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ABOUT ADMISSIBLE BOUNDARY CONDITIONS FOR EULER AND  
PARABOLIZED NAVIER-STOKES EQUATIONS

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Dedicated to Gérard Wanders on the occasion of his 60th birthday

**Abstract**

We consider the parabolized approximation of Navier-Stokes equations for the two-dimensional steady flow of an incompressible or isentropic fluid. First, the equations and a perturbative method to get them are presented; then, the notion of admissible boundary conditions in the sense of Friedrichs systems of differential equations is introduced. Finally, various admissible conditions for the parabolized Navier-Stokes equations and, as a by-product, for Euler equations are exhibited.

**1. Preliminaries**

Parabolized Navier-Stokes (PNS) equations are used to describe the high-speed (e.g. supersonic) steady flow of a viscous compressible gas over a blunt body when there is a preferred direction, in which the component of the displacement velocity of the fluid is positive [1]. For numerical purposes, it is essential to have boundary conditions (BC) for these

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equations such that the resulting problem is well-posed; to our knowledge, this issue has never been addressed in the case of a bounded domain.

Consider the steady state Navier-Stokes equations for an incompressible fluid, without external force :

$$\begin{cases} (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{0}, \\ \operatorname{div} \mathbf{u} = 0, \end{cases} \quad \begin{matrix} (1.1a) \\ (1.1b) \end{matrix}$$

where, in cartesian coordinates  $\mathbf{x} = (x, y)$ ,  $\mathbf{u} = (u, v)$  is the displacement velocity,  $p$  the pressure and  $\nu^{-1} > 0$  the Reynolds number. Throughout this paper,  $\nu$  will be assumed to be positive. The PNS equations are obtained from (1.1) by neglecting the diffusion in the  $x$ -direction, i.e. :

$$\begin{cases} (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \partial_y^2 \mathbf{u} + \nabla p = \mathbf{0}, \\ \operatorname{div} \mathbf{u} = 0, \end{cases} \quad \begin{matrix} (1.2a) \\ (1.2b) \end{matrix}$$

**Remark 1.1 :** *Like Prandtl's boundary-layer equations, the PNS equations are simplified Navier-Stokes equations, but they are valid in a larger region than the boundary-layer.*

For a compressible fluid, we restrict ourselves to the isentropic case [2], where the equation of state ( $p = A\rho^\gamma$ ,  $\rho$  : mass density,  $A, \gamma$  : constants) allows to decouple the mechanical conservation laws from the thermodynamical one. The PNS equations are obtained like above by neglecting the second-order derivatives with respect to  $x$  and read

$$\begin{cases} \rho \mathbf{u} \cdot \nabla \mathbf{u} + \partial_x p - \nu \partial_y^2 \mathbf{u} = \mathbf{0}, \\ \rho \mathbf{u} \cdot \nabla \mathbf{v} + \partial_y p - \frac{4}{3} \nu \partial_y^2 \mathbf{v} = 0, \\ \rho \operatorname{div} \mathbf{u} + a^{-2} \mathbf{u} \cdot \nabla p = 0, \\ p = a^2 \rho; \end{cases} \quad \begin{matrix} (1.3a) \\ (1.3b) \\ (1.3c) \\ (1.3d) \end{matrix}$$

we have assumed for simplicity that the sound speed  $a = \sqrt{p\gamma/\rho}$  is constant.

Except in the next section, we shall work in the bounded domain  $\Omega = (0, 1) \times (0, 1)$  with boundary  $\partial\Omega = \Gamma_- \cup \Gamma_0 \cup \Gamma_+$ , where

$$\begin{aligned} \Gamma_- &= \{(x, y) \in \partial\Omega \mid x = 0, 0 < y < 1\}, \quad \Gamma_+ = \{(x, y) \in \partial\Omega \mid x = 1, 0 < y < 1\}, \\ \Gamma_0 &= \Gamma_1 \cup \Gamma_2, \quad \Gamma_1 = \{(x, y) \in \partial\Omega \mid y = 0\}, \quad \Gamma_2 = \{(x, y) \in \partial\Omega \mid y = 1\}. \end{aligned}$$

This geometry corresponds to the flow over a flat plate lying on the positive  $x$ -axis; more general situations can be handled by replacing  $(x, y)$  by curvilinear coordinates, the type of the equations being unchanged.

## 2. Parabolized Oseen's equations

We add to eqs (1.2) a right hand side  $\mathbf{f}$  which may arise from inhomogeneous BC and set

$$\mathbf{q} = \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix}, \quad A_x(\mathbf{u}) = \begin{pmatrix} u & 0 & 1 \\ 0 & u & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_y(\mathbf{u}) = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad D = \text{diag} [1,1,0]; \quad (2.1)$$

then, we get the system

$$A_x(\mathbf{u}) \partial_x \mathbf{q} + A_y(\mathbf{u}) \partial_y \mathbf{q} - \nu D \partial_y^2 \mathbf{q} = \mathbf{f}. \quad (2.2)$$

These equations can be linearized by replacing  $\mathbf{u}$  in the matrices by a given velocity  $\mathbf{c} = (c,d)$ ,  $c > 0$ ; the same procedure applied to the Navier-Stokes equations (1.1) yields Oseen's equations

$$P(\partial_x, \partial_y) \mathbf{q} \equiv A_x(\mathbf{c}) \partial_x \mathbf{q} + A_y(\mathbf{c}) \partial_y \mathbf{q} - \nu D (\partial_x^2 + \partial_y^2) \mathbf{q} = \mathbf{f}. \quad (2.3)$$

We intend to show, using the method developped in [3] for the case  $d \equiv 0$ , that the linearized version of System (2.2) is an approximation of (2.3) when  $\nu$  and  $s = d/c$  are close to zero. For simplicity, we assume that  $\mathbf{c}$  is constant and look for a solution of (2.3), in the domain  $\Omega_\infty = \mathbb{R}_+ \times \mathbb{R}$ , of the type

$$\mathbf{q}(x,y) = \sum_{k=1}^4 \int_{\mathbb{R}} \exp(i\mu y + \lambda_k(\mu)x) \hat{\mathbf{q}}_j(\mu) d\mu + \mathbf{q}_0(x,y),$$

with  $\mathbf{q}_0$  a particular solution of the inhomogeneous system; the generalized eigenvalues  $\lambda_k$ , such that the matrix  $P(\lambda, i\mu) = \lambda A_x + i\mu A_y - \nu(\lambda^2 + (i\mu)^2) D$  is singular, determine the behavior of  $\mathbf{q}$  as  $x \rightarrow \infty$ . The first two eigenvalues are  $\lambda_1 = |\mu|$ ,  $\lambda_2 = -|\mu|$  and the other ones have the asymptotic expansion

$$\lambda_3 = -i\mu s - \frac{\mu^2}{c} (1+s^2)\nu + O(\nu^2), \quad \lambda_4 = c\nu^{-1} + i\mu s + O(\nu), \quad \nu \rightarrow 0.$$

In order to get the approximation, we drop  $\lambda_4$  (responsible for a divergent behavior when  $x \rightarrow \infty$ ), we keep  $\lambda_1$  (the divergence of which will be killed by a regularity condition) and  $\lambda_2$ , but we replace  $\lambda_3$  by its asymptotic expansion up to the order  $\nu$ . These new eigenvalues are the roots of the determinant of the matrix  $P_s(\lambda, i\mu) = \lambda A_x + i\mu A_y - \nu(1+s^2)(i\mu)^2 D$ , which is associated to the system

$$P_s(\partial_x, \partial_y) \mathbf{q} \equiv A_x(\mathbf{c}) \partial_x \mathbf{q} + A_y(\mathbf{c}) \partial_y \mathbf{q} - \nu(1+s^2) D \partial_y^2 \mathbf{q} = \mathbf{f}. \quad (2.4)$$

Consider the problems of solving (2.3) or (2.4) with the BC  $\mathbf{u}|_{x=0} = \mathbf{0}$ ; then, the Fourier technique of [3] allows us to prove that, for sufficiently regular data  $\mathbf{f}$ , both problems have a solution with unique velocities  $\mathbf{u}$ , resp.  $\mathbf{u}_s$ , which belong to  $H^1(\Omega_\infty)^*$  and one has the estimate

$$\|\partial_y(\mathbf{u} - \mathbf{u}_s)\|_{L^2(\Omega_\infty)^2} + \|\mathbf{u} - \mathbf{u}_s\|_{L^2(\Omega_\infty)^2} = O(\nu^2), \quad \nu \rightarrow 0.$$

This result shows in what sense System (2.4) is an approximation of the Oseen equations; it remains to establish a bound for the difference  $\mathbf{e} = \mathbf{u}_s - \mathbf{u}_0$  of the velocities satisfying (2.4) with  $s \neq 0$  and  $s = 0$ . This can be done only in a finite domain, e.g. the unit square  $\Omega$ . We assume that there exists a unique solution  $\mathbf{q}_s \in H^1(\Omega)^3$  of (2.4) satisfying (for instance) the BC

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_- \cup \Gamma_0, \quad p = 0 \quad \text{on } \Gamma_+ \quad (2.5)$$

(this holds if  $\mathbf{f}$  is regular enough [4]). From the first two equations (2.4), we get an equation for  $\mathbf{e}$ ; taking the dot product of this latter with  $\mathbf{e}$ , integrating over  $\Omega$ , performing some integrations by parts and using the equation for  $p_s - p_0$  obtained from the last equation (2.4), yields, with the help of the Cauchy-Schwartz inequality:

$$\|\partial_y \mathbf{e}\|_{L^2(\Omega)^2} \leq s^2 \|\partial_y \mathbf{u}_s\|_{L^2(\Omega)^2};$$

finally, Poincaré's inequality  $\|\mathbf{e}\|_{L^2(\Omega)^2} \leq \alpha \|\partial_y \mathbf{e}\|_{L^2(\Omega)^2}$ ,  $\alpha > 0$ , implies that

$$\|\partial_y(\mathbf{u}_s - \mathbf{u}_0)\|_{L^2(\Omega)^2} + \|\mathbf{u}_s - \mathbf{u}_0\|_{L^2(\Omega)^2} = O(s^2), \quad s \rightarrow 0.$$

Consequently, the linear PNS equations (2.4) with  $s = 0$  can be viewed, for  $s^2 = o(1)$  (e.g.  $s = O(\nu)$ ) as an approximation of Oseen's equations (2.3).

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\* For an open domain  $\Omega \subset \mathbb{R}^2$ ,  $H^1(\Omega)$  denotes the Sobolev space of functions  $\Omega \rightarrow \mathbb{R}$  which, together with their first-order derivatives, are in  $L^2(\Omega)$ .

### 3. Admissible boundary conditions

Admissible BC were first introduced within the theory of linear first-order differential equations system [5], e.g. in our case :

$$\mathcal{A}q(x) \equiv A_x(x)\partial_x q(x) + A_y(x)\partial_y q(x) + K(x)q(x) = f(x), x \in \Omega, \quad (3.1)$$

where the unknown  $q$  and the r.h.s.  $f$  take values in  $\mathbb{R}^p$  and the matrix-valued functions  $A_x, A_y$  are assumed to be symmetric. With the help of a matrix  $M$  defined on  $\partial\Omega$  and of the matrix

$$B(x) = (n_x A_x + n_y A_y)(x), x \in \partial\Omega, \quad (3.2)$$

( $n$  : outward unit normal to  $\partial\Omega$ ), homogeneous Dirichlet BC for (3.1) are laid down by requiring that

$$q(x) \in \text{Ker}(B-M)(x), x \in \partial\Omega. \quad (3.3)$$

The BC are called *admissible* iff

$$\begin{cases} \cdot \text{ the matrix } M + M^t \text{ is positive semi-definite } (M+M^t \geq 0) \text{ on } \partial\Omega, & (3.4a) \\ \cdot \text{ Ker}(B-M) + \text{Ker}(B+M) = \mathbb{R}^p \text{ on } \partial\Omega. & (3.4b) \end{cases}$$

**Example :** The matrix  $M = \sqrt{B^2}$  generates admissible BC.

Finally, the system (3.1) is said to be *positive* iff the matrix  $C = K + K^t - \partial_x A_x - \partial_y A_y$  is positive definite in  $\Omega$ .

We quote hereafter, in an informal way, the basic results of [5]. A *symmetric positive system with admissible BC* has at least one solution  $q \in L^2(\Omega)^p$ , i.e.

$$\int_{\Omega} q^t \mathcal{A}^* \varphi dx = \int_{\Omega} f^t \varphi dx \quad \forall \varphi \in C^1(\bar{\Omega})^p \text{ with } \varphi \in \text{Ker}(B+M^t) \text{ on } \partial\Omega, \\ f \in L^2(\Omega)^p,$$

where  $\mathcal{A}^*$  is the formal adjoint of  $\mathcal{A}$ ; moreover, if  $q$  is regular enough (e.g. in  $H^1(\Omega)^p$ ), the solution is unique and satisfies the BC in the sense of traces. These results led us to consider admissible BC for (linearized) PNS equations.

#### 4. The incompressible case

It is quite usual to infer BC for nonlinear equations from those of their linear version (see [6] for instance). Hence, we consider the linearized PNS system (2.4) with  $s = 0$  in the domain  $\Omega$ ; introducing the unknowns  $q = \partial_y u$  and  $r = \partial_y v$ , it takes the standard form (3.1) with

$$\mathbf{q} = \begin{pmatrix} \mathbf{u} \\ p \\ q \\ r \end{pmatrix}, \quad A_x = \left( \begin{array}{c|c} \bar{A}_x & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right), \quad A_y = \left( \begin{array}{c|c} \bar{A}_y & \begin{matrix} -v & 0 \\ 0 & -v \\ 0 & 0 \end{matrix} \\ \hline \begin{matrix} -v & 0 & 0 \\ 0 & -v & 0 \end{matrix} & \mathbf{0} \end{array} \right),$$

$$K = \text{diag } [0, 0, 0, v, v], \quad (4.1)$$

where  $\bar{A}_x$  and  $\bar{A}_y$  are the matrices for the corresponding Euler system ( $v=0$ ), defined by (2.1) with  $\mathbf{c}(\mathbf{x})$  in place of  $\mathbf{u}$ . It is easy to check that equations (3.1) and (4.1) yield a symmetric positive system if  $\text{div } \mathbf{c} < 0$ ; unfortunately, this is unphysical since  $\mathbf{c}$  must mimic  $\mathbf{u}$ . However, it is shown in [4] that most admissible BC in the present case lead to the same results we would obtain if the system were positive.

It is worthwhile noticing that, since the matrix  $B$  (3.2) is given by

$$B = \left( \begin{array}{c|c} \bar{B} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right) = \left( \begin{array}{c|c} n_x \bar{A}_x & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right) \quad \text{on } \Gamma_- \cup \Gamma_+,$$

we obtain, from admissible BC for the Euler system generated with the matrix  $\bar{M}$ , admissible BC for the PNS system with the help of

$$M = \left( \begin{array}{c|c} \bar{M} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right) \quad \text{on } \Gamma_- \cup \Gamma_+; \quad (4.2)$$

of course, written down in function of  $u, v, p$  these BC coincide.

In order to describe the physical situation of the flow over a flat plate, we make the following hypotheses on the given velocity  $\mathbf{c}$ :

$$\mathbf{c} \in C^1(\bar{\Omega})^2, \quad \text{div } \mathbf{c} = 0 \text{ in } \Omega, \quad (4.3a)$$

$$\mathbf{c} \cdot \mathbf{n} = 0 \text{ on } \Gamma_0, \quad \mathbf{c} \cdot \mathbf{n} > 0 \text{ on } \Gamma_+, \quad \mathbf{c} \cdot \mathbf{n} < 0 \text{ on } \Gamma_-. \quad (4.3b)$$

First, we want to show that the standard BC (2.5) are admissible. On  $\Gamma_2$ , one has  $B = A_y$  and the matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & -v & 0 \\ 0 & 0 & 1 & 0 & -v \\ 0 & -1 & 0 & 0 & 0 \\ v & 0 & 0 & 0 & 0 \\ 0 & v & 0 & 0 & 0 \end{pmatrix} \quad (4.4)$$

generates the admissible BC  $\mathbf{u} = \mathbf{0}$ ; on  $\Gamma_1$ , since  $B = -A_y$ , we get the same BC by replacing  $M$  by  $-M$ . By setting

$$\bar{M} = \begin{pmatrix} c & 0 & -1 \\ 0 & c & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

in (4.2), we get the admissible BC  $\mathbf{u} = \mathbf{0}$  on  $\Gamma_-$  and also  $p = 0$  on  $\Gamma_+$ . We remark that the condition  $\mathbf{u} = \mathbf{0}$  on  $\Gamma_0$  does not depend from the hypothesis  $\mathbf{c} \cdot \mathbf{n} = 0$  there.

Other simple admissible conditions for the PNS system are for instance :

$$\begin{cases} c u + p = 0, v = 0 \text{ on } \Gamma_- , \\ u = 0 \text{ on } \Gamma_+ , \end{cases} \quad (4.5a)$$

$$(4.5b)$$

with

$$\bar{M} = \begin{pmatrix} c & 0 & 1 \\ 0 & c & 0 \\ -1 & 0 & 0 \end{pmatrix} \text{ in eq. (4.2) ;}$$

$$u = 0, -p + v \partial_y v = 0 \text{ on } \Gamma_0, \quad (4.6)$$

with the matrix  $M$  (4.4) on  $\Gamma_1$  and  $-M$  on  $\Gamma_2$ . This latter BC is a zero-strain condition, frequently used for the Navier-Stokes equations.

It is also interesting to look at the BC given by the choice  $M = \sqrt{B^2}$ ; an easy computation of the eigenvalues and eigenvectors of  $B$  yields :

$$\begin{cases} u + \frac{1}{2} (\sqrt{c^2+4} - c) p = 0, v = 0 \text{ on } \Gamma_- , \\ u - \frac{1}{2} (c + \sqrt{c^2+4}) p = 0 \text{ on } \Gamma_+ . \end{cases} \quad (4.7a)$$

$$(4.7b)$$

## 5. The isentropic case

This section deals with the linearized version of System (1.3) obtained by adding a r.h.s. due to inhomogeneous BC, setting  $\rho = 1$  and replacing  $\mathbf{u} \cdot \nabla$  by  $\mathbf{c} \cdot \nabla$ , where  $\mathbf{c} = (c, d)$  satisfies (4.3); the underlying physics is the isentropic flow of a *weakly*



*compressible fluid*, the density of which ( $\rho = 1 + \rho_1$ ,  $\rho_1 \ll 1$ ) is almost constant. The resulting system can be put into the standard form (3.1), with the new unknowns  $q = \partial_y u$ ,  $r = \partial_y v$ , by setting

$$q = \begin{pmatrix} u \\ p \\ q \\ r \end{pmatrix}, A_x = \begin{pmatrix} \bar{A}_x & 0 \\ 0 & 0 \end{pmatrix}, A_y = \left( \begin{array}{ccc|ccc} & & & & -v & 0 \\ & & & & 0 & -4/3 v \\ & & & & 0 & 0 \\ \hline -v & 0 & 0 & & & \\ 0 & -4/3 v & 0 & & & \\ & & & 0 & & \end{array} \right),$$

$$K = \text{diag} [0, 0, 0, v, 4/3 v], \quad (5.1)$$

where the matrices

$$\bar{A}_x = \begin{pmatrix} c & 0 & 1 \\ 0 & c & 0 \\ 1 & 0 & c a^{-2} \end{pmatrix}, \quad \bar{A}_y = \begin{pmatrix} d & 0 & 0 \\ 0 & d & 1 \\ 0 & 1 & d a^{-2} \end{pmatrix}$$

are those of the corresponding Euler problem. Here again, admissible BC for the Euler system on  $\Gamma_- \cup \Gamma_+$ , defined with the help of a matrix  $\bar{M}$ , are also admissible for the PNS system and given by the matrix  $M$  (4.2).

**Remark 5.1 :** *As far as BC are concerned, the assumption of weak compressibility does not play any role.*

Compared to the incompressible case, the compressible problem has a new feature : there are two regions in the flow; with regard to the PNS equations, one must distinguish between the "supersonic" zone in which  $c > a$  and the "subsonic" zone where  $c < a$ . With the change of variables  $q = \tilde{q} \exp(\alpha x)$ , we get from (3.1) the equivalent system

$$A_x \partial_x \tilde{q} + A_y \partial_y \tilde{q} + (K + \alpha A_x) \tilde{q} = \exp(-\alpha x) f. \quad (5.3)$$

The following conditions are sufficient to insure that the symmetric system (5.3), (5.1) and (5.2) is positive :

- (i) For  $c > a$  :  $\alpha > 0$  if  $\text{div } c = 0$  ;  $\alpha = \frac{\max \text{div } c}{\min (c-a)}$  if  $\text{div } c \neq 0$ .

(ii) For  $c < a$ :  $\max \frac{\text{div } \mathbf{c}}{2(c+a)} < \alpha < \min \frac{\text{div } \mathbf{c}}{2(c-a)}$  if  $\text{div } \mathbf{c} < 0$ .

In the case  $c < a$ ,  $\text{div } \mathbf{c} = 0$ , admissible BC yield again the same existence and uniqueness results as for a positive system [4].

We have also to distinguish between two parts of the boundary, namely :

$$\Gamma_s = \{ \mathbf{x} \in \partial\Omega \mid c(\mathbf{x}) > a \}, \quad \Gamma_i = \{ \mathbf{x} \in \partial\Omega \mid c(\mathbf{x}) < a \}.$$

Standard BC for the PNS system are given by

$$\begin{cases} c u + p = 0, v = 0 \text{ on } \Gamma_- \cap \Gamma_i, & (5.4a) \\ \mathbf{u} = \mathbf{0}, p = 0 \text{ on } \Gamma_- \cap \Gamma_s, & (5.4b) \\ \mathbf{u} = \mathbf{0} \text{ on } \Gamma_0, & (5.4c) \\ p = 0 \text{ on } \Gamma_+ \cap \Gamma_i; & (5.4d) \end{cases}$$

thus, we see that on the supersonic inflow  $\Gamma_- \cap \Gamma_s$ , every unknown has to be prescribed, whereas on the supersonic outflow  $\Gamma_+ \cap \Gamma_s$  no condition is required. The BC (5.4) are admissible, given by the following matrices :

$$\begin{aligned} \bar{M} &= \begin{pmatrix} c & 0 & -1 \\ 0 & c & 0 \\ 1 & 0 & 2/c - c/a^2 \end{pmatrix} \text{ in (4.2), on } (\Gamma_- \cup \Gamma_+) \cap \Gamma_i, \\ \bar{M} &= \begin{pmatrix} c & 0 & -1 \\ 0 & c & 0 \\ 1 & 0 & c/a^2 \end{pmatrix} \text{ in (4.2), on } \Gamma_- \cap \Gamma_s, \\ M &= \begin{pmatrix} 0 & 0 & 0 & -v & 0 \\ 0 & 0 & 1 & 0 & -4/3 v \\ 0 & -1 & 0 & 0 & 0 \\ v & 0 & 0 & 0 & 0 \\ 0 & 4/3 v & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (5.5)$$

on  $\Gamma_2$  and  $-M$  on  $\Gamma_1$ .

On the horizontal boundaries, the matrices  $M$  (5.5) on  $\Gamma_1$  and  $-M$  on  $\Gamma_2$  define the admissible zero-strain condition

$$\partial_y u = 0, \quad -p + \frac{4}{3}v \partial_y v = 0 \text{ on } \Gamma_0. \quad (5.6)$$

Finally, the matrix  $M = \sqrt[3]{B^2}$  generates the conditions

$$(c + \lambda_+) u + p = 0, \quad v = 0 \text{ on } \Gamma_- \cap \Gamma_i, \quad (5.7a)$$

(and of course  $u = 0, p = 0$  on  $\Gamma_- \cap \Gamma_s$ ),

$$u + (\lambda_- - c) p = 0 \text{ on } \Gamma_+ \cap \Gamma_i, \quad (5.7b)$$

with  $\lambda_{\pm} = \frac{1}{2}(c(1+a^{-2}) \pm \sqrt{c^2(1+a^{-2})^2 - 4(c^2a^{-2}-1)})$ .

**Remark 5.2 :** The BC on  $(\Gamma_- \cup \Gamma_+) \cap \Gamma_i$  look like those for the incompressible PNS system; in this latter case  $a$  is infinite and  $\Gamma_s = \emptyset$ . For instance, the BC (5.4a), (5.4d) become (4.5) and (5.4c) is also admissible when  $a$  tends to infinity. However, it is very important to notice that the condition  $u = 0$  on  $\Gamma_-$  is not admissible. The matrix  $M$  defining this condition would be such that  $(0,0,1,0,0)^t \in \text{Ker}(B-M)$  and consequently its third diagonal element should be equal to  $-c a^{-2}$ , thus preventing  $M$  be positive semi-definite.

**Remark 5.3 :** Some of the boundary conditions proposed in this paper coincide with results of [6], where time dependent compressible Euler equations are studied.

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