\underline{x}' = \frac{\partial h}{\partial \underline{p}} = \frac{1}{p_o} e^{y_o} \frac{\partial \varepsilon}{\partial \underline{y}} = p_o \frac{\partial \varepsilon}{\partial \underline{y}} \tag{56a}$$

$$\begin{aligned} \underline{y}' = \frac{1}{p_o}[\underline{p}' - \underline{y}\, p_o'] &= -\frac{1}{p_o}\left[\frac{\partial h}{\partial \underline{x}} + \frac{1}{2} e^{\gamma x_o} \underline{\varphi} - \underline{y} \frac{\partial h}{\partial x_o} \right] \\ &= -p_o \left[\frac{\partial \varepsilon}{\partial \underline{x}} + \frac{1}{2} e^{\gamma q_o} \underline{\varphi} - \underline{y} \frac{\partial \varepsilon}{\partial q_o} \right] \end{aligned} \tag{56b}$$

$$q_o' = x_o' - \frac{2}{\gamma}\frac{p_o'}{p_o} = p_o \left[4 + \frac{2}{\gamma} \frac{\partial \varepsilon}{\partial q_o} \right] . \tag{56c}$$

Using the time coordinate T, $-dT := dx_o = 4p_o d\tau$, these are (2n-1) autonomous equations for the (2n-1) coordinates (q_o , $\underline{x}$, $\underline{y}$). Since the transformation $\eta(\xi)$ is constructed such that the new Hamiltonian separates in the y_o-dependent part, the Hamiltonian constraint $h(\xi) = 0$ holding in the 2n-dimensional phase space reduces to the new constraint

$$\varepsilon(q_o, \underline{x}, \underline{y}) = 0 \tag{57a}$$

for the (2n-1) quantities $\underline{x}$, $\underline{y}$, and q_o. Solving (57a) for q_o,

$$q_o = \frac{1}{\gamma} \ln\left(4 \frac{|\underline{y}|^2 - 1}{\Psi(\underline{x})}\right) \tag{57b}$$

and using the time T, the right hand sides of the equations (56a,b) become only dependent on $\underline{x}$ and $\underline{y}$:

$$\frac{d\underline{x}}{dT} = \underline{y} \quad ; \quad \frac{d\underline{y}}{dT} = \frac{1}{2}(1 - |\underline{y}|^2)\left(\gamma\underline{y} - \frac{\underline{\nabla}\Psi + \underline{\varphi}}{\Psi}(\underline{x})\right) . \tag{58}$$

These are now the 2n-2 autonomous first-order differential equations. Using finally $q_o = x_o - 1/\gamma \ln(p_o{}^2) = -T - 1/\gamma \ln[(-dT/4d\tau)^2]$ the constraint (57) can be written in the form

$$e^{-T}\left(-\frac{dT}{4d\tau}\right)^{-2/\gamma} = \left(4 \frac{|\underline{y}|^2 - 1}{\Psi(\underline{x})}\right)^{1/\gamma} ,$$

which for a given solution $\underline{x}(T)$, $\underline{y}(T)$ of (58) can be integrated with respect to T and yields (54). Δ

Some comments are reasonable:

Using the original time coordinate t, $dt = \sqrt{g}\, d\tau$ and the relation $\sqrt{g} = \exp(\sqrt{n}/2\ x_n) = \exp(1/2\gamma\ x_o) = \exp(-T/2\gamma)$, we can also write

$$t(T) = -\int^T 2e^{-T'/(2\sqrt{n(n-1)})} \left| \frac{|\underline{y}(T')|^2 - 1}{\Psi(\underline{x}(T'))} \right|^{1/2} dT' . \tag{59}$$

The cosmological singularity is achieved as $x_o \propto \ln\sqrt{g} \to -\infty$ i.e. as $T \to +\infty$. If $\sqrt{g}$ has no local maximum then the long time behavior may be discussed considering (58) for $T \to -\infty$ whereas otherwise T is only an admissible time for $T \in [T_o,\infty[$.

If the original system can be written in Hamiltonian form then $\underline{\varphi}(\underline{x})$ vanishes and the last term in (58) is a logarithmic gradient. If, on the other hand, the Lie group is not unimodular, we still have from (50,51) the orthogonality between the extra force $\underline{\varphi}(\underline{x})$ and the "velocity" $\underline{y}$ with respect to the Eukliedian metric of IR^{n-1}:

$$(\underline{\varphi}(\underline{x}), \underline{y}) = 0 . \tag{60}$$

The last term in (58) may also be written as

$$\left(\frac{\underline{\nabla}\Psi + \underline{\varphi}}{\Psi}\right)_i = \sum_{j=1}^{n} A_{ij} \frac{\bar{R}^j}{R} . \tag{61}$$

This is obtained after multiplying (23) with A, $A_{ij}\ \partial R/\partial X_j = \partial R/\partial x_i = A_{ij}$ R^j $-2/g\ A_{ij}\ F^j$, using (48a), $gR = \exp(\gamma\, x_o)\ \Psi(\underline{x})$ and $g = \exp(x_o/\gamma)$:

$$\frac{e^{\gamma x_o}}{g}\frac{\partial\Psi}{\partial x_i} + \frac{e^{\gamma x_o}}{g}\varphi_i = A_{ij}\,\bar{R}^j \quad , \quad i = 1 \ldots n-1 \quad .$$

The system (58) also possesses a Ljapunov function to which we shall pay attention in the following

Proposition 5.3.
Let the function $L : (\underline{x},\underline{y}) \in IR^{2n-2} \to L(\underline{x},\underline{y}) \in IR$ be defined as

$$L(\underline{x},\underline{y}) := \frac{|\underline{y}|^2 - 1}{\Psi(\underline{x})} \tag{62}$$

and let (for $T>T_o$) $\Gamma(T) = (\underline{x}(T),\underline{y}(T))$ be a trajectory in the (n-1) $\times$ (n-1)-dimensional phase space of (58) for which $L(\Gamma(T_o)) \neq 0,\infty$. Then

$$L\,|_\Gamma \geq 0 \quad (\text{ and } \neq 0 \text{ for } T \neq \infty) , \tag{63a}$$
$$L^\circ|_\Gamma \leq 0 \quad (\text{ and } = 0 \text{ only for } \underline{y} = 0 \text{ or } T = \infty) \tag{63b}$$

(where $L|_\Gamma := L(\Gamma(T))$ and $L^\circ|_\Gamma$ denotes the orbital derivative of L along Γ), i.e. L is a Ljapunov function belonging to the semi flow induced by the vector field W on the phase space.

Proof: Multiplying the second equation of (58) by $\underline{y}$ we obtain together with $(\underline{\varphi},\underline{y})=0$ and $d\Psi/dT = (\underline{\nabla}\Psi,\underline{y})$ (where " $\circ$ " denotes the derivative with respect to T):

$$2\,\underline{y}\,\underline{y}^\circ = (\underline{y}^2)^\circ = (1-\underline{y}^2)\left(\gamma\underline{y}^2 - \frac{\Psi^\circ}{\Psi}\right)$$

or the integral equation

$$\frac{\underline{y}^2-1}{\Psi(\underline{x})}(T) = \frac{\underline{y}^2-1}{\Psi(\underline{x})}(T_o)\,\exp\left[-\gamma\int_{T_o}^{T}\underline{y}^2(T')\,dT'\right] , \tag{64}$$

where we have used $\mathrm{sig}(|\underline{y}|^2-1) = \mathrm{sig}\Psi(\underline{x})$ which follows for any physically relevant solution of (58) from the constraint (55,57). From (64) we can conclude that $L|_\Gamma$ is continuous, positive and that it does not vanish at finite times $T \geq T_o$. Differentiating (64) we also obtain (63b) ,

$$\frac{dL}{dT}\Big|_\Gamma = -\gamma(\underline{y}^2 L)|_\Gamma \leq 0 \tag{65}$$

where the "=" sign only holds for $L=0$ or $\underline{y} = 0$. Δ

Using equation (64) we can write the transformation (59) to the cosmological time in the form (with $t(\infty) = 0$)

$$t(T) \propto -\int^{T} \exp\left[-\frac{\gamma}{2}\int^{T'}\left(\underline{y}^2(T'') + \frac{1}{n-1}\right)dT''\right]dT' , \tag{66}$$

which is as an example helpful for the discussion of the generalized Kasner solution which we shall discuss in the following

<u>Proposition 5.4.</u>
Let $\sigma_i(t)$ be the Kasner "functions" defined as

$$\sigma_i(t) \;:=\; \frac{1}{2}\ln\left(\frac{g_i(t)\,/\,g_{io}}{t\,/\,t_o}\right) \tag{67}$$

and let the generalized Kasner solution be defined as usual , $g_i/g_{io} = t^{2\sigma_i}$ where the constants σ_i satisfy the two relations

$$\sum^{n} \sigma_i = \sum^{n} \sigma_i^2 = 1 \quad . \tag{68}$$

Then the functions (67) in the time coordinate T read

$$\sigma_i(T) = \left[\sum_j^{n-1} A_{ji}\, x_j(T) - \frac{T}{\sqrt{n(n-1)}}\right] /\; 2\ln t(T) \quad , \; i = 1 \ldots n \quad , \tag{69}$$

and any solution of (59) satisfying

$$\underline{v} = \underline{e} \quad , \quad \underline{e} = \text{const with} \quad |\,\underline{e}\,| = 1 \tag{70}$$

is a generalized Kasner solution (68).

Proof : Using the transformation $X = A^T x$ with $A_{ni} = 1/\sqrt{n}\ \forall i$ and $-T = x_o = \sqrt{(n-1)}\, x_n$ we obtain (69) from (67) :

$$g_i = \exp(X_i) = \exp\left[\sum_j^{n-1} A_{ji}\, x_j(T) - \frac{T}{\sqrt{n(n-1)}}\right] = \exp\left[\,2\sigma_i \ln t(T)\,\right] \; .$$

For $\Psi = 0$ we have $|\underline{v}| = 1$ from the integral equation (64) and thus $\underline{v} = \underline{e}$ with $|\underline{e}| = 1$ from (59). This is indeed the Kasner solution since (66) yields in that case $\ln(t/t_o) = -(T-T_o)/\,2\gamma$ and together with (59b) and $\underline{v} = \underline{e}$ we have $\underline{x} = \underline{x}_o + \underline{e}\,(T-T_o)$. Inserting these equations in (69) the time-dependence cancels out and the Kasner exponents become

$$\sigma_i = -\gamma\left[\sum_j^{n-1} A_{ji}\, e_j - \frac{1}{\sqrt{n(n-1)}}\right] = -\gamma\,(AE)_i \tag{71}$$

where $E \in \mathbb{R}^n$ is defined as $E_j := e_j$ for $j=1...(n-1)$ and $E_n := -1/\sqrt{(n-1)}$. The relations $A^TA = 1$, $\Sigma_j A_{ij} = \sqrt{n}\, \delta_{ij}$ and $E^2 = |\underline{e}|^2 + (n-1)^{-1} = 1/\gamma^2$ guarantee the Kasner conditions to be fulfilled :

$$\Sigma_i \sigma_i = -\gamma \Sigma_{ij} A_{ij} E_j = -\gamma \sqrt{n}\, \Sigma_j \delta_{nj} E_j = -\gamma \frac{-\sqrt{n}}{\sqrt{n-1}} = 1 ,$$

$$\Sigma_i \sigma_i^2 = \gamma^2 (AE, AE) = \gamma^2 (E, E) = \gamma^2 / \gamma^2 = 1 . \qquad \Delta$$

II.6. Conclusion

Introducing volume and anisotropy coordinates and rescaling the canonical momenta we have shown that the explicite time dependence of the modified Hamiltonian equations can be eliminated. Our system consists in the (2n-2) autonomeous first-order differential equations (D),

$$\frac{d\underline{x}}{dT} = \underline{y} \quad ; \quad \frac{d\underline{y}}{dT} = \frac{1}{2}(1 - |\underline{y}|^2)\left(\gamma \underline{y} - \frac{\nabla\Psi + \varphi}{\Psi}(\underline{x})\right) , \qquad (D)$$

for which we can also find a Ljapunov function $L(\underline{x},\underline{y})$ (62). Any solution of the diagonal field equations is obtained from a solution of (D) by the transformation (A) and the integration (I):

$$g_i(T) = \exp\left[\sum_j^{n-1} A_{ji}\, x_j(T)\right] \exp\left[-\frac{T}{\sqrt{n(n-1)}}\right] \qquad (A)$$

$$t(T) = -\int^T dT' \exp -\frac{\gamma}{2}\int^{T'} dT'' \left[|\underline{y}(T'')|^2 + \frac{1}{n-1}\right] . \qquad (I)$$

In the next Part we shall use these relations to discuss the behavior of the (n+1)-dimensional homogeneous models near the cosmological singularity.

III. Cosmological models with product topology

The field equations for cosmological vacuum models of the form $\mathrm{IR} \times R_c{}^d \times R_c{}^D$, where $R_c{}^d$ and $R_c{}^D$ are d- and D-dimensional Riemannian spaces of constant curvature can be reduced to a two-dimensional autonomous dynamical system. Discussing the phase portrait we show that all solutions exhibit a Kasner-like behavior near the cosmological singularity. For all decompositions $n = d + D$ $(d,D \neq 1)$, these models have d contracting and D expanding scales or vice versa as $t \to 0$.

III.1. Introduction

In Part II we have introduced coordinates which are adapted to the symmetries of the internal curvature and to the scaling properties of the Hamiltonian. The field equations for an (n+1)-dimensional homogeneous vacuum model were reduced to a system of (2n-2) autonomeous first-order differential equations. In order to obtain a better understanding of the dynamics described by this system, we shall in the following discuss the qualitative behavior of some highly symmetric spacetimes.

The cosmological models that we shall treat now have a product topology of the form $\mathrm{IR} \times R_c{}^d \times R_c{}^D$ where $R_c{}^d$ and $R_c{}^D$ are d- and D-dimensional Riemannian spaces of constant curvature, respectively. Most models studied in the literature are assumed to have a product topology [9-14]. The cases where, additionally, $R_c{}^d$ and $R_c{}^D$ are isotropic spaces are especially discussed in [9,10,13,14]. The qualitative behavior of the solutions near the cosmological singularity is found to be dependent on the assumptions on the energy momentum tensor. For the vacuum models $\mathrm{IR}^3 \times S^D$ and $S^3 \times S^D$ the solutions exhibit a line-like singularity with D expanding and d contracting scales as $t \to 0$ [13]. In our formulation the advantage of writing the field equations in terms of anisotropy and volume coordinates reflects in the fact that the phase space becomes two-dimensional.

In the second section we shall first reduce the general system to the two-dimensional case and then give an analytic solution for the case where either $R_c{}^d$ or $R_c{}^D$ is flat.

The different possibilities for the behavior of the solutions are analysed in the third section. The aim is to show that for all initial conditions (up to a set of vanishing measure) the trajectories approach a Kasner-like solution. Whenever the curvatures of the subspaces have the same signs, the two exact solutions (Kasner solutions) $\Gamma_+(T) = (x_0+T,1)$ and $\Gamma_-(T) = (x_0-T,-1)$ divide the phase plane into the connected component $S := \{ (x,y) \mid |y| \leq 1 \}$ and its disconnected complement in IR^2. The fact that every trajectory has to remain in S or $\mathrm{IR}^2 \setminus S$ at all times together with the existence of a Ljapunov function causes the solutions either to approach the boundary of S or to perform an infinite number of oscillations. We shall show that the logarithmic derivative of the curvature function $\Psi(x)$ determines which of these two possibilities is realized. Finally we shall extend our arguments to the case where $R_c{}^d$ and $R_c{}^D$ have different curvature and show that the oscillatory behavior can be excluded in all cases.

In the last section we shall extend the discussion to a toy model which still has a two-dimensional phase space but nevertheless is closely related to the general case where no symmetry restrictions on the Lie group G are made. We shall consider two cases and show that oscillatory solutions can not exist for $n > 3$ in the first case and $n > 9$ in the second case. These are exactly the critical spatial dimensions separating the chaotic and the regular regime in homogeneous and in inhomogeneous cosmological models [15-18,20]. Our toy model may thus serve as a help for an intuitive understanding of the dynamics of the full equations, which we shall discuss in Part IV.

III.2. The system and the phase plane

III.2.1. Reduction to a two-dimensional system

In Part II (chapter II.5.) we have introduced the 2(n-1) first-order differential equations :

$$\frac{d\underline{x}}{dT} = \underline{y} \quad ; \quad \frac{d\underline{y}}{dT} = \frac{1}{2}(1 - |\underline{y}|^2)\left(\gamma \underline{y} - \frac{\nabla\Psi + \underline{\varphi}}{\Psi}(\underline{x}) \right) . \tag{1}$$

Having solved these equations for the anisotropy vector $\underline{x} \in IR^{n-1}$ we obtain the solution of the field equations performing the transformation (2) and the integration (3):

$$g_i(T) = \exp\left[\sum_j^{n-1} A_{ji}\, x_j(T)\right] \exp\left[-\frac{T}{\sqrt{n(n-1)}}\right] \tag{2}$$

$$t(T) = -\int^{T} dT' \exp -\frac{\gamma}{2}\int^{T'} dT'' \left[|\underline{y}(T'')|^2 + \frac{1}{n-1}\right] \quad . \tag{3}$$

We have also mentioned that any solution satisfying $\underline{y} = \underline{e}$ with an arbitrary constant unit vector $\underline{e}$ is a generalized Kasner solution (Proposition II.5.4.).

Let us now discuss the system (1) for cosmological models with a product topology of the form $IR \times R_c{}^d \times R_c{}^D$ where $R_c{}^d$ and $R_c{}^D$ are d- and D-dimensional Riemannian spaces of constant curvature, respecti- vely. Considering the transformation (II.42) $x = AX$ and the scale factors $g_1 = \exp(X_1) = ... = g_d = \exp(X_d)$, $g_{d+1} = \exp(X_{d+1}) = ... = g_n = \exp(X_n)$ we see that the only non-vanishing components of x are x_d and x_n. The function $\exp(x_d)$ describes the evolution of the ratio of the two scale factors of $R_c{}^d$ and $R_c{}^D$ (II.33) whereas x_n is related to the new time coordinate T through $-T = x_o = \sqrt{(n-1)}\, x_n$. The vector $\underline{x} \in IR^{n-1}$ thus reduces to the the scalar x_d where from now on we shall suppress the index d and often use κ instead of x_d :

$$\kappa := e^{\omega x_d} \quad , \quad \omega := \sqrt{n/dD} \quad . \tag{4}$$

In general, the functions $r_d, r_D, r_{dD}, r_{Dd}, f_d$ and f_D defined in section II.3. depend only on $x_1 ... x_{d-1}, x_{d+1} ... x_{n-1}$. In our case where the only non-vanishing component of $\underline{x} \in IR^{n-1}$ is x_d they reduce to constants :

$$\begin{aligned} r_d &= \text{const} \quad , \quad r_D = \text{const} \\ r_{dD} &= r_{Dd} = 0 \quad , \quad f_d = f_D = 0 \quad . \end{aligned} \tag{5}$$

r_{dD} and r_{Dd} are vanishing since for product topologies no structure constants of the Lie group G with both sorts of indices $\mu,\nu \in 1...d$; $I,J \in d+1...n$

do exist (II.28c,d). We thus obtain from (5) and (II.47)

$$\Psi(x_d) = \kappa^{-\alpha} (r_d + \kappa r_D) \quad , \quad \varphi(x_d) = 0 \; . \tag{6}$$

The system (1) reduces to two autonomeous differential equations of first order for x and y

$$x^{\cdot} = y \;,\; y^{\cdot} = \frac{1}{2}(1-y^2)[\gamma y - l(x)] \;, \tag{7}$$

where we have introduced the logarithmic derivative $l(x)$ of $\Psi(x)$. The discussion of the solutions of the field equations for the scale factors $g^{(d)}(t)$ and $g^{(D)}(t)$ belonging to the isotropic subspaces ${R_c}^d$ and ${R_c}^D$ with the curvatures r_d and r_D is thus reduced to the treatment of the trajectories in a phase plane. Having found a solution $x(T)$ of (7) the two scale factors $g_1 = ... = g_d =: g^{(d)}(t)$ and $g_{d+1} = ... = g_n =: g^{(D)}(t)$ and the cosmological time t are obtained in parametric form (see chapter II.6.):

$$g^{(d)}(T) = \kappa^{D/n} \exp\left[-\frac{T}{\sqrt{n(n-1)}}\right] \tag{8a}$$

$$g^{(D)}(T) = \kappa^{-d/n} \exp\left[-\frac{T}{\sqrt{n(n-1)}}\right] \tag{8b}$$

$$t\,(T) = -\int^{T} dT' \exp -\frac{\gamma}{2}\int^{T'} dT'' \left[y^2(T'') + \frac{1}{n-1}\right] \quad , \tag{9}$$

where we have used the definition of A (II.42) and $\Sigma A_{ji}x_j = A_{di}x_d = D/n \ln(\kappa)$ (j=1...n-1) if $i \in 1...d$ ($\Sigma A_{ji}x_j = -d/n \ln(\kappa)$ if $i \in d+1...n$). We remind that T is only a well defined time coordinate either in the expanding ($T \in [T_0,\infty[$) or in the contracting regime of the n-dimensional volume.

The Kasner functions are defined as in the general case (II. 67,68). Now there exist two generalized Kasner solutions $y_d = e_d = \pm 1$. The corresponding Kasner exponents are obtained from $\sigma_i = -\gamma\,[\,A_{di}y_d - 1/\sqrt{(n(n-1))}]$ (II.71),

$$i \in 1...d \;:\; \sigma^{(d)}_{-,+} := \sigma_i = \frac{1}{n}\left[1 \;-,+\; \sqrt{\frac{D}{d}(n-1)}\right]$$

$$i \in d+1...n:\; \sigma^{(D)}_{+,-} := \sigma_i = \frac{1}{n}\left[1 \;+,-\; \sqrt{\frac{d}{D}(n-1)}\right] \tag{10}$$

and satisfy the conditions (II.68):

$$d\,\sigma_{-,+}^{(d)} + D\sigma_{+,-}^{(D)} = d\,[\sigma_{-,+}^{(d)}]^2 + D[\sigma_{+,-}^{(D)}]^2 = 1 \ .$$

If $d \neq 1$ and $D \neq 1$ there are always d positive and D negative (or D positive and d negative) exponents. Near the singularity any generalized Kasner solution of $IR \times R_c{}^d \times R_c{}^D$ thus consists either in d expanding and D contracting (y_d=-1) or in D expanding and d contracting (y_d=+1) scales. Since we shall show that any solution of (7) approaches either y_d=-1 or y_d=+1, this behavior is generic for all solutions of $IR \times R_c{}^d \times R_c{}^D$ (with d,D >1). Although solutions with n expanding dimensions can not exist in homogeneous vacuum models of the above type, they may occur in more realistic models with non-vanishing energy-momentum tensor $T_{\mu\nu}$ [9,13, 14]. To prove the above statement, we shall consider how the shape of $l(x)$ affects the phase portrait of (7). We shall especially distinguish the cases where $l(x)$ is bounded ($\Psi(x)$ has no zeros, i.e. $R_c{}^d$ and $R_c{}^D$ have both positive or negative curvature) and where $l(x)$ has a singularity (i.e. $R_c{}^d$ and $R_c{}^D$ have different curvature). As a starting point we consider the case where either $R_c{}^d$ or $R_c{}^D$ is flat and (7) can be solved analytically.

III.2.2. An analytic solution for $r_d = 0$

In order to become familiar with the plane dynamical system (7), we shall first treat the case where either $R_c{}^d$ or $R_c{}^D$ is flat, i.e. r_d = 0 or r_D = 0. As $T \to \infty$ ($t \to 0$) the solutions exhibit the expected Kasner behavior $y := y_d \to \pm 1$ which we shall also prove in the next section for the remaining cases where none of the curvatures vanish.

Let us now solve (7) for $r_d = 0$ and $r_D \neq 0$. The logarithmic derivative $l(x)$ then reduces to a constant l_o. Together with (6) and $\kappa = \exp(\omega x)$ we have

$$l(x) = \frac{[r_D\, \kappa^{1-\alpha}]_{,x}}{r_D\, \kappa^{1-\alpha}} = \omega\,(1-\alpha) =: l_o > 0 \quad , \tag{11}$$

where α = D/n and $\omega = (n/Dd)^{1/2}$ (II.32).

Proposition 2.1.

Let $\Gamma(T) = (x(T),y(T))$ be a solution of

$$x^{\cdot} = y \qquad y^{\cdot} = \frac{1}{2}(1-y^2)[\gamma y - l_o] \tag{12}$$

with $y(T_o) \neq l_o/\gamma = (d / D(n-1))^{1/2}$, and let S be the strip

$$S := \{ (x,y) \mid |y| \leq 1 \} \tag{13}$$

in the phase plane IR^2. Then

i) $\Gamma(T_o) \in (\notin) S \Rightarrow \Gamma(T) \in (\notin) S \quad \forall T$,

ii) $y(T) \to \pm 1$ as $T \to \infty$,

iii) $y(T) \to l_o/\gamma$ as $T \to -\infty$, for $\Gamma(T_o) \in S$,

iv) $y(T) \to \pm\infty$ as $T \to T^* > -\infty$, for $\Gamma(T_o) \notin S$.

Proof: i) Since l_o is finite, no solution of (12) can intersect one of the special solutions $\Gamma_{\pm}(T) := (x_o \pm T , \pm 1)$ for finite T.

ii) The second equation of (12) can be integrated with respect to T and yields

$$\left(\frac{y-l_o/\gamma}{y_o-l_o/\gamma}\right)^2 \left(\frac{y_o-1}{y-1}\right)^{1+l_o/\gamma} \left(\frac{y_o+1}{y+1}\right)^{1-l_o/\gamma} = \exp[(1-(l_o/\gamma)^2)\ \gamma(T-T_o)] \tag{14a}$$

for $D\neq1$, i.e. $1-l_o/\gamma > 0$, whereas for $D=1$, i.e. $1-l_o/\gamma = 0$ we obtain

$$\exp\left[\frac{2}{y-1} - \frac{2}{y_o-1}\right] \left(\frac{y-1}{y_o-1}\right)\left(\frac{y_o+1}{y+1}\right) = \exp[2\gamma(T-T_o)] \ . \tag{14b}$$

The right hand sides of (14) tend to infinity as $T\to\infty$ and so must the left hand sides, which in both cases is only possible if $y\to+1$ or $y\to-1$.

iii) For $\Gamma(T_o) \in S$ we have $|y(T)| \leq 1 \ \forall T$ and we obtain $y \to l_o/\gamma$ as $T \to -\infty$ in (14a) and $y \to 1$ ($= l_o/\gamma$, $D=1$) in (14b).

iv) If $\Gamma(T_o) \notin S$ then $|y(T)| \geq 1 \ \forall T$ and the minimum of the left hand sides in (14) is easily seen to be strictly positive and to be attained only if $y = +\infty$ or $y = -\infty$. Since the r.h.s. in (14) are monotonically decreasing to zero as $T \to -\infty$, there exists a finite time $T^* < T_o$ with $y(T^*) = +\infty$ if

$y(T_o) > 1$ and $y(T^*) = -\infty$ if $y(T_o) < -1$. The qualitative behavior of the solutions is shown in fig.2.1. Δ

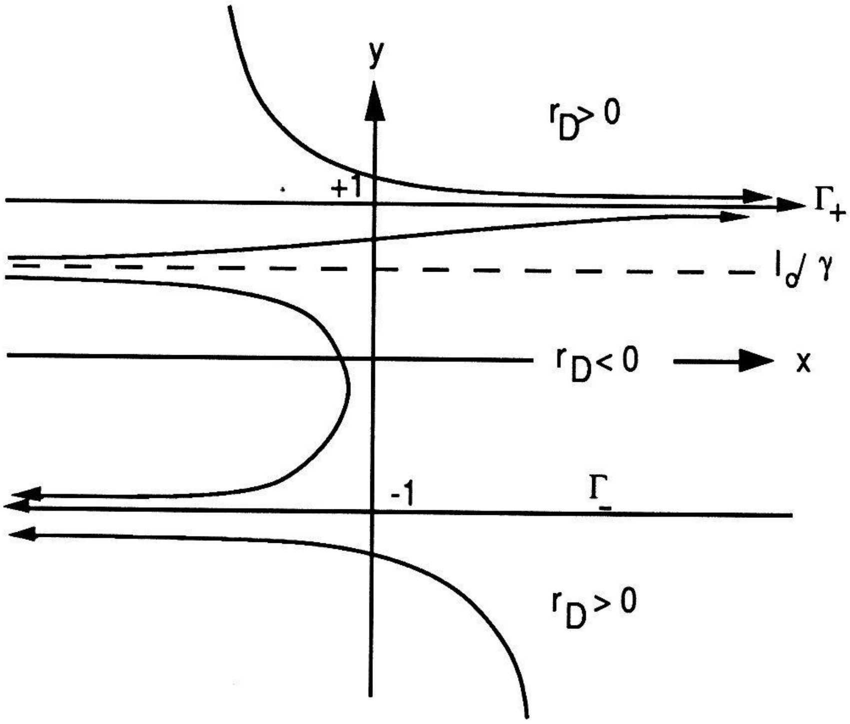

fig.2.1.

Qualitative behavior of the solutions of (12). As T increases, all trajectories either tend to y = -1 or y = 1. As T decreases, the solutions inside the strip $|y| < 1$ approach the dashed line whereas the trajectories lying outside the strip reach $|y| = \infty$ after a finite time interval.

The physically relevant solutions are those with $sig(y^2-1) = sig(\Psi(x)) = sig(r_D)$ (Proposition II.5.3.) and we thus obtain the following

<u>Corollary 2.2.</u>
Let $R_c{}^d$ be flat and $R_c{}^D$ have non-vanishing curvature. Then
i) every solution approaches one of the two possible generalized Kasner solutions $y = \pm 1$ as $T \to \infty$ (i.e. $t \to 0$):

$$g^{(d)} \to t^{2\sigma^{(d)}_{-,+}} \quad , \quad g^{(D)} \to t^{2\sigma^{(D)}_{+,-}} \quad , \qquad (15)$$

ii) if $R_c{}^D$ has negative curvature, the scale of the flat space tends to a constant whereas the scale of $R_c{}^D$ diverges as $T \to -\infty$ (i.e. $t \to \infty$),
iii) if the curvature of $R_c{}^D$ is positive, the n-dimensional volume reaches its maximal extension after a finite time $t^*>t_o$ (i.e. $T^*<T_o$).

Proof: The first and the second statement follow from Proposition 2.1.ii) and from Proposition 2.1.iii) and (8), respectively, since $g^{(d)}(T) \rightarrow \exp\{T[D\omega l_o/\gamma n - 1/\sqrt{n(n-1)}]\} = \text{const}$ and $g^{(D)}(T) \rightarrow \exp\{T[-d\omega l_o/\gamma n - 1/\sqrt{n(n-1)}]\} = \exp\{-T/\gamma D\}$. The last statement is due to Proposition 2.1.iv). Δ

III.3. The properties of the two-dimensional system

Let us now analyse the two-dimensional dynamical system (7) for a class of functions $l(x)$ as general as possible. We shall first assume that $l(x)$ is bounded for finite values of $|x|$. This holds true whenever the curvatures of the two subspaces have the same sign (see(6)).

Proposition 3.1.

Let $\Gamma(T)$ be a solution of (D) and let $l(x) : x \in \mathbb{R} \rightarrow l(x) \in \mathbb{R}$ be a differentiable function satisfying the conditions

$$|l(x)| < \infty \quad \forall x \ , \qquad \lim_{|x|\rightarrow\infty} l(x) \neq 0$$
$$l(x) > (<,=)\ 0 \text{ for } x > (<,=)\ x_o \ , \quad l_x(x_o) \neq 0 \ . \tag{16}$$

Then

i) $\Gamma(T_o) \in (\notin)\ S \ \Rightarrow \ \Gamma(T) \in (\notin)\ S \quad \forall\ T$,

ii) $P_o := (x_o,0)$ is the only critical point of (7) and P_o is a stable node or a spiral point as $T \rightarrow -\infty$,

iii) $L(x,y)$ is a Ljapunov function of the system :

$$L(x,y) := (y^2-1) \exp\left[-\int_{x_o}^{x} l(x')\,dx' \right] \qquad , \tag{17}$$

iv) P_o is an asymptotically stable solution of (D) as $T \rightarrow -\infty$,

v) (7) has no periodic solutions, no limit cycles and no bounded solutions.

Proof: i) Since $l(x)$ is bounded, no solution of (12) can intersect one of the special solutions $\Gamma_{\pm}(T)$ of (D) for finite T.

ii) The function $l(x)$ has exactly one zero at $x = x_o$. P_o is thus the only critical point and the linearized system has the eigenvalues

$$\lambda_{1,2} = \frac{\gamma}{4}\left[1 \pm \sqrt{1 - \frac{8}{\gamma^2} l_x(x_o)}\right] .$$

Since the derivative of $l(x)$ at x_o is strictly positive, the eigenvalues are unequal and $Re(\lambda_{1,2}) > 0$. Thus P_o is either a stable node ($0 < l_x(x_o) \leq \gamma^2/8$) or a stable focus ($l_x(x_o) > \gamma^2/8 > 0$).

iii) No solution of (D) (except $\Gamma_\pm(T)$) can have $|y(T)| = 1$ at a finite time T. Since $l(x)$ is bounded the integral $\int l(x')dx'$ is finite for finite x-intervals. For a solution $\Gamma(T) \in S$ we have $L|_\Gamma < 0$ (if $T \neq \infty$) and the orbital derivative of L along Γ is positive semidefinite , $(dL/dT)|_\Gamma = -\gamma y^2 L|_\Gamma \geq 0$ and vanishes only if $y = 0$ or $T = \infty$. The functions L and dL/dT are satisfying definiteness conditions of opposite signs and L is thus a Ljapunov function of the two-dimensional system (see also Proposition II.5.3.).

iv) The asymptotic stability theorem [42] states that if there exists a positive definite function $L(x)$ which has an infinitesimal upper bound and if dL/dT is negative then $x(T) = x_o$ is an asymptotically stable solution. In our case P_o is thus an asymptotically stable solution as $T \to -\infty$.

v) Since any trajectory in $\mathbb{R}^2 \setminus S$ has index zero (there are no critical points in $\mathbb{R}^2 \setminus S$) there exist no periodic solutions and no limit cycles in $\mathbb{R}^2 \setminus S$. Let us now consider the trajectories in S: At any finite time T we have $L|_{\Gamma(T)} < 0$ and the function $(dL/dT)|_{\Gamma(T)}$ is strictly positive on every trajectory with $\Gamma(T_o) \neq P_o$ (up to the set of vanishing measure on the T-axis where $\Gamma(T)$ intersects the x-axis and $(dL/dT)|_{\Gamma(T)}$ vanishes). There are thus no bounded solutions as $T \to \infty$ except of the instable one $\Gamma(T) = P_o$. We can also introduce the level set $\gamma(L)$ of L,

$$\gamma(L_o) := \{ (x,y) \in S \mid L(x,y) = L_o \} \tag{18}$$

and use a similar argument as in the Poincaré-Bendixon theorem : Every solution has to intersect each of the closed level curves $\gamma(L)$ with $L \geq L(\Gamma(T_o))$ exactly once. The statement is now a consequence of $\gamma(L_o + \Delta L) \supset \gamma(L_o) \supset \dots \supset P_o = \gamma(-\infty)$ for $\Delta L > 0$. Δ

Next we shall show that the solutions of (7) in S either wind infinitely often around the critical point P_o or approach asymptotically one of the Kasner solutions $\Gamma_{\pm}(T)$ as $T \rightarrow \infty$. The behavior of $l(x)$ for large values of $|x|$ determines which of the two cases is realised.

Proposition 3.2.

Let $\gamma(\Phi)$ denote the set of the orthogonal trajectories to $\gamma(L)$ (18), Q_i ($i=1...4$) the i'th quadrant of the phase plane with respect to P_o and $n(\Gamma,T_o)$ the winding number of Γ with respect to P_o in the interval $[T_o,\infty[$:

$$n(\Gamma,T_o) := -\frac{1}{2\pi}\int_{T_o}^{\infty} \varphi^{\cdot}(T)\,dT \tag{19}$$

where $\mathrm{tg}(\varphi) := y/(x-x_o)$. Let the assumptions on $l(x)$ be the same as in Proposition 3.1. (16) and let $\Gamma(T) \in S\ \forall\ T$ be a solution of (7). Then

i) $\Gamma(T)$ intersects the trajectories $\gamma(\Phi)$ in the same order as $c_\rho(T) := P_o - \rho(\cos T, \sin T)$ ($\rho<1$) ,

ii) $\Gamma(T)$ passes the Q_i in cyclic order,

iii) either there exists a T' with $\Gamma(T) \in Q_i\ \forall\ T>T'$ or $n(\Gamma,T_o) = \infty\ \forall\ T_o<\infty$.

Proof: The orthogonal field to $\gamma(L)$ points in the direction of $(l(x)^{-1}, (y^2-1)/2y) =: \underline{\nabla}\Phi$ since $\underline{\nabla} \wedge (l(x)^{-1}, (y^2-1)/2y) = 0$ and $(\underline{\nabla}\Phi, \underline{\nabla}L) = 0$. The variation of Φ along the circle $c_\rho(T)$ is thus

$$d\Phi\big|_{c(T)} = (\underline{c}^{\cdot}, \underline{\nabla}\Phi)\,dT = \frac{1-y^2}{2y\,l(x)}\left[\, l(x)\,(x-x_o) + \frac{2y^2}{1-y^2} \,\right] dT \ .$$

Since $|y| \le 1$ and $l(x)\,(x-x_o) \ge 0$ by the assumptions on $l(x)$, we have

$$\mathrm{sig}[\,d\Phi\big|_{c(T)}] = \mathrm{sig}\,[\,y\,(x-x_o)\,] = -(-1)^i \quad \text{in } Q_i \ .$$

On the other hand, the variation of Φ along a solution Γ is obtained by using (7):

$$d\Phi\big|_{\Gamma(T)} = (\underline{\Gamma}^{\cdot}, \underline{\nabla}\Phi)\,dT = \frac{1}{y\,l(x)}\,[\,y^2 + \varepsilon^2\, l(x)\,\{\, l(x) - \gamma y \,\}\,]\,dT \ ,$$

with $\varepsilon := (1-y^2)/2 \leq 1$. The expression in brackets is positive for all $(x,y) \in S \backslash P_o$. This is clear for $l(x)-\gamma y \geq 0$ and $l(x) \geq 0$, whilst for $l(x) \leq 0$ we have $[...] \geq y^2(1-\varepsilon^2 y^2) + \varepsilon^2 \gamma y\, l(x) > 0$. If $l(x)-\gamma y \leq 0$, we can write the bracket in the form $\varepsilon^2 l(x)^2 + y(y - l(x)\, \gamma\varepsilon^2) \geq \varepsilon^2 l(x)^2 + y^2(1 - \gamma^2\varepsilon^2) > 0$, since $\gamma,\varepsilon < 1$ and $|y| \leq 1$. We thus obtain again

$$\mathrm{sig}[\, d\Phi\,|_{\Gamma(T)}] = \mathrm{sig}\,[\, y\, l(x)\,] = -(-1)^i \quad \text{in } Q_i \quad .$$

ii) Since every $\gamma(\Phi)$ contains the critical point P_o, $c_p(T)$ intersects all $\gamma(\Phi)$ whilst passing the quadrants in cyclic order and so does $\Gamma(T)$ by i).
iii) Except of P_o there exist no critical points in Q_2. Using the first equa- tion of (7), we have for $\Gamma \in Q_2$: $dx/dT = y < 0$ which can not hold for all times without Γ crossing the y-axis and entering Q_3. The analogue behavior holds in Q_4 and thus every trajectory that does not wind infinitely often around P_o stays either in Q_1 or Q_3 for $\forall\ T > T'$. Δ

Next we shall consider the question of how the behavior of $l(x)$ for $|x| \to \infty$ determines which of the two possibilities $n(\Gamma,T_o) = \infty$ or $\Gamma(T) \in Q_{1,3}\ \forall\ T > T_o$ is realised. If the graph of $l(x)/\gamma$ is located outside the strip S for large values of $|x|$ then the winding number of Γ around P_o turns out to be infinite (later we shall see that this can not happen in the cosmological models, where $l(x)$ is the logarithmic derivative of $\Psi(x)$).

<u>Proposition 3.3.</u>
Let the assumptions on $l(x)$ be the same as in Proposition 3.1.(16) and let there exist an $R \in \mathrm{IR}$ such that

$$|l(x)| \geq \gamma \quad \text{for } \forall\ x : |x-x_o| > R > 0 \quad . \tag{20}$$

Then every solution $\Gamma(T) \in S$ has infinite winding number $n(\Gamma,T_o)\ \forall\ T_o$ (fig.3.1.).

Proof: Using Proposition 3.2. we only have to show that no solution can stay in Q_1 or Q_3 as $T \to \infty$. Let $\Gamma(T) \in Q_1\ \forall\ T > T'$. Since there is no critical point in Q_1 and since $dx/dT = y > 0$, there exists a time T'' such that $|x(T) - x_o| > R$, $\forall T > T''$. From the assumption on $l(x)$ we obtain for the second equation of (D) : $dy/dT < (1-y^2)(\,\gamma y - l(x)\,)/2 \leq 0$

$\forall\, T > T''$ and Γ intersects the x-axis after a finite time. The same argument also shows that $\Gamma(T) \notin Q_3 \;\forall\, T > T'$. Δ

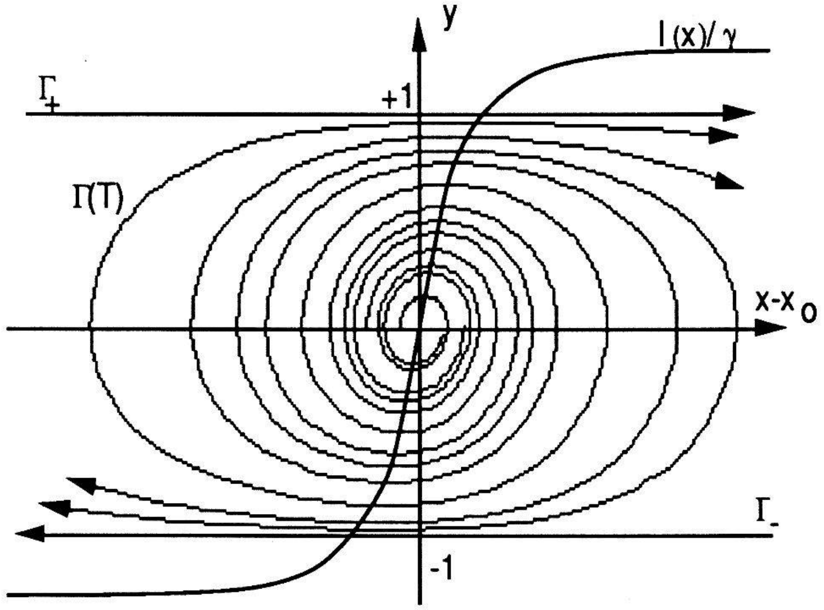

fig.3.1.

The solutions for a function $l(x)$ which is located outside the strip S for large $|x|$ have infinite winding numbers. The arrows point in the direction of increasing T and the cosmological singularity is at $T = \infty$.

Let us now discuss the case where $l(x)/\gamma$ has a horizontal asymptotic which is located inside S. We shall show that every trajectory enters the region between this asymptotic and the boundary of S, stays in this region and approaches either Γ_+ or Γ_- as $T \to \infty$.

<u>Proposition 3.4.</u>

Let the assumptions on $l(x)$ be the same as in Proposition 3.1.(16) and let there exist two positive, finite constants $R_+ , R_- \in \mathbb{R}$ such that

$$l(x) \le -\gamma \quad \text{for } \forall\, x : \; x < x_0 - R_- ,$$
$$l(x) < \gamma \quad \text{for } \forall\, x : \; x > x_0 + R_+ .$$

Let us define $l_+ := \lim l(x)$ as $x \to \infty$ and let S_+ be the semi strip (fig.3.2.)

$$S_+ := \{ (x,y) \in S \mid x > x_0 + R_+, \; \gamma y \ge (l_+ + \gamma)/2 \} . \tag{21}$$

Then every solution $\Gamma(T) \in S$ fulfils :

i) $\Gamma(T') \in S_+ \Rightarrow \Gamma(T) \in S_+ \;\; \forall\, T > T'$,

ii) $\exists\, T' : \Gamma(T') \in S_+$,
iii) $\Gamma(T) \to \Gamma_+(T)$ as $T \to \infty$.

Proof: Since $l(x) \to l_+ < \gamma$ as $x \to \infty$ we can choose R_+ such that $l(x) < (l_+ + \gamma)/2$ for $x > x_o + R_+$. Let us define the set with boundary $\gamma(L_o)$ (18) (fig.3.2.) :

$$S_{Lo} := \{ (x,y) \in S \mid L(x,y) \leq L_o \} .$$

i) For $\Gamma(T) \in S_+$ we have the inequalities

$$x^{\cdot} = y > 0 \quad , \quad y^{\cdot} \geq \frac{1}{2}(1-y^2)\left[\frac{1}{2}(l_+ + \gamma) - l(x)\right] \geq 0 \quad ,$$

which obviously guarantee $\Gamma(T) \in S_+$ at all later times.
ii) Let L_o be such that $S_{Lo} \cap S_+ \neq \{\ \}$. Since $L|_{\Gamma(T)}$ is increasing on every solution $\Gamma(T) \neq P_o$ there exists a time T_c such that $\Gamma(T)$ remains outside S_{Lo} for all times $T > T_c$. Thus $\Gamma(T)$ can not wind infinitely often around P_o without entering S_+ where it has to stay by i), i.e. $n(\Gamma) \neq \infty$. From Proposition 3.2. we thus see that either $\Gamma(T) \in Q_1$ or $\Gamma(T) \in Q_3$ as $T \to \infty$. Using the same arguments as in Proposition 3.3. it is easy to show that $\Gamma(T)$ can neither stay in Q_3 nor in $Q_1 \backslash S_+$ and thus $\Gamma(T) \in S_+$ as $T \to \infty$.
iii) Let $\Gamma(T) \in S_+\ \forall\ T > T_o$. Then we have $\gamma y - l(x) \geq (l_+ + \gamma)/2 - l(x) > 0$ and thus we obtain the inequality

$$\frac{2\,dy}{(1-y^2)\,[\,\gamma y - (l_+ + \gamma)/2\,]} \geq dT \quad ,$$

which after integration yields

$$\left[\frac{2\,\gamma y - (\gamma + l_+)}{2\,\gamma y_o - (\gamma + l_+)}\right]^{4\gamma} \left(\frac{1-y_o}{1-y}\right)^{3\gamma + l_+} \left(\frac{1+y_o}{1+y}\right)^{\gamma - l_+} \geq \exp\left[\frac{1}{2}(\gamma - l_+)(3\gamma + l_+)(T - T_o)\right] .$$

Since $(\gamma - l_+)(3\gamma + l_+)$ is strictly positive, all solutions $\Gamma(T) \in S_+$ fulfil $y \to +1$ as $T \to \infty$. Δ

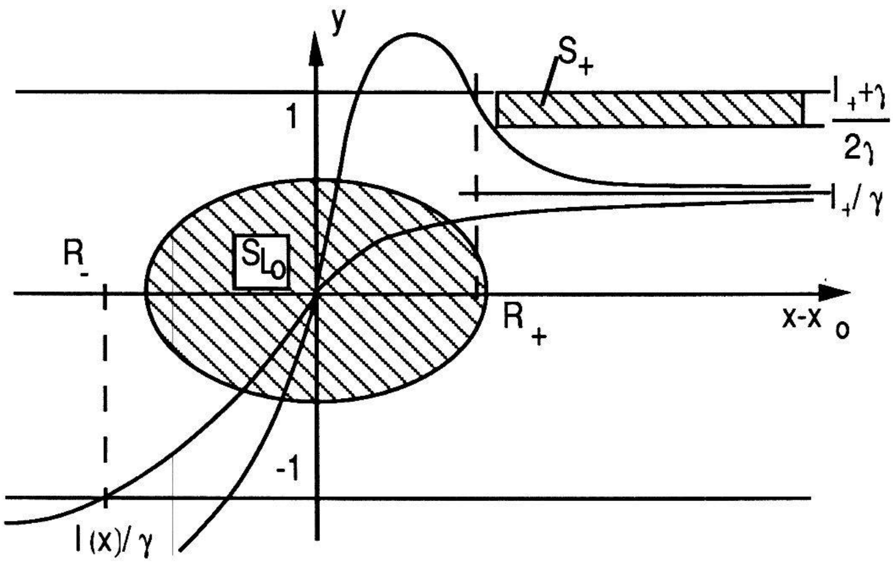

fig.3.2.

The region S_{Lo} and the strip S_+. S_+ is not empty if the function $l(x)$ has a horizontal asymptotic inside the strip S. Any trajectory then enters the dashed semi strip after a finite time interval.

Until now we have considered solutions of (7) for bounded functions $l(x)$. Using $l(x) = \Psi_{,x}/\Psi$ and (6) this corresponds to the situation where Ψ has no zeroes, i.e. r_d and R_D have the same sign. Since $\mathrm{sig}\Psi = \mathrm{sig}(y^2-1)$ the solutions inside the strip S are those with $r_d < 0$ and $r_D < 0$. We thus have the following

<u>Corollary 3.5.</u>

Let $R_c{}^d$ and $R_c{}^D$ have negative curvature and let $n = d+D \geq 3$. Then

i) as $T \to \infty$ (i.e. $t \to 0$), every solution of (7) approaches one of the two generalized Kasner solutions (see 10,15):

$$g^{(d)} \to t^{2\sigma_{-,+}^{(d)}} \quad , \; g^{(D)} \to t^{2\sigma_{+,-}^{(D)}} \quad . \tag{22}$$

ii) as $T \to -\infty$ (i.e. $t \to \infty$), the ratio of the scale factors approaches the constant value

$$\frac{g^{(d)}}{g^{(D)}} \to \frac{D\, r_d}{d\, r_D} \quad .$$

Proof: i) From (6) we obtain the expression

$$l(x) = \frac{\Psi_{,x}}{\Psi} = \omega\left[\frac{e^{\omega x}}{r_d/r_D + e^{\omega x}} - \alpha\right] \tag{23}$$

and $l(x)$ fulfils the assumptions (16) of Proposition 3.1. Additionally, $l(x)$ has the horizontal asymptotics

$$\lim_{x \to +\infty} l(x) = \omega(1-\alpha) , \quad \lim_{x \to -\infty} l(x) = -\omega\alpha . \tag{24}$$

Since we have

$$\begin{aligned} -\omega\alpha &= -\sqrt{D/nd} > -\sqrt{(n-1)/n} = -\gamma , \quad d \neq 1 , \\ \omega(1-\alpha) &= \sqrt{d/nD} < \sqrt{(n-1)/n} = \gamma , \quad D \neq 1 , \end{aligned} \tag{25}$$

there exists for all choices of d and D with $d + D = n \geq 3$ at least one non-empty set S_+ or S_- (S_- is the analogue to S_+ in Q_3 (21)). If neither $d=1$ nor $D=1$ then even both sets are nonempty. Using Proposition 3.4 we know that every solution has to approach either Γ_+ or Γ_- as $T \to \infty$ which are the Kasner solutions with Kasner indices $\sigma^{(d)}{}_-$ and $\sigma^{(D)}{}_+$ or $\sigma^{(d)}{}_+$ and $\sigma^{(D)}{}_-$, respectively.

ii) As $T \to -\infty$, every solution approaches the critical point P_o (see Proposition 3.1.). Since x_o is the zero of $\Psi_{,x}$, we obtain $\kappa_o = \alpha\, r_d / [(1-\alpha)\, r_D] = D r_d / d\, r_D$. Δ

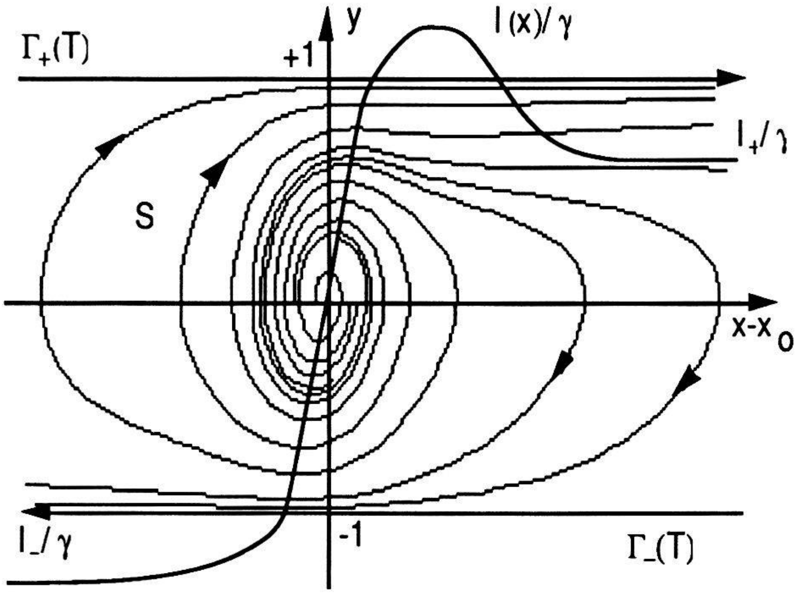

fig.3.3.

If the function $l(x)$ has a horizontal asymptotic in the strip S then all solutions tend towards the Kasner solution $(x,y) = (T,1)$ as $T \to \infty$.

Fig.3.3. shows the behavior of the solutions in the case where the

asymptotic l_-/γ is located outside S whilst l_+/γ is inside. Thus all $\Gamma(T) \to \Gamma_+(T)$ as $T \to \infty$ $(t \to 0)$ and $\Gamma(T) \to P_o$ as $T \to -\infty$ $(t \to \infty)$.

Next we shall discuss the solutions of (7) which remain outside the strip S for all times. These correspond to the situation where both subspaces have positive curvature. Since this part of the phase plane is unbounded and contains no critical points the situation is simpler than before.

Proposition 3.6.

Let the assumptions on $l(x)$ be the same as in Proposition 3.1.(16) and let $|l(x)| \leq \gamma$. Then every solution $\Gamma(T) \in \mathrm{IR}^2 \setminus S$ of (7) has the following properties:

i) $y(T)$ is monotonically decreasing (increasing) in the region $y \geq 1$ $(y \leq 1)$,

ii) $y(T) \to +1$ (-1) for $y(T_o) \geq 1$ $(y(T_o) \leq -1)$,

iii) $\exists\, T^* < T_o :\ |y(T^*)| = \infty$.

Proof: i) We consider the case $y \geq 1$. The second equation of (7) yields the inequality

$$dy/dT = {}^1/_2(1-y^2)\,(\gamma y - l(x)) \leq {}^\gamma/_2\,(1-y^2)\,(y-1) \leq 0$$

which holds along any trajectory in the region $y \geq 1$ and thus $dy/dT \leq 0$.

ii) Integrating the second inequality above we obtain

$$\exp\left[\frac{2}{y-1} - \frac{2}{y_o-1}\right]\left(\frac{y-1}{y_o-1}\right)\left(\frac{y_o+1}{y+1}\right) \geq \exp[\,2\gamma\,(T-T_o)\,] \quad .$$

We thus have $y \to 1_+$ as $T \to \infty$.

iii) Let us define the time $T' := -T$ and integrate the inequality with respect to T':

$$\exp\left[\frac{2}{y-1} - \frac{2}{y_o-1}\right]\left(\frac{y-1}{y_o-1}\right)\left(\frac{y_o+1}{y+1}\right) \leq \exp[\,2\gamma\,(T_o'-T')\,] \quad .$$

Since the left hand side is always greater or equal to the positive constant $(y_o+1)/(y_o-1)\,\exp[-2/(y_o-1)] = \exp(2\gamma T'_o)$ (and equal only if $y = \infty$), the inequality becomes wrong as soon as $\exp(-2\gamma T') < 1 =: \exp(-2\gamma T'^*)$

and we thus have $y(T'^{*}) = \infty$ at a finite time $T'^{*} > T'_o$ (i.e. $T^{*} < T_o$). Δ

Using this Proposition we obtain the following

Corollary 3.7.
Let $R_c{}^d$ and $R_c{}^D$ have positive curvature. Then
i) as $T \to \infty$ (i.e. $t \to 0$), every solution of (7) approaches one of the two generalized Kasner solutions (22),
ii) The n-dimensional volume reaches its maximum after a finite time t^{*}.

Proof: The function $l(x)$ has the same properties as in Corollary 3.5., but r_d and r_D now are positive constants. Since $l(x)$ is monoto- nically increasing, the horizontal asymptotics (24) are the lower and upper bounds of $l(x)$. Using (25) the assumption $|l(x)| \leq \gamma$ in Proposition 3.6. is fulfilled and the corollary immediately follows from Proposition 3.6.ii) and iii). Δ

Let us finally consider the case where the curvatures of $R_c{}^d$ and $R_c{}^D$ have different signs. The function $\Psi(x)$ then has a zero and its logarithmic derivative $l(x)$ is no longer bounded.

Proposition 3.8.
Let $l(x) : \mathrm{IR} \setminus \{x_p\} \to \mathrm{IR}$ be a differentiable function with

$$\lim_{x\to\pm\infty} l(x) = l_{\pm} \quad , \quad |l_{\pm}| \leq \gamma \quad ; \quad \lim_{x\to x_p^{\pm}} l(x) = \pm\infty \quad .$$

Then every solution $\Gamma(T)$ of (7) satisfies

i) $\Gamma(T) \to \Gamma_{+}(T)$ or $\Gamma_{-}(T)$ as $T \to \infty$,
ii) $\exists\, T^{*} < T_o \ : \ y(T) \to -\infty$ as $T \to T^{*}_{-}$.

Proof: We shall not go into the details of the proof since these are similar to the arguments used in the preceding Propositions. The only difference consists in the zero of $\Psi(x)$. Multiplying the second equation in (7) with y and using $y l(x) = (d\Psi/dT)/\Psi$ we obtain the equation in the integrated form

$$y^2(T) \;=\; 1 + \frac{y_o^2 - 1}{\Psi_o}\;\Psi(x)\,\exp\left[-\gamma\int_{T_o}^{T} y^2(T')\,dT'\right] \quad ,$$

where $(y_o{}^2-1)/\Psi_o$ has to be positive (Proposition II.5.3.). Whenever $\Psi(x(T))$ vanishes for finite times, $y^2(T)$ becomes equal to one and the trajectory intersects the boundary of the strip S.
Going backwards in T-time it is easy to see that all solutions intersect $y=-1$ and enter the region $Q_2{}^+ := \{\,(x,y) \in Q_2 \mid (x,y) \notin S\,\}$. In $Q_2{}^+$ the trajectories approach $y = -\infty$ after a finite time T^* which can be shown as in Proposition 3.6.
As $T \to \infty$, i.e. as the scales approach the cosmological singularity, the solutions can behave in two different ways: If there exists a strip $S_- \in Q_3$ (i.e. if the asymptotic l_-/γ is located in side S) and if the trajectory enters S_- then it stays there and approaches the Kasner solution $\Gamma_-(T)$. If on the other hand, the solution does not enter S_- it has to leave Q_3, to enter Q_4 and to intersect $y = +1$ at the point P_+. Subsequently $\Gamma(T)$ remains in $Q_1{}^+ := \{\,(x,y) \in Q_1 \mid (x,y) \notin S\,\}$ and approaches the other Kasner solution $\Gamma_+(T)$ which is easily seen using $l_+/\gamma < 1$. Δ

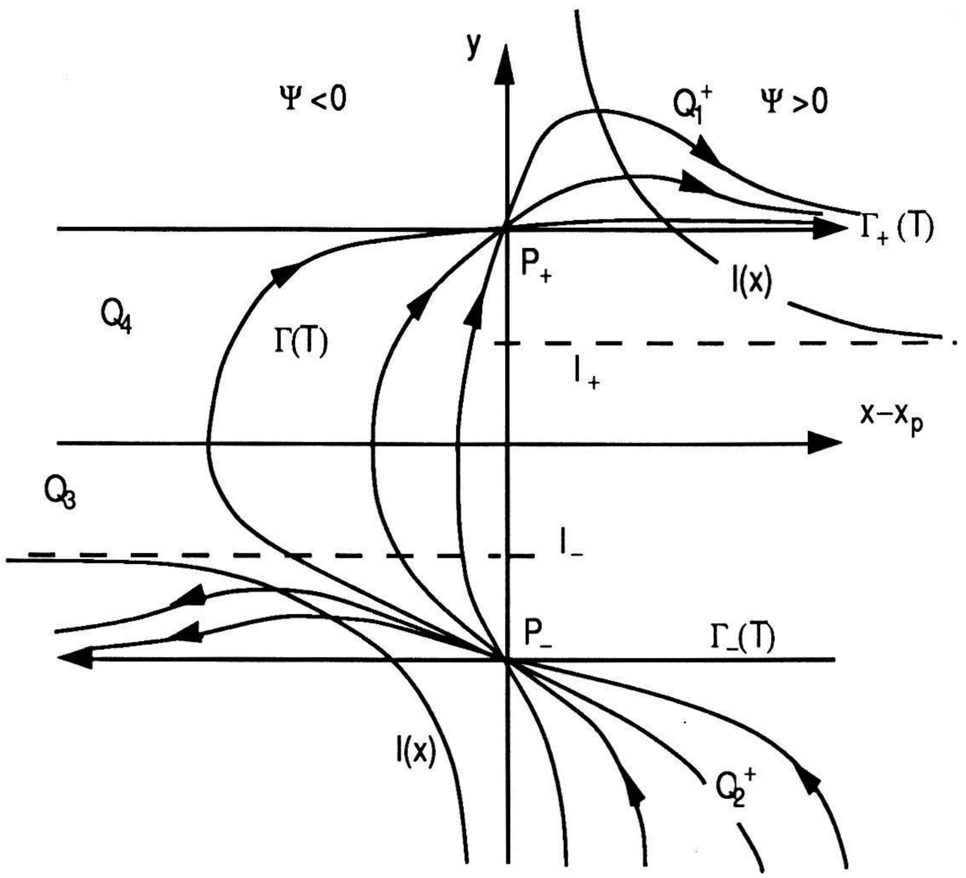

fig.3.4.

If the function l(x) has a pole then the solutions intersect the boundary of the strip S. Since the asymptotics of l(x) are located inside the strip S, all solutions either tend towards (x(T),y(T)) = (T,1) or (-T,-1).

The behavior of the solutions is illustrated in fig.3.4. According to the condition $\text{sig}\Psi(x) = \text{sig}(y^2-1)$ the trajectories are located inside thestrip S in the region where $\Psi(x)$ is negative whereas they have to stay outside S if $\Psi(x)$ is positive.

Corrolary 3.9.

Let $R_c{}^d$ and $R_c{}^D$ have different non-vanishing curvature. Then

i) as $T \to \infty$ (i.e. $t \to 0$), every solution of (7) approaches one of the two generalized Kasner solutions (22),

ii) the average scale factor (or equivalently the n-dimensional volume) reaches its maximum after a finite time t^*.

Proof: Let $r_d < 0$ and $r_D > 0$. Then the function $l(x)$ is given by (23) where r_d/r_D now is a negative constant. Together with $l_+ = \omega(1-\alpha) \leq \gamma$, $l_- = -\omega\alpha \geq -\gamma$ and $x_p = 1/\omega \ln(-r_d/r_D)$, we have $l(x) \to l_\pm$ as $x \to \pm\infty$, $|l_\pm| \leq \gamma$ and $l(x) \to \pm\infty$ as $x \to x_{p\pm}$. The corollary thus follows from Proposition 3.8. Δ

III.4. Summary

In the preceding sections we have shown that the generic vacuum solutions of cosmological models, consisting in a product of two isotropic Riemannian subspaces with arbitrary dimensions, approach a generalized Kasner solution as $t \to 0$. Since the latter are defined as $g_i \propto t^{2\sigma_i}$, there are only two possibilities satisfying $\Sigma\, \sigma_i = \Sigma\, \sigma_i{}^2 = 1$. Thus, if neither d=1 nor D=1, there always exist d contracting and D expanding scales near the cosmological singularity (or vice versa). In the exceptional case where either $R_c{}^d$ or $R_c{}^D$ is one-dimensional, there is a solution with (n-1) dimensions approaching a finite value whereas the one-dimensional scale factor vanishes as $t \to 0$. Discussing the long time behavior, we obtain a constant ratio of the scale factors if both subspaces do not have positive curvature. If either r_d or r_D is positive then the n-dimensional volume is bounded and reaches its maximum after a finite cosmic time t^*.

We have shown that the shape of the logarithmic derivative of Ψ with respect to x for large values of $|x|$ determines the behavior of the solutions as $t \to 0$. The reason why all models of the form $\mathbb{R} \times R_c{}^d \times R_c{}^D$ exhibit only Kasner-like solutions near the singularity is thus simply due to the fact that at least one of the asymptotics of $l(x)/\gamma$ is located inside the strip S. Kasner-like solutions could not exist if both asymptotics of $l(x)/\gamma$ were located outside (or on the boundary) of S, as we have demonstrated in Proposition 3.3.

Before we turn to the discussion of the general equations (1) we shall briefly consider a "toy model" which also has a two-dimensional phase space. But instead of vanishing r_{dD} and r_{Dd} we assume these to be negative constants.

III.5. A toy model

As shown in Part II, the function $\Psi(x) : \mathbb{R}^{n-1} \to \mathbb{R}$ can be written in the form

$$Y(x) = \kappa^{-\alpha} [r_d + \kappa r_D + \kappa^2 r_{dD} + \kappa^{-1} r_{Dd}] \tag{26}$$

where r_d, r_D,...depend on $x_1 \dots x_{d-1}, x_{d+1} \dots x_{n-1}$ and $\kappa := \exp(\omega x_d)$. In the preceding section we have considered product topologies for which r_{dD}, r_{Dd} vanish and r_d, r_D are constants. In order to make a step towards the general homogeneous models, let us now assume that the "mixed curvatures" r_{dD} and r_{Dd} are also non-vanishing constants. Having in view the general situation where r_{dD} and r_{Dd} are negative semidefinite functions (II.28) of $x_1 \dots x_{d-1}, x_{d+1} \dots x_{n-1}$, we restrict the discussion of (7,26) to the case where $r_{dD}, r_{Dd} \le 0$. We shall distinguish between the situations where both constants r_{dD} and r_{Dd} are strictly negative and where r_{Dd} vanishes. In the first case, oscillating solutions can not exist if $n \ge 10$, whilst in the second case the critical dimension is $n = 3$. In the general situation the Levi-Malcev theorem [37] renders a decomposition $n = d + D$ possible, such that the function $r_{Dd}(x_1 \dots x_{d-1}, x_{d+1} \dots x_{n-1})$ identically vanishes for $n > 3$. The non-existence of chaotic solutions in homogeneous models with $n > 3$ corresponds thus to the non-existence of oscillating solutions in the toy model with $r_{Dd} = 0$.

Let us now discuss the trajectories of the system (7,26). The function $\Psi(x)$ has maximally two zeroes since $r_{dD}, r_{Dd} \le 0$. The logarithmic derivative $l(x)$ still has the horizontal asymptotics l_+ and l_- as $x \to \pm\infty$.

If $\Psi(x)$ has exactly one zero or no zero at all (i.e. $l(x)$ has one pole or is bounded), the situation is completely analogue to the discussion in section 3. The question of whether the trajectories approach a Kasner solution or not can be answered by investigating whether one of the asymptotics of $l(x)/\gamma$ is located inside the strip S.

The new case where $l(x)$ has two poles is related to the situation where $l(x)$ is bounded, i.e. $\Psi(x) < 0$ (as far as the behavior of $T \to \infty$ ($t \to 0$) is considered. Since the region where $\Psi(x)$ is positive is bounded by the zeroes x_1 and x_2 , the trajectories must be located inside the strip S for $x < x_1$ and $x > x_2$, and again the asymptotics of $l(x)/\gamma$ determine whether a Kasner solution is approached or not (fig.5.1.). (The long-time behavior (i.e. $T \to -\infty$) is different from the cases discussed up to now, since the critical point $(x_0,0)$ is located in the "forbidden" region $\{(x,y) \mid x \in [x_1,x_2] , |y| < 1 \}$ of the phase plane).

Using the fact that (independent of the number of zeroes of $\Psi(x)$) the asymptotics of $l(x)/\gamma$ determine the behavior of $\Gamma(T)$ as $T \to \infty$ and writing $l(x)$ in the form

$$l(x) = \omega\left[-\alpha + \frac{r_D\,\kappa^2 + 2r_{dD}\,\kappa^3 - r_{Dd}}{r_d\,\kappa + r_D\,\kappa^2 + r_{dD}\,\kappa^3 + r_{Dd}} \right] , \tag{27}$$

we can now easily discuss the behavior of the solutions of (7) as $T \to \infty$.

Corrolary 5.1.

Let $r_{dD}, r_{Dd} < 0$. Then as $T \to \infty$

i) no solution of (7,27) can approach one of the generalized Kasner solutions $\Gamma_+(T)$, $\Gamma_-(T)$ if $n \le 9$,

ii) all solutions approach either $\Gamma_+(T)$ or $\Gamma_-(T)$ if $n \ge 10$ and $d,D \ne 1$.

Proof: i) The asymptotics of $l(x)$ (27) fulfill the inequalities:

$$l_+ := \lim_{x\to\infty} l(x) = \omega(2-\alpha) = \frac{2d+D}{\sqrt{ndD}} \ge \frac{\sqrt{8}}{\sqrt{n}} ,$$

$$l_- := \lim_{x\to-\infty} l(x) = \omega(-1-\alpha) = -\frac{2D+d}{\sqrt{ndD}} \le -\frac{\sqrt{8}}{\sqrt{n}} . \tag{28}$$

For $n \le 9$ we have $|l_\pm/\gamma| \ge [(8/n)\ (n/(n-1))]^{1/2} \ge 1$ and neither l_+/γ nor l_-/γ are located inside the strip S. No solution can thus approach $\Gamma_+(T)$ or $\Gamma_-(T)$ as $T \to \infty$ $(t \to 0)$.

ii) The asymptotic l_+/γ is located inside S if $l_+/\gamma < 1$. Using $D = n - d$ this is equivalent to $d^2 + d\,(3 - n) + n < 0$. For $n > 10$ this inequality holds whenever $2 \le d \le n-5$. On the other hand, $l_-/\gamma > -1$ holds if $2 \le D \le n-5$ or equivalently $5 \le d \le n-2$. Thus for all $d \in [2\ ,\ n-2]$ either the negative or the positive asymptotic of $l(x)/\gamma$ is located inside the strip S and the solutions of (7,27) approach either $\Gamma_+(T)$ or $\Gamma_-(T)$ as $T \to \infty$ (i.e. $t \to 0$). (As is easily seen, the same statement holds for $n = 10$ if $d \in [3\ ,\ n-3]$). Δ

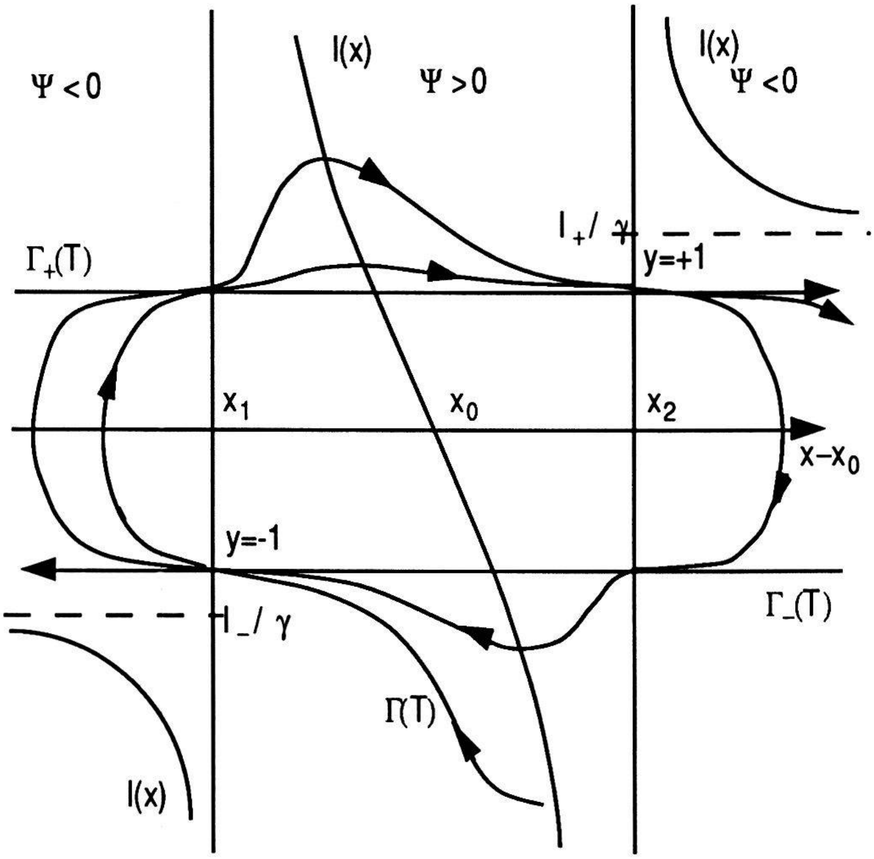

fig.5.1.

If l(x) has two poles and both asymptotics are located outside the strip S then the trajectories can not approach a Kasner solution. The behavior is similar to the case where l(x) has no poles (fig.3.1.).

Let us now consider $r_{Dd} = 0$ which will be the relevant case in all higher-dimensional homogeneous models.

<u>Corrolary 5.2.</u>

Let $r_{dD} < 0$, $r_{Dd} = 0$. Then for arbitrary choices of n,d,D $(d \ne 1)$, all solutions of (7,27) approach one of the generalized Kasner solutions $\Gamma_+(T)$, $\Gamma_-(T)$ as $T \to \infty$ $(t \to 0)$.

Proof: Instead of (28) the asymptotics of $I(x)$ (27) fulfill the inequalities :

$$I_+ := \lim_{x\to\infty} I(x) = \omega(2-\alpha) = \frac{2d+D}{\sqrt{ndD}} \geq \frac{\sqrt{8}}{\sqrt{n}} ,$$

$$I_- := \lim_{x\to-\infty} I(x) = \omega(-\alpha) = -\frac{\sqrt{D}}{\sqrt{nd}} \overset{d\neq 1}{>} -\frac{\sqrt{n-1}}{\sqrt{n}} = -\gamma . \qquad (29)$$

Depending on n the positive asymptotic of $I(x)/\gamma$ can be located inside or outside the strip S, whereas the negative asymptotic I_-/γ is always ($d\neq 1$) located inside S. Thus every solution which does not approach $\Gamma_+(T)$ has to approach $\Gamma_-(T)$ (after a certain number of oscillations, depending on the initial conditions). Δ

In the general case which we shall discuss in the next Part, the function $\Psi(\underline{x})$ may be written in the form (26) where r_d, r_D, r_{dD}, and r_{Dd} are functions of $x_1 \ldots x_{d-1}, x_{d+1} \ldots x_{n-1}$. Whenever $n > 3$, there exists a decomposition $d + D = n$ with $d \neq 1$ such that the structure constants of G appearing in $r_{Dd}(\ldots)$ vanish (II.28). The above Corollary, stating that the plane toy model with vanishing constant r_{Dd} has no oscillating solutions, then finds its generalization in the fact that the general model with the identically vanishing function $r_{Dd}(\ldots)$ has no chaotic solutions for $n > 3$. Moreover, the first Corollary 5.1. states that if all terms in $\Psi(\underline{x})$ exist, the critical dimension is increased to $n = 10$. This is in agreement with the supposition that chaos does not occur in inhomogeneous cosmological models with $n \geq 10$ [16-18].

IV. Regular and chaotic solutions in (n+1)-dimensional cosmological models

Homogeneous models in an arbitrary number of dimensions are discussed. Approaching the cosmological singularity, the generic solu- tions of the field equations are either oscillating functions or behave like general Kasner solutions. Using the (2n-2)-dimensional autonomous system derived in Part II, a geometrical condition which guarantees the existence of regular solutions is found. It is shown that in homogeneous cosmology chaos may only occur in (3+1) dimensions. The arguments are also generalized to inhomogeneous models where the critical number of spatial dimensions is found to be n=10.

IV.1. Introduction

The behavior of the scale factors near the initial singularity has been discussed for higher-dimensional cosmological models in [8,15-18,20]. Apart from the cases treating a product topology [9-14,36] there exist essentially two different approaches to the general problem.

The first approach [15-17] bases on the discussion of the ergodic properties of the generalized "mixmaster" map [4,5], which is the discrete dynamical system describing the change of Kasner exponents. In order to discuss the problem within this framework, one has to assume that the generalized Kasner solutions are admissible approximations during successive time intervals (Kasner epochs). If the number of space dimensions increases n=10 then there exists a non-empty set of Kasner exponents in which no further transitions can take place and the solutions thus behave regular as $t \rightarrow 0$.

The second approach uses the Hamiltonian formulation [18], describing the motion of the "universe point" in a time-dependent poten- tial. The evolution is replaced by a sequence of free propagations inter- rupted by collisions with the moving potential walls [19]. (In order to obtain a bouncing law, these are usually assumed to be infinitely steep.) Again the critical dimension is found to be $n = 10$, since for $n \geq 10$ the velocity of the equipotentials exceeds the velocity of the "universe point", and there exists a last bounce as $t \rightarrow 0$. As was pointed out in

[18], these arguments hold for inhomogeneous as well as for homogeneous models. The fact that the critical number of space dimensions in homo- geneous cosmo- logical models is decreased to $n = 3$ is due to the absence of some equipotential walls.

In the following we shall propose a third approach which also yields the same critical dimensions in homogeneous as well as in inhomogeneous cosmological models. The advantage of this method is that it works without the approximations mentioned above. We shall give a condition rendering it possible to decide whether or not there exists a set $X \times Y$ in the phase space where the solutions behave monotonic. The solutions entering this region approach a generalized Kasner solution as $t \to 0$. Since we do not replace the field equations by a discrete dynamical system, the question whether the solutions in fact enter the set $X \times Y$ seems to be more difficult to answer within our framework than it is for the mixmaster map [15].

In the second and third section we shall discuss the behavior of the modulus $y := |\underline{y}|$, where $\underline{y} \in \mathbb{R}^{n-1}$ is the velocity vector of the system (Part II, eq(58)). Using an integral equation for y, we shall show that either the modulus $y(t)$ is an oscillating function as $t \to 0$ or the vector $\underline{y}(t)$ approaches a fixed point $\underline{e} \in S^{n-2}$, i.e. the solutions approach a generalized Kasner solution .

In order that the cosmological model behaves regular as $t \to 0$, a necessary condition concerning the Lie group G must be fulfilled. Roughly speaking, the combination of all hyperspheres $c_{kij}(\underline{y}) := |\underline{y} - \underline{\alpha}_{kij}| - |\underline{\alpha}_{kij}| = 0$ coupling to non-vanishing structure constants $C^k{}_{ij}$ must not contain the hypersphere S^{n-2} ($\underline{\alpha}_{kij}$ are constant vectors of R^{n-1}). This criterion is derived in the fourth section and, in the fifth section, it is shown to hold whenever G has at least a two-dimensional subgroup.

For the cases in which the condition mentioned above holds we shall construct an invariant set $X \times Y$ in the phase space in section six. Every solution entering this set approaches a generalized Kasner solution. To close the discussion of the homogeneous cases, we use the Levi Malcev decomposition [20,21] for Lie algebras to show that $X \times Y$ is not empty for all Lie groups except SO(3) and SO(2,1).

In section eight we finally extend the discussion to the inhomogeneous models. Since our criterion only affects the geometrical configuration of the hyperspheres $c_{kij}(\underline{y})=0$ coupling to the non-vanishing structure constants $C^k{}_{ij}$, it still holds in the case where $C^k{}_{ij}$ depend on the spatial coordinates. In order that the model behaves regular near the cosmological singularity, we re-derive n=10 for the critical number of spatial dimensions.

IV.2. The behavior of the modulus of the momentum

Let us now consider the full system (II.58)

$$\frac{d\underline{x}}{dT} = \underline{y} \quad ; \quad \frac{d\underline{y}}{dT} = \frac{1}{2}(1-|\underline{y}|^2)\left(\gamma\underline{y} - \frac{\nabla\Psi+\underline{\varphi}}{\Psi}(\underline{x})\right) \tag{1}$$

with the functions $\underline{\varphi}: \underline{x} \in \mathbb{R}^{n-1} \to \underline{\varphi}(\underline{x}) \in \mathbb{R}^{n-1}$ and $\Psi: \underline{x} \in \mathbb{R}^{n-1} \to \Psi(\underline{x}) \in \mathbb{R}$. As we have shown in Part II, $\underline{\varphi}(\underline{x})$ stands perpendicular to $\underline{y}$ with respect to the Eukliedian metric of $\mathbb{R}^{n-1}$, $(\underline{\varphi}(\underline{x}), \underline{y}) = 0$. The function $\Psi(\underline{x})$ can be written in the form

$$\Psi(\underline{x}) = -t_i \exp[2\gamma(\underline{\alpha}_i, \underline{x})] - s_{ij}{}^k \exp[2\gamma(\underline{\alpha}_{kij}, \underline{x})] \tag{2}$$

where we have introduced the constant vectors $\underline{\alpha}_i, \underline{\alpha}_{kij} \in \mathbb{R}^{n-1}$ with components $(\underline{\alpha}_i)_t$, $(\underline{\alpha}_{kij})_t$, defined as

$$\begin{aligned}(\underline{\alpha}_i)_t &:= -(2\gamma)^{-1} A^T{}_{it}, \\ (\underline{\alpha}_{kij})_t &:= -(2\gamma)^{-1}[A^T{}_{it} + A^T{}_{jt} - A^T{}_{kt}], \quad t \in \{1...n-1\}.\end{aligned} \tag{3}$$

A is the $n \times n$ matrix introduced in (II.42.) and $\gamma = \sqrt{(n-1)}/\sqrt{n}$ (II.32.). The constants t_i and $s_{ij}{}^k$ are defined in terms of the structure constants of the Lie group G (II.24):

$$\begin{aligned}t_i &= (C_i)^2 + 1/2\ C^k{}_{il}C^l{}_{ik}, \\ s_{ij}{}^k &= 1/4\ (C^k{}_{ij})^2 \geq 0 .\end{aligned} \tag{4}$$

Using $X_i = A^T{}_{it}x_t + A^T{}_{in}x_n$, where t runs from 1 to (n-1) and $A^T{}_{in} = 1/\sqrt{n}\ \forall i$, expression (2) for $\Psi(\underline{x})$ follows from (II.22) and (3):

$$\bar{R} = -t_i\, e^{-X_i} - s_{ij}^{\ k}\, e^{X_k - X_i - X_j} = e^{-x_n/\sqrt{n}}\, \Psi(\underline{x}) .$$

Instead of the second (n-1) differential equations for the vector $\underline{y} \in \mathrm{IR}^{n-1}$, we can also discuss the equations for the modulus $y(T) := |\underline{y}(T)| \in \mathrm{IR}^+$ and the unit vector $\hat{a}(T) := \underline{y}(T)/y(T) \in S^{n-2}$. We shall show that the evolution of the modulus y is closely related to the behavior of the d'th component y_d in the two-dimensional model discussed in the previous part. Although the motion of $\hat{a}(T)$ on the hypersphere S^{n-2} can be very complicated, $\hat{a}(T)$ has to approach a fixed point $\underline{e}$ on S^{n-2} when- ever the modulus $y \to 1$ as $T \to \infty$ (i.e. $t \to 0$). Thus, in order to decide whether the trajectories behave like generalized Kasner solutions, we shall be able to restrict ourselves to the discussion of the modulus $y(T)$. Before we show that $y(T)$ is either an oscillating function in T or approaches the constant value $y = 1$ as $T \to \infty$, we first derive an equation for $y(T)$:

Proposition 2.1.

Let the functions $c_i, c_{kij} : \underline{y} \in \mathrm{IR}^{n-1} \to c_i(\underline{y}), c_{kij}(\underline{y}) \in \mathrm{IR}$ be defined as

$$\begin{aligned} c_i(\underline{y}) &:= |\,\underline{y} - \underline{\alpha}_i\,|^2 - |\,\underline{\alpha}_i\,|^2 \\ c_{kij}(\underline{y}) &:= |\,\underline{y} - \underline{\alpha}_{kij}\,|^2 - |\,\underline{\alpha}_{kij}\,|^2 \end{aligned} \tag{5}$$

and let Φ denote the mapping $\Phi : \Gamma(T) \in \mathrm{IR}^{n-1} \times \mathrm{IR}^{n-1} \to \Phi(T) \in \mathrm{IR}$:

$$\Phi(T) := \Psi(\underline{x}(T)) \exp\left[-\gamma \int^T y^2(T')\, dT'\right] , \tag{6}$$

defined on every trajectory $\Gamma(T) = (\, \underline{x}(T)\, ,\, \underline{y}(T)\,)$ of (1). Then every solution of (1) fulfils the equation

$$y^2(T) = 1 + \Phi(T) \tag{7}$$

and together with the definition (3), $\Phi(T)$ may also be written as

$$\Phi(T) := -t'_i \exp\left[-\gamma \int^T c_i(\underline{y}(T'))\, dT'\right] - s'_{ij}{}^{k} \exp\left[-\gamma \int^T c_{kij}(\underline{y}(T'))\, dT'\right] \tag{8}$$

where t'_i and $s'_{ij}{}^k$ are constants with $\mathrm{sig}(\, t'_i\, ,\, s'_{ij}{}^k\,) = \mathrm{sig}(\, t_i\, ,\, s_{ij}{}^k\,)$.

Proof: Differentiating (7) with respect to T we obtain from (1), $d(y^2)/dT = (\underline{y}\ , d\underline{y}/dT)$, $(\underline{\varphi}(\underline{x})\ , \underline{y}) = 0$ and $(\underline{\nabla}\Psi, \underline{y}) = d\Psi/dT$:

$$\frac{d\Phi/dT}{\Phi} = \frac{2\,\underline{y}\,\dot{\underline{y}}}{y^2-1} = -\left(\gamma y^2 - \frac{1}{\Psi}(\underline{y}\ , \underline{\nabla})\,\Psi\right) = \frac{d\Psi/dT}{\Psi} - \gamma y^2\ , \qquad (9)$$

in agreement with definition (6) for Φ. Using (6), (2) and $\underline{x}(T) = \int \underline{y}(T')\, dT' + \underline{c}onst$, we can write

$$\Phi(T) = -t'_i \exp\left[-\gamma\int^T (y^2 - 2\,\underline{\alpha}_i\underline{y})\,dT'\right] - s'_{ij}{}^{k} \exp\left[-\gamma\int^T (y^2 - 2\,\underline{\alpha}_{kij}\underline{y})\,dT'\right]$$

which is identical with (8) since $y^2 - 2\,\underline{\alpha}_i\underline{y} = |\,\underline{y} - \underline{\alpha}_i\,|^2 - |\,\underline{\alpha}_i\,|^2 = c_i(\underline{y})$. The constants t'_i and $s'_{ij}{}^k$ differ from t_i and $s_{ij}{}^k$ in some exponential factors depending only on initial conditions. Δ

Before we apply (7) and (8) to discuss the behavior of $y(T)$, let us take a closer look at the functions $c_i(\underline{y})$, $c_{kij}(\underline{y})$ defined in (5).

<u>Proposition 2.2.</u>
Let $c_i(\underline{y})$, $c_{kij}(\underline{y})$ be the functions defined in (5). Then $\{ c_i(\underline{y}) = 0 \}$ and $\{c_{kij}(\underline{y}) = 0 \}$ are hyperspheres in $\mathbb{R}^{n-1}$. They all contain the origin $\underline{y} = \underline{0}$ and, if $i \neq j$, their radius is either R_1 or R_2 :

$$\begin{aligned} R_1 &:= R(c_i) = R(c_{kij}) = 1/2 && \text{if } k \in \{ i\, , j \}\ , \\ R_2 &:= R(c_{kij}) = 1/2\,[\,1 + 2/\gamma^2\,]^{1/2} && \text{if } k \notin \{ i\, , j \}\ . \end{aligned} \qquad (10)$$

Proof: The centres of $\{ c_i(\underline{y}) = 0 \}$ and $\{ c_{kij}(\underline{y}) = 0 \}$ are $\underline{\alpha}_i$ and $\underline{\alpha}_{kij}$. The radii $|\underline{\alpha}_i|$ and $|\underline{\alpha}_{kij}|$ are computed using the definitions (3) and the relations $\Sigma\, A^T{}_{it}\, A_{tj} = (A^TA)_{ij} - A^T{}_{in}\, A_{nj} = \delta_{ij} - 1/n$ for the matrix A defined in (II.42) (the sum over t runs from 1 to (n-1)):

$$|\,\underline{\alpha}_i\,|^2 = \frac{1}{(2\gamma)^2}\sum_{t=1}^{n-1} A^T{}_{it}\, A^T{}_{it} = \frac{1}{(2\gamma)^2}\left(1 - \frac{1}{n}\right) = (2\gamma)^{-2}\gamma^2 = \frac{1}{4} = R_1^2\ .$$

For the modulus of $\underline{\alpha}_{kij}$ we either obtain R_1 ($k \in \{i,j\}$) or R_2 ($k \notin \{i,j\}$):

$$|\underline{\alpha}_{kij}|^2 = \frac{1}{(2\gamma)^2}\left[\sum_{t=1}^{n}(A^T_{kt}-A^T_{it}-A^T_{jt})(A_{tk}-A_{ti}-A_{tj}) - (A^T_{kn}-A^T_{in}-A^T_{jn})(A_{nk}-A_{ni}-A_{nj})\right]$$

$$= (2\gamma)^{-2}\,[3-2\,\delta_{ki}-2\,\delta_{kj}+2\,\delta_{ij}-1/n\,] \overset{(i\neq j)}{=} (2\gamma)^{-2}\,[\gamma^2+2\,(1-\delta_{ki}-\delta_{kj})\,]\,.$$

If neither i nor j are equal to k we have $|\underline{\alpha}_{kij}|^2 = (2\gamma)^{-2}[\gamma^2+2] = [1+2/\gamma^2]/4 = R_2{}^2$. If $k \in \{i,j\}$ but $i \neq j$ the expression $(1-\delta_{ki}-\delta_{kj})$ vanishes and thus $|\underline{\alpha}_{kij}|^2 = R_1{}^2$. Δ

Next we shall show that every solution with non-oscillating modulus $y(T)$ (as $T \to \infty$) approaches $y = 1$. We thus assume that there exists a time T^* such that $y(T)$ is a monotonic function in T for $T > T^*$ and, choosing T^* big enough we thus have either $y(T) \geq 1$ or $y(T) \leq 1$ $\forall\, T > T^*$. Let us first consider the simpler case $y(T) \geq 1$ $\forall\, T > T^*$:

<u>Proposition 2.3.</u>
Let $\Gamma(T)$ be a solution of (1,2) with monotonic modulus $y(T) \geq 1$ $\forall\, T > T^*$. Then $y(T) \to 1$ as $T \to \infty$.

Proof: We show that the assumption $\lim y(T) \neq 1$ for $T \to \infty$ leads to a contradiction. Since $y(T) \geq 1$ we obtain

$$\begin{aligned} c_i(\underline{y}) &= |\,\underline{y}-\underline{\alpha}_i\,|^2 - |\,\underline{\alpha}_i\,|^2 \geq (\,|\,\underline{y}\,| - |\,\underline{\alpha}_i|\,)^2 - |\,\underline{\alpha}_i|^2 \\ &= y\,(\,y-2\,|\,\underline{\alpha}_i|\,) = y\,(\,y-1\,) \geq 0 \end{aligned}$$

where we have used $R_1 = 1/2$ (10) in the last equality. According to the assumptions, $y(T)$ is monotonic for $T > T^*$ and does not approach $y = 1$. The inequality $y\,(y-1) \geq 0$ thus yields

$$\int^{\infty} c_i\,(\underline{y}\,(T'))\,dT' \geq \int^{\infty} y\,(T')\,[\,y\,(T')-1\,]\,dT' = \infty\,. \tag{11}$$

Now we are able to give an upper bound for the limes of $\Phi(T)$. Using (11) in (8) we obtain

$$\lim_{T\to\infty} \Phi(T) = -s'^{\,k}_{ij} \lim_{T\to\infty} \exp\left[-\gamma \int^{T} c_{kij}(\underline{y}(T'))\,dT'\right] \leq 0 \tag{12}$$

where the last inequality is due to the fact that the s_{ij}^k are quadratic expressions in terms of the structure constants (4) and that $\mathrm{sig}(s'^{\,k}_{ij}) = \mathrm{sig}(s_{ij}^k)$. Since $y(T) \geq 1 \ \forall\ T > T^*$ and $\lim (y(T)) \neq 1$, equation (7) yields $\lim \Phi(T) = \lim (y^2(T) - 1) > 0$ which is in contradiction to (12). Thus, if there exists a time T^* such that the modulus of $\underline{y}$ is a monotonic function with $y \geq 1\ \forall T > T^*$ then y has to approach the constant value $y=1$. Δ

Next we shall show that the same statement also holds in the situation where $y(T) \leq 1\ \forall\ T > T^*$.

Proposition 2.4.

Let $\Gamma(T)$ be a solution of (1,2) with monotonic modulus $y(T) \leq 1\ \forall\ T > T^*$. Then $y(T) \to 1$ as $T \to \infty$.

Proof: We first show that (1) has no stable critical points as $T \to \infty$ (i.e. $t \to 0$). As already mentioned in Part I, all critical points are located within the region $|y| < 1$. Their coordinates are $P_c = (\underline{x}_c, \underline{0})$ where $\underline{x}_c$ has to be a solution of $(\underline{\nabla}\Psi + \underline{\varphi})(\underline{x}_c) = 0$. The $2(n-1) \times 2(n-1)$ matrix M corresponding to the linearized system (1) at P_c is

$$M = \begin{pmatrix} 0_{n-1} & \mathbb{1}_{n-1} \\ -\frac{1}{2} l_{ij}(\underline{x}_c) & \frac{\gamma}{2}\mathbb{1}_{n-1} \end{pmatrix}$$

where $l_{ij} := \nabla_j\{(\nabla_i\Psi + \varphi_i)/\Psi\}$. The quadratic form $Q(\theta) := (\theta, M\theta)$, $\theta = \theta_1 \times \theta_2$, $Q(\theta) = (\theta_1 \times \theta_2, M\,\theta_1 \times \theta_2) = (\theta_1, \theta_2) - 1/2\,(\theta_2, l\,\theta_1) + \gamma/2\,|\theta_2|^2$ is not negative semi-definite, thus M has at least one eigenvalue with strictly positive real part and the critical points of (1) are instable as $T \to \infty$, i.e. $t \to 0$.

Let us now consider a solution $\Gamma(T)$ for which $y(T)$ is a monotonic and bounded function with $y(T) \leq 1$ for $T > T^*$. Thus we have $dy/dT \to 0$ and $y(T) \to y_\infty \in [0,1]$ as $T \to \infty$. Multiplying (1) by $\underline{y}$, the equation for $y^2(T)$ becomes

$$\frac{d(y^2)}{dT} = (1-y^2)\left(\gamma y^2 - \frac{d\Psi/dT}{\Psi}\right) \quad . \tag{13}$$

Let $y_\infty \neq 1$. Since the l.h.s. of (13) has to vanish as $T \to \infty$ and since $f(T) := \gamma\ (1-y(T)^2) \to f_\infty = \gamma\ (1-y_\infty{}^2) > 0$, the second bracket on the r.h.s. of (13) also must approach zero as $T \to \infty$. The function $g(T) := -(1-y(T)^2)\ (d\Psi/dT)/\Psi$ is thus bounded as $T \to \infty$ and $g(T) \to g_\infty = -\gamma\ y_\infty{}^2\ (1-y_\infty{}^2) < 0$. From the integral equation

$$y^2(T) = \left[\ y_0^2 + \int_{T_0}^{T} g(s)\, e^{-F(s)}\, ds\ \right] e^{F(T)} \quad , \quad F(T) := \int_{T_0}^{T} f(s)\, ds \ ,$$

following from (13), we can see that for unbounded $F(T)$ every solution $y^2(T)$ (with $y^2(T_0) \neq -\int_{T_0} g(s)\ \exp(-F(s))\ ds$) is not bounded as $T \to \infty$. Since we have $f(T) \to f_\infty > 0$ by the assumption $y_\infty \neq 1$, the function $F(T)$ is unbounded in our case and thus (up to a set of initial conditions with vanishing measure) $y(T)^2 \to \infty$ in contradiction to $y(T)^2 \in [0,1]\ \forall\ T$. The only possibility to avoid this contradiction for a bounded and monotonic modulus $y(T)$ is to assume $y(T) \to 1$ as $T \to \infty$. Δ

IV.3. The behavior of the momentum $\underline{y}(T)$ near the singularity

Having shown that the modulus $y(T)$ approaches $y=1$ whenever it is a nonoscillating function for $T > T^*$, we shall next turn to the behavior of the direction $\hat{a}(T) = \underline{y}(T)/y(T)$. The important observation is that $\hat{a}(T)$ tends towards a fixed point $\underline{e}$ on S^{n-2} whenever $\underline{y}(T)$ approaches S^{n-2} (i.e. the modulus $y(T) \to 1$). In order to show this we first have to discuss the integrals of the functions $c_i(\underline{y}(T))$ and $c_{kij}(\underline{y}(T))$ defined in (5). In the generic case (with $y \to 1$) all integrals $\int^T c_i(\underline{y}(T')dT'$ diverge as $T \to \infty$. The situations for which exactly one of the integrals (for instance $\int^T c_b(\underline{y}(T')dT'$) is bounded correspond to instable solutions with $\underline{y} \to 2\underline{\alpha}_b$.

Proposition 3.1.

Let $\Gamma(T)$ be a solution of (1,2) with monotonic modulus $y(T)$ for $T > T^*$. Then

i) either $\int^\infty c_i(T)\, dT = \infty$ for all i or there exists exactly one bounded integral $\int^\infty c_b(T)\, dT$.

ii) $dc_i(T)/dT \to 0 \ \forall i$ if no bounded integral $\int^\infty c_b(T)\, dT$ exists and $dc_b(T)/dT \to 0$ if $\int^\infty c_b(\underline{y}(T')dT'$ has an upper bound.

Proof: i) Let $i \neq j$. From definition (5) we obtain $c_i + c_j \geq 2y^2 - 2y\, |\underline{\alpha}_i + \underline{\alpha}_j|$. Since

$$|\,\underline{\alpha}_i + \underline{\alpha}_j\,|^2 = \frac{1}{4} + \frac{1}{4} + \frac{2}{(2\gamma)^2}(\delta_{ij} - 1/n\ ) \stackrel{i\neq j}{=} \frac{n-2}{2(n-1)} < \frac{1}{2}$$

and since $y(T)$ approaches $y=1$ monotonically as $T \to \infty$, T_o can be chosen such that $y \geq (1+1/\sqrt{2})/2$ for $T \geq T_o$ and we obtain

$$\int_{T_o}^{\infty} (c_i + c_j)\, dT \geq \int_{T_o}^{\infty} 2y\,(y - 1/\sqrt{2}\,)\, dT \geq \frac{1}{4}\int_{T_o}^{\infty} dT = \infty\ . \tag{14}$$

Thus there exists mostly one index b with $\lim \int^\infty c_b(T)\, dT \neq +\infty$.

ii) Since $\int^\infty c_b(T)\, dT$ is the only bounded integral and since $\Phi(T) \to 0$ as $y \to 1$ (7), we obtain from (8) :

$$t'_b \exp\left[-\gamma\int^T c_b\, dT'\right] + \sum s'^{\,k}_{ij} \exp\left[-\gamma\int^T c_{kij}\, dT'\right] \to 0\ , \tag{15}$$

where the sum contains at least one non-vanishing term as $T \to \infty$. Using the differential equation (1), definition (6) and equation (7), we obtain after a differentiation of (5) with respect to T:

$$\begin{aligned}\frac{dc_b}{dT} &= (\underline{\alpha}_b - \underline{y}\,)\left(\gamma\underline{y}\,\Phi(T) - \underline{\nabla}\Psi \exp\left[-\gamma\int^T y^2\, dT'\right]\right)\\ &\to -\,(\underline{\alpha}_b\,,\,\underline{\nabla}\Psi)\,\exp\left[-\gamma\int^T y^2\, dT'\right]\ ,\ \text{as } T \to \infty\ .\end{aligned} \tag{16}$$

The second step is a consequence of $\Phi(T) \to 0$, $y \to 1$ and of $dy/dT \to 0$, implying $d\Phi/dT \to 0$ and thus also $\exp(-\gamma \int^T y^2 dT')\ (\underline{y}, \underline{\nabla}\Psi) = \exp(-\gamma \int^T y^2 dT') \cdot (d\Psi/dT) \to 0$. Now we can use (2) to calculate the gradient of Ψ and again (5) in the exponents to write

$$\frac{dc_b}{dT} \to 2\gamma \{\ t'_i\, (\underline{\alpha}_i, \underline{\alpha}_b)\ e^{-\gamma \int c_i\, dT'} + s'_{ij}{}^{k}\, (\underline{\alpha}_{kij}, \underline{\alpha}_b)\ e^{-\gamma \int c_{kij}\, dT'}\ \}\,. \tag{17}$$

From (17) we can conclude that $dc_b/dT \to 0\ \forall\, b$ if all integrals tend to infinity as $T \to \infty$.

If, on the other hand, there exists one bounded integral $\int^\infty c_b(T)\, dT$, only the term with $i = b$ can give a non-vanishing contribution to the first sum as $T \to \infty$. Using (15) and $(\underline{\alpha}_b, \underline{\alpha}_b) = 1/4$ we obtain

$$\frac{dc_b}{dT} \to 2\gamma\ s'_{ij}{}^{k}\ e^{-\gamma \int c_{kij}\, dT'} \left[(\underline{\alpha}_{kij}, \underline{\alpha}_b) - \frac{1}{4} \right]\,. \tag{18}$$

The bracket always ($\forall$ k,i,j,b) vanishes or is negative since

$$(\underline{\alpha}_{kij}, \underline{\alpha}_b) = \frac{1}{(2\gamma)^2} \left[\delta_{ib} + \delta_{jb} - \delta_{kb} - \frac{1}{n} \right] \le \frac{1}{(2\gamma)^2} (1 - 1/n\ ) = \frac{1}{4}\,,$$

where the inequality is due to the fact that no $\underline{\alpha}_{kij}$ with $i=j$ exists ($s_{ij}{}^{k} = 0$ for $i=j$). If a bounded integral $\int^\infty c_{kij}(T)dT$ with k,i,j such that $(\underline{\alpha}_{kij}, \underline{\alpha}_b) \neq 1/4$ would exist then $-dc_b(T)/dT$ would approach a non-vanishing positive value and the integral $\int^\infty c_b(T)dT$ could not be bounded, which is in contradiction to the assumption. The only integrals $\int^\infty c_{kij}(T)\, dT$ that can co-exist with $\int^\infty c_b(T)dT$ are those with $(\underline{\alpha}_{kij}, \underline{\alpha}_b) = 1/4$, i.e. $b \in \{i,j\}$, $b \neq k$. Thus either the bracket in (18) vanishes or the factor in front of the bracket approaches zero as $T \to \infty$. Δ

In the preceding Proposition we have seen that the solutions with $y \to 1$ as $T \to \infty$ admit either none or exactly one bounded integral $\int^\infty c_b(T)dT$. The first case corresponds to the generic situation whereas the second case occurs only for special initial conditions. More precisely, we have the following

Proposition 3.2.

Let $\Gamma(T)$ be a solution of (1,2) with monotonic modulus $y(T)$ for $T > T^*$ and for witch $\exists$ b, such that $\int^T c_b(\underline{y}(T'))dT' \leq B^2 < \infty$. Then

i) $\underline{y} \to 2\underline{\alpha}_b$ as $T \to \infty$,

ii) $\Gamma(T)$ is not a generic solution.

Proof: i) Using $dc_b/dT \to 0$ (Proposition 3.1.) and $\int^T c_b(T')dT' \leq B^2$, we first show by contradiction that $c_b \to 0$ as $T \to \infty$: Let $\lim c_b(T) \neq 0$. Then $\exists\, \delta > 0$ such that $\forall\, T_o\, \exists\, T_1$ with $c_b(T_1) = \delta$ (or $-\delta$). Since $\lim(dc_b/dT) = 0$ we can choose T_o such that $|dc_b/dT| \leq \delta^2/(6B^2)$. Let $T_2 := T_1 + 6B^2/\delta$. For $T > T_1$ we have $c_b(T) = c_b(T_1) + dc_b/dT(T^*)\,(T - T_1) \geq \delta - \delta^2/(6B^2)\,(T - T_1)$. Integrating this inequality in the interval $[T_1, T_2]$ we obtain $\int c_b(T')dT' \geq \delta\Delta T\,[1 - \delta\,\Delta T/(12B^2)] = 3B^2 > 2B^2$ in contradiction to $\int^T c_b(T')dT' \leq B^2$. From $c_b(T) \to 0$ we now obtain $2\underline{\alpha}_b\underline{y} \to y^2 \to 1$ and thus with $|\underline{\alpha}_b| = 1/2$, $\underline{y} \to 2\underline{\alpha}_b$. This shows that for a solution with bounded $\int^T c_b(T')dT'$ the magnitude of the oscillations of each component has to vanish as $\underline{y}$ approaches S^{n-2}. In this case the only possible limit for $\underline{y}$ is $2\underline{\alpha}_b$ which is the only common point of the spheres $c_b(\underline{y}) = 0$ and S^{n-2}.

ii) Since $\Phi(T)$ must tend to zero as $y \to 1$ (7), the asymptotically non-vanishing term $t'_b \exp[-\gamma \int^T c_b(T')\,dT']$ in (8) must be compensated by some terms of the form $s'_{bj}{}^k \exp[-\gamma \int^T c_{kbj}(T')dT']$. Using (5) we obtain $t'_b + s'_{bj}{}^k \exp[2\gamma\, \underline{x}\,(\underline{\alpha}_j - \underline{\alpha}_k)] \to 0$, which is only possible for a set of initial conditions with vanishing measure (t'_b and $s'_{bj}{}^k$ depend on initial conditions ; $t'_b := t_b \exp(2\gamma\underline{\alpha}_b\, \underline{x}_o)$). Δ

As we shall see later, these special solutions correspond to instable Kasner solutions . Considering the Bianchi type IX model in (3+1) dimensions as an example, we sshall find three solutions (fig.7.1.) with $\underline{y} \to 2\underline{\alpha}_b$ as $T \to \infty$. In the time-dependent Hamiltonian formulation, these are exactly the special solutions where the "universe point" is moving on a straight line inside one of the three infinitely thin channels. In agreement with Proposition 3.2, the directions of the channels are $2\underline{\alpha}_1 = (\sqrt{3}/2\ ,\ -1/2)$, $2\underline{\alpha}_2 = (-\sqrt{3}/2\ ,\ -1/2)$ and $2\underline{\alpha}_3 = (0\ ,\ 1)$.

Let us now discuss the generic cases with $y \to 1$ as $T \to 0$:

Corollary 3.3.

Let $\Gamma(T)$ be a generic solution of (1,2) with monotonic modulus $y(T)$ for

$T>T^*$. Then

$$\lim_{T\to\infty} \underline{y}(T) = \underline{e} \in S^{n-2} \ . \tag{19}$$

Proof: Using Propositions 3.1. and 3.2. we have $dc_i/dT \to 0 \ \forall i$ as $T \to \infty$ and thus $(\underline{\alpha}_i, d\underline{y}/dT) \to 0 \ \forall i$. Since the vectors $\underline{\alpha}_i$ form a (not orthogonal) basis of $\mathbb{R}^{n-1}$, we obtain $d\underline{y}/dT \to 0$ and also $d\hat{a}/dT \to 0$, using $y \to 1$ and $dy/dT \to 0$ (Propositions 2.3. and 2.4.). The unit vector $\hat{a}$ thus approaches (not necessarily in a monotonic way) a fixed point $\underline{e}$ on the hypersphere S^{n-2}. (The case where $\hat{a}$ is oscillating around $\underline{e}$ with an asymptotically non-vanishing magnitude, but such that $d\hat{a}/dT \to 0$, will be ruled out later by a stability consideration.) Δ

IV.4. A geometrical condition for $|y(t)| \to 1$

Until now we have assumed a monotonic behavior of the modulus $y(T)$ for $T > T^*$ and have shown that in this case the vector $\underline{y}(T)$ approaches a fixed point $\underline{e} \in S^{n-2}$. In this section we shall show that the solutions can only exhibit this behavior if a necessary condition concerning the configuration of the hyperspheres $c_{kij}(\underline{y}) = 0$ is fulfilled. In order to find this condition, we first investigate the set of points $\underline{e} \in S^{n-2}$ which can be approached by the vector $\underline{y}(T)$.

<u>Proposition 4.1.</u>

Let D_t (D_s) be the combination of the inner of all hyperspheres $c_i(\underline{y}) = 0$ ($c_{kij}(\underline{y}) = 0$) appearing in expression (8) for Φ,

$$D_t := \bigcup_i \{ c_i(\underline{y}) < 0 \} \ , \quad \forall i \quad \text{with} \quad t_i \neq 0 \ ,$$
$$D_s := \bigcup_{ijk} \{ c_{kij}(\underline{y}) < 0 \} \ , \ \forall ijk \quad \text{with} \quad s_{ij}^{\ k} \neq 0 \ , \tag{20}$$

and let E be the set of all points of S^{n-2} which are located outside (or on the boundary) of all hyperspheres $c_{kij}(\underline{y}) = 0$:

$$E := \{ \underline{e} \in S^{n-2} \ | \ \underline{e} \notin D_s \} . \tag{21}$$

Then for almost all solutions $\Gamma(T)$ of (1), $\underline{y}(T)$ can only approach points belonging to E as $T \to \infty$.

Proof: Let $\lim \underline{y}(T) \neq 2\underline{\alpha}_i \ \forall i$ and $|\underline{y}(T)| \to 1$. Then there exists an $\varepsilon>0$ such that $c_i(\underline{y}) \geq 0 \ \forall \ \underline{y} \in U_\varepsilon(\underline{e})$ and thus $\exists \ T^*(\varepsilon)$ such that $c_i(\underline{y}(T)) \geq 0 \ \forall T \geq T^*$. All integrals $\int^T c_i(\underline{y}(T'))\, dT'$ thus tend to infinity as $T \to 0$. Let us now assume that the unit vector $\underline{e} \notin E$ and that, nevertheless, $\underline{y} \to \underline{e}$ as $T \to \infty$. Then $\underline{e}$ is a point of D_s and there exists an $\varepsilon>0$ and an index triple m,n,l such that $c_{lmn}(\underline{y}) < 0 \ \forall y \in U_\varepsilon(\underline{e})$ and thus $\exists \ T^*(\varepsilon)$ such that $c_{lmn}(\underline{y}(T)) < 0 \quad \forall T \geq T^*$. This yields the following estimation:

$$\int_{T_o}^{\infty} c_{lmn}(T')\,dT' \;\leq\; (T^*-T_o)\,c_{max} - \int_{T^*}^{\infty} |c_{lmn}(T')|\,dT' \ ,$$

which is finite, since the maximum of $c_{lmn}(\underline{y}(T))$ over all $T \in [T_o, T^*]$ is finite. Thus, expression (8) for Φ contains at least one term in the second sum which does not vanish as $T \to \infty$. Since all terms of the first sum vanish ($\int^T c_i(\underline{y}(T'))\, dT' \to \infty$) as $T \to \infty$ and since all terms of the second sum are positive, we obtain $\lim \Phi(T) \neq 0$ and thus from equation (7) $\lim|y(T)| \neq 1$, which is in contradiction to the assumption that $\underline{y}(T)$ approaches a point of S^{n-2}. Δ

Let us now show that the solutions approaching a point $\underline{e} \in E$ are stable. In order to see this, we consider the trajectories $\underline{x} = \underline{x}_o + \underline{e}T$. Since $|\underline{e}| = 1$, these are exact solutions of (1) for all directions $\underline{e} \in S^{n-2}$.

<u>Proposition 4.2.</u>
The generalized Kasner solutions (II.70) , $\Gamma(T) = (\underline{x}_o + \underline{e}T \ , \underline{e})$, $|\underline{e}| = 1$ are
i) instable solutions of (1) if $\underline{e} \notin E$,
ii) stable solutions of (1) if $\underline{e} \in E \setminus \partial E$, $\underline{e} \neq \underline{\alpha}_i$.

Proof: The 2(n-1) $\times$ 2(n-1) matrix M corresponding to the non-autonomeous linearised system for $(\underline{u},\underline{v}) := (\underline{x} - \underline{x}_o - \underline{e}T \ , \underline{y} - \underline{e})$ at (0,0) is

$$M(T) = \begin{pmatrix} 0_{n-1} & \mathbb{1}_{n-1} \\ 0_{n-1} & \underline{e} \otimes \left(\frac{\nabla\Psi}{\Psi}(T) - \gamma\underline{e}\right) \end{pmatrix} \quad . \tag{22}$$

The only non-vanishing eigenvalue of M is equal to the trace and the corresponding eigenspace is spanned by $\underline{e}$ (The equation $(\underline{e} \otimes \underline{b})\, \underline{v} = \lambda\, \underline{v} = (\underline{b}, \underline{v})\, \underline{e}$ implies $\underline{v} = \mu\, \underline{e}$ and thus $\lambda = (\underline{e}, \underline{b}) = \mathrm{tr}\,(\underline{e} \otimes \underline{b})$). Using (2) we obtain

$$\lambda(T) = 2\gamma \left[\frac{t'_i\, (\underline{e},\underline{\alpha}_i)\, \exp[2\gamma(\underline{e},\underline{\alpha}_i)\, T] \;+\; s'^{\,k}_{ij}\, (\underline{e},\underline{\alpha}_{kij})\, \exp[2\gamma(\underline{e},\underline{\alpha}_{kij})\, T]}{t'_i\, \exp[2\gamma(\underline{e},\underline{\alpha}_i)\, T] \;+\; s'^{\,k}_{ij}\, \exp[2\gamma(\underline{e},\underline{\alpha}_{kij})\, T]} - \frac{1}{2} \right] . \tag{23}$$

i) If $\underline{e} \notin E$ then there exists at least one c_{lmn} with $c_{lmn}(\underline{e}) < 0$, thus $(\underline{e},\underline{\alpha}_{lmn}) > 1/2$. Let l,m,n be such that $(\underline{e},\underline{\alpha}_{lmn}) \geq (\underline{e},\underline{\alpha}_{kij})$ $\forall k,i,j$. Since $(\underline{e},\underline{\alpha}_i) < |\underline{\alpha}_i| = 1/2$ $\forall i$, the terms with the exponents $[2\gamma\ (\underline{e},\underline{\alpha}_{lmn})\ T]$ dominate as $T \to \infty$ and we have

$$\lambda(T) \;\to\; 2\gamma\, [\, (\underline{e}\, ,\, \underline{\alpha}_{lmn}) \;-\; 1/2 \,] \quad , \text{ as } T \to \infty \; . \tag{24}$$

Since $(\underline{e},\underline{\alpha}_{lmn}) > 1/2$, there exists a time T^*, such that $\lambda(T) > 0$ $\forall T > T^*$ and thus $(\underline{u},\underline{v}) = (0,0)$ i.e. $(\, \underline{x}\, ,\underline{y}\,) = (\, \underline{x}_0 + \underline{e}T\, ,\, \underline{e}\,)$ is not a stable solution of (1) as $T \to \infty$.

ii) Let $\underline{e} \in E \setminus \partial E$. Then $c_{kij}(\underline{e}) > 0$ $\forall$ k,i,j and thus $(\underline{e},\underline{\alpha}_{kij}) < 1/2$. Since $\lambda(T) \to 2\gamma\, [(\underline{e}\, ,\, \underline{\alpha}) - 1/2]$ as $T \to \infty$, where $(\underline{e}, \underline{\alpha}) := \max\, \{(\underline{e}\, ,\, \underline{\alpha}_{kij})\, ,\, (\underline{e}\, ,\, \underline{\alpha}_i)\} < 1/2$, there exists a time T^*, such that $\lambda(T) < 0$ $\forall T > T^*$. The equation for the component of $\underline{v}$ in the direction of $\underline{e}$ reads

$$d(\underline{v},\underline{e})\, /\, dT \;=\; \lambda(T)\ (\underline{v},\underline{e}) \;+\; o(v^2) \quad , \text{ where } \lambda(T) < 0 \ \ \forall T > T^*.$$

Thus we obtain $(\underline{y},\underline{e}) \to 1$ as $T \to \infty$ and, together with $y \to 1$ we have $\underline{y} \to \underline{e}$. Δ

As a consequence of the preceding Propositions, the solutions of (1) can behave in two qualitatively different ways as $T \to \infty$. Which way is realized depends on the existence of a non-empty set $E \in S^{n-2}$:

<u>Corollary 4.3.</u>

Let $E \in S^{n-2}$ be defined as in Proposition 4.1. and let $\Gamma(T) = (\, \underline{x}(T), \underline{y}(T)\,)$ be a solution of (1,2).

i) If $E \setminus \partial E = \{\ \}$, then $\Gamma(T)$ cannot approach a generalized Kasner solution (up to a set of initial conditions with vanishing measure) and

the modulus of the "velocity" vector $\underline{v}(T)$ is an oscillating function with non-vanishing magnitude as $T \to \infty$ (i.e. $t \to 0$).
ii) If $E \setminus \partial E \neq \{ \}$, then $\Gamma(T)$ can approach a generalized Kasner solution,

$$(\underline{x}, \underline{v}) = (\underline{x}_0 + \underline{e}T, \underline{e}) \text{ with } \underline{e} \in E \setminus \partial E.$$

The condition that $E \setminus \partial E$ is not empty is at least necessary for a generic solution to approach a generalized Kasner solution. If we assume that there exists a time T^* with $\hat{a}(T^*) \in E \setminus \partial E$ then we can show that $\hat{a}(T) \in E \setminus \partial E \ \forall T > T^*$ (see section 7) and the condition also becomes sufficient. This assumption must still be justified by discussing the ergodic properties of the system (1,2). We shall however not treat this question here since it seems to be more difficult than the corresponding problem for the mixmaster map [33].

IV.5. A consequence of the Levi-Malcev decomposition for the Lie algebra $\mathfrak{g}$

In the preceding sections we have reduced the problem of whether a solution behaves regular as $T \to \infty$ to the question of whether there exists a region $E \in S^{n-2}$ such that $c_{kij}(\underline{e}) \geq 0$ for $\underline{e} \in E$ and all k,i,j with $s_{ij}{}^k \neq 0$. This condition depends only on the configuration of the hyperspheres coupling to the non-vanishing constants $s_{ij}{}^k$ which are de- fined as the squares of the structure constants $C^k{}_{ij}$ of $\mathfrak{g}$. The following Proposition shows, that $E \setminus \partial E$ is not empty, whenever $\mathfrak{g}$ has at least a two-dimensional subalgebra. Let us again use small Latin letters for indices running from 1 to n, Greek and capital Latin letters for indices between 1...d and (d+1)...n, respectively.

Proposition 5.1.
Let G_d be a d-dimensional Lie subgroup of G with $d \geq 2$. Let $\underline{e}_d$ denote the "south pole" of S^{n-2} with respect to the d-axis, $\underline{e}_d := (0,\ldots,0,-1,0,\ldots,0)$ and let D_t and D_s be defined as in Proposition 4.1. (18) and

$$\Omega := \{ \underline{v} \in \mathbb{R}^{n-1} \mid |\underline{v} - \underline{e}_d| < \rho \} , \tag{25}$$

where $\rho := 2 - [\,3 + 1/\sqrt{2}\,]^{1/2} > 0$. Then
i) $E \setminus \partial E$ is not empty ,
ii) $\Omega \cap D_t = \Omega \cap D_s = \{\}$.

Proof: We show that none of the hyperspheres $c_i(\underline{y}) = 0$, $c_{kij}(\underline{y}) = 0$ contain the point $\underline{e}_d$. Let us define δ and Δ as

$$\delta := (2\,\gamma)^{-1} A_{d\mu} = \frac{1}{2}\sqrt{\frac{D}{d(n-1)}} \quad , \quad \Delta := -(2\,\gamma)^{-1} A_{dM} = \frac{1}{2}\sqrt{\frac{d}{D(n-1)}} \; . \tag{26}$$

Denoting the d´th components of the centres $\underline{\alpha}_i$ and $\underline{\alpha}_{kij}$ with α_i and α_{kij}, we obtain the following possibilities using (3) and (26) :

$$\alpha_{\kappa\mu\nu} = \alpha_{K\mu N} = \alpha_{\mu} = -\delta \tag{27a}$$
$$\alpha_{\kappa\mu N} = \alpha_{KMN} = \alpha_{M} = \Delta \tag{27b}$$
$$\alpha_{\kappa MN} = [\,\delta + 2\Delta\,] \tag{27c}$$
$$\alpha_{K\mu\nu} = -[\,\Delta + 2\delta\,] \; . \tag{27d}$$

If g_d is a d-dimensional ($d \geq 2$) subalgebra of g then all structure constants of the form $C^K{}_{\mu\nu}$ vanish and thus

$$s_{\mu\nu}{}^{K} = 0 \;, \quad \forall\, K \in [d+1\,, d+D] \;, \quad \forall\, \mu\nu \in [\,1\,, d\,] \; . \tag{28}$$

(Due to the Levi-Malcev decomposition, such a subalgebra always exists for $n > 4$; see also section 7.) In this case $\Psi(x)$ (2) contains no terms with exponents $2\,\gamma\,(\,\underline{\alpha}_{K\mu\nu}\,,\underline{x}\,)$ and thus the terms with $\int^T c_{K\mu\nu}(T')dT'$ are also absent in the corresponding expression for Φ (8). The most negative d-component the vector $\underline{\alpha}_{kij}$ can have is thus not $-[\,\Delta + 2\delta\,]$ as in the general case, but only $-\delta$ (27a,d) :

$$(\,\underline{\alpha}_{kij}\,)_d \geq -\delta = -\frac{1}{2}\sqrt{\frac{D}{d(n-1)}} \overset{d\geq 2}{\geq} -\frac{1}{2}\sqrt{\frac{n-2}{2(n-1)}} > -\frac{1}{\sqrt{8}} \; . \tag{29}$$

Since all hyperspheres contain the point $\underline{y} = 0$ and since by (29) the d´th component of all centres is greater than -1/2, none of the hyperspheres can contain the "south pole" $\underline{e}_d$ of S^{n-2}. More precisely, there exists a neighborhood Ω of $\underline{e}_d$ such that no element of Ω is contained in either one of the sets $c_{kij}(\underline{y}) \leq 0$ or $c_i(\underline{y}) \leq 0$. Let $\underline{y} \in \Omega$, then

$$\begin{aligned} c_{kij}(\underline{y}) &= |\,\underline{y} - \underline{\alpha}_{kij}|^2 - |\,\underline{\alpha}_{kij}|^2 = y^2 - 2\,(\underline{\alpha}_{kij}, \underline{y} - \underline{e}_d) - 2\,(\underline{\alpha}_{kij}, \underline{e}_d) \\ &> (1-\rho)^2 - 2\,R_2\,|\,\underline{y} - \underline{e}_d| + 2\,(\underline{\alpha}_{kij})_d \geq (1-\rho)^2 - 2\,\rho - 2/\sqrt{8} = 0, \end{aligned}$$

where we have used (25),(29) and $R_2 \leq 1$ for $n \geq 3$. For $\underline{y} \in \Omega$ we thus obtain $c_{kij}(\underline{y}) > 0$ since the spheres $c_{K\mu\nu}$ do not exist. This proves ii). Defining E as $S^{n-2} \cap \Omega$, we also obtain i). Δ

IV.6. Construction of an invariant set in the phase space

Let us now assume that g has a subalgebra g_d of dimension $d \geq 2$. Any solution $\Gamma(T)$ with $\underline{y}(T) \in \Omega\ \forall T > T^*$ then fulfills $c_{kij}(T) > 0$ and $c_i(T) > 0\ \forall T > T^*$ for all hyperspheres appearing in (8). We thus obtain $|\underline{y}| \rightarrow 1$ from equation (7) and the solution approaches a Kasner solution as $T \rightarrow \infty$ (i.e as $t \rightarrow 0$).

It remains to show that $\underline{y}(T)$ stays in Ω if $\underline{y}(T^*) \in \Omega$ for a (sufficiently late) time T^*. In order to see this, we have to consider the whole $(n-1) \times (n-1)$-dimensional phase space. Using the representation (II.47) for $\Psi(x)$ we are able to construct an invariant set $X \times Y \in \mathrm{IR}^{n-1} \times \mathrm{IR}^{n-1}$ with $Y \in \Omega$ and to show explicitly that $\underline{y}(T)$ approaches a fixed point $\underline{e} \in E$ as $T \rightarrow \infty$.

In the second part we have written $\Psi(\underline{x})$ in the form

$$\Psi(\underline{x}) = \kappa^{-\alpha}\,[\,r_d + \kappa\,r_D + \kappa^2\,r_{dD} + \kappa^{-1}\,r_{Dd}\,]\ , \tag{30}$$

where r_d, r_D etc. depend on $x_1,\ldots,x_{d-1},x_{d+1},\ldots,x_{n-1}$ and $\kappa := \exp(\omega x_d)$. The advantage of writing $\Psi(x)$ like this consists, on the one hand, in the explicit x_d-dependence and, on the other hand, in the fact that now the terms which are proportional to κ^{-1} are vanishing: The function r_D has been defined in the Part II (section 3). It is proportional to $s_{\mu\nu}{}^K \exp[\,X_k - X_\mu - X_\nu\,]$, where $X = A^T x$. Since all terms in r_{Dd} now contain a vanishing factor $s_{\mu\nu}{}^K = (C^K{}_{\mu\nu})^2/4$, the terms in (30), which are proportional to κ^{-1} do not appear if g has a subalgebra g_d which is at least two-dimensional.

Let us now construct the invariant set $X \times Y$. We shall assume that the region where $\Psi(x)$ is positive is simply connected and bounded in IR^{n-1}. (All candidates for an irregular behavior of the scales, as the Bianchi types VIII and IX, belong to this category. The other cases in which $\Psi(x)$ is positive for a set of directions $\underline{x}/|\underline{x}|$ with non-vanishing measure correspond to models with open potentials for which the solutions behave like Kasner solutions as $T \to \infty$). Let us define M as

$$M := \max_{\Psi(\underline{x})>0} |\underline{x}|$$

Before we construct $X \times Y$, we give the following estimation for the logarithmic derivative of $\Psi(\underline{x})$ with respect to x_d :

<u>Proposition 6.1.</u>
Let $\Psi(\underline{x}) = \kappa^{-\alpha} [\, r_d + \kappa\, r_D + \kappa^2\, r_{dD}\,]$, $\underline{z} := \underline{x} \setminus x_d \in \mathrm{IR}^{n-2}$ and $M < \infty$. Let Ω be defined as in (25) and

$$\Omega_- := \{\, \underline{y} \in \Omega \mid |\underline{y}| \le 1 \,\},$$

$$X_\varepsilon := \left\{\, \underline{x} \in \mathrm{IR}^{n-1} \mid x_d < -M - \frac{1}{\omega} \ln\left(1 + \frac{\varepsilon}{2\omega}\right) \right\}. \tag{31}$$

Then there exists an $\varepsilon > 0$ such that

i) $l_d(\underline{x}) := (\partial\Psi/\partial x_d) / \Psi > -(\omega\alpha + \varepsilon)$, for all $\underline{x} \in X_\varepsilon$, (32)

ii) $[\, \gamma\, y_d - l_d(\underline{x})\,] < 0$, for all $(\underline{x},\underline{y}) \in X_\varepsilon \times \Omega_-$. (33)

Proof: Let $\varepsilon := [\, \gamma - \omega\alpha\,]/2$, where $\gamma = [(n-1)/n]^{1/2}$, $\alpha = D/n$ and $\omega = [\, n/Dd\,]^{1/2}$ (II.32). Since the dimension of the subgroup G_d is $d \ge 2$, we have $\gamma/\omega\alpha \ge \sqrt{2}\, [(n-1)/(n-2)]^{1/2} > 1$ and thus $\varepsilon > 0$. For the radius ρ of Ω (25) we obtain

$$\rho - 1 < -\frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) \le -\frac{1}{2}\left(1 + \sqrt{\frac{D}{d(n-1)}}\right) = -\frac{1}{2}\left(1 + \frac{\omega\alpha}{\gamma}\right) = -\frac{\omega\alpha}{\gamma}\left(1 + \frac{\varepsilon}{\omega\alpha}\right).$$

If the first statement holds, we can use this estimation to prove the second one :

$$\gamma\, y_d - l_d(\underline{x}) \;<\; \gamma(\rho-1) + \omega\alpha + \varepsilon \;<\; -\,\omega\alpha\,(1 + \varepsilon/\omega\alpha) + \omega\alpha + \varepsilon \;=\; 0 \;.$$

Let us now prove i). If G consists in a product of two groups then the factors which depend only on structure constants with both sorts of indices vanish, i.e. $r_{dD}(\underline{z}) = r_{Dd}(\underline{z}) = 0$. Since M is finite, the functions $r_d(\underline{z})$ and $r_D(\underline{z})$ are negative semidefinite. The logarithmic derivative of $\Psi(\underline{x})$ with respect to x_d becomes

$$l_d\,(\underline{x}) = \omega\left[\,-\alpha + \frac{r_D\,(\underline{z})\;\kappa}{r_d\,(\underline{z}) + r_D\,(\underline{z})\;\kappa}\,\right] \geq -\,\omega\,\alpha > -\,(\,\omega\,\alpha\, +\, \varepsilon\,)\;.$$

In the case where $\mathfrak{g}$ has a subalgebra $\mathfrak{g}_d$, the function $r_{dD}(\underline{z})$ does not vanish identically but is negative semidefinite by the definition (II.28c). We can then write $\Psi(\underline{x})$ in the form

$$\Psi(\underline{x}) \;=\; r_{dD}(\underline{z})\;\; \kappa^{-\alpha}\,[\,\kappa - \kappa_1(\underline{z})\,]\,[\,\kappa - \kappa_2(\underline{z})\,]\;, \tag{34}$$

where $\kappa_1(\underline{z})$ and $\kappa_2(\underline{z})$ are bounded functions with $e^{-\omega M} \leq \kappa_1(\underline{z}) \leq \kappa_2(\underline{z}) \leq e^{\omega M}$, since $\Psi(\underline{x}) \leq 0$ for $|\,x_d| > M$. For $\underline{x} \in X_\varepsilon$ we thus obtain from (31)

$$\left[1 - \frac{\kappa_2\,(\underline{z})}{\kappa}\right]^{-1} \geq \left[1 - \frac{\kappa_1\,(\underline{z})}{\kappa}\right]^{-1} \geq \;[1 - e^{-\omega\,M}\,e^{-\omega\,x_d}\,]^{-1} \;>[1 - e^{\ln(\,1 + 2\omega/\varepsilon\,)}\,]^{-1} \;=\; -\,\frac{\varepsilon}{2\omega}\;.$$

The logarithmic derivative of $\Psi(\underline{x})$ with respect to x_d now satisfies the inequality (32):

$$l_d\,(\underline{x}) \;=\; \omega\left\{\,-\alpha + \left[1 - \frac{\kappa_1\,(\underline{z})}{\kappa}\right]^{-1} + \left[1 - \frac{\kappa_2\,(\underline{z})}{\kappa}\right]^{-1}\right\} > \omega\left[\,-\,\alpha\, - 2\,\frac{\varepsilon}{2\omega}\,\right] = -\,(\,\omega\alpha + \varepsilon\,).\;\; \Delta$$

Using inequality (33) we can now give an invariant set in the phase space $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$.

<u>Corollary 6.2.</u>

Let $\quad Y := \{\, \underline{y} \in \mathbb{R}^{n-1} \;\;|\;\; |\underline{y}| \;\leq 1\;\;,\;\; y_d \;\leq\; -1 + \rho^2/2 \;\}$

$\quad\quad\; X \;:=\; X_\varepsilon \;\;,\;\; \text{with} \;\;\; \varepsilon := \;[\,\gamma - \omega\alpha\,]/2\;.$ (35)

Then $\quad \Gamma(T^*) \;\in\; X \times Y \;\;\Rightarrow\; \Gamma(T) \in X \times Y \;\;\; \forall\; T > T^*\;.$

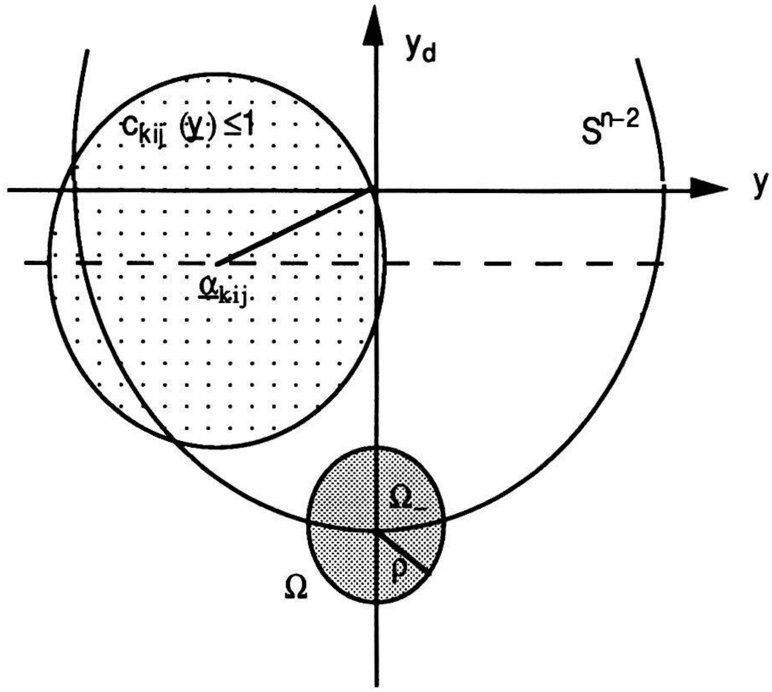

fig.6.1.

The hypersphere Ω contains no points with $c(y) \le 0$ since all centres of the c-spheres are located above the dashed hyperplane. Ω contains the subset Y, which is a part of the invariant set X x Y.

Proof: If $\underline{y} \in Y$ then we have $y_d < 0$ and thus from (1) $dx_d/dT < 0$, i.e. $\underline{x}$ remains in X and $|\underline{y}|$ remains smaller than one. Since Y is a subset of Ω_-, we obtain $(\gamma y_d - l_d(\underline{x})) < 0$ for $(\underline{x},\underline{y}) \in X \times Y$ from (33), and the derivative of y_d is smaller (or equal) than zero :

$$\frac{dy_d}{dT} = \frac{1}{2}(1 - y^2)(\gamma y_d - l_d(\underline{x})) \le 0 \ .$$

Thus, if $(\underline{x},\underline{y}) \in X \times Y$, the d-components of $\underline{x}$ and $\underline{y}$ cannot increase, and we obtain $(\underline{x},\underline{y})(T) \in X \times Y \ \forall \ T > T^*$ if $(\underline{x},\underline{y})(T^*) \in X \times Y$. Δ

Since $\Gamma(T)$ remains in the set $X \times Y$ and since $\Omega \supset \Omega_- \supset Y$, we see that $\underline{y}(T) \in \Omega \ \forall \ T > T^*$ and thus $\underline{y}(T) \to \underline{e} \in E$ as $T \to \infty$. A similar Corollary can also be proved in the case where $\underline{y}(T^*) \in \Omega_+ := \{\underline{y} \in \Omega \mid |\underline{y}| \ge 1\}$.

IV.7. Conclusions

Using the results of the preceding sections, we can now show that chaotic solutions in (n+1)-dimensional homogeneous models can only occur for n = 3. If n > 3, the scales behave regular approaching the cosmo- logical singularity.

Corollary 7.1.

Let G be the Lie group of an (n+1)-dimensional homogeneous model. If $n > 3$, there exists a non-empty set Σ, such that any generic solution approaches a generalized Kasner solution with Kasner indices $\sigma_i \in \Sigma$.

Proof: Since for $n \geq 4$, any real n-dimensional Lie algebra has a subalgebra, there always exists a decomposition $n = d + D$, such that $C^K{}_{\mu\nu} = 0 \;\; \forall \; \mu,\nu \in [1,d]$ and $\forall \; M \in [d+1,n]$. (This is a consequence of the Levi-Malcev decomposition [21] for a Lie algebra into a semisimple and a solvable part [20].) We can thus apply the preceding Propositions to construct a non-empty invariant set $X \times Y$ (35). Every solution $\Gamma(T)$ of (1) entering $X \times Y$ then approaches a generalized Kasner solution with $\underline{y}(T) \to \underline{e} \in E = S^{n-2} \cap \Omega$ as $T \to \infty$, i.e.as $t \to 0$.

The Kasner exponenets are parametrized in terms of $\underline{e}$ (II.71) and fulfil $\Sigma^n \sigma_i^2 = \Sigma^n \sigma_i = 1$, since $|\underline{e}| = 1$ (II.68). Using (25), we have $|\underline{e}-\underline{e}_d| < \rho$ and together with (II.71) we obtain the set Σ :

$$\Sigma = \left\{ \sigma_i \;|\; \sigma_i = -\gamma \left[\sum_j^{n-1} A_{ji}\, e_j - \frac{1}{n\gamma} \right] , \; |\underline{e}| = 1 \; , \; e_d \in \left[-1, -1 + \frac{\rho^2}{2} \right] \right\} . \quad \Delta \tag{36}$$

In Part II we have discussed the models consisting in a product of two isotropic subspaces. These had a two-dimensional phase space with coordinates (x_d, y_d). The condition for the scales to approach a Kasner solution consisted in the existence of a horizontal asymptotic of $l(x_d)/\gamma$ inside the strip $S = \{ (x_d, y_d) \;|\; |y_d| \leq 1 \}$. The toy models discussed in III.5. satisfied this condition if r_{Dd} vanished (III.29), since then

$$\lim_{x_d \to -\infty} l(x_d)/\gamma = -\sqrt{D/d(n-1)} \;\geq\; -\sqrt{1/2}\,\sqrt{(n-2)/(n-1)} \;>\; -1 \; . \tag{37}$$

This estimation obviously corresponds to the estimation (29) for the d-components of the centres $\underline{\alpha}_{kij}$ (if g has subalgebra which is at least two-dimensional) :

$$2\,(\underline{\alpha}_{kij})_d \;\geq\; -2\,\delta \;\overset{d \geq 2}{\geq}\; -\sqrt{\frac{n-2}{2(n-1)}} \; .$$

The region in the plane toy model where y_d can approach the value -1 now corresponds to the set $X \times Y$ where the vector $\underline{y}$ can approach a constant unit vector $\underline{e} \in E$.

Let us finally consider the (3+1)-dimensional homogeneous models (Bianchi types) as an illustration.

Corollary 7.2.

The only (n+1)-dimensional homogeneous models which cannot approach a stable Kasner solution as $t \to 0$ are those which correspond to one of the Lie groups SO(3) or SO(2,1) (i.e. the Bianchi type VIII and IX models).

Proof: Using Corollary 7.1., the only candidates are based on 3-dimensional Lie groups. We have to show that the only groups for which the set $E \setminus \partial E$ in S^1 (20,21) is empty are SO(3) and SO(2,1).

$$E := \{ \underline{e} \in S^1 \mid c_{kij}(\underline{e}) \geq 0 , \ \forall\ k,i,j \text{ with } s_{ij}{}^k \neq 0 \} .$$

For n=3, the hyperspheres S^{n-2}, $c_{kij}(\underline{y})=0$ and $c_i(\underline{y})=0$ reduce to circles. Since the radius of all circles $c_i(\underline{y})=0$ is $R_1 = 1/2$ (10) and since all $c_i(\underline{y})=0$ contain the origin $\underline{y} = 0$, we have $c_i(\underline{e}) \geq 0\ \forall\ \underline{e} \in S^1$. Let us thus consider the circles with radius $R_2 = 1/2\,[1 + 2/\gamma^2]^{1/2} = 1$ $(\gamma^2=2/3)$. These couple to the constants $s_{ij}{}^k$ with $k \notin \{i,j\}$, $i\neq j$ (Proposition 2.2). For n=3 there exist at most three circles of this kind, $c_{123}(\underline{y}) = 0$, $c_{231}(\underline{y}) = 0$ and $c_{312}(\underline{y}) = 0$. Their centres are located on S^1 and on a triangle with equal sides, since $|\underline{\alpha}_{123}| = 1$ and since

$$(\underline{\alpha}_{123}\,,\underline{\alpha}_{312}) = |\underline{\alpha}_2|^2 - |\underline{\alpha}_1|^2 - |\underline{\alpha}_3|^2 + 2(\underline{\alpha}_1\,,\underline{\alpha}_3) = -\frac{1}{4} - \frac{1}{2n\,\gamma^2} = -\frac{1}{2} .$$

If at least one of the structure constants $C^1{}_{23}$, $C^2{}_{31}$, $C^3{}_{12}$ vanishes then not all three circles are present and the set $E \setminus \partial E$ obviously is not empty. The only cases in which $E \setminus \partial E = \{\}$ corresponds thus to the Lie groups with structure constants $C^1{}_{23} \neq 0$, $C^2{}_{31} \neq 0$ and $C^3{}_{12} \neq 0$ which are SO(3) and SO(2,1). Δ

Fig.7.1. illustrates this situation. The combination of the sets $c_{kij} \leq 0$ contains all points of the set $|\underline{y}| \leq 1$. The centres are $\underline{\alpha}_{123}$ =

$(-\sqrt{3}/2\ ,\ 1/2)$, $\underline{\alpha}_{231} = (\sqrt{3}/2\ ,\ 1/2)$ and $\underline{\alpha}_{321} = (0\ ,\ -1)$. The cenres of the small circles (radius $R_1=1/2$) are $\underline{\alpha}_1 = (\sqrt{3}/4\ ,\ -1/4)$, $\underline{\alpha}_2 = (-\sqrt{3}/4\ ,\ -1/4)$ and $\underline{\alpha}_3 = (0\ ,\ 1/2)$. The set $E \setminus \partial E$ is empty, and E consists in the three points $2\underline{\alpha}_1$, $2\underline{\alpha}_2$ and $2\underline{\alpha}_3$. As already mentioned, there exist three instable solutions with Kasner indices $(\sigma_1,\sigma_2,\sigma_3) = (2/3,\ -1/3,\ 2/3)$ for $\underline{y} \to 2\underline{\alpha}_1$, $(-1/3,\ 2/3,\ 2/3)$ for $\underline{y} \to 2\underline{\alpha}_2$ and $(2/3,\ 2/3,\ -1/3)$ for $\underline{y} \to 2\underline{\alpha}_3$. For each of these cases there exist two identical contracting scale factors and one expanding scale as $t \to 0$.

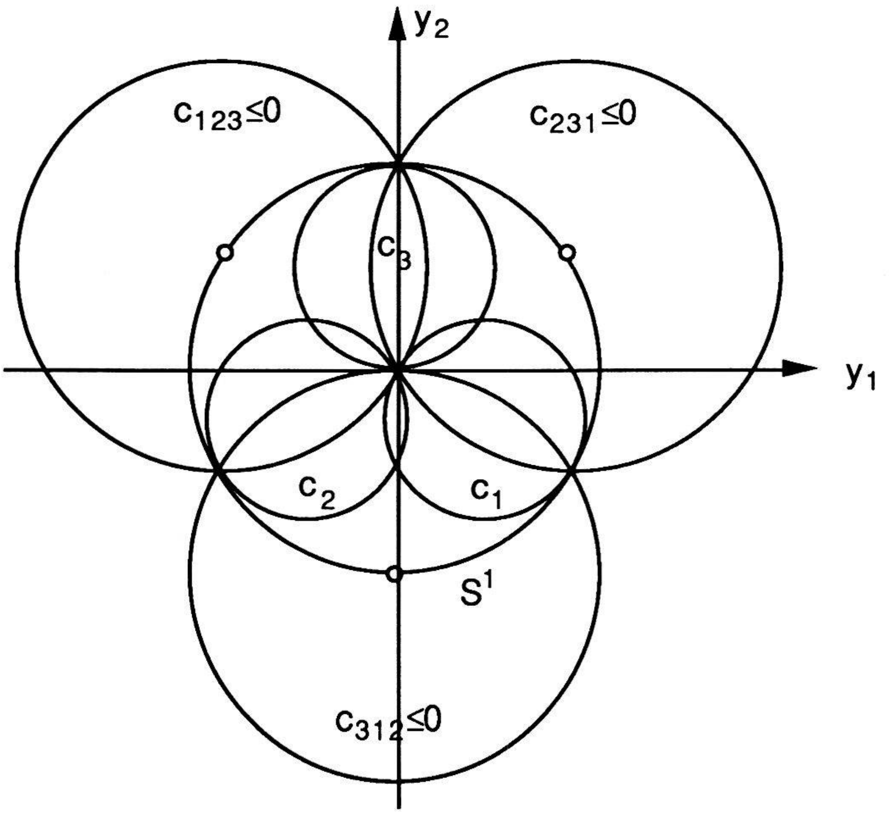

fig.7.1. SO(3) and SO(2,1) are the only Lie groups for which the combination of all sets $c_{kij} \leq 0$ contains the whole hypersphere (circle) $|\underline{y}| \leq 1$.

Using $\kappa = \exp(\omega x_d) = \exp[\sqrt{(3/2)}\ x_2]$, the potential $\Psi(\underline{x})$ (2, II.37) may be written in the form (30) ,

$$\Psi(\underline{x}) = \kappa^{-\alpha} [\pm 2\mathrm{ch}(x_1/\sqrt{2}) + (1-\mathrm{ch}(\sqrt{2}x_1))\ \kappa \ -1/2\ \kappa^{-1}] \quad ,(38)$$

where the upper (lower) sign holds for SO(3) (SO(2,1)) and the term which is proportional to κ^{-1} now does not vanish. As is well known [19] and easily seen from (38), there exist three directions in the (x_1,x_2)-plane where the equipotentials of $\Psi(\underline{x})$ are not closed . In the time-dependent Hamiltonian formulation of the problem (II,section 4.1.), the "universe point" can only move in one of these directions without colliding with a potential wall. The directions are $2\underline{\alpha}_1, 2\underline{\alpha}_2$ and $2\underline{\alpha}_3$, in agreement with the directions of the instable Kasner solutions mentioned above.

IV.8. A comment on inhomogeneous models

Let us finally extend the discussion to inhomogeneous cosmological models. The arguments are of the same kind as in the preceding sections. The structure terms $C^k{}_{ij}$ are still constants with respect to the time coordinate, but they now depend on the spatial coordinates. The explicit form of these functions does not affect the following discussion. The important point is that in contrast to the homogeneous case, there now exists no decomposition $n = D + d$, such that all $C^K{}_{\mu\nu}$ with $\mu\nu \in [1,d]$ and $K \in [d+1,n]$ vanish. The most negative d-component a centre $\underline{\alpha}_{kij}$ of a hypersphere $c_{kij}(\underline{v}) = 0$ can now have is no longer $-\delta$ but $-(2\delta+\Delta)$ (27). Thus, even for $n>3$, there may exist some hyperspheres containing the "south pole" $\underline{e}_d$ of S^{n-2}. In order to find the critical number of dimensions, we again have to answer the geometrical question whether there exists a set E on S^{n-2} which has no common points with D_s (21). (D_s is the combination of the sets $\{ c_{kij}(\underline{v}) < 0 \}$, where now all k,i,j must be taken into account (20).)

Proposition 8.1.

Let $s_{ij}{}^k \neq 0 \ \forall k\neq i\neq j\neq k$ and let D_s be the combination of all corresponding sets $\{ c_{kij}(\underline{v}) < 0 \}$. Let $E := \{ \underline{e} \in S^{n-2} \mid \underline{e} \notin D_s \}$.
If $n \geq 10$, there exists at least one decomposition $n = D + d$, such that $E \setminus \partial E$ is not empty.

Proof: Let us show that either a whole neighborhood of the "south pole" or of the "north pole" (with respect to the d-component) of S^{n-2} is located outside the set D_s. This is the case, if either the most negative d-component of all centres $\underline{\alpha}_{kij}$ is greater than $-1/2$ or if the most positive d-component is smaller than +1/2. Using equation (27), we obtain the conditions

$$-[\,\Delta + 2\delta\,] > -1/2 \quad \text{or} \quad [\,\delta + 2\Delta\,] < +1/2 \quad .$$

Together with (26) and D+d=n we obtain the following inequalities :

$$d^2 - d(3+n) + 4n = \left[d - \frac{3+n}{2}\right]^2 + \frac{1}{4}(9-n)(n-1) < 0 \ ,$$
$$d^2 + d(3-n) + 4n = \left[d + \frac{3-n}{2}\right]^2 + \frac{1}{4}(9-n)(n-1) < 0 \ .$$

If $n \leq 9$, none of these inequalities is fulfilled and the "south pole" as well as the "north pole" are contained inside the set D_s. If, on the other hand, $n \geq 10$ then the first inequality holds for all decompositions with $[n+3 - \{(n-9)(n-1)\}^{1/2}] < 2d < [n+3 + \{(n-9)(n-1)\}^{1/2}]$. Δ

Since the choice of the decomposition n=D+d only concerns the definition of the coordinates introduced in Part II, section 4.2., we have shown, that for $n \geq 10$ there always exists a point $\underline{e} \in S^{n-2}$, a real number $\rho > 0$ and a set $\Omega := \{\, \underline{y} \in R^{n-1} \mid |\underline{y} - \underline{e}| < \rho \,\}$, such that $\Omega \cap D_s = \{\,\}$. Approaching the initial singularity in (n+1)-dimensional ($n \geq 3$) cosmological models, the generic solutions of the field equations behave thus

- chaotic if $n < 10$ and monotonic if $n \geq 10$ in inhomogeneous cosmological models ;

- chaotic if $n = 3$ and $G = SO(3)$ or $SO(2,1)$ and Kasner-like in all other homogeneous cosmological models with an arbitrary number of dimensions.

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