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Perturbation Theory for Many Fermion Systems

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Abstract

Perturbation theory is analysed for a class of standard many Fermion systems. At positive temperature we show that perturbation theory is finite to all orders and exponentially bounded. At zero temperature an expansion is developed whose coefficients are bounded by $C^n n!$. The Fermi surface is studied in perturbation theory.

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I. Introduction

In this paper we study perturbation theory about the spherically symmetric independent electron approximation for a crystal at temperature zero.

We start by describing the d-dimensional independent electron approximation. Imagine a finite crystal with "ions" fixed at the points of $\mathbf{Z}^d \cap [-\frac{L}{2},\frac{L}{2}]^d$ (with L a positive integer) and with N freely moving electrons. Each electron moves in a common electric field $-\vec{\nabla}U(x)$ which is periodic with respect to $\mathbf{Z}^d$. Imposing periodic boundary conditions on the Laplacian, the Hamiltonian for this system of electrons is

$$\sum_{i=1}^{N} [-\frac{1}{2m}\Delta_i + U(x_i)].$$

In this paper, we shall, for pedagogical reasons, consider only the spherically symmetric case in which U = 0. The general case will be treated in another paper.

The independent electron approximation is the limit of these finite systems as N and L tend to infinity with the density $\rho = \frac{N}{L^d}$ held fixed. This limit is described by a statistical mechanical Fock space

$$F = \bigoplus_{n,m=0}^{\infty} F_{n,m}.$$

$F_{0,0} = \mathbf{C}$ is identified with set of, all multiples of Ω, the fermionic ground state. The state Ω, normalized by $\|\Omega\|=1$, should be thought as the Fermi sea in which, by the Pauli exclusion principle, each single particle electron state of energy less than or equal to $\mu = \frac{2\pi}{m}\left[\frac{d}{4}\,\Gamma(\frac{d}{2})\rho\right]^{2/d}$ is filled with two electrons. The bounding sphere is called the Fermi surface. For each $k \in \mathbf{R}^d$ and $\sigma \in \{+,-\}$, $a_{k,\sigma}$ is the operator which when

$\frac{k^2}{2m} < \mu$ creates a hole of momentum $-k$ and spin σ and when $\frac{k^2}{2m} > \mu$ annihilates an electron of momentum k and spin σ. Conversely, the adjoint $a^+_{k,\sigma}$ annihilates a hole when $\frac{k^2}{2m} < \mu$ and creates a particle when $\frac{k^2}{2m} > \mu$. To be technically precise we should smear $a_{k,\sigma}$ and $a^+_{k,\sigma}$ with Schwartz space test functions $\phi_\sigma(k)$ and specify the domains of the resulting operators $\sum_\sigma \int d^dk\ \phi_\sigma(k)\ a^+_{k,\sigma}$ but these details do not play a role in our formulation of the model and so are neglected. The anticommutation relations are

$$\{a_{k,\sigma}, a_{k',\sigma'}\} = \{a^+_{k,\sigma}, a^+_{k',\sigma'}\} = 0$$
$$\{a_{k,\sigma}, a^+_{k',\sigma'}\} = \delta_{\sigma,\sigma'}\delta(k-k').$$

Now, $F_{n,m}$ is the closed span of all states of the form

$$a^+_{p_1,\sigma_1} \cdots a^+_{p_n\sigma_n}\ a_{h_1,\tau_1} \cdots a_{h_m\tau_m}\ \Omega \qquad \text{(I.1)}$$

(suitably smeared against test functions). Here $|p_i| > \sqrt{2\mu m}$ for $1 \le i \le n$ and $|h_j| < \sqrt{2\mu m}$ for $1 \le j \le m$. The Hamiltonian describing the independent electron approximation is

$$H_o = \sum_{\sigma\in\{+,-\}} \int \frac{d^dk}{(2\pi)^d}\ \epsilon(k)\ a^+_{k,\sigma}\ a_{k,\sigma}$$

$$\epsilon(k) = k^2/2m$$

We have

$$H_o\Omega = 2 \int_{|k|\le\sqrt{2m\mu}} \frac{d^dk}{(2\pi)^d}\ \epsilon(k)\ \Omega$$

and (I.1) is a generalized eigenstate of eigenvalue

$$\sum_{i=1}^{n} \epsilon(p_i) - \sum_{j=1}^{m} \epsilon(h_j) + 2 \int_{|k|\le\sqrt{2m\mu}} \frac{d^dk}{(2\pi)^d}\ \epsilon(k)$$

The number operator

$$N = \sum_{\sigma\in\{+,-\}} \int \frac{d^dk}{(2\pi)^d}\ a^+_{k,\sigma}\ a_{k,\sigma}$$

also has (I.1) as a generalized eigenstate with eigenvalue

$$n - m + 2 \int_{|k|\le\sqrt{2m\mu}} \frac{d^dk}{(2\pi)^d} .$$

As usual it will be convenient to deal with correlation functions like the Schwinger functions, that is the imaginary time Green's functions, rather than operators. The free Schwinger functions are defined as expectations of the free field operators

$$\begin{aligned}
\psi(x,\sigma) &= \int \frac{d^dk}{(2\pi)^d}\, a_{k,\sigma}\, e^{ik\cdot x} \\
\psi^+(x,\sigma) &= \int \frac{d^dk}{(2\pi)^d}\, a^+_{k,\sigma}\, e^{-ik\cdot x} \\
\psi(x,\tau,\sigma) &= e^{K_o\tau}\,\psi(x,\sigma)\,e^{-K_o\tau} = \int \frac{d^dk}{(2\pi)^d}\, a_{k,\sigma}\, e^{ik\cdot x}\, e^{-e(k)\tau} \\
\bar\psi(x,\tau,\sigma) &= e^{K_o\tau}\,\psi^+(x,\sigma)\,e^{-K_o\tau} = \int \frac{d^dk}{(2\pi)^d}\, a^+_{k,\sigma}\, e^{-ik\cdot x}\, e^{e(k)\tau}
\end{aligned}$$

where

$$K_o = H_o - \mu N$$

and

$$e(k) = \epsilon(k) - \mu = \frac{k^2}{2m} - \mu$$

The free n-point Schwinger function is

$$S^o_n\,(\overset{(-)}{\xi}_1,\dots,\overset{(-)}{\xi}_n) = (-1)^{n/2} \langle\, \Omega, T\, \overset{(-)}{\psi}(\xi_1)\dots\overset{(-)}{\psi}(\xi_n)\,\Omega\,\rangle$$

where $\xi_i = (x_i,\tau_i,\sigma_i)$, (-) denotes that the bar - may be present or absent. As usual T is the time ordering operator which orders the smallest τ to the right and introduces a compensating sign $(-1)^P$, the signature of the permutation required to restore the original order. When there are coinciding times the $\bar\psi$'s are ordered to the left of ψ's. Note that if n is not even S^o_n vanishes identically. For convenience we shall also denote the free 2-point function S^o_2 by C.

Clearly $C(\xi_1,\xi_2)$ is translation invariant and therefore is the kernel of a convolution operator. Direct calculation yields

$$
\begin{aligned}
C(\xi_1,\xi_2) &= S_2^o(\xi_1,\bar{\xi}_2) \\
&= \delta_{\sigma_1,\sigma_2} \int \frac{d^d k}{(2\pi)^d}\, e^{ik.(x_1-x_2)}\, e^{-(\epsilon(k)-\mu)(\tau_1-\tau_2)} \\
&\qquad \times \begin{cases} -\chi(|k|-\sqrt{2m\mu}) & \text{if } \tau_1>\tau_2 \\ \chi(\sqrt{2m\mu}-|k|) & \text{if } \tau_1\le\tau_2 \end{cases} \\
&= \delta_{\sigma_1,\sigma_2} \int \frac{d^{d+1}k}{(2\pi)^{d+1}}\, \frac{e^{i\langle k,\xi_1-\xi_2\rangle_-}}{ik_o-e(\underline{k})}
\end{aligned}
\tag{I.2}
$$

where

$$k = (k_o,\underline{k}) \in \mathbf{R}^{d+1}$$

$$\langle\, k,\xi\,\rangle_- = -k_o\tau + \underline{k}\cdot x.$$

and

$$\chi(t) = \begin{cases} 1 & \text{if } t > o \\ 0 & \text{if } t \le o \end{cases}$$

is the Heavyside function. The integral $\int d^{d+1}k\, e^{i\langle k,\xi_1-\xi_2\rangle_-}[ik_o-e(\underline{k})]^{-1}$ is not absolutely convergent so the last equality requires some explanation. For $\tau_1 - \tau_2 \neq 0$, the integral $\int dk_o\, e^{-ik_o(\tau_1-\tau_2)}[ik_o-e(\underline{k})]^{-1}$ is conditionally convergent and a contour integration yields the previous expression. The special case $\tau_1 - \tau_2 = 0$ is defined by the limit $\tau_1 - \tau_2 \to 0$ with $\tau_1 - \tau_2 < 0$.

Observe that the (partial) Fourier transform

$$\int d^d x\, e^{-ik\cdot x}\, C((x,\tau,\sigma_1),(o,o,\sigma_2))$$

is supported in $\{k|\frac{k^2}{2m} \ge \mu\}$ when $\tau > 0$ and in $\{k|\frac{k^2}{2m} \le \mu\}$ when $t \le 0$. The discontinuity in the partial Fourier transform at the Fermi surface $\frac{k^2}{2m} = \mu$ is reflected in a singularity of the full Fourier transform

$$\tilde{C}(k,\sigma_1,\sigma_2) = \int d^d x\, d\,\tau\, e^{-i\langle k,\xi\rangle}\, C(\xi,(o,o,\sigma_2))$$

$$= \delta_{\sigma_1,\sigma_2} \frac{1}{ik_o - e(\underline{k})}$$

on $\{k_o=0, e(\underline{k})=0\}$. Thus the singular support of $\tilde{C}$ has codimension 2. It will follow that the behaviour of the infrared end of the model is largely independent of dimension in contrast to conventional field theory models.

The Schwinger function S_n^o is easily expressed, using the anticommutation relations, in terms of C,

$$S_{2n}^o\,(\xi_1,\bar{\xi}_1,\xi_2,\bar{\xi}_2,\ldots,\xi_n,\bar{\xi}_n) = \det\,[C(\xi_i,\bar{\xi}_j)]\,. \tag{I.3}$$

Consequently, S_{2n}^o may be represented as the Grassman integral:

$$S_n^o\,(\overset{(-)}{\xi_1},\ldots,\overset{(-)}{\xi_n}) = \int \overset{(-)}{\psi}\,(\xi_1)\ldots\overset{(-)}{\psi}\,(\xi_n)\; d\mu_C(\psi,\bar{\psi})\,.$$

The Grassmann "measure" $d\mu_C(\psi,\bar{\psi})$ is a linear functional on polynomials in ψ and $\bar{\psi}$, where now by abuse of notation $\overset{(-)}{\psi}\,(\xi)$, $\xi \in \mathbf{R}^{d+1}\, x\{+,-\}$ are the generators of an infinite dimensional Grassmann algebra and no longer refer in any way to the field operators introduced above. The measure is defined by requiring:

(a) that (I.3) be satisfied

(b) that $\int \overset{(-)}{\psi}\,(\xi_1)\;\ldots\;\overset{(-)}{\psi}\,(\xi_n)\; d\mu_C(\psi,\bar{\psi}) = 0$ if the number of ψ's differ from the number of $\bar{\psi}$'s

(c) that $\overset{(-)}{\psi}\,(\xi)\;\overset{(-)}{\psi}\,(\zeta) = -\overset{(-)}{\psi}\,(\zeta)\;\overset{(-)}{\psi}\,(\xi)$ and

(d) that $\int \cdot\; d\mu_C$ be linear.

We consider a spin-independent two-body interaction with potential $\lambda\, V(x-y)\, \delta_{\alpha\alpha'}\delta_{\beta\beta'}$ (where α,α',β and β' are spin indices). In general, V is assumed to be an even function in $L^\infty(\mathbf{R}^d) \cap L^1(\mathbf{R}^d)$. However, as we shall see, the screened Coulomb potential $V(x) = \frac{e^{-m|x|}}{|x|}$ is also admissable in two or more dimensions.

The interacting system is described in the Fock space representation by the Hamiltonian

$$H = H_0 + \frac{1}{2} \sum_{\alpha,\beta\in\{+,-\}} \int \prod_{i=1}^{4} \frac{d^d k_i}{(2\pi)^d} (2\pi)^d \, \delta^{(d)}(k_1+k_2-k_3-k_4)$$
$$\lambda \hat{V}(k_3-k_1) \, a^+_{k_1,\alpha} \, a^+_{k_2,\beta} \, a_{k_4,\beta} \, a_{k_3,\alpha}.$$

Similarly K_o is replaced by $K = H - \mu N$. To get the interacting Schwinger functions the free measure $d\mu_C(\psi,\bar{\psi})$ is replaced by the formal interacting measure

$$\frac{1}{Z} e^{-\frac{\lambda}{2}\mathcal{V}} \, d\mu_C(\psi,\bar{\psi})$$

where

$$\mathcal{V} = \sum_{\alpha,\beta} \int d\tau \, d\sigma \, dx \, dy \, \bar{\psi}((x,\tau,\alpha))\psi((x,\tau,\alpha))\delta(\tau-\sigma) \, V(x-y)$$
$$\bar{\psi}((y,\sigma,\beta)) \, \psi((y,\sigma,\beta)).$$

and

$$Z = \int e^{-\frac{\lambda}{2}\mathcal{V}} \, d\mu_C(\psi,\bar{\psi}).$$

Since $e^{-\frac{\lambda}{2}\mathcal{V}}$ is not a polynomial in ψ and $\bar{\psi}$ it is far from clear that integrals against this formal measure exist. As a first step towards the construction of this measure we study perturbation theory for the Schwinger functions. The most naive perturbation expansion is derived by expanding

$$e^{-\lambda/2\,\mathcal{V}} = \sum_{m=o}^{\infty} \frac{1}{m!} \left(-\frac{\lambda}{2}\mathcal{V}\right)^m.$$

Then the numerator of the interacting Schwinger function

$$S_{2n}(\xi_1,\bar{\zeta}_1,\ldots,\xi_n,\bar{\zeta}_n)$$
$$= \frac{\int \prod_{i=1}^{n} [\psi(\xi_i) \, \bar{\psi}(\bar{\zeta}_i)] \, e^{-\lambda/2\,\mathcal{V}} \, d\mu_C(\psi,\bar{\psi})}{\int e^{-\lambda/2\,\mathcal{V}} \, d\mu_C(\psi,\bar{\psi})} \tag{I.4}$$

becomes the formal power series

$$\sum_{m=o}^{\infty} \frac{(-1)^m}{m!} \left(\frac{\lambda}{2}\right)^m \int \prod_{i=1}^{n} [\psi(\xi_i)\ \bar{\psi}\ (\bar{\zeta}_i)]\mathcal{V}^m\, d\mu_C(\psi,\bar{\psi}).$$

The coefficient of λ^m is then the integral of a monomial and consequently can be evaluated explicitly in terms of C. The result is conveniently represented in terms of labelled Feynman graphs.

An m^{th} order labelled graph contributing to the integral

$$\int \prod_{i=1}^{n} [\psi(\xi_i)\ \bar{\psi}\ (\bar{\zeta}_i)]\mathcal{V}^m\, d\mu_C(\psi,\bar{\psi})$$

is constructed from three kinds of vertices. There are n external hole vertices labelled $\xi_1,\ldots,\xi_n$, n external particle vertices labelled $\bar{\zeta}_1,\ldots,\bar{\zeta}_n$ and m internal generalized vertices which are represented as

$$\chi_j = (x_j,\tau_j,\alpha_j) \bullet\!\!\sim\!\!\sim\!\!\sim\!\!\bullet (y_j,\sigma_j,\beta_j) = \eta_j$$

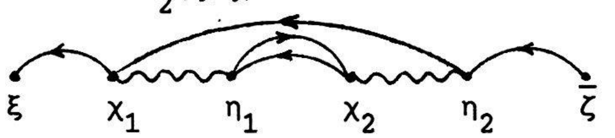
The intermediate squiggle $\sim\!\sim\!\sim$ is called an interaction line. These vertices are then connected by particle lines $\longrightarrow$ in such a way that

a) each external hole vertex is connected to precisely one line and the arrow of that line points towards the external hole vertex $\xi_i \bullet\!\!\leftarrow$

b) each external particle vertex is connected to precisely one line and the arrow of that line points away from the external particle vertex $\leftarrow\!\!\bullet\ \bar{\zeta}_i$

c) each end of each $\bullet\!\!\sim\!\!\sim\!\!\sim\!\!\bullet$ is connected to precisely two lines, one incoming, one outgoing

An important graph contributing to the second order term of the perturbation expansion of $S_2(\xi,\bar{\zeta})$ is



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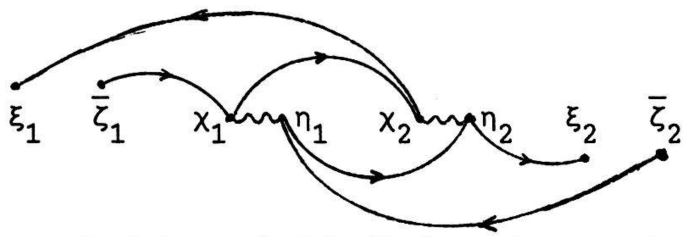
and an important graph contributing to the second order term of $S_4(\xi_1,\bar{\zeta}_1,\xi_2,\bar{\zeta}_2)$ is

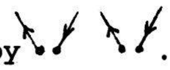


Altogether there are (n+2m)! such labelled graphs G. The value Val(G;C) is by definition

$$\mathrm{Val}(G) = \mathrm{sgn}(G) \prod_{j=1}^{m} [\sum\int d^{d+1}\chi_j \sum\int d^{d+1}\eta_j] \prod_{j=1}^{m} [V(\chi_j - \eta_j)\, \delta\, (\chi_j^{d+1} - \eta_j^{d+1})]$$
$$\prod_{\ell \in P_G} C((x_\ell, \tau_\ell)\ ,\ \overline{(x_\ell, \tau_\ell)})$$

Here by an abuse of notation $V(\chi_j - \eta_j)$ is the potential evaluated at the difference of the spatial ($\mathbf{R}^d$) components of χ_j and η_j; $(x_\ell, \tau_\ell) \in \mathbf{R}^{d+1}$ and $\overline{(x_\ell, \tau_\ell)} \in \mathbf{R}^{d+1}$ are the space-time components of the end-points of the particle line ℓ

$(x_\ell, \tau_\ell, \alpha_\ell)$ $\qquad$ $\overline{(x_\ell, \tau_\ell, \alpha_\ell)}$

and P_G denotes the set of all particle lines of G. Each $(x_\ell, \tau_\ell, \alpha_\ell)$ is a ξ_i or χ_j or η_j and each $\overline{(x_\ell, \tau_\ell, \alpha_\ell)}$ is a $\bar{\zeta}_i$ or χ_j or η_j. Each $\sum \int d^{d+1} \chi_j$ integrates over the space-time components of χ_j and sums over the spin component of χ_j. The sign sgn(G) is determined as follows. Consider the auxiliary graph G' gotten by replacing each  by . Permute the vertices of G' to write it as a graph of the form . Then sgn(G) is just the signature of this permutation. When n = 1 i.e. we are considering a 2-point Schwinger function sgn(G) reduces to $(-1)^b$ where b is the number of independent loops of G' i.e. the first Betti number of G'.

It is important for us to reexpress Val(G) by taking its Fourier ransform

$$\widetilde{\mathrm{Val}}(G) = \prod_{j=1}^{n} \left[\int d^{d+1}\xi_j\ e^{-i\langle \xi_j, p_j \rangle_-} \int d^{d+1}\bar{\zeta}_j\ e^{-i\langle \bar{\zeta}_j, q_j \rangle_-} \right] \mathrm{Val}(G)$$

Then

$$\widetilde{\mathrm{Val}}(G) = \mathrm{sgn}(G) \int \prod_{\ell \in P_G} \frac{d^{d+1}k_\ell}{(2\pi)^{d+1}} \frac{1}{i(k_\ell)_o - e(\underline{k}_\ell)} \prod_{j=1}^{m} \frac{d^{d+1}\mu_j}{(2\pi)^{d+1}} \tilde{V}(\underline{\mu}_j) \prod_{v \in N_G} (2\pi)^{d+1} \delta\Big(\sum_{\ell \in L_v} w_\ell\Big) \quad . \tag{I.5}$$

Here, N_G is the set of all vertices (nodes) of G and L_v is the set of all particle and interaction lines attached to the vertex v. The way to interpret this formula is the following: Momentum p_j enters the graph through the j^{th} external hole vertex; momentum q_j enters through the j^{th} external particle vertex; k_ℓ is the momentum flowing through the particle line ℓ; μ_j is the momentum flowing through the j^{th} interaction line and the d+1-dimensional delta function $\delta(\sum_{\ell \in L_v} w_\ell)$ enforces conservation of momentum at the vertex v.

Expression (I.5) can be simplified by eliminating most of the delta functions. The result is interpretted as follows. Choose a maximal set Z_G of independent closed loops of G. That is, a basis for the first homology group of G. Select a distinguished external vertex. Construct a set of 2n - 1 paths each joining a different external vertex to the distinguished one. View momentum μ_L as circulating in the loop $L \in Z_G$ and momentum $p_j(q_j)$ as flowing in the path connecting the j^{th} hole (particle) vertex to the distinguished external vertex. Then

$$\widetilde{\mathrm{Val}}(G) = \mathrm{sgn}(G)(2\pi)^{d+1} \delta\Big(\sum_{j=1}^{n} p_j + \sum_{j=1}^{n} q_j\Big) \int \prod_{L \in Z_G} \frac{d\mu_L}{(2\pi)^{d+1}} \prod_{\ell \in P_G} \frac{1}{i(k_\ell)_o - e(\underline{k}_\ell)} \prod_{j=1}^{m} \tilde{V}(\underline{\mu}_j) . \tag{I.6}$$

Here, k_ℓ is the signed sum of those mementa $\{\mu_L, p_j, q_j\}$ flowing through the particle line ℓ and μ_j is the signed sum of those momenta flowing through the interaction line j.

Using (1.3) it is easy to see that the integral

$$\int \prod_{i=1}^{n} \; [\psi(\xi_i)\bar{\psi}(\bar{\zeta}_i)]\mathcal{V}^m d\mu_C(\psi,\bar{\psi}) \;=\; \sum_G \mathrm{Val}(G)$$

where the sum runs over all labelled graphs of order m described above. The formal perturbation expansion for the numerator of (I.4) is now complete.

However, we are really interested in the perturbation expansion of S_{2n} itself. One can show that taking the quotient of the two formal power series has the effect of restricting the class of graphs to those for which each connected component contains at least two external vertices. Note that in any connected graph the number of external particle and hole vertices are the same. Thus

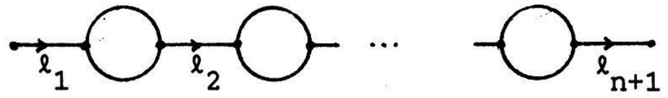
$$S_{2n} \sim \sum_{m=o}^{\infty} \frac{(-1)^m}{m!} \left(\frac{\lambda}{2}\right)^m \sum_G \mathrm{Val}(G) \tag{I.7}$$

where the sums are restricted to those described above.

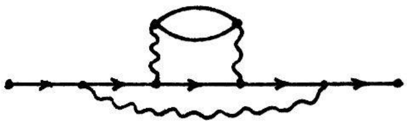
There are a number of questions concerning the formal power series (I.7). Is each Val(G) finite? If so how big is it? These questions cannot be immediately answered because the covariance C defined in (1.2) is a rather complicated function: in three dimensions for $|\underline{x}|$ and/or τ large

$$C((\underline{x},\tau,\alpha),(o,o,\beta)) \sim -\delta_{\alpha,\beta}\,\frac{2\sqrt{2\mu}}{(2\pi)^2}\;\frac{\cos\sqrt{2\mu}|x| + \sqrt{2\mu}(\tau/|x|)\sin\sqrt{2\mu}|x|}{|\underline{x}|^2 + 2\mu\,\tau^2}$$

The first question is easily answered. One can construct graphs G for which Val(G) is infinite. In fact let G be any graph containing a subgraph of the form



with $n \geq 1$. Suppose furthermore that the lines $\ell_1,\ldots,\ell_{n+1}$ lie on a closed loop L in G. An explicit example of such a graph is



Choosing L as one of the elements of Z_G, $\widetilde{\mathrm{Val}}(G)$ becomes an integral of the form

$$\int \cdots \int \frac{d^{d+1}k_L}{(2\pi)^{d+1}} \frac{1}{[i(k_L)_o - e(\underline{k}_L)]^{n+1}} f(k_L,\ldots).$$

The function $[ik_o - e(\underline{k})]^{-(n+1)}$ has a nonintegrable singularity on the Fermi surface $k_o = 0$, $|\underline{k}| = \sqrt{2m\mu}$ when $n \geq 1$ for all dimensions d. This follows from the observation that near the singularity $[ik_o - e(\underline{k})]^{-1}$ behaves like

$$\frac{1}{|k_o| + \sqrt{\frac{2\mu}{m}}\,\Big||\underline{k}| - \sqrt{2m\mu}\Big|}\,.$$

It turns out that there is no obstruction to the finiteness of Val(G) other than that above. This is true even for the singular, screened Coulomb potential $\frac{e^{-m|x|}}{|x|}$ in dimension $d \geq 2$.

Roughly speaking there are two complementary mechanisms for generating infinities: the large $(k_o,\underline{k})$ behaviour of $[ik_o - e(\underline{k})]^{-1}$ (the ultraviolet regime) and the singularity of $[ik_o - e(\underline{k})]^{-1}$ on the Fermi surface $k_o = o$, $|\underline{k}| = \sqrt{2m\mu}$ (the infrared regime). We will consider these two regimes separately. We shall see in section III that, for all d, there are no ultraviolet divergences at all.

On the other hand we have already given an example of an infrared divergence. There is a good reason for this divergence. We are attempting to expand the physical Schwinger functions, in particular S_2, in terms of $C = S_2^0$. The latter has a singularity at $k_o = o$, $\epsilon(\underline{k}) = \mu$. The divergence above reflects the fact that the singular surface, i.e. the physical Fermi surface, moves with λ. These infinities can be eliminated by simply adjusting $\mu = \mu(\lambda)$ in C in such a way that the singularity remains fixed at the physical Fermi surfaces as λ varies. This is implemented in section VI using renormalization methods from quantum field theory. The formal power series expansion for $\mu(\lambda)$ is chosen so that every two point subgraph $T(k)$ of every graph G arises in combination with a new two-legged vertex:

$$\bullet\!\xleftarrow{k}\! T(k) \!\xleftarrow{k}\!\bullet \;-\; T((0,|\underline{k}|=\sqrt{2m\mu})\bullet\!\xleftarrow{k}\!\bullet\!\xleftarrow{k}\!\bullet$$
$$= \frac{1}{[ik_o-e(\underline{k})]^2}\;\{T(k)\;-\;T(0,|\underline{k}|=\sqrt{2m\mu}))\}.$$

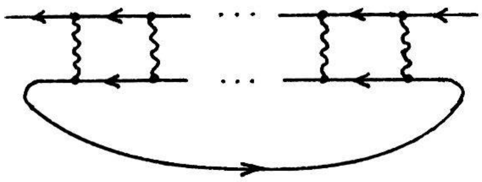
The zero of $T(k)-T(0,|\underline{k}|=\sqrt{2m\mu})$ regularizes the singularity in $\frac{1}{[ik_o-e(\underline{k})]}$.

Unlike the problem of ultraviolet renormalization $\mu(\lambda)$ remains finite order by order in perturbation theory. When the periodic potential U, determining the independent electron approximation, is non-constant so that $\epsilon(\underline{k})$ is not spherically symmetric it does not suffice to adjust just μ with λ. The entire Fermi surface must be held fixed. Clearly this complicates the analysis and so for pedagogical reasons the more general case is considered in another paper.

We shall show in section VII that each graph contributing to the renormalized perturbation expansion is finite. Let us briefly consider the size of the coefficient λ^m in the expansion of S_{2n}. We have already seen that there are $(n+2m)! \sim (m!)^2$ graphs in the sum

$$\frac{(-1)^m}{m!}\,(\frac{\lambda}{2})^m \sum_{\substack{m^{th}\text{ order}\\ \text{graphs}}} Val(G).$$

Even if $Val(G) \sim C^m$ for all G one would expect the sum over m to diverge. However the situation is still worse. There are special graphs like



whose magnitude grows like $m!$. Nonetheless we shall show in section VII that

$$\left| \frac{(-1)^m}{m!} \left(\frac{\lambda}{2}\right)^m \sum_{\substack{m^{th}\text{ order} \\ \text{graphs}}} \mathrm{Val}(G) \right| \le C_n |\lambda|^m K^m m!$$

and consequently the formal power series is locally Borel summable. That is the Borel transform

$$B\{S_n\} \equiv \sum_{m=o}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{\lambda}{2}\right)^m \sum_{\substack{m^{th}\text{ order} \\ \text{graphs}}} \mathrm{Val}(G)$$

has a strictly positive radius of convergence. This of course does not suffice to prove the existence of functions S_n, $n \ge 1$ having these formal power series as asymptotic expansions and satisfying the appropriate positivity and regularity conditions. To do so requires an appropriate version of asymptotic freedom, stability and the Pauli exclusion principle.

In section III we discuss perturbation theory at positive temperature and show that the value of every graph is finite and indeed exponentially bounded. In a forthcoming paper we discuss the limit temperature $T \to 0$. One constructs two perturbation expansions: one at $T = 0$ as above and the other by taking the limit as $T \to 0$ of a $T > 0$ expansion as in [FW]. It is shown that these expansions are the same. In the spherically symmetric case this was claimed in [KL,LW]. The approach in this paper also applies to the nonspherically symmetric case and leads to the same result i.e. the $T = 0$ expansion is graph by graph equal to the limit of the positive temperature expansions as $T \to 0$. This procedure avoids the anomolous diagrams introduced in [KL, FW p.281].

In section VIII we discuss a condition, motivated by perturbation theory, for the existence of an interacting Fermi surface. There is an important difference between one and higher dimensions.

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II. Length Scale Expansions and Properties of the Covariance

Our analysis of graphs requires that we resolve the singularity in $\tilde{C}(k)$. This is done by decomposing momentum space into shells around the Fermi surface. There is also an ancillary decomposition that will be explained in a moment.

Without loss of generality we shall suppress spin. This is possible since each spin sum can be majorized by a harmless factor of 2. There is one spin sum for each closed loop of particle lines. Consequently there are at most 2m spin sums for a graph of order m. Thus the total effect of spin sums is majorized by a factor of 4^m in the final bound of Val(G).

For notational simplicity we set the mass m=1. Thus

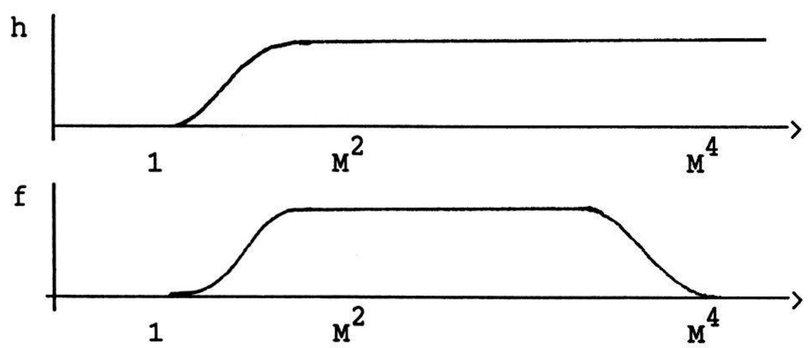
$$\tilde{C}(k) = \left[ik_0 - \left(\frac{k^2}{2} - \mu\right) \right]^{-1} . \tag{II.1}$$

The primary scale decomposition is introduced in the following way. We start by constructing a particular C^∞ partition of unity on $(0,\infty)$. Fix a number $M > 1$ which will control the slice width. Let h be a montone C^∞ function obeying

$$h(x) = \begin{cases} 0 & \text{for } x \le 1 \\ 1 & \text{for } x \ge M^2 \end{cases}$$

and let

$$f(x) = h(x)\left[1 - h(x/M^2)\right] = \begin{cases} h(x) & \text{for } x \le M^2 \\ 1-h(x/M^2) & \text{for } x \ge M^2 \end{cases}$$



Then

$$1 = h(x) + \sum_{i=-\infty}^{-1} f(xM^{-2i}) \qquad \text{for } x > 0.$$

Set

$$U(\xi) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{e^{i\langle k,\xi\rangle_-}}{ik_0 - e(\underline{k})} h(k_0^2 + e(\underline{k})^2)$$

$$C^{(j)}(\xi) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{e^{i\langle k,\xi\rangle_-}}{ik_0 - e(\underline{k})} f\left(M^{-2j}\left[k_0^2 + e(\underline{k})^2\right]\right), \; j<0$$

We have

$$C(\xi) = U(\xi) + I(\xi)$$

where

$$I(\xi) = \sum_{j=-\infty}^{-1} C^{(j)}(\xi) \; .$$

Recall that the covariance C as defined in section I is a convolution operator. So we have written it here as a function of the single argument

$$\xi = (\tau, \underline{x}) \in \mathbf{R} \times \mathbf{R}^d .$$

The estimates on $C^{(j)}$ contained in the following Lemma will be used repeatedly.

<u>Lemma II.1</u> Fix an integer $N \ge 1$. Then for all $j < 0$

1) $|C^{(j)}(\xi)| \le \text{const}\; M^j [1 + |\underline{x}|]^{\frac{1-d}{2}} \left[1 + (M^j|\xi|)^N\right]^{-1}$

2) $|\partial_\tau^m e(\tfrac{1}{i}\nabla)^n \, C^{(j)}(\xi)| \le (\text{const})^{m+n} M^{j(1+m+n)} [1+|\underline{x}|]^{\frac{1-d}{2}} \left[1 + (M^j|\xi|)^N\right]^{-1}$

3) $|\partial_\tau^m e(\tfrac{1}{i}\nabla)^n \, C^{(j)^{*\alpha}} * C^{(j+1)^{*\beta}}|$

$$\le (\text{const})^{\alpha+\beta+m+n} M^{j(2+m+n-\alpha-\beta)} [1+|\underline{x}|]^{\frac{1-d}{2}} \left[1 + (M^j|\xi|)^N\right]^{-1}$$

4) $C^{(j_1)} * C^{(j_2)} * \ldots * C^{(j_m)} = 0$ if $|j_\alpha - j_{\alpha'}| > 1$ for any α, α' .

Here the constant depends on N, μ and the dimension d and $\phi^{*\alpha}$ denotes the convolution $\phi * \phi * \ldots * \phi$ with α ϕ's. Recall that $e(k) = \frac{k^2}{2} - \mu$ so that $e(\frac{1}{i}\nabla) = -\frac{1}{2}\Delta - \mu$.

Proof We first bound

$$|C^{(j)}(\xi)| \le \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{1}{|ik_0 - e(\underline{k})|} f(M^{-2j}[k_0^2 + e(\underline{k})^2])$$
$$\le M^{-j} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} f(M^{-2j}[k_0^2 + e(\underline{k})^2])$$
$$\le \text{const } M^{-j} M^{2j} = \text{const } M^j$$

since in the support of $f(M^{-2j}[k_0^2 + e(\underline{k})^2])$ we must have $|k_0| \le M^2M^j$ and $|e(\underline{k})| \le M^2M^j$ i.e. $\sqrt{\mu}\left|\frac{|\underline{k}|}{\sqrt{2}} - \sqrt{\mu}\right| \le M^2M^j$.

Next we bound, for any even N,

$$(M^j|\xi|)^N C^{(j)}(\xi) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{f(M^{-2j}[k_0^2 + e(\underline{k})^2])}{ik_0 - e(\underline{k})} \times$$
$$M^{jN}\left(-\frac{d^2}{dk_0^2} - \Delta\right)^{\frac{N}{2}} e^{i\langle k,\xi\rangle_-}$$
$$= \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{i\langle k,\xi\rangle_-} M^{jN} \left(-\frac{d^2}{dk_0^2} - \Delta\right)^{\frac{N}{2}} \frac{f(M^{-2j}[k_0^2+e(\underline{k})^2])}{ik_0 - e(\underline{k})} \tag{II.2}$$

As above the support of the integrand has volume const M^{2j} and $|ik_0-e(\underline{k})^2| \le M^{-j}$. Further, the Laplacian Δ acts as $\frac{d^2}{d\rho^2} + \frac{d-1}{\rho}\frac{d}{d\rho}$, $\rho = |\underline{k}|$ since the quotient in brackets is spherically symmetric. Observe that each derivative $\frac{d}{d\rho}$, $\frac{d}{dk_0}$ yields an extra M^{-j}. If $\frac{d}{d\rho}$ acts

$$\frac{d}{d\rho}[ik_0-e(\rho)]^{-n} = -n[ik_0 - e(\rho)]^{-n-1}\rho$$

the additional $[ik_0 - e(\rho)]^{-1}(-n\rho)$ is bounded by const. M^{-j} since $0 < \text{const} \le |\rho| \le \text{const}$. If $\frac{d}{d\rho}$ acts

$$\frac{d}{d\rho} f^{(n)}(M^{-2j}[k_0^2 + e(\rho)^2])$$
$$= f^{(n+1)}(M^{-2j}[k_0^2 + e(\rho)^2])\ M^{-2j}e(\rho)2\rho$$

the additional derivative on f has no effect (since n < N) while $|2\ e(\rho)\rho M^{-2j}| \le \text{const}\ M^{-j}$ on the support of f. The action of $\frac{d}{dk_0}$ is similar. Altogether

$$|(M^j|\xi|)^N C^{(j)}(\xi)| \le \text{const}\ M^{2j}M^{-j}\ .$$

When d = 1

$$(M^j|\xi|)^N C^{(j)}(\xi) = \int \frac{dk_0}{2\pi}\frac{dk_1}{2\pi}\ e^{ik_1\underline{x}}\ e^{-ik_0\tau}\ M^{jN}$$
$$\times\ \left(-\frac{d^2}{dk_0^2} - \frac{d^2}{dk_1^2}\right)^{N/2} \frac{f(M^{-2j}[k_0^2 + e(k_1)^2])}{ik_0 - e(k_1)}$$

and the right hand side is estimated in essentially the same way as above.

Now make the change of variables $\underline{k} = \rho\underline{k}'$, $|\underline{k}'| = 1$ in (II.2) to yield

$$(M^j|\xi|)^N C^{(j)}(\xi)$$
$$= \int_{-\infty}^{\infty} \frac{dk_0}{(2\pi)^{d+1}} \int_0^{\infty} d\rho\rho^{d-1} \int_{S^{d-1}} d\sigma(\underline{k}')\ e^{i\rho\underline{k}'\cdot\underline{x}}\ e^{-ik_0\tau}\ M^{jN}$$
$$\left(-\frac{d^2}{dk_0^2} - \frac{d^2}{d\rho^2} - \frac{d-1}{\rho}\frac{d}{d\rho}\right)^{\frac{N}{2}} \frac{f(M^{-2j}[k_0^2 + e(\rho)^2])}{ik_0 - e(\underline{k})}$$

For d ≥ 2

$$\int_{S^{d-1}} d\sigma(\underline{k}')e^{i\underline{k}'\cdot\rho\underline{x}} = \text{const}\ (\rho|\underline{x}|)^{1-\frac{d}{2}}\ J_{\frac{d}{2}-1}(\rho|\underline{x}|)$$

where the constant is $2^{\frac{d}{2}-1}\ \Gamma(\frac{d}{2})\omega_d$. Thus

$$(M^j|\xi|)^N C^{(j)}(\xi)$$

$$= \text{const} \int_{-\infty}^{\infty} dk_0 \int_0^{\infty} d\rho\, \rho^{d-1} e^{-ik_0\tau} (\rho|\underline{x}|)^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(\rho|\underline{x}|)\, M^{jN}$$

$$\left(-\frac{d^2}{dk_0^2} - \frac{d^2}{d\rho^2} - \frac{d-1}{\rho}\frac{d}{d\rho}\right)^{\frac{N}{2}} \frac{f(M^{-2j}[k_0^2 + e(\rho)^2])}{ik_0 - e(\rho)}$$

We bound the right hand side as above using, in addition,

$$(\rho|\underline{x}|)^{\frac{1-d}{2}} \le \text{const } |\underline{x}|^{\frac{1-d}{2}}$$

$$\left| (\rho|\underline{x}|)^{1/2} J_{\frac{d}{2}-1}(\rho|\underline{x}|) \right| \le \text{const.}$$

It follows that

$$|(M^j|\xi|)^N C^{(j)}(\xi)| \le \text{const } M^j |\underline{x}|^{\frac{1-d}{2}}.$$

Combining the estimates

$$|C^{(j)}(\xi)| \le \text{const } M^j$$

$$|(M^j|\xi|)^N C^{(j)}(\xi)| \le \text{const } M^j$$

and

$$|\underline{x}|^{\frac{d-1}{2}} |(M^j|\xi|)^N C^{(j)}(\xi)| \le \text{const } M^j$$

yields the first part of the Lemma.

To prove the second part it suffices to observe that

$$\partial_\tau e^{i\langle k,\xi\rangle_-} = -ik_0 e^{i\langle k,\xi\rangle_-}$$

$$e(\tfrac{1}{i}\nabla)\, e^{i\langle k,\xi\rangle_-} = e(\underline{k}) e^{i\langle k,\xi\rangle_-}$$

and that, on the support of f, $|k_0|, |e(\underline{k})| \le \text{const } M^j$.

The convolution

$$C^{(j_1)} * C^{(j_2)} * \ldots * C^{(j_m)}(\xi) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{e^{i\langle k,\xi\rangle_-}}{[ik_0 - e(\underline{k})]^m} \prod_{\alpha=1}^{m} f(M^{-2j_\alpha}[k_0^2 + e(\underline{k})^2]$$

Part four is an immediate consequence of the fact that $f(M^{-2j_\alpha}[k_0^2 + e(\underline{k})^2])$

and $f(M^{-2j_{\alpha'}}[k_0^2+e(\underline{k})^2])$ have disjoint support when $|j_\alpha - j_{\alpha'}|>1$. The estimate of part three is proven just as the estimate of part 1 with the one difference that $[ik_0-e(\underline{k})]^{-1}$ is replaced by $[ik_0-e(\underline{k})]^{-\alpha-\beta}$. This results in the replacement of M^{-j} by $M^{-j(\alpha+\beta)}$ in the estimate. ∎

The decomposition of the covariance $C = U + \sum_{j=-\infty}^{-1} C^{(j)}$ yields a decomposition

$$\psi = \sum_{j=-\infty}^{0} \psi^{(j)}$$
$$\bar{\psi} = \sum_{j=-\infty}^{0} \bar{\psi}^{(j)}$$

of the Grassmann variables and also a product decomposition

$$d\mu_C(\psi,\bar{\psi}) = d\mu_U(\psi^{(0)},\bar{\psi}^{(0)}) \prod_{j=-\infty}^{-1} d\mu_{C^{(j)}}(\psi^{(j)},\bar{\psi}^{(j)})$$

of the Grassman gaussian measure. The factor $d\mu_U$ is the ultraviolet end of the model and the remaining product $\prod_{j=-\infty}^{-1} d\mu_{C^{(j)}}$ is the infrared end. This allows us to isolate and study problems in the two regimes separately. In the next section we shall investigate perturbation theory in the ultraviolet end.

We could of course decompose the ultraviolet regime into slices $U = \sum_{j=0}^{\infty} C^{(j)}$ as we have done in the infrared regime. But as we shall see the real problems are at the infrared end. For the study of perturbation theory this decomposition is not necessary.

On the other hand at the infrared end a finer decomposition than the one we have just introduced is necessary. Each $C^{(j)}$ must itself be expanded into j pieces. Roughly speaking this is done to take in to account the asymmetry in the dependence of C on τ and $\underline{x}$ that one sees, for example, in the three dimensional asymptotics

$$C(\tau,\underline{x}) \sim \text{const}\ \frac{\cos\sqrt{2\mu}|\underline{x}| + \sqrt{2\mu}\frac{\tau}{|\underline{x}|}\sin\sqrt{2\mu}\ |\underline{x}|}{|\underline{x}|^2 + 2\mu\tau^2} .$$

We do this in detail in Section V.

III Ultraviolet bounds

In this section we replace the full covariance C by its ultraviolet end U and prove that for any connected graph

$$\|\mathrm{Val}(G;U)\| \le (\text{const})^{|G|} .$$

Here Val(G;U) denotes the value of the graph G as defined in the introduction, but with C replaced by U; $\|\cdot\|$ is the L^1 norm (but with one external vertex fixed at 0 to break translation invariance)

$$\|f\| = \int d\xi_1 \ldots d\xi_n\ |f(\xi_0,\xi_1, \ldots ,\xi_n)|_{\xi_0=0}$$

and $|G|$ is the order of G i.e. the number of interaction squiggles.

First, we derive such bounds in any dimension d and for any regular two-body potential $V\in L^1(\mathbf{R}^d)\cap L^\infty(\mathbf{R}^d)$. Then, using a different technique, we obtain the same bound for the screened Coulomb potential $\frac{e^{-m|\underline{x}|}}{|x|}$, m > 0 in dimension d ≥ 2.

We began by separating U into a regular part R and singular part S.

Lemma III.1

$$U(\xi) = R(\xi) + S(\xi)$$

where

$$S(\xi) = -g(\xi)(2\pi\tau)^{-d/2}e^{\mu\tau}e^{-\underline{x}^2/(2\tau)} \begin{cases} 1 , & \tau > 0 \\ 0 , & \tau \le 0 \end{cases} .$$

Here g is a smooth function of compact support. The regular part $R = U - S$ is in $\mathcal{S}(\mathbf{R}^{d+1})$.

Proof.

Define $g(\xi)$ to be a C^∞ function that is 1 for $|\xi| < 1$ and zero for $|\xi| > 2$. We first show that $(1 - g(\xi))U(\xi)$ is in $\mathcal{S}(\mathbf{R}^{d+1})$. To do so it suffices to show that $\Delta^N U(\xi)$ is bounded and rapidly decaying for all N,

$|\xi| > 1$. Write,

$$U(\xi) = \frac{1}{(\xi^2)^{N'}} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{h(k_0^2 + e(\underline{k})^2)}{ik_0 - e(\underline{k})} (-\Delta_k)^{N'} e^{i\langle k,\xi\rangle_-}$$

$$= \frac{1}{(\xi^2)^{N'}} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{i\langle k,\xi\rangle_-} (-\Delta_k)^{N'} \frac{h(k_0^2 + e(\underline{k})^2)}{ik_0 - e(\underline{k}} \quad .$$

If N' is sufficiently large the integrand is L^1 so that $U(\xi)$ is bounded for $|\xi| > 1$ and rapidly decreasing. To prove the same for $\Delta^N U(\xi)$ we observe that

$$\Delta^N U(\xi) = \Delta^N \frac{1}{(\xi^2)^{N'}} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{i\langle k,\xi\rangle_-} (-\Delta_k)^{N'} \frac{h(k_0^2 + e(\underline{k}))}{ik_0 - e(\underline{k})}$$

is bounded for $|\xi| > 1$ and N' sufficiently larger than N. It follows that $(1 - g(\xi))U(\xi) \in \mathcal{S}(\mathbb{R}^{d+1})$. We place $(1 - g)U$ in R.

We further decompose

$$g(\xi)U(\xi) = g(\xi)\int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{e^{i\langle k,\xi\rangle_-}}{ik_0 - e(\underline{k})}$$

$$+ g(\xi)\int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{e^{i\langle k,\xi\rangle_-}}{ik_0 - e(\underline{k})} [h(k_0^2 + e(\underline{k})) - 1]$$

The first term above is

$$g(\xi)\int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{e^{i\langle k,\xi\rangle_-}}{ik_0 - e(\underline{k})} = g(\xi)\int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} e^{-e(k)\tau} \begin{cases} -\chi(e(k)) & \tau > 0 \\ \chi(-e(k)) & \tau \le 0 \end{cases}$$

$$= g(\xi)\int \frac{d^d k}{(2\pi)^2} e^{ik\cdot x} e^{-e(k)\tau} \chi(-e(k))$$

$$+ g(\xi)\int \frac{d^d k}{(2\pi)^d} e^{ik\cdot n} e^{-(\frac{k^2}{2} - \mu)\tau} \begin{cases} -1 & \text{if } \tau > 0 \\ 0 & \text{if } \tau \le 0 \end{cases}$$

$$= g(\xi) \int_{|k| < \sqrt{2\mu}} \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} e^{-e(k)\tau}$$

$$- g(\xi)e^{\mu\tau}(2\pi\tau)^{-d/2}\, e^{-\frac{x^2}{2\tau}} \begin{cases} 1 & \text{if} \quad \tau > 0 \\ 0 & \text{if} \quad \tau \le 0 \end{cases}.$$

we now need only observe that $g(\xi) \displaystyle\int_{|k|<\sqrt{2\mu}} \frac{d^d k}{(2\pi)^d}\, e^{ik\cdot x}\, e^{-e(k)\tau}$ and

$g(\xi)\displaystyle\int\frac{d^{d+1}k}{(2\pi)^{d+1}}\, \frac{e^{i\langle k,\xi\rangle_-}}{ik_0 - e(\underline{k})}\, [h(k_0^2 + e(\underline{k})) - 1]$ are Schwartz class since

$[ik_0 - e(\underline{k})]^{-1}$ is locally integrable. ∎

Recall that the 2n-point Schwinger function restricted to the ultraviolet regime has the formal asymptotic expansion

$$S^U_{2n} \sim \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\lambda}{2}^m \sum_G \mathrm{Val}(G;U)$$

Where the sum $\sum_G \mathrm{Val}(G;U)$ is over graphs having n external particle vertices, n external hole vertices and m interaction squiggles. Each connected component of G must contain an external vertex. For convenience let Γ^m_n be the set of such graphs that are <u>connected</u>.

As mentioned above

$$\mathrm{Val}(G;U) = \mathrm{sgn}(G) \int \prod_v d^{d+1}\xi_v \prod_{\ell\in I_G} \mathcal{V}(\xi_{u_\ell} - \xi_{w_\ell}) \prod_{\ell\in P_G} U(\xi_{u_\ell} - \xi_{w_\ell})$$

where the product $\prod_v$ is over all internal vertices of G, I_G is the set of all interaction squiggles, P_G is the set of all particle lines, u_ℓ and w_ℓ are the vertices at the two ends of ℓ and

$$\mathcal{V}((x,\tau)) = V(x)\delta(\tau) .$$

<u>THEOREM III.2</u> For any $G\in\Gamma^m_n$

$$\int d\bar{\zeta}_1 d\xi_2 d\bar{\zeta}_2 \cdots d\xi_n d\bar{\zeta}_n\, |\mathrm{Val}(G;U)(0,\bar{\zeta}_1,\cdots\xi_n,\bar{\zeta}_n)| \le (\mathrm{const})^m$$

where the constant const depends on n,μ,V.

<u>Proof</u> Substitute $U = R + S$ to obtain

$$\mathrm{Val}(G;U) = \mathrm{sgn}(G) \sum_{\sigma\subset P_G} \int \prod_v d^{d+1}\xi_v \prod_{\ell\in I_G} \mathcal{U} \prod_{\ell\in\sigma} S \prod_{\ell\in P_G\setminus\sigma} R .$$

There are (const)^m terms in this sum so it suffices to bound them individually. So fix any $\sigma\subset P_G$.

The factors S and R are bounded

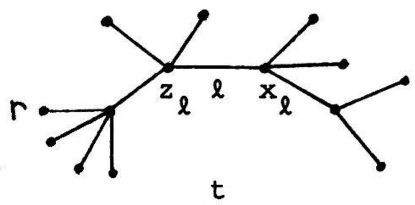
$$|R(\xi)| \le \text{const } [1 + |x|]^{-d-1} [1 + |\tau|]^{-2}$$

$$|S(\xi)| \le \text{const } g(\tau)\tau^{-d/2} e^{-x^2/(2\tau)} \begin{cases} 1 & , \tau > 0 \\ 0 & , \tau \le 0 \end{cases} .$$

Now we bound the integrals over the spatial variables followed by the integrals over the time (τ) variables. We bound multiple integrals by using the fact that for any tree t we have <u>the tree identity</u>

$$\int \prod_{\substack{v\in t\\ v\neq r}} dy_v \prod_{\ell\in t} f_\ell(x_\ell - z_\ell) = \prod_{\ell\in t} [\int dy f_\ell(y)] . \tag{III.1}$$

Here the product $\prod_{\substack{v\in t\\ v\neq r}}$ runs over all vertices v of the true t except for one distinguished vertex r called the root. The product $\prod_{\ell\in t}$ runs over all lines of the tree; x_ℓ and z_ℓ are the vertices at the ends of ℓ with x_ℓ being the further of the two from the root. For example.



We apply the tree identity to bound the spatial integrals by constructing a tree t which is a connected subgraph of G (including both particle lines and squiggles) and which contains all the vertices of G. Such a tree is called a spanning tree for G. It is important to observe that if G contains a closed loop of S-lines, i.e. lines $\ell\in\sigma$, the integral is zero. This is because the sum of the time differences for neighbouring vertices around the loop is zero. Hence one time difference must obey

$\tau_i - \tau_{i+1} \le 0$. But then $S(\tau_i - \tau_{i+1}) = 0$. So we may assume the integral is non-zero. Then include all S-lines in the tree t. Add enough particle lines and squiggles to get a spanning tree. Designate the external particle vertex corresponding to ξ_1 to be the root.

For all lines of G not in t bound the x-dependence by

$$[1 + |x_\ell - z_\ell|]^{-d-1} \le 1 \qquad \text{for } \ell \in P_G \setminus (\sigma \cup t)$$

$$|V(x_\ell - z_\ell)| \le \text{const} \qquad \text{for } \ell \in I_G \setminus t.$$

We can now use the tree identity and

$$\int d^d y |V(y)| \le \text{const}$$

$$\int d^d y [1 + |y|]^{-d-1} \le \text{const}$$

$$\int d^d y\ e^{-y^2/(2\tau)} \le \text{const}\ \tau^{d/2}$$

to estimate the spatial integrals by

$$\Big|\int \prod_v d^{d+1}\xi_v \prod_{\ell\in I_G} \mathcal{L} \prod_{\ell\in\sigma} S \prod_{\ell\in P_G\setminus\sigma} R\Big|$$

$$\le \text{const}^m \int \prod_v d^1\tau_v \prod_{\ell\in I_G} \delta(\tau_{u_\ell} - \tau_{w_\ell}) \prod_{\ell\in\sigma} g(\tau_{u_\ell} - \tau_{w_\ell}) \prod_{\ell\in P_G\setminus\sigma} [1 + |\tau_{u_\ell} - \tau_{w_\ell}|]^{-2}.$$

Note for each $\ell \in \sigma$ the $\tau^{d/2}$ arising from the integral $\int d^d y e^{-y^2/(2\tau)} = \text{const}\ \tau^{d/2}$ cancels the $\tau^{-d/2}$ in S.

Construct another tree t' to perform the τ-integrals. This time we include all interaction squiggles in t'. We add particle lines as necessary to get a spanning tree. Once again we place the root at the ξ_1 vertex. For all lines not in t' we bound

$$|g(\tau_{u_\ell} - \tau_{w_\ell})| \le \text{const}$$

$$[1 + |\tau_{u_\ell} - \tau_{w_\ell}|]^{-2} \le 1.$$

Finally, we use the tree identity again and

$$\int d\tau\ \delta(\tau) = 1$$

$$\int d\tau\ g(\tau) \le \text{const}$$

$$\int d\tau\,[1 + |\tau|]^{-2} \le \text{const}$$

to obtain

$$\Big|\int \prod_{v} d^{d+1}\xi_v \prod_{\ell\in I_G} \; \prod_{\ell\in\sigma} S \prod_{\ell\in P_G\backslash\sigma} R\Big| \le (\text{const})^m .$$

∎

We now present a more powerful technique for estimating the values of graphs. It is essential for sections V-VII. The method we used to prove Theorem III.2 required that $V(x)$ be bounded and so is inadequate for (the physically interesting) screened Coulomb interaction. The new technique is applied to this case by decomposing $V(x)$ into a sum of bounded, integrable potentials.

<u>Lemma III.3</u>

1) $$e^{-m|x|}/|x| = \sum_{j=0}^{\infty} V^{(j)}(x)$$

$$\text{with } |V^{(j)}(x)| \le \text{const } M^j e^{-\text{const } M^j|x|}$$

2) $$U(\xi) = \sum_{j=0}^{\infty} U^{(j)}(\xi)$$

with $|U^{(j)}(x,\tau)| \le \text{const } M^{dj} e^{-\text{const}(M^{2j}\tau + M^j|x|)} g(|\xi|) \begin{cases} 1 & \tau>0 \\ 0 & \tau\le 0 \end{cases}$ for $j\ge 1$

and $|U^{(0)}(x,\tau)| \le \text{const}[1+|\tau|]^{-2}[1+|x|]^{-d-1}$.

<u>Proof</u>

1) $$\frac{e^{-m|x|}}{|x|} = \frac{2}{(2\pi)^2}\int d^3k\,\frac{e^{ik\cdot x}}{k^2+m^2} = \frac{2}{(2\pi)^2}\int d^3k\, e^{ik\cdot x}\int_0^\infty d\alpha\, e^{-\alpha(k^2+m^2)}$$

$$= \frac{1}{2\sqrt{\pi}}\int_0^\infty d\alpha\,\alpha^{-3/2}\, e^{-(x^2/4\alpha)-\alpha m^2}$$

$$= \frac{1}{2\sqrt{\pi}}\int_1^\infty d\alpha\,\alpha^{-3/2} e^{-(x^2/4\alpha)-\alpha m^2} + \sum_{j=1}^{\infty}\frac{1}{2\sqrt{\pi}}\int_{M^{-2j}}^{M^{2(1-j)}} \alpha\alpha\,\alpha^{-3/2}e^{-(x^2/4\alpha)-\alpha m}$$

Now set

$$V^{(0)} = \frac{1}{2\sqrt{\pi}} \int_1^\infty d\alpha\ \alpha^{-3/2} e^{-x^2/(4\alpha) - \alpha m^2}$$

$$\le \frac{1}{2\sqrt{\pi}}\ e^{-m|x|} \int_1^\infty d\alpha\ \alpha^{-3/2} = \text{const}\ e^{-m|x|}$$

$$V^{(j)} = \frac{1}{2\sqrt{\pi}} \int_{M^{-2j}}^{M^{2(1-j)}} d\alpha\ \ \alpha^{-3/2} e^{-x^2/(4\alpha) - \alpha m^2}$$

$$\le \frac{1}{2\sqrt{\pi}}\ M^{-2j}[M^2 - 1]\ M^{3j}\ e^{-\frac{M^{2j}x^2}{4M^2}}$$

$$\le \text{const}\ M^j\ e^{-\text{const}M^j|x|}$$

2)
$$\tau^{-d/2} e^{-x^2/(2\tau)} = \frac{1}{\Gamma(\frac{d}{2})} \int_0^\infty d\alpha\ \alpha^{\frac{d-2}{2}}\ e^{-\alpha\tau}\ e^{-\frac{x^2}{2\tau}}$$

$$= \frac{1}{\Gamma(\frac{d}{2})} \int_0^{M^2} d\alpha\ \alpha^{\frac{d-2}{2}}\ e^{-\alpha\tau} e^{-\frac{x^2}{2\tau}} + \sum_{j=1}^{\infty} \frac{1}{\Gamma(\frac{d}{2})} \int_{M^{2j}}^{M^{2(j+1)}} d\alpha\ \alpha^{\frac{d-2}{2}} e^{-\alpha\tau} e^{-\frac{x^2}{2\tau}}$$

Let

$$U^{(0)}(\xi) = R(\xi) - g(|\xi|)(2\pi)^{-d/2} e^{\mu\tau} \frac{1}{\Gamma(\frac{d}{2})} \int_0^{M^2} d\alpha\ \alpha^{\frac{d-2}{2}} e^{-\alpha\tau} e^{-\frac{x^2}{2\tau}} \begin{cases} 1 & \tau > 0 \\ 0 & \tau \le 0 \end{cases} .$$

So

$$|U^{(0)}(\xi)| \le \text{const}\ [1 + |\tau|]^{-2} [1 + |x|]^{-d-1} + \text{const}\ g(|\xi|)$$

$$\le \text{const}\ [1 + |\tau|]^{-2} [1 + |x|]^{-d-1}.$$

Also set, for $j \ge 1$,

$$U^{(j)}(\xi) = -g(|\xi|)(2\pi)^{-d/2} e^{\mu\tau} \frac{1}{\Gamma(d/2)} \int_{M^{2j}}^{M^{2(j+1)}} d\alpha\ \ \alpha^{\frac{d-2}{2}} e^{-\alpha\tau} e^{-x^2/2\tau} \begin{cases} 1 & \tau > 0 \\ 0 & \tau \le 0 \end{cases}$$

So

$$|U^{(j)}(\xi)| \le \text{const}\ g(|\xi|) M^{2j} (M^2 - 1) M^{(j+1)(d-2)} e^{-M^{2j}\tau} e^{-x^2/2\tau} \begin{cases} 1 & \tau > 0 \\ 0 & \tau \le 0 \end{cases}$$

$$\le \text{const}\ g(|\xi|)M^{dj}e^{-1/2\ M^{2j}\tau}e^{-1/2\ M^{j}|x|}\begin{cases}1 & \tau > 0\\ 0 & \tau \le 0\end{cases}$$

since $1/2\ M^{2j}\tau + \frac{x^2}{2\tau} \ge \frac{1}{2}M^j|x|$ for all $\tau > 0$.

∎

Fix a graph $G\epsilon\Gamma_n^m$. We now estimate Val(G;u) with the screened Coulomb interaction by expanding

$$\begin{aligned}\mathrm{Val}(G) &= \mathrm{sgn}(G)\int \prod_v d^{d+1}\xi_v \prod_{\ell\epsilon I_G} \mathcal{V} \prod_{\ell\epsilon P_G} U\\ &= \mathrm{sgn}(G)\int \prod_v d^{d+1}\xi_v \prod_{\ell\epsilon I_G} [\delta(\tau_{u_\ell} - \tau_{w_\ell}) \sum_{j_\ell=0}^{\infty} V^{(j_\ell)}(x_{u_\ell} - x_{w_\ell})]\\ &\qquad \prod_{\ell\epsilon P_G} [\sum_{j_\ell=0}^{\infty} U^{(j_\ell)}(\xi_{u_\ell} - \xi_{w_\ell})]\\ &= \sum_{\substack{j_\ell \ge 0\\ \ell\epsilon I_G \cup P_G}} \mathrm{sgn}(G)\int \prod_v d^{d+1}\xi_v \prod_{\ell\epsilon I_G} \mathcal{V}^{(j_\ell)} \prod_{\ell\epsilon P_G} U^{(j_\ell)}\end{aligned}$$

where $\mathcal{V}^{(j)}(\xi) = \delta(\tau)V^{(j)}(x)$. We think of each term in this sum as the value of a labelled graph $G^J, J = \{j_\ell | \ell\epsilon I_G \cup P_G\}$ in which each particle and interaction line $\ell\epsilon I_G \cup P_G$ is assigned the scale label j_ℓ. The label designates the covariance (or interaction) assigned to the line.

Let G^J be a labelling of G. We associate a tree $t(G^J)$ to G^J as follows. The union

$$F(J)(=F(G^J)) = \bigcup_{j\ge 0} \{\text{connected components of } \{\ell\epsilon G^J | j_\ell \ge j\}\} \cup \{\text{vertices of } G\} \tag{III.2}$$

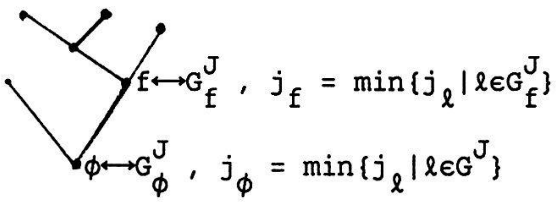
is called the forest in G determined by J. For each j, $\{\ell\epsilon G^J | j_\ell \ge j\}$ is the subgraph of G^J consisting of lines whose scales are at least j. A vertex is considered to be a trivial subgraph of G. Then {vertices} can be thought of as the limit as j tends to infinity of the subgraph

$\{\ell \in G^J \mid j_\ell \geq j\}$. The forest F has the property that for all $g,h \in F$ either $g \subseteq h$ or $h \subseteq g$ or $h \cap g = \phi$. Therefore, partially ordering F by inclusion one obtains the tree $t(J) = t(G^J)$ whose forks are in one to one correspondence with the elements of F. In general a forest in G is any set of connected subgraphs of G satisfying the trichotomy above. The class of all forests is called $\mathfrak{F}$.

For each fork f in the tree t(J) we define the scale

$$j_f = \min\{j_\ell \mid \ell \in G_f^J\} \tag{III.3}$$

where G_f^J is the element of $\mathfrak{F}$ corresponding to the fork f. When f corresponds to a vertex, $j_f = \infty$. Clearly, the scale j_f increases as f is moved up through the tree. Notice the first fork ϕ corresponds to $G_\phi^J = G^J$, the whole labelled graph.



Let us denote by $\mathfrak{J}$ the set of all trees $\mathcal{S}(t)$, $t \in \mathfrak{J}$, the set of all allowed assignments of scales to the forks of t. An assignment $\{j_f \mid f \in t\}$ is allowed if $j_f > j_{f'}$ whenever f lies above f' in the tree t. Finally, the assignment of scales on $t(G^t)$ defined above is denoted by $s(J) = s(G^J)$.

We shall show in a moment that $\mathrm{Val}(G^J)$ decays exponentially in j_ℓ with the result that the sum over J in

$$\mathrm{Val}(G) = \sum_J \mathrm{Val}(G^J).$$

converges and obeys the bound of Theorem III.2. To do this we prove a lemma which exhibits the exponential decay in a general setting. Suppose that G is any connected graph at all (arbitrarily many lines joining arbitrarily many vertices) and that the line ℓ of G has a covariance C_ℓ obeying

$$|C_\ell(y)| \le K M^{\delta_\ell j_\ell} g(M^{\alpha j_\ell}|y|) , \ y\in\mathbf{R}^d, \tag{III.4}$$

with

$$\|g\|_1 \le 1, \ \|g\|_\infty \le 1.$$

With the same notation as before

$$\mathrm{Val}(G;C_\ell) = \int \prod_v d^d y_v \prod_\ell C_\ell(u_\ell - w_\ell). \tag{III.5}$$

<u>Lemma III.4</u> Let $\xi_1,\ldots\xi_n$ be the external vertices of G. Then

$$\int d^d\xi_2\ldots d^d\xi_n |\mathrm{Val}(G)(0,\xi_2,\ldots,\xi_n)| \le K^{L(G)} \prod_{\substack{f\in t(G^J) \\ G_f^J \text{ nontrivial}}} M^{D_f(j_f - j_{\pi(f)})}$$

where L(G) is the number of lines of G, $J = \{j_\ell | \ell\in G\}$, $\pi(f)$ is the fork immediately preceeding f in the tree, $j_{\pi(\phi)} = 0$ and

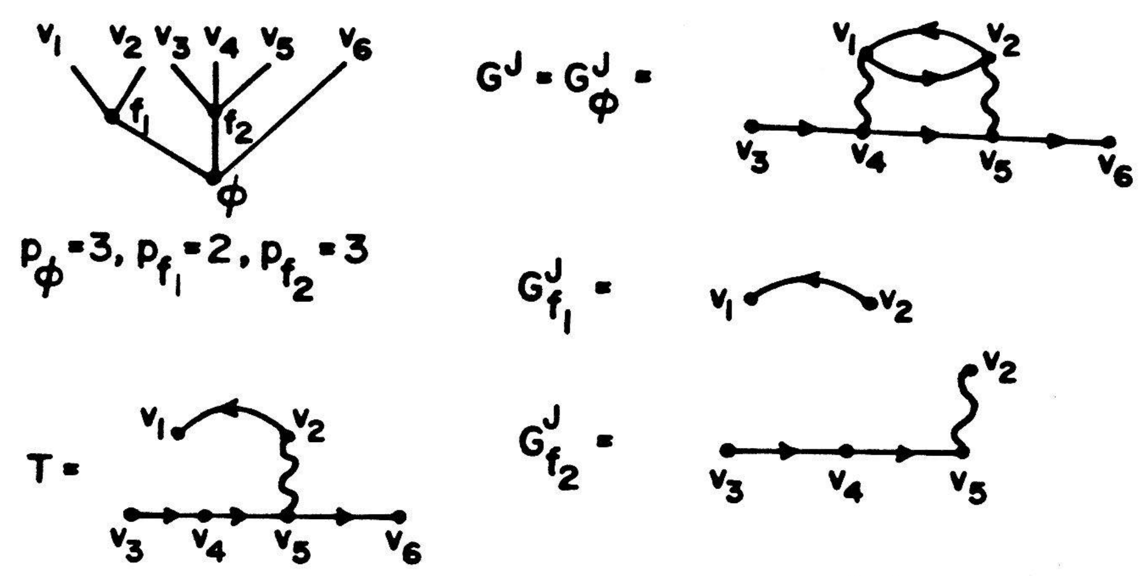
$$D_f = \sum_{\ell\in G_f^J}\delta_\ell - \alpha d(V(G_f^J) - 1).$$

where V(G) is the number of vertices of G.

<u>Proof.</u> We start by constructing a spanning tree T in G. This is done inductively from high to low scale in such a way that T G_f^J is connected for each f. Since T is a spanning tree, it has V(G) - 1 lines. It also has the following important properties:

(1) If $f_1,\ldots,f_{p_f}$ are the forks (including trivial forks) immediately above f in $t(G^J)$ then $G_{f_1}^J,\ldots,G_{f_{p_f}}^J$ are connected by $p_f - 1$ lines of T. These lines all have scale j_f.

(2) $\sum_{f\in t(G^J)} (p_f - 1) = V(G) - 1$. As in (1) p_f is the number of forks immediately above f.



We bound $\int d^d\xi_2 \dots d^d\xi_n |\mathrm{Val}(G)|$ by applying $|C_\ell(y)| \le KM^{\delta_\ell j_\ell}$ for ℓ <u>not</u> in T and then applying the tree identity with distinguished vertex $r = \xi_1$ and

$$\int d^d y |C_\ell(y)| \le KM^{\delta_\ell j_\ell} M^{-\alpha d j_\ell}.$$

Therefore,

$$\int d^d\xi_2 \dots d^d\xi_n |\mathrm{Val}(G)| \le K^{L(G)} \prod_{\ell\in G} M^{\delta_\ell j_\ell} \prod_{f'} M^{-\alpha d j_{f'}(p_{f'}-1)}.$$

The last factor is obtained from property (1) by noting that precisely $(p_{f'} - 1)$ integrals $\int d^d y \dots$ are performed using lines of scale $j_{f'}$.

The last step is to manipulate $\prod_{\ell\in G} M^{\delta_\ell j_\ell} \prod_{f'} M^{-\alpha d j_{f'}(p_{f'}-1)}$ into $\prod_f M^{D_f(j_f - j_{\pi(f)})}$. To do this we observe that if G_{f_ℓ} is the smallest element of $t(G^J)$ containing ℓ then

$$j_\ell = j_{f_\ell} = \sum_{f\le f_\ell} (j_f - j_{\pi(f)})$$

and that

$$M^{-\alpha d j_{f'}(p_{f'}-1)} = \prod_{f\le f'} M^{-\alpha d(p_{f'}-1)(j_f - j_{\pi(f)})}.$$

Consequently,

$$\prod_{\ell\in G} M^{\delta_\ell j_\ell} \prod_{f'} M^{-\alpha d j_{f'}(p_{f'}-1)}$$

$$= \prod_{f} \prod_{\ell \in G_f^J} M^{\delta_\ell (j_f - j_{\pi(f)})} \prod_{f' \geq f} M^{-\alpha d(p_{f'} - 1)(j_f - j_{\pi(f)})}$$

where we have interchanged the orders of the products $\prod_{\ell} \prod_{f \leq f_\ell}$ (note that $f \leq f_\ell$ if and only if $\ell \epsilon G_f^J$) and $\prod_{f} \prod_{f \leq f'}$. Finally, by (2),

$$\sum_{f' \geq f} (p_{f'} - 1) = V(G_f^J) - 1.$$

∎

We now apply Lemmas III.3 and III.4 to estimate $\mathrm{Val}(G^J)$ for the screened Coulomb interaction $V(x) = \dfrac{e^{-m|x|}}{|x|}$ in d-dimensions. As before let U be the ultraviolet end of the covariance C. See (I.2) and Lemma 3.1.

<u>Theorem III.2'</u> Let $d \geq 2$. For any $G \epsilon \Gamma_n^m$

$$\int d^{d+1}\bar{\zeta}_1 \ldots d^{d+1}\xi_n d^{d+1}\bar{\zeta}_n |\mathrm{Val}(G;U,V)| \leq (\mathrm{const})^m.$$

<u>Proof</u> As before,

$$\mathrm{Val}(G) = \sum_{J} \mathrm{Val}(G^J).$$

By the estimates of Lemma III.3

$$\delta(\tau)|V^{(j)}(x)| \leq [\delta(\tau)][\mathrm{const} M^j e^{-\mathrm{const} M^j |x|}]$$

for $j \geq 1$

$$|U^{(j)}(x,\tau)| \leq \mathrm{const}[M^j e^{-\mathrm{const} M^{2j}|\tau|} \begin{cases} 1 & \tau > 0 \\ 0 & \tau \leq 0 \end{cases} M^{(d-1)j} e^{-\mathrm{const} M^j |x|} g(|x|)$$

and

$$|U^{(0)}(x,\tau)| \leq \mathrm{const}\ [1 + |\tau|]^{-2}\ [1 + |x|]^{-d-1}.$$

The resulting estimate on $\int|\mathrm{Val}(G^J)|$ becomes a product of spatial (x) and temporal (τ) integrals. We now apply Lemma III.4 to each factor. For the temporal integral the covariance C_ℓ of Lemma III.4 is either $M^j e^{-\mathrm{const} M^{2j}|\tau|} \begin{cases} 1 & \tau > 0 \\ 0 & \tau \leq 0 \end{cases}$ for $j \geq 1$ particle lines, $[1 + |\tau|]^{-2}$ for $j = 0$ particles lines and $\delta(\tau)$ for interaction lines.

Hence, interaction lines effectively have scale ∞ and must be integrated first collapsing each  of G to a point  . The result of collapsing all particle lines of G is called the quotient graph of G by $\{\sim\}$ and is denoted $G/\{\sim\}$. So, the temporal factor is bounded by

$$(\mathrm{const})^m \prod_{f \in t(G^J/\{\sim\})} M^{D_f^{temp}(j_f - j_{\pi(f)})}$$

where

$$D_f^{temp} = L(G_f^J/\{\sim\}) - 2\,[V(G_f^J/\{\sim\}) - 1]$$

Let V_i and V_e denote the number of  and vertices respectively of G_f^J . Let E denote the number of external particle lines of G_f^J (i.e. the number of particle lines of G that touch G_f^J but are not in it). Furthermore, G_f^J may not have external interaction squiggles.

Then,

$$L(G_f^J/\{\sim\}) = \frac{2V_i + V_e - E}{2}$$

$$V(G_f^J/\{\sim\}) = \frac{V_i}{2} + V_e$$

and $D_f^{temp} = 2 - \frac{E}{2} - \frac{3}{2} V_e$. If $j_f > 0$ it is impossible for G_f^J to have just two external particle lines and no external interaction lines. Otherwise, G_f^J would have a particle line $U^{(j_\ell)}(x,\tau)$ with $j_\ell \geq 1$ and $\tau \leq 0$, in which case $U^{(j_\ell)}(x,\tau) = 0$. Hence, $D_f^{temp} \leq 0$ and the temporal factor is bounded by $(\mathrm{const})^m$.

The spatial factor is bounded in the same way by

$$(\mathrm{const})^m \prod_{f \in t(G^J)} M^{D_f^{space}(j_f - j_{\pi(f)})}$$

where

$$D_f^{space} = (\#\text{interaction lines}) + (d-1)(\#\text{particle lines}) - d(V(G_f^J) - 1)$$

$$= \frac{V_i - I}{2} + (d-1)\,\frac{2V_i + V_e - E}{2} - d(V_i + V_e - 1)$$

Here, V_i, V_e, and E have the same meaning as above and I is the number of external interaction lines of G_f^J. Simplifying, we obtain

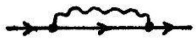
$$D_f^{\text{space}} = d - \frac{1}{2}V_i - \frac{(d+1)V_e}{2} - \frac{I}{2} - \frac{(d-1)}{2}E$$

$$= -d + 2 - (E + V_e - 4)\frac{d-1}{2} - \frac{I}{2} - \frac{1}{2}V_i - V_e\,.$$

We now show that, when $d \geq 2$ and $j_f > 0$, $D_f^{\text{space}} \leq -\epsilon(E + I) - V_e$ for some $\epsilon > 0$. When $E + V_e > 4$ this is obvious. When $E + V_e = 4$, $D_f^{\text{space}} \leq -\frac{I}{2} - \frac{V_i}{2} - V_e \leq -\frac{1}{2} - \frac{I}{2} - V_e$ and the claim is again obvious. When $E + V_e = 2$,

$$D_f^{\text{space}} = 1 - \frac{1}{2}(V_i + I) - V_e$$

$$\leq -\frac{1}{4} - \frac{1}{12}(V_i + I) - V_e$$

unless $I + V_i \leq 2$, $V_e = 0$. But the only nontrivial graph satisfying $E = 2$, $V_e = 0$, $I + V_i \leq 2$ is  . This graph as well as all graphs G_f^J having $j_f > 0$, $E = 0$ is ruled out by the support property for $U^{(j_\ell)}(x,\tau) = 0$ for $\tau \leq 0$, $j_\ell > 0$.

The temporal factor was bounded by $(\text{const})^m$ and now the spatial factor has been bounded by

$(\text{const})^m \prod_{f \in t(G^J)} M^{-\epsilon(E+I+V_e)(j_f - j\pi(f))}$. Consequently

$$\left|\int \mathrm{Val}(G^J)\right| \leq (\text{const})^m \prod_{f \in t(G^J)} M^{-\epsilon(E+I+V_e)(j_f - j_{\pi(f)})} \tag{III.6}$$

It remains to sum over J.

First we rearrange the product to get

$$|\int \mathrm{Val}(G^J)| \le (\mathrm{const})^m M^{-2\epsilon n j_\phi} \prod_v \prod_{\substack{\ell,\ell' \text{hooked} \\ \text{to } v}} M^{-\frac{\epsilon}{4}|j_\ell - j_{\ell'}|}$$

where v runs over all vertices in $G \in \Gamma_n^m$. The factor $M^{-2\epsilon n j_\phi}$ is exactly $M^{-\epsilon V_e (j_t - j_{\pi(f)})}$ for $f = \phi$. The other factors are constructed as follows. For each $f \in t(G^J)$ one factor of $M^{-\epsilon(j_f - j_{\pi(f)})}$ is assigned to each external line of G_f^J. Fix any vertex v of G_f^J and let us determine the net factor assigned to any line ℓ emanating from v. If ℓ_v is the line of highest scale emanating from v, then ℓ receives factors $M^{-\epsilon(j_f - j_{\pi(f)})}$ from forks in a linear subtree of $t(G^J)$ starting with $j_f = j_{\ell_v}$ and ending with a fork obeying $j_{\pi(f)} = j_\ell$. The product of all these factors assigned to ℓ is exactly $M^{-\epsilon(j_{\ell_v} - j_\ell)}$.

Hence,

$$\prod_{f \in t(G^J)} M^{-\epsilon(E+I)(j_f - j_{\pi(f)})} = \prod_v \prod_{\substack{\ell \text{ hooked to} \\ v}} M^{-\epsilon (j_{\ell_v} - j_\ell)}$$

$$\le \prod_v \prod_{\substack{\ell,\ell' \text{ hooked} \\ \text{to } v}} M^{-\frac{\epsilon}{4}|j_\ell - j_{\ell'}|}$$

Finally, we perform the sum. This is done by ordering the lines ℓ of G^J such that $j_{\ell_1} = j_\phi$ (there are at most (const m) choices of such a starting line) and ℓ_{i-1} and ℓ_i share a vertex. Then we bound

$$|\int \mathrm{Val}(G)| \le \sum_J (\mathrm{const})^m M^{-2\epsilon n j_\phi} \prod_v \prod_{\substack{\ell,\ell' \text{hooked} \\ \text{to } v}} M^{-\frac{\epsilon}{4}|j_\ell - j_{\ell'}|}$$

$$\le (\mathrm{const})^m \sum_J M^{-j\ell_1} \prod_{i=2}^{L(G)} M^{-\frac{\epsilon}{4}|j_{\ell_i} - j_{\ell_{i-1}}|}$$

$$\leq (\mathrm{const})^m$$

since $L(G) \leq \mathrm{const}\ m$ and $\sum_j M^{-\frac{\epsilon}{4}|j|} \leq \mathrm{const}.$ ∎

IV. Positive Temperature

Let us recall the standard model for electrons in a crystal at positive temperature. The Fock space as well as free and interacting Hamiltonians are the same as those of the introduction. Now however the expected value of an observable A is given by the normalized trace

$$\langle A\rangle_\beta := \frac{\mathrm{Tr}\{e^{-\beta(H-\mu N)}A\}}{\mathrm{Tr}\{e^{-\beta(H-\mu N)}\}}$$

rather than by the expectation $\langle\Omega, A\Omega\rangle$ in the fermionic ground state Ω. The positive temperature Schwinger functions are formally defined by

$$S_n(\overset{(-)}{\xi}_1, \ldots, \overset{(-)}{\xi}_n) = (-1)^{n/2}\langle T\overset{(-)}{\psi}_{\sigma_1}(\xi_1)\cdots\overset{(-)}{\psi}_{\sigma_n}(\xi_n)\rangle_\beta$$

As before the free Schwinger functions (i.e. when $H = H_0$) are given by the determinant

$$S^o_{2n}(\xi_1, \bar{\zeta}_1, \xi_2\bar{\zeta}_2, \ldots, \xi_n, \bar{\zeta}_n; \beta) = \det\,[S^0_2(\xi_i, \bar{\zeta}_j; \beta)] \tag{IV.1}$$

where by direct calculation

$$S^0_2(\xi_1, \xi_2; \beta) = \delta_{\sigma_1,\sigma_2} \int \frac{d^dk}{(2\pi)^d}\, e^{ik(x_1-x_2)}\, e^{-(\epsilon(k)-\mu)(\tau_1-\tau_2)} \times \begin{cases} -1+n_k & \text{if } \tau_1>\tau_2 \\ n_k & \text{if } \tau_1\leq\tau_2 \end{cases}.$$

Here, $n_k = \left[e^{\beta e(k)} + 1\right]^{-1}$, $e(k) = \epsilon(k)-\mu$ and $\epsilon(k) = k^2/(2m)$ is the dispersion relation for the spherically symmetric independent electron approximation.

To start with, S^0_2 is defined for $\tau_i \in [0,\beta]$. But

$$C(x,\tau) = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot x} e^{-e(k)\tau} \begin{cases} -1+n_k & \tau>0 \\ n_k & \tau\le 0 \end{cases}$$

satisfies

$$\begin{aligned} C(x,\tau+\beta) &= \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot x} e^{-e(k)(\tau+\beta)} (-1+n_k) \\ &= \int \frac{d^dk}{2\pi)^d} e^{ik\cdot x} e^{-e(k)\tau} (-n_k) \\ &= -C(x,\tau) \end{aligned}$$

for $-\beta < \tau \le 0$. Therefore C extends to a function on $\mathbf{R}^{d+1}$ with period 2β in τ satisfying

$$C(x,\tau+\beta) = -C(x,\tau).$$

Consequently,

$$S_2^0(\xi_1,\xi_2) = \delta_{\sigma_1,\sigma_2} C(x_1-x_2,\tau_1-\tau_2)$$

extends to a function on $\mathbf{R}^{d+1} \times \mathbf{R}^{d+1}$ periodic in τ_1 and τ_2 with period 2β. It now follows from the determinant identity (IV.1) that all free Schwinger functions have periodic extensions of the same kind.

At positive temperature the dichotomy

$$\begin{cases} -1 + n_k & \text{if } \tau > 0 \\ n_k & \text{if } \tau \le 0 \end{cases}$$

replaces

$$\begin{cases} -\chi(e(k)) & \text{if } \tau > 0 \\ \chi(-e(k)) & \text{if } \tau \ge 0 \end{cases}$$

with the result that $C_\beta(x,\tau)$ is a rapidly decreasing function of x. Therefore the large distance, i.e. infrared end, of the model is completely regular. This is the crucial difference between zero and positive temperature.

The interacting positive temperature Schwinger functions have the same formal perturbation theory

$$S_{2n} \sim \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{\lambda}{2}\right)^m \sum_G \mathrm{Val}_\beta(G)$$

as their zero temperature counterparts with the modification that C is replaced by C_β and "time" is restricted to $[0,\beta)$. We shall show that the value of each graph satisfies

$$|\mathrm{Val}_\beta(G)| \le (\mathrm{const})^m$$

The constant depends on β and diverges as β tends to infinity. As mentioned in the introduction, we shall show in another paper that the renormalized value of each graph converges, as β tends to infinity, to its renormalized value at zero temperature.

<u>Lemma IV.1</u>

$$\begin{aligned} C_\beta(\xi) &= R_\beta(\xi) + S_\beta(\xi) \\ &= \sum_{j=0}^{\infty} U_\beta^{(j)}(\xi) \end{aligned}$$

where

$$S_\beta(\xi) = -(2\pi\tau)^{-\frac{d}{2}} e^{\mu\tau} e^{-\frac{1}{2}\frac{x^2}{\tau}} \begin{cases} 1 & \tau \bmod 2\beta \in (0, \frac{\beta}{20d}] \\ -1 & \tau \bmod 2\beta \in (-\beta, (-1 + \frac{1}{20d})\beta] \\ 0 & \text{otherwise} \end{cases}$$

$$|U^{(j)}(x,\tau)| \le \mathrm{const}\ M^{dj} e^{-\mathrm{const}(M^{2j}[\tau]+M^j|x|)} \begin{cases} 1 & \text{if } [\tau]\in(0,\frac{\beta}{20d}] \\ 0 & \text{otherwise} \end{cases}$$

$$|U^{(0)}(x,\tau)| \le \mathrm{const}\ [1+|x|]^{-d-1}$$

The constants depend on β. Here $[\tau]$ denotes the representative of τ modulo β in $[-\beta/2, \beta/2)$.

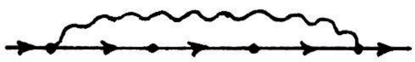
<u>Proof</u>. Since $C(x,\tau+\beta) = -C(x,\tau)$ it suffices to consider $\tau \in (0,\beta]$.

For $\tau \in (0,\beta]$, $e^{-e(k)\tau} n_k$ is a Schwartz class function of $k \in \mathbb{R}^d$ and hence has a Schwartz class Fourier transform and

$$- \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot x} e^{-e(k)\tau} = -e^{\mu\tau}(2\pi\tau)^{-d/2} e^{-\frac{1}{2}\frac{x^2}{\tau}}$$

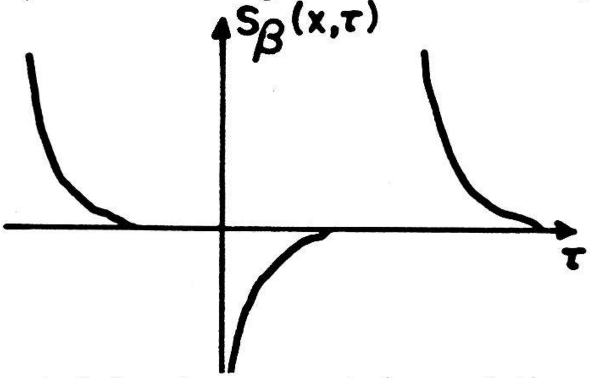
We put the portion of this for $\tau \in (0, \frac{\beta}{20d})$ into S_β and the rest in R_β. Lemma III.3 part 2) is used to decompose G_β into $\sum_{j=0}^{\infty} U_\beta^{(j)}$. ∎

At temperature zero the ultraviolet end of the covariance U was decomposed into a regular part R and singular part S. The latter vanished identically for $\tau \le 0$. This property implied that the value of any graph containing a closed loop of S particle lines or a string



of S-lines having two vertices connected by an interaction line is zero.

Now the singular part of C_β does not have a strict support property. It vanishes for τ mod β small and negative as in the figure.



Consequently graphs containing loops or strings of the above type may have nonzero values provided the loop or string is long enough. The decomposition into regular and singular, in particular the choice of the interval $[0, \frac{\beta}{20d}]$, ensures that the loop or strings contains at least 20d vertices. There is considerable freedom in the choice of the denominator 20d.

<u>Theorem IV.2</u>. Let $d \ge 1$ and V be any $L^\infty \cap L^1$ pair interaction or $d \ge 2$ and $V = \frac{1}{|x|}e^{-m|x|}$. For any $G \in \Gamma_n^m$

$$\int d^{d+1}\bar{\zeta}_1 \ldots d^{d+1}\xi_n d^{d+1}\bar{\zeta}_n |\mathrm{Val}_\beta(G)| \le (\mathrm{const})^m.$$

Proof. We follow the proof of Theorem III.2'. In factoring the bound on $\int|\mathrm{Val}(G^j)|$ into spatial and temporal integrals we use

$$\delta(\tau)|V^{(j)}(x)| \le \mathrm{const}\ [\delta(\tau)]\ [M^j e^{-\mathrm{const}M^j|x|}]$$

$$|U^{(j)}(x,\tau)| \le \mathrm{const}\left[M^{\frac{9}{10}j}\ e^{-\mathrm{const}M^{2j}[\tau]}\begin{cases}1 & [\tau]\in(0,\frac{\beta}{20d}]\\ 0 & \text{otherwise}\end{cases}\right]$$

$$\times\left[M^{(d-\frac{9}{10})j}\ e^{-\mathrm{const}M^j|x|}\right]$$

$$|U^{(0)}(x,\tau)| \le \mathrm{const}\ [1+|x|]^{-d-1}.$$

If $V \in L^\infty \cap L^1$ then $V^{(j)} = 0$ for $j \ge 1$. This case is easy, so we concentrate on $d \ge 2$ and $V = \frac{1}{|x|}\ e^{-m|x|}$.

The temporal factor is bounded by

$$(\mathrm{const})^m \sum_{f\in t(G^J/\{\sim\})} M^{D_f^{\mathrm{temp}}(j_f - j_{\pi(f)})}$$

where now

$$D_f^{\mathrm{temp}} = \frac{9}{10}L(G_f^J/\{\sim\}) - 2\ [V(G_f^j/\{\sim\}-1]$$

$$= 2 - \frac{9}{20}E - \frac{1}{10}V_i - \frac{31}{20}V_e.$$

Similarly the spatial factor is now bounded by

$$(\mathrm{const})^m \prod_{f\in t(G^J)} M^{D_f^{\mathrm{space}}(j_f - j_{\pi(f)})}$$

where

$$D_f^{\mathrm{space}} = (\#\ \text{interaction lines}) + (d - \tfrac{9}{10})(\#\ \text{particle lines}) - d\ (V(G_f^J)-1)$$

$$= d - \frac{d-\frac{9}{10}}{2}E - \frac{1}{2}I - \frac{2}{5}V_i - (\frac{d}{2}+\frac{9}{20})V_e$$

$$= \frac{9}{5} - d - \left(\frac{d - \frac{9}{10}}{2}\right)(E-4) - \frac{1}{2}I - \frac{2}{5}\, V_i - \left(\frac{d}{2} + \frac{9}{20}\right)V_e .$$

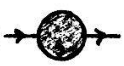
We now verify that $D_f^{temp} \le 0$ for any graph G_f^J with nonzero value and $j_f > 0$ so that the temporal factor is bounded by $(\text{const})^m$. When $E > 4$ this is obvious. When $E = 4$, $D_f^{temp} = \frac{1}{5} - \frac{1}{10}\, V_i - \frac{31}{20}\, V_e$ but V_i must be at least 2. When $E = 2$, $D_f^{temp} = \frac{11}{10} - \frac{1}{10}\, V_i - \frac{31}{20}\, V_e$. Now G_f^J may have external interaction squiggles so, as remarked just before the statement of the theorem $V_i \ge 20d \ge 20$. Finally $E = 0$ cannot occur.

The last step is to show that $D_f^{space} \le -\epsilon(E+I) - \frac{1}{4}\, V_e$ for any graph G_f^J with nonzero value and $j_f > 0$. When $E \ge 4$ this is obvious.
When $E = 2$, $D_f^{space} = \frac{9}{10} - \frac{I}{2} - \frac{2}{5}\, V_i - \left(\frac{d}{2} + \frac{9}{20}\right) V_e$. to violate the desired bound we must have $I + V_i \le 2$, $V_e = 0$. As before the only nontrivial graph satisfying $E = 2$, $V_e = 0$ and $I + V_i \le 2$ is  which still takes the value zero. Finally for $E = 0$, $D_f^{space} = d - \frac{1}{2}I - \frac{2}{5}V_i - \left(\frac{d}{2} + \frac{9}{20}\right)V_e$. In order to have $E = 0$, either $V_e \ge 2$ or there must be a closed loop of at least 20d particle lines so that $V_i \ge 20\, d$. In either case $D_f^{space} = -\frac{1}{2}I - \frac{1}{4}V_e$.

The proof is completed just as in section III. ∎

V. Infrared Convergent Graphs

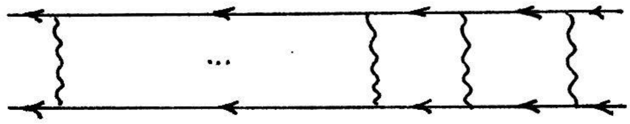
Our discussion of the ultraviolet end of the model is complete. We have seen that the ultraviolet value of every graph is finite and furthermore grows geometrically with the size of the graph. On the other hand, it was pointed out in the introduction that there are graphs whose values are infinite.

Let us recall that any graph containing a subgraph of the form 

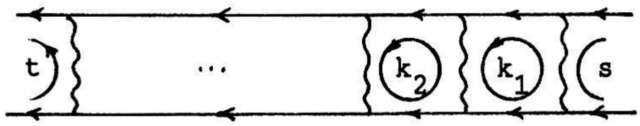
is infinite. Here the shaded disc denotes any graph having precisely 2 external particle lines and no external interaction lines. Such graphs are called mass subgraphs. Graphs containing no mass subgraphs are called convergent graphs. Indeed we shall show that their values are finite.

In addition to divergences there is another important phenomenon hidden in the infrared end. Convergent graphs are no longer geometrically bounded. They can grow like $n_F!$ where n_F is the number of four legged subdiagrams.

Consider the graph



in Γ_2^m. In the Fourier representation momentum flowing in and out of the graph is conserved. So we may view it a as a function of three independent external momenta s,t and q. If we denote the momenta flowing around the m-1 internal loops $k_1,\ldots,k_{m-1}$



the value of this graph is

$$\int \prod_{j=1}^{m-1} \frac{dk_j}{(2\pi)^{d+1}}\, \tilde{C}(\tfrac{q}{2}+s)\tilde{C}(\tfrac{q}{2}-s) \prod_{j=1}^{m-1} \tilde{V}(k_j-k_{j-1})\tilde{C}(\tfrac{q}{2}+k_j)\tilde{C}(\tfrac{q}{2}-k_j)$$

$$\tilde{V}(t-k_{m-1})\tilde{C}(\tfrac{q}{2}+t)\tilde{C}(\tfrac{q}{2}-t) \qquad \text{(V.1)}$$

where $k_0:=s$ and, as before, $\tilde{C}(k) = \frac{1}{ik_0-e(\underline{k})}$. It is implicit here that $V(x)$ is rapidly decaying and consequently $\tilde{V}$ is smooth. Thus in the infrared end, that is when all momenta are restricted to a common ball, $\tilde{V}$ cannot aggravate singularities of the integrand. So in this example we take $V(x) = \delta(x)$ i.e. $\tilde{V}(k) = 1$. Then the integral factorizes to yield

$$\tilde{C}(\tfrac{q}{2}+s)\tilde{C}(\tfrac{q}{2}-s)\tilde{C}(\tfrac{q}{2}+t)\tilde{C}(\tfrac{q}{2}-t)\left[\int_{|k|\le\text{const}}\frac{dk}{(2\pi)^{d+1}}\,\tilde{C}(\tfrac{q}{2}+k)\tilde{C}(\tfrac{q}{2}-k)\right]^{m-1}$$

To further simplify this example we restrict ourselves to three dimensions. We have

$$\int\frac{d^4k}{(2\pi)^4}\,\tilde{C}(\tfrac{q}{2}+k)\tilde{C}(\tfrac{q}{2}-k)$$

$$=\int\frac{d^4k}{(2\pi)^4}\left[i(k_0+q_0/2)-e(\underline{k}+\underline{q}/2)\right]^{-1}\left[i(q_0/2-k_0)-e(\underline{q}/2-\underline{k})\right]^{-1}$$

$$=\int_{\substack{e(\underline{k}+\underline{q}/2)e(\underline{k}-\underline{q}/2)>0\\ |\underline{k}|\le\text{const}}}\frac{d^3k}{(2\pi)^3}\left[-\operatorname{sign} e(\underline{k}+\underline{q}/2)\right]\left[i\,q_0-e(\underline{k}+\underline{q}/2)-e(\underline{k}-\underline{q}/2)\right]^{-1}$$

Explicit calculation yields that for small q

$$\int_{|k|\le\text{const}}\frac{d^4k}{(2\pi)^4}\,\tilde{C}(\tfrac{q}{2}+k)\tilde{C}(\tfrac{q}{2}-k)=\text{const}\,\ln\{|\underline{q}|+\frac{iq_0}{2\sqrt{2\mu}-|\underline{q}|}\}+0(1)\qquad(\text{V.2})$$

has an integrable logarithmic singularity at the origin. This does not affect the convergence of our graph, which is to be regarded as a tempered distribution. However applying this distriubtion to test functions $\tilde{f}_1(\frac{q}{2}+s)\ \tilde{f}_2(\frac{q}{2}-s)\ \tilde{f}_3(\frac{q}{2}+t)\ \tilde{f}_4(\frac{q}{2}-t)$ results in a value of the order of m! because of the singularity $\left[\ln\{|\underline{q}|+\frac{iq_0}{2\sqrt{2\mu}-|\underline{q}|}\}\right]^{m-1}$.

Such m!'s are typical of strictly renormalizeable field theories. For constructive purposes it is essential to understand the behaviour of four legged graphs in great detail.

Our first step towards bounding infrared graphs is to modify and refine the tree of subgraphs of a labelled graph.

The infrared end of the covariance C is

$$I(\xi) = \sum_{j=-\infty}^{-1} C^{(j)}(\xi)$$

where by Lemma II.1.1

$$|C^{(j)}(\xi)| \le \text{const } M^j [1 + |\underline{x}|]^{\frac{1-d}{2}} \phi(M^j|\xi|)$$

with $\phi \in L^\infty \cap L^1$. Infrared divergences arise when covariances and/or interactions decay too slowly to be integrable. Thus, there is no need to decompose the interaction $\delta(\tau)V(|x|)$ since it is L^1 even when $V(|x|)$ is the screened Coulamb interaction (for $d \ge 2$). The interaction should be thought of as being of scale zero.

Expanding, as in Section III, we obtain

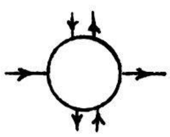
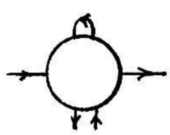
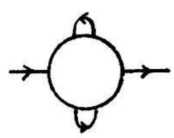
$$\begin{aligned} \mathrm{Val}(G;I) &= \mathrm{sgn}(G)\int \prod_V d^{d+1}\xi_v \prod_{\ell\in I_G} \mathcal{V} \prod_{\ell\in P_G} \left[\sum_{j_\ell=-\infty}^{-1} C^{(j_\ell)}(\xi_{u_\ell}-\xi_{w_\ell})\right] \\ &= \sum_{\substack{j_\ell\le -1\\ \ell\in P_G}} \mathrm{sgn}(G) \int \prod_V d^{d+1}\xi_v \prod_{\ell\in I_G} \mathcal{V} \prod_{\ell\in P_G} C^{(j_\ell)}(\xi_{u_\ell} - \xi_{w_\ell}) \end{aligned}$$

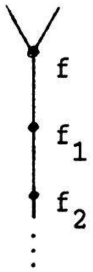
where $\mathcal{V}(\xi) = \delta(\tau)V(x)$. Once again $\mathrm{Val}(G;I)$ is the sum of values of labelled graphs G^J, $J = \{j_\ell | \ell\in P_G\}$. The forest $F(G^J)$ and $t(G^J)$ determined by J are defined as before. Namely

$$F(G^J) = \bigcup_{j<0} \{\text{connected components of } \{\ell\in G^J | j_k \ge j\}\} \cup \{\text{vertices of } G\}$$

and $t(G^J)$ is the set of elements of $F(G^J)$ ordered by inclusion.

We now slightly modify trees and forests in order to simplify the combinatorial structure by preventing the formulation of two-forks $\vert$. To the contrary suppose the fork $f \in t(G^J)$ lies at the top of a string of two forks. Then $G_f^J \subset G_{f_1}^J$ ...

$G_f^J =$  $G_{f_1}^J =$  $G_{f_2}^J =$ 



all have the same sets of vertices. The graph $G^J_{f_1}$ is formed by connecting one or more pairs of external legs of G^J_f and so on. The lines formed in this way are called Wick lines.

Consider the equivalence classes of labellings J of G which generate the same set of Wick particle lines and assign the same scale j_ℓ to each non-Wick line ℓ. The labellings differ only in the scale assigned to each Wick line. For each such equivalence class we construct a new kind of labelling $\mathcal{J}$ of G. Pick any representative. Each non-Wick line is given the scale j_ℓ common to every element of the class. Each Wick line ℓ is given the <u>soft</u> scale which is one plus the maximum of all scales assigned to this line in the class. Furthermore assign the label s to each Wick line. Let us define the soft covariance

$$C_s^{(j)} = \sum_{j'<j} C^{(j)} .$$

It follows that

$$\begin{aligned} \mathrm{Val}(G;I) &= \sum_J \mathrm{Val}(G^J;I) \\ &= \sum_{\mathcal{J}} \mathrm{Val}(G^{\mathcal{J}};I) \end{aligned}$$

where the sum is over all labellings generated by equivalence classes and

$$\mathrm{Val}(G^{\mathcal{J}};I) = \mathrm{sgn}(G)\int \prod_v d^{d+1}\xi_v \prod_{\ell\in I_G} \mathcal{V}_\ell \prod_{\substack{\ell\in P_G \\ \ell\ \mathrm{soft}}} C_s^{(j_\ell)}(\xi_{u_\ell} - \xi_{w_\ell}) \prod_{\substack{\ell\in P_G \\ \ell\ \mathrm{hard}}} C^{(j_\ell)}(\xi_{u_\ell} - \xi_{w_\ell}) . \tag{V.4}$$

Here a hard line is one without a soft label.

For each labelling $\mathcal{J}$ generated by an equivalence class there is a forest

$$F(G^{\mathcal{J}}) = \bigcup_{j\le -1} \{\text{connected components of } \{\ell\in G^{\mathcal{J}} \mid j_\ell \ge j,\ \ell \text{ hard or soft}\}\}$$
$$\cup \{\text{vertices of } G\}$$

and a tree $t(G^{\mathcal{J}})$, as always, the elements of $F(G^{\mathcal{J}})$ ordered by inclusion. Now, $t(G^{\mathcal{J}})$ has no 2-forks. Observe that any fork $G_f^{\mathcal{J}}$ of $t(G^{\mathcal{J}})$ is a connected graph, connected by interaction lines and hard particle lines.

The idea is to block the sum

$$\begin{aligned} \mathrm{Val}(G;I) &= \sum_{\mathcal{J}} \mathrm{Val}(G^{\mathcal{J}};I) \\ &= \sum_{t\in\mathfrak{T}} \sum_{F\in\mathfrak{F}} \sum_{s\in\mathcal{S}(t)} \sum_{\substack{\mathcal{J} \\ F=F(\mathcal{J}) \\ t=t(\mathcal{J}) \\ s=s(\mathcal{J})}} \mathrm{Val}(G^{\mathcal{J}};I). \end{aligned} \tag{V.5}$$

Here $\mathfrak{T}$ is the set of all allowed trees i.e. tree without 2-forks, $\mathcal{S}(t)$ is the set of all allowed assignments of scales j_f to the forks of t, i.e. if $f > f'$ then $j_f > j_{f'}$, $\mathfrak{F}$ is set of allowed forests of subgraphs of G and $F(\mathcal{J})$, $t(\mathcal{J})$, and $s(\mathcal{J})$ are functions giving the forest, tree and scale assignments generated by the labelling $\mathcal{J}$. Note that for any given graph the number of possible trees, forests and assignments of hard/soft labels is finite. The only infinite sum in the blocking is that over $s\in\mathcal{S}(t)$. It will be controlled by exponential decay between the scales as in section III.

The size of a graph must be measured differently in the infrared end because its value is typically not an integrable function of its external vertices. For convergent graphs we shall bound the integral of Val(G;I) against L^1 test functions. to do this we need an infrared version of Lemma III.4.

Suppose G is a general connected graph, not necessarily arising from our model. It contains external vertices, that are integrated against test functions, internal vertices that are integrated over $\mathbb{R}^d$, internal lines, but no external lines. Internal lines ℓ may be hard lines, which carry covariances C_ℓ satisfying

$$|C_\ell(y)| \le KM^{\delta_\ell j_\ell} g(M^{j_\ell}|y|),\ y \in \mathbb{R}^d,\ \delta_\ell > 0 \tag{V.6}$$

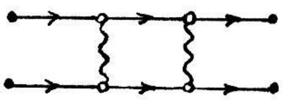
$$\|g\|_1 \le 1, \qquad \|g\|_\infty \le 1.$$

Or they may be soft lines carrying covariances obeying

$$|C_\ell(y)| \le KM^{\delta_\ell j_\ell},\ \delta_\ell > 0\ .$$

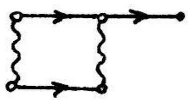
The bounds on C_ℓ induce a hard/soft labelling of G.

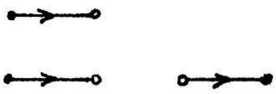
By definition a subgraph G_f^γ is nontrivial if it isn't a vertex. Each subgraph G_f must be connected by hard lines. A line ℓ of G is an external line of G_f^γ if it is hooked to a vertex of G_f^γ but is not a line of G_f^γ. For instance

G = 

• = external vertex

∘ = internal vertex

G_f^γ = 

The external lines of G_f^γ are  .

<u>Lemma V.1</u> Let G be a general graph as above with external vertices ξ_j, $j \le n$. Suppose that each internal vertex v is dimensionless in the sense that

$$\frac{1}{2} \sum_{\ell \text{ hooked to } v} \delta_\ell = d.$$

This is the case for our model.

Let f_j, $j \le n$ be in $L^1(\mathbb{R}^d)$. Then

$$\left|\int \prod_j f_j(\xi_j)d\xi_j \; \mathrm{Val}(G;C_\ell)(\xi_1\ldots,\xi_n)\right|$$

$$\le \prod_j \|f_j\|_1 K^{L(G)} \prod_{\substack{f\in t(G^\gamma)\\ f>\emptyset\\ G^\gamma_f \text{ nontrivial and contains}\\ \text{no external vertices}}} M^{D_f(j_f-j)_{\pi(f)}}$$

$$\prod_{\substack{f\in t(G^\gamma)\\ t>\emptyset\\ G^\gamma_f \text{ nontrivial and contains}\\ \text{an external vertex}}} M^{\Delta_f(j_f-j_{\pi(f)})}$$

$$\prod_{\substack{f\in t(G^\gamma)\\ G^\gamma_f \text{ an external vertex}}} M^{\Delta_f(0-j_{\pi(f)})}$$

Here,

$$\Delta_f = -\frac{1}{2}\sum_{\substack{\text{external lines}\\ \ell \text{ of } G^\gamma_f}} \delta_\ell \tag{V.7a}$$

and as before

$$D_f = \sum_{\ell\in G^\gamma_f} \delta_\ell - d(V(G^\gamma_f)-1) \tag{V.7b}$$

Since internal vertices are dimensionless

$$D_f = d + \Delta_f \tag{V.7c}$$

The distinction between the ultraviolet estimate and this one is the integration of external vertices against L^1 test functions yielding better exponential decay. Indeed, a decay rate proportional to the number of external vertices, since Δ_f is always strictly negative. Roughly speaking we may use the estimate $\int d^dy|f(y)| = 0(1)$ in place of the estimate $\int d^dy\; g(M^j|y|) = 0(M^{-dj})$ $(j < 0)$ whenever j is the position of an external vertex.

Proof We follow the same strategy as in Lemma III.4 but, construct n connected tree T_j, $j \le n$, of hard lines, each containing precisely one external vertex, rather than a single spanning tree. They are constructed by induction in the following way. For each maximal element $f \in T(G^{\mathcal{Y}})$ choose any connected tree of hard lines whose union contains all the vertices of $G_f^{\mathcal{Y}}$ such that each of these trees contains precisely one external vertex of $G_f^{\mathcal{Y}}$. If $G_f^{\mathcal{Y}}$ has no external vertices choose one tree. Extend these connected trees working down through the forks $f \in t(G^{\mathcal{Y}})$. There is no obstruction to this inductive procedure because closed loops are never formed and because, by hypothesis every $G_f^{\mathcal{Y}}$ is connected by hard lines.

Apply

$$|C_\ell(y)| \le KM^{\delta_\ell j_\ell}$$

to every line not in $\bigcup_j T_j$. Apply the tree identity to each T_j, with T_j's external vertex as distinguished vertex, to perform the integrals

$$\Pi_j \int_{v \in T_j} d^d y_v \cdots ,$$

This yields

$$|\int \prod_j f(\xi_j) d\xi_j \; \mathrm{Val}(G;C_\ell)(\xi_1,\cdots,\xi_n)|$$

$$\le \prod_j \|f_j\|_1 \; K^{L(G)} \prod_\ell M^{\delta_\ell j_\ell} \prod_{f'} M^{-dj_{f'}[p_{f'} - \max(1,p_{f'}^e)]}$$

As in Lemma III.4 $p_{f'}$ is the number of forks immediately above f' (i.e. obeying $\pi(f") = f'$) in $t(G^{\mathcal{Y}})$. In addition $p_{f'}^e$ is the number of those forks that are external i.e. for which the corresponding graph $G_{f"}^{\mathcal{Y}}$ contains an external vertex.

We must now manipulate $\pi_\ell M^{\delta_\ell j_\ell} \prod_{f'} M^{-dj_{f'}[p_{f'},\cdots]}$ into the desired form. We do so in two stages. The first stage mimics the corresponding step of Lemma III.4:

$$M^{\delta_\ell j_\ell} = M^{\delta_\ell j_{f\ell}} = M^{\delta_\ell j_\phi} \prod_{\phi<f\le f_\ell} M^{\delta_\ell (j_f - j_{\pi(f')})}$$

$$M^{-dj_{f'}[p_{f'},-1]} = M^{-dj_\phi [p_{f'},-1]} \prod_{\phi<f\le f'} M^{-d[p_{f'},-1](j_f - j_{\pi(f)})}$$

$$M^{dj_{f'}[p^e_{f'},-1]_+} = M^{dj_\phi [p^e_{f'},-1]_+} \prod_{d<f\le f'} M^{d[p^e_{f'},-1]_+(j_f - j_{\pi(f)})}$$

where $[x]_+ = \max(x,0)$. Consequently

$$\prod_\ell M^{\delta_\ell j_\ell} \prod_{f'} M^{-dj_{f'}[p_{f'},-\max(0,p^e_{f'})]}$$

$$= \prod_{\ell\in G^\gamma} M^{\delta_\ell j_\phi} \prod_{f,\text{nontrivial}} M^{-dj_\phi [p_f - 1 - (p^e_f - 1)_+]}$$

$$\prod_{\phi<f} \prod_{\ell\in G^\gamma_f} M^{\delta_\ell (j_f - j_{\pi(f)})} \prod_{\substack{f'\ge f\\ f'\text{nontrivial}}} M^{-d[p_{f'},-1](j_f - j_{\pi(f)})} \prod_{\substack{f'\ge f\\ f'\text{nontrivial}}} M^{d(p^e_{f'},-1)_+(j_f - j_{\pi(f)})}$$

$$= M^{j_\phi [D_\phi + d(E_\phi - 1)]} \prod_{\substack{f>\phi\\ f\text{ nontrivial}}} M^{(j_f - j_{\pi(f)})[D_f + d(E_f - 1)_+]}$$

where E_f is the number of external vertices in G^γ_f. The result of this manipulation is not helpful. Those exponents having $E_f > 1$ will typically be positive since $j_f - j_{\pi(f)} > 0$ and usually $D_f > -dE_f$. On the other hand the exponent $j_\phi[D_\phi + d(E_\phi - 1)]$ is very negative.

The second stage redistributes $M^{j_\phi [D_\phi + d(E_\phi - 1)]}$ up the tree $t(G^\gamma)$. Note that, since all internal vertices are dimensionless

$$
\begin{aligned}
& D_\phi + d(E_\phi - 1) \\
&= \sum_\ell \delta_\ell - d(V(G^\gamma) - E_\phi) \\
&= \sum_{\text{vertices } v} \Big\{ \sum_{\substack{\ell \text{ hooked} \\ \text{to } v}} \frac{1}{2}\delta_\ell \Big\} - d \sum_{\substack{\text{internal} \\ \text{vertices}}} 1 \\
&= \sum_{\substack{\text{external} \\ \text{vertices}}} \Big\{ \sum_{\substack{\ell \text{ hooked} \\ \text{to } v}} \frac{1}{2}\delta_\ell \Big\} \\
&= -\sum_{\substack{v \in t(G^\gamma) \\ v,\text{trivial},\text{external}}} \Delta_V
\end{aligned}
$$

Now, for each trivial, external $v \in t(G^\gamma)$, i.e. for which G^γ_v consists of a single external vertex,

$$
-j_\phi \Delta_v = \sum_{\phi<f<v} (j_f - j_{\pi(f)})\Delta_v + (0 - j_{\pi(v)})\Delta_v .
$$

Hence,

$$
M^{j_\phi[D_\phi + d(E_\phi - 1)]} \prod_{\substack{f>\phi, \\ f \text{ nontrivial}}} M^{(j-j_{\pi(f)})[D_f + d(E_f - 1)_+]}
$$

$$
= \prod_{\substack{f>0 \\ f \text{ not trivial,} \\ \text{not external}}} M^{(j-j_{\pi(f)})D_f} \prod_{\substack{v \text{ trivial,} \\ \text{external}}} M^{-j_{\pi(v)}\Delta_v} \times
$$

$$
\times \prod_{\substack{f>\phi \\ f \text{ nontrivial} \\ f \text{ external}}} M^{(j_f - j_{\pi(f)})\{D_f + d(E_f - 1) + \sum_{\substack{v \text{ external} \\ v \in G^\gamma_f}} \Delta_v\}}
$$

and the lemma follows from

$$D_f + d(E_f - 1) = \sum_{\substack{v \text{ vertex} \\ v \in G_f^{\gamma}}} \{ \sum_{\substack{\ell \in G_f^{\gamma} \\ \ell \text{ hooked} \\ \text{to } v}} \frac{1}{2} \delta_\ell \} - d \sum_{\substack{\text{internal} \\ \text{vertices} \\ \text{of } G_f^{\gamma}}} 1$$

$$= \sum_{\substack{v \in G_f^{\gamma} \\ v \text{ external}}} \{ \sum_{\substack{\ell \in G_f^{\gamma} \\ \ell \text{ hooked} \\ \text{to } v}} \frac{1}{2} \delta_\ell \} - \sum_{\substack{v \in G_f^{\gamma} \\ v \text{ internal}}} \{ \sum_{\substack{\ell \in G, \\ \ell \notin G_f^{\gamma} \\ \ell \text{ hooked to } v}} \frac{1}{2} \delta_\ell \}$$

We now wish to apply Lemma V.1 to estimate $\mathrm{Val}(G^{\gamma};I)$ in preparation for our bound on Val $(G;I)$. The bound of Lemma II.1.1 is not of the form (V.6), because of the $[1 + |x|]^{\frac{1-d}{2}}$, so we cannot immediately apply Lemma V.1. This is easy to correct even without foolishly discarding the $[1+|x|]^{\frac{1-d}{2}}$:

<u>Lemma V.2</u> For $d > 1$

1) $[1+|\underline{x}|]^{\frac{1-d}{2}} \leq \sum_{k=-\infty}^{-1} \text{const } M^{k\frac{d-1}{2}} e^{-M^k[1+|\underline{x}|]}$

2) $|C^{(j)}(\xi)| \leq \sum_{k=j}^{-1} \text{const } M^j M^{k\frac{d-1}{2}} g(M^k|\underline{x}|) g(M^j|\tau|)$

with $\|g\|_\infty \leq 1$, $\|g\|_1 \leq 1$.

3) $|C_s^{(j)}(\xi)| \leq \text{const } M^j M^{j\frac{d-1}{2}} + \sum_{k=j+1}^{-1} \text{const } M^j M^{k\frac{d-1}{2}} g(M^k|x|)$

<u>Proof</u>

1) As usual

$$[1+|\underline{x}|]^{\frac{1-d}{2}} = \text{const} \int_0^\infty dt\ t^{\frac{d-3}{2}} e^{-[1+|\underline{x}|]t}$$

$$= \sum_{k=-\infty}^{-2} \text{const} \int_{M^k}^{M^{k+1}} dt\ t^{\frac{d-3}{2}} e^{-[1+|\underline{x}|]t} + \int_{M^{-1}}^{\infty} dt\ t^{\frac{d-3}{2}} e^{-[1+|\underline{x}|]t}$$

$$\le \sum_{k=-\infty}^{-2} \text{const}\ (M-1)M^k (M^{k+1})^{\frac{d-3}{2}} e^{-[1+|\underline{x}|]M^k}$$

$$+ \text{const}\ e^{-[1+|\underline{x}|]M^{-1}} .$$

2) From part 1) and Lemma II.1.1

$$|C^{(j)}(\xi)| \le \text{const}\ M^j \sum_{k=-\infty}^{-1} M^{k\frac{d-1}{2}} e^{-[1+|\underline{x}|]M^k} [1+(M^j|\xi|)^N]^{-1}.$$

When $k > j$ we simply bound $[1+(M^j|\xi|)^N]^{-1} \le [1+(M^j|\tau|)^N]^{-1}$. All the terms with $k \le j$ may be bounded by

$$\text{const}\ M^j \sum_{i=-\infty}^{j} M^{i\frac{d-1}{2}} [1+(M^j|\xi|)^N]^{-1}$$

$$\le \text{const}\ M^j M^{j\frac{d-1}{2}} [1+(M^j|\xi|)^N]^{-1},$$

since $d > 1$, and lumped into a single term with $k= j$.

3) The soft covariance $C_s^{(j)}$ is, by definition, $\sum_{k=-\infty}^{j-1} C^{(j)}$ and so obeys

$$|C_s^{(j)}(\xi)| \le \sum_{i=-\infty}^{j-1} \text{const}\ M^i [1+|\underline{x}|]^{\frac{1-d}{2}} [1+(M^i|\xi|)^N]^{-1}$$

$$\le \text{const}\ M^j [1+|\underline{x}|]^{\frac{1-d}{2}} .$$

We may now continue as in part 2) lumping all terms with $k \le j$ into a single const $M^j M^{j\frac{1-d}{2}}$. ∎

A good way to get some intuition regarding bounds on graphs in the infrared end is to consider only the terms with $k = j$ in Lemma V.2 (they will indeed turn out to be dominant) and to collapse interaction lines $\sim\!\sim\!\sim$ to points (they are of scale zero and so look like delta functions to the infrared end). This gives a toy model with vertex  and with lines having $\delta_\ell = \frac{d+1}{2}$. For this toy model a graph G_f^γ with v internal vertices  (no external vertices $\bullet\!\!\rightarrow\!\!-$ or $\bullet\!\!\leftarrow\!\!-$) and E external lines has

$$D_f = \frac{d+1}{2}\left[\frac{4V-E}{2}\right] - (d+1)(V-1)$$

$$= \frac{d+1}{4}(4-E).$$

This is negative for $E > 4$, zero (i.e. marginal) for $E = 4$ and positive for $E = 2$. In the infrared end marginal subgraphs do not produce divergences, though they do produce m!'s. See (V.1), (V.2). Hence we call a graph convergent if it contains no internal $E = 2$ (i.e. mass) subgraphs.

<u>Lemma V.3</u> Let $G \in \Gamma_n^m$ contain no mass subgraphs that are free of external vertices. Then

$$\left|\int \prod_{j=1}^{n} f_j(\xi_j)d\xi_j \mathrm{Val}(G^\gamma;I)(\xi_1,\dots,\xi_n)\right|$$

$$\le \prod_j \|f_j\|_1 K^{L(G)} \prod_{\substack{f\in t(G^\gamma)\\ f>\phi\\ f \text{ nontrivial}\\ f \text{ not external}}} M^{D_f^1(j_f-j_{\pi(f)})} \prod_{\substack{f\in t(G^\gamma)\\ f>\phi\\ f \text{ nontrivial}\\ f \text{ external}}} M^{\Delta_f^1(j_f-j_{\pi(f)})}$$

$$\prod_{\substack{v\in t(G^\gamma)\\ v \text{ trivial}\\ v \text{ external}}} M^{\Delta_v^1(0-j_{\pi(v)})} \tag{V.8}$$

where $\quad D_f^d = \frac{d}{4}(4-E_f)$

$$\Delta_f^d = -\frac{d}{4}E_f$$

E_f = the number of external lines of $G_f^{\mathcal{J}}$ that are internal lines of G.

We remark that D_f^d and Δ_f^d are specializations of the D_f and Δ_f of (V.7) to the toy model in d dimensions having internal vertices  , external vertices $\bullet\!\!\longleftarrow$, $\longrightarrow\!\!\bullet$ and $\delta_\ell = \frac{d}{2}$. We also remark that since mass subgraphs are forbidden (unless they contain external vertices) $\Delta_f^d <$ 0 and $D_f^d \le 0$ with equality only for $E_f = 4$.

<u>Proof</u>.

We start by applying

$$\left|\int \prod_{j=1}^{n} f_j(\xi_j)d\xi_j \; \mathrm{Val}(G^{\mathcal{J}};I)(\xi_1,\ldots,\xi_n)\right|$$

$$\le \prod_{j=1}^{n} \|f_j\|_1 \; \sup|\mathrm{Val}(G^{\mathcal{J}};I)(\xi_1,\ldots,\xi_n)|$$

and then Lemma V.2 parts 2) and 3). This yields

$$|\mathrm{Val}(G^{\mathcal{J}};I)(\xi_1,\ldots,\xi_n)|$$
$$\le \sum_{k_\ell=j_\ell}^{-1} K^{L(G)} \; \mathrm{Val}(G^{\mathcal{J},\mathcal{K}})(\xi_1,\ldots,\xi_n)|$$

where the hard particle lines of $G^{\mathcal{J},\mathcal{K}}$ have covariance

$$M^{-\frac{1}{2}(k_\ell-j_\ell)}[M^{\frac{d}{2}k_\ell}g(M^{k_\ell}|\underline{x}|)][M^{\frac{1}{2}j_\ell}g(M^{j_\ell}|\tau|)],$$

the soft particle lines of $G^{\mathcal{J},\mathcal{K}}$ have covariance

$$M^{-\frac{1}{2}(k_\ell-j_\ell)}[M^{\frac{d}{2}k_\ell}\,g(M^{k_\ell}|\underline{x}|)][M^{\frac{1}{2}j_\ell}] \quad \text{if } k_\ell > j_\ell$$

and

$$M^{-\frac{1}{2}(k_\ell - j_\ell)}[M^{\frac{d}{2}k_\ell}][M^{\frac{1}{2}j_\ell}] \quad \text{if } k_\ell = j_\ell$$

and the interaction lines of $G^{\mathcal{J},\mathcal{K}}$ have covariance

$$M^{-\frac{1}{2}(0-0)}[M^{d0}V(M^0|x|)][M^{0x1}\delta(M^0\tau)].$$

All the zeroes in the interaction line covariance are to emphasize that they should be thought of as having $j_\ell = k_\ell = 0$. Hence we get the factorization

$$\mathrm{Val}(G^{\mathcal{J},\mathcal{K}}) = [\prod_\ell M^{-\frac{1}{2}(k_\ell - j_\ell)}]\mathrm{Val}^d(\mathcal{K})\ \mathrm{Val}^1(\mathcal{J}).$$

The final factor $\mathrm{Val}^1(\mathcal{J})$ arises from the τ integrals. It is the value of a graph having the same lines and vertices as $G^{\mathcal{J}}$, the same scale assignment j_ℓ as $G^{\mathcal{J}}$, the same hard/soft assighments as $G^{\mathcal{J}}$ but artificially living in dimension 1 with $\delta_\ell = \frac{1}{2}$ for particle lines and $\delta_\ell = 1$ for interaction lines. Applying Lemma V.1 to $\mathrm{Val}^1(\mathcal{J})$ yields the factors $\prod M^{D_f^1(j_f - j_{\pi(f)})}$ $\prod M^{\Delta_f^1(j_f - j_{\pi(f)})}$ $\prod M^{\Delta_v^1(0-j_{\pi(v)})}$ in the statement of Lemma V.3. In this regard note that interaction lines are always of scale 0 so that they never occur as external lines of any $G_f^{\mathcal{J}}$. Furthermore all interaction lines may be placed in the integration trees of Lemma V.1 so that it does not matter that $\delta(\tau)$ violates $\|\delta(\tau)\|_\infty = 1$.

The factor $\mathrm{Val}^d(\mathcal{K})$ arises from the $\underline{x}$ integrals. It is the value of a graph $G^{\mathcal{K}}$ having the same lines and vertices as $G^{\mathcal{J}}$, but living in dimension d, having scale assignments k_ℓ rather than j_ℓ having $\delta_\ell = \frac{d}{2}$ (resp. δ_ℓ=d) for particle (resp. interaction) lines and having as soft lines only those lines that are soft in $\mathcal{J}$ and that in addition have $k_\ell = j_\ell$. The forest $F(G^{\mathcal{K}})$ of subgraphs of $G^{\mathcal{K}}$ and the tree $t(G^{\mathcal{K}})$ can be quite different from $F(G^{\mathcal{J}})$ and $t(G^{\mathcal{J}})$. However, because $k_\ell \geq j_\ell$ with equality for k-soft lines, each $G_f^{\mathcal{K}}$ is connected by hard lines and we may apply Lemma V.1 to obtain

$$|\mathrm{Val}^d(\mathcal{K})| \leq \mathrm{const}^{L(G)} \, \Pi \, M^{D_f^d(k_f - k_{\pi(f)})} \, \Pi \, M^{\Delta_f^d(k_f - k_{\pi(f)})} \, \Pi \, M^{\Delta_f(0 - k_{\pi(f)})}$$

$$\leq \mathrm{const}^{L(G)}.$$

The last inequality is an immediate consequence of $k_f - k_{\pi(f)} \geq 0$, $(0 - k_{\pi(f)}) \geq 0$, $D_f^d \leq 0$ (recall $E_f \geq 4$ if $G_f^{\mathcal{K}}$ contains no external vertices) and $\Delta_f \leq 0$.

Combining the above bounds on $\mathrm{Val}^d(\mathcal{K})$ and $\mathrm{Val}^1(\mathcal{J})$ with

$$\sum_{k_\ell \geq j_\ell} M^{-\frac{1}{2}(k_\ell - j_\ell)} \leq \mathrm{const}$$

yields the desired bound.

∎

Theorem V.4. Let $G \in \Gamma_n^m$ contain no mass subgraphs without external vertices. Then

$$\left|\int \prod_{j=1}^{n} f_j(\xi) d\xi_j \, \mathrm{Val}(G;I)(\xi_1, \ldots, \xi_n)\right| \leq K^{L(G)} n_4! \prod_{j=1}^{n} \|f_j\|_1$$

where

$$n_4 = \max_{F \in \mathfrak{F}} \#\{f \mid G_f \text{ has four external lines, } G_f \neq \sim\!\sim\}$$

is the maximum number of four-legged subgraphs in any forest of subgraphs of G.

Proof. Given a labelling $G^{\mathcal{J}}$ of G denote by $F_4(\mathcal{J})$ the forest $\{G_f \mid G_f$ has four external lines, $G_f \neq \sim\!\sim \} \subset F(\mathcal{J})$. Denote by $\mathfrak{F}_4$ the set of all $F_4(\mathcal{J})$'s for G. We block

$$\begin{aligned}\mathrm{Val}(G;I) &= \sum_{\mathfrak{d}} \mathrm{Val}(G^{\mathfrak{d}};I) \\ &= \sum_{F_4 \in \mathfrak{F}_4} \sum_{\substack{\mathfrak{d} \\ F_4 = F_4(\mathfrak{d})}} \mathrm{Val}(G^{\mathfrak{d}};I).\end{aligned}$$

By [deCR, Lemma A.2] $|\mathfrak{F}_4| \le 8^{L(G)}$ so it suffices to consider any fixed $F_4 \in \mathfrak{F}_4$.

Furthermore each G_f in F_4 must contain at least one line λ_f whose scale j_f is exactly j_f. The sum over choices of $\{\lambda_f \in G_f | G_f \in F_4\}$ contains at most $\binom{L(G)}{|F_4|} \le 2^{L(G)}$ terms so again it suffices to consider any fixed choice of λ_f's.

Another factor of $2^{L(G)}$ takes care of the assignment of hard/soft labels to the lines of G.

The strategy for tackling the sum over scale assignments $\{j_\ell | \ell \in G\}$ is similar to that used in Theorem III.2'. There are two notable differences between the bound of Lemma V.3 and its analogue in Theorem III.2'. The latter contains a factor $M^{-\epsilon V_e (j_\phi - j_{\pi(\phi)})} = M^{-2\epsilon n |j_\phi|}$ (used to sum over the scale assigned to the lowest scale line j_{ℓ_1}) that is absent in the former. The latter also contains a factor $M^{-\epsilon(E+I+V_\ell)(j_f - j_{\pi(f)})}$ for every nontrivial $f \in t(G^J)$ while the former is missing such factors for each $G_f \in F_4$.

The first difference is easily handled. Our graph $G = G_\phi$ contains at least one external vertex v. Hence the tree $G^{\mathfrak{d}}$ contains a connected linear subtree $\phi = f_1 = \pi(f_2) < f_2 = \pi(f_3) < \cdots < f_{p-1} = \pi(f_p) < f_p = v$. Since every G_{f_i}, $2 \le i \le p$ contains the external vertex v the bound (V.8) contains the factors

$$\prod_{i=2}^{p-1} M^{-\frac{d}{4} E_{f_i}(j_{f_i}-j_{f_{i-1}})} \; M^{-\frac{d}{4}(0-j_{f_{p-1}})} .$$

We can use half of each of these factors to give $M^{-\frac{1}{8}(0-j_\phi)} = M^{-\frac{1}{8}|j_\phi|}$.

The second difference means in effect that (V.8) does not contain exponential decay factors allowing us to sum over the j_{λ_i}'s. But each j_{λ_i} must obey $-1 \le j_{\lambda_i} < j_\phi$, so the sum over all values of all the j_{λ_i}'s contains at most $|j_\phi|^{|F_4|} \le |j_\phi|^{n_4}$ terms.

The sum over the j_ℓ's, $\ell \ne \lambda_f$, $\ell \ne \ell_1$ (where ℓ_1 has been chosen so that $j_{\ell_1} = j_\phi$) goes essentially as Lemma III.2'. The only modification is that we work inductively on F_4. Select any $G_f \in F_4$ which does not contain any proper subgraph. As in Theorem III.2' we may rearrange a portion of (V.8) to get $\prod_{v \in G_f} \prod_{\substack{\ell,\ell' \text{hooked to } v \\ \ell,\ell' \in G_f}} M^{-\frac{\epsilon}{4}|j_\ell - j_{\ell'}|}$. The sum over j_ℓ, $\ell \in G_f$ is then bounded just as in the last paragraph of Theorem III.2' with the role of the 'first line' j_{ℓ_1} played by j_{λ_f} (which is being held fixed).

To proceed by induction simply collapse G_f to a point and repeat. This ultimately brings us to G_ϕ. Here the first line does not have its scale $j_{\ell_1} = j_\phi$ held fixed. Instead we have

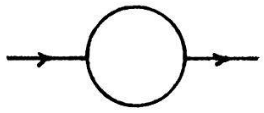
$$\sum_{j_\phi} |j_\phi|^{n_4} M^{-\frac{1}{8}|j_\phi|} \le \text{const}^{n_4} (n_4)! \le \text{const}^{L(G)} (n_4)!$$

∎

VI. The Formalism of Renormalization

In this section we develop a new (renormalized) perturbation expansion in which a λ dependent portion $\delta\mu$ of the chemical potential is moved from the covariance into the interaction with the result that every graph has a finite value. The function $\delta\mu$ is constructed as a formal power series whose coefficients are finite sums of graphs. The values of these graphs are also finite. In section VII we estimate the values of the graphs occurring in this renormalized expansion and verify that they are indeed finite.

We have already observed in section 1 that graphs containing two-legged subdiagrams



diverge because $[ik_0-e(\underline{k})]^{-2}$ has a nonintegrable singularity on the Fermi surface $k_0=0$, $|\underline{k}| = \sqrt{2m\mu}$. These divergences reflect the deformation of the Fermi surface with the change of λ as we will now illustrate.

Consider a model in which the electron-electron interaction $V = 0$ but in which the chemical potential, which we denote μ, varies linearly with λ

$$\mu_0(\lambda) = \mu + \lambda\delta\mu.$$

at the rate $\delta\mu$. The Fourier transform of the two-point Green's function in this model is

$$\left[ik_0 - \frac{1}{2m}\underline{k}^2 + \mu_0(\lambda) \right]^{-1}$$

and has the perturbation expansion

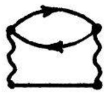
$$\left[ik_0 - \frac{1}{2m}\underline{k}^2 + \mu_0(\lambda) \right]^{-1} = \sum_{n=0}^{\infty} \left[ik_0 - \frac{1}{2m}\underline{k}^2 + \mu \right]^{-1} (-\lambda\delta\mu)^n \left[ik_0 - \frac{1}{2m}\underline{k}^2 + \mu \right]^{-n} \tag{VI.1}$$

All terms in this geometric series save $n = 0$, contain non-integrable singularities. They fail to be tempered distributions even though both the "interacting" and "free" Schwinger functions $\left[ik_0 - \frac{1}{2m}\underline{k}^2 + \mu_0(\lambda)\right]^{-1}$ and $\left[ik_0 - \frac{1}{2m}\underline{k}^2 + \mu\right]^{-1}$ are locally integrable and the "strength of interaction" $\lambda\delta\mu$ is finite. The difficulty here, of course, is that the distributions have different singular supports. Expanding one in powers of the other is not a good idea.

Let V be any two-body potential and consider the two parameter family of models determined by the chemical potential (which we now call μ_0 rather than μ) and coupling constant λ. For each μ_0 and λ suppose that the Fermi surface is given by $k_0 = 0$, $|\underline{k}| = \sqrt{2m\mu(\lambda,\mu_0)}$.

To circumvent the difficulty illustrated above we parametrize the models by λ and μ rather than by λ and μ_0. That is μ_0, which determines the position of the free Fermi surface, is replaced by μ, which is determined by the position of the interacting Fermi surface. Precisely, the function $\mu = \mu(\lambda,\mu_0)$ is inverted to obtain $\mu_0 = \mu_0(\lambda,\mu)$. Define $\delta\mu(\lambda,\mu)$ by $\mu_0(\lambda,\mu) = \mu + \lambda\delta\mu(\lambda,\mu)$. The new perturbation expansion is now generated by taking derivatives with respect to λ keeping μ rather than μ_0 fixed.

The new expansion may be determined without knowing $\mu(\lambda,\mu_0)$ ahead of time. One determines $\delta\mu(\lambda,\mu)$ inductively, order by order in λ, by requiring that the inverse of the two point function have a zero at $k_0 = 0$, $|\underline{k}| = \sqrt{2m\mu}$. If we write the inverse as $ik_0 - \frac{1}{2m}\underline{k}^2 + \mu - \sum(k,\mu,\lambda)$ the <u>proper self-energy</u> $\sum$ is the sum of all amputated, one particle irreducible two point Feynman diagrams. "Amputated" means that the two particle lines hooked to the external vertices are removed and "one particle irreducible" means that the diagram remains connected whenever a single <u>electron</u> line is cut. Here are two examples

The condition (in perturbation theory) that the Fermi surface is at $|\underline{k}| = \sqrt{2m\mu}$ can bé formulated in terms of the self energy as

$$\tilde{\Sigma}(k,\mu,\lambda)\Big|_{k_0=0,\ |\underline{k}|\ =\ \sqrt{2m\mu}} = 0 \tag{VI.2}$$

(as a formal power series in λ).

Equation (VI.2) is neatly combined with the decomposition of lines into scales and graphs into forests through the effective potential and tree expansion which we now define.

Recall the decomposition of the free two point function given in Section 2:

$$C = \sum_{j=-\infty}^{0} C^{(j)}$$

where now $U := C^{(0)}$. As before, there is a corresponding decomposition of the fields and free measure

$$\begin{aligned} d\mu_C(\psi,\bar{\psi}) &= \prod_{j=-\infty}^{0} d\mu_{C^{(j)}}(\psi^{(j)},\bar{\psi}^{(j)}) \\ \psi = \sum_{j=-\infty}^{0} \psi^{(j)}, &\quad \bar{\psi} = \sum_{j=-\infty}^{0} \bar{\psi}^{(j)} \end{aligned} \tag{VI.3}$$

To simplify notation write $\Phi := (\psi,\bar{\psi})$.

The effective potential at scale r, $-1 \geq r \geq -\infty$, is, by definition,

$$\mathcal{G}^{r}(\Phi^e) := \log \frac{1}{Z_r} \int \exp\Big[(-\tfrac{\lambda}{2}\mathcal{V} + \delta\mathcal{V})(\Phi^e + \sum_{j>r} \Phi^{(j)}) \prod_{j>r} d\mu_{C^{(j)}}(\Phi^j) \tag{VI.4}$$

where Z_r is a constant that will be chosen later, $\mathcal{V}$ is the usual quartic interaction and

$$\delta\mathcal{V}(\Phi) := \delta\mu(\lambda,\mu) \int d^d\underline{x}\,d\tau\ \bar{\psi}(\underline{x},\tau)\psi(\underline{x},\tau).$$

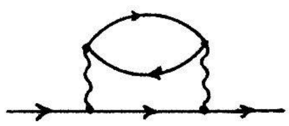
The term $\delta\mathcal{V}(\Phi)$ is the portion of the chemical potential that has been placed in the interaction. The coefficient $\delta\mu(\lambda,\mu)$ is still to be determined. The "test functions" Φ^e are Grassmann valued (i.e.

anticommuting) just like the Φ^j's.

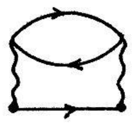
Expanding the effective potential in Φ^e (or equivalently taking functional derivatives) one obtains

$$\mathcal{V}^r(\Phi^e) = \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{k=1}^{n} \Big(\int d^d\underline{x}_k d\tau_k \sum_{\alpha_k} \Phi^e(\underline{x}_k, \tau_k, \alpha_k)\Big)\, G_n^r(\underline{x}_1, \tau_1, \alpha_1; \ldots; \underline{x}_n, \tau_n, \alpha_n) \tag{VI.5}$$

The coefficients G_n^r are functions much like the n-point Schwinger functions S_n. In fact, for $r = -\infty$, $G_n^{-\infty}$ is the connected, amputated n-point Green's function. (See [FHRW p.3]) Perturbatively, $G_n^{-\infty}$ is obtained from S_n by (i) dropping all disconnected graphs from S_n (ii) removing lines hooked to external vertices of S_n and (iii) when n=2 dropping the graph  from S_2. The combinatorial coefficients are unchanged. For example the graph in $G_2^{-\infty}(\xi_1, \xi_2)$ corresponding to



is



There are simple, explicit formulae expressing the S_n's in terms of the $G_n^{-\infty}$'s and vice versa. For example

$$S_2(\xi_1, \xi_2) = C(\xi_1, \xi_2) + \int d\zeta_1 d\zeta_2 C(\xi_1, \zeta_1) C(\xi_2, \zeta_2) G_2^{-\infty}(\zeta_1, \zeta_2)$$

$$G_4^{-\infty}(\xi_1, \xi_2, \xi_3, \xi_4) = \int \prod_{i=1}^{4} \Big[d\zeta_i C^{-1}(\xi_i, \zeta_i)\Big] \{S_4(\zeta_1, \ldots, \zeta_4)$$

$$- S_2(\zeta_1, \zeta_2) S_2(\zeta_3, \zeta_4) + S_2(\zeta_1, \zeta_3) S_2(\zeta_1, \zeta_4) - S_2(\zeta_1, \zeta_4) S_2(\zeta_2, \zeta_3)\}.$$

It follows from these remarks that we may consider G_n^r, $n \geq 2$, rather than

S_n, $n \geq 2$.

The decomposition of graphs into scales arises naturally if we express $\mathcal{Y}^h$ inductively:

$$\mathcal{Y}^0(\Phi^e) = -\frac{\lambda}{2}\mathcal{V}(\Phi^e) + \delta\mathcal{V}(\Phi^e)$$

$$\mathcal{Y}^{h-1}(\Phi^e) = \log \int \exp \mathcal{Y}^h(\Phi^e + \Phi^{(h)})\, d\mu_{C^{(h)}}(\Phi^{(h)}) + \log \frac{Z_h}{Z_{h-1}}, \qquad \text{(VI.6)}$$

Apply the identity

$$\log \int e^{\mathcal{U}}\, d\mu_{C^{(h)}}(\Phi^{(h)}) = \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{E}_h^T(\mathcal{U}, \ldots, \mathcal{U}) \quad \text{(n arguments)}$$

where

$$\mathcal{E}_h^T(\mathcal{M}_1, \ldots, \mathcal{M}_n) := \frac{\partial^n}{\partial\lambda_1 \ldots \partial\lambda_n} \log \int \exp\left[\sum \lambda_i \mathcal{M}_i\right] d\mu_{C^{(h)}}(\Phi^{(h)})\Big|_{\lambda_1 = \ldots = \lambda_n = 0}. \qquad \text{(VI.7)}$$

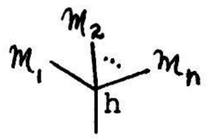
with the result that

$$\mathcal{Y}^{h-1} = \mathcal{E}_h^T(\mathcal{Y}^h) + \sum_{n=2}^{\infty} \frac{1}{n!} \mathcal{E}_h^T(\mathcal{Y}^h, \ldots, \mathcal{Y}^h) + \text{const}$$

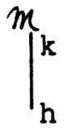
Successive application of the last identity to the leading terms $\mathcal{E}_h^T(\mathcal{Y}^h)$ yields

$$\mathcal{Y}^r = \mathcal{E}_{r+1}^T \cdots \mathcal{E}_0^T(\mathcal{Y}^0) + \sum_{h=r+1}^{0} \mathcal{E}_{r+1}^T \cdots \mathcal{E}_{h-1}^T \sum_{n=2}^{\infty} \frac{1}{n!} \mathcal{E}_h^T(\mathcal{Y}^h, \ldots, \mathcal{Y}^h)$$
$$+ \text{const} \qquad \text{(VI.8)}$$

Introducing a tree notation for $\mathcal{E}_h^T$:



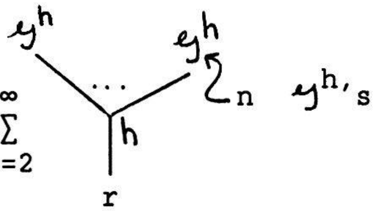
$$:= \frac{1}{n!} \mathcal{E}_h^T(\mathcal{M}_1, \ldots, \mathcal{M}_n)$$



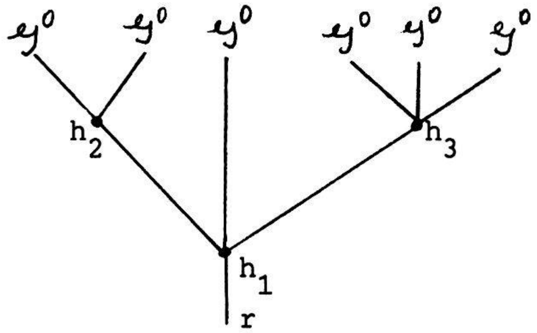
$$:= \chi(k>h)\, \mathcal{E}_{h+1}^T\, \mathcal{E}_{h+2}^T \cdots \mathcal{E}_{k-1}^T(\mathcal{M})$$

$$\overset{\mathcal{M}}{\Big|}_{h} \quad := \chi(0>h)\ \mathcal{E}^T_{h+1}\,\mathcal{E}^T_{h+2}\cdots\mathcal{E}^T_0(\mathcal{M})$$

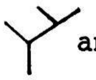
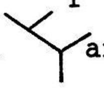
(VI.8) becomes

$$\mathcal{Y}^r = \overset{\mathcal{Y}^0}{\Big|}_{r} + \sum_{h=r+1}^{0}\ \sum_{n=2}^{\infty}$$  $$\sum_n \mathcal{Y}^{h}\text{'s} \quad + \text{const}$$

Iterating (VI.8), $\mathcal{Y}^r(\Phi^e)$ becomes the sum of all trees



The root frequency is r, each fork bifurcates upwards into two or more branches, the fork frequencies h_i increase monotonically up the tree and finally there is a leaf $\mathcal{Y}^0(\Phi^e + \sum_{j>r}\Phi^j)$ at the end of each topmost branch.

The trivial tree $\overset{\mathcal{Y}^0}{\Big|}_{r}$ is included in the sum. Note that each tree is planar (e.g.  and  are distinct) and has a distinguished root.

The truncated expectation $\mathcal{E}^T_h(\mathcal{M}_1,\ldots,\mathcal{M}_n)$ can be evaluated graphically in the following way. If each $\mathcal{M}_i$ is a monomial

$$\mathcal{M}_i = \int d\xi_1\ldots d\xi_{p_i}\ M_i\ (\xi_1,\ldots,\xi_{p_1})\ \Phi^{(h)}(\xi_1)\ldots\Phi^{(h)}(\xi_{p_i})$$

then $\mathcal{E}^T_h(\mathcal{M}_1,\ldots,\mathcal{M}_n)$ is the sum of all connected graphs built from n generalized vertices. The i^{th} vertex has p_i legs and takes the value $M_i(\xi_1,\ldots,\xi_{p_i})$. The lines of the graph are evaluated using the covariance $C^{(h)}(\xi_i,\xi_j)$.

$$\mathcal{E}^T_h(\mathcal{M}_1,\ldots,\mathcal{M}_n) = \sum_{G,\ \text{connected}} \int \prod_{\substack{1\le i\le n\\ 1\le k\le p_i}} d\xi_{i,k} \prod_{i=1}^{n} M_i(\xi_{i,1},\ldots,\xi_{i,p_i}) \prod_{\ell\in G} C^{(h)}(\xi_{u_\ell},\xi_{w_\ell}). \tag{VI.9}$$

The legs of $\mathcal{M}_i$ are $\Phi^{(h)}$'s and hence have all been integrated out to become parts of lines of G.

We have expressed $\mathcal{V}^r$ as a sum of trees and each tree vertex as a sum of Feynman graphs. To express an entire tree as a sum of Feynman graphs view $\mathcal{V}^{h-1}(\Phi^e)$ as being expanded in powers Φ^e and $\mathcal{V}^h(\Phi^{(h)}+\Phi^e)$ as being expanded in powers of $\Phi^{(h)}$ and Φ^e with all monomials Wick ordered (for the same reason as in section V). That is

$$\mathcal{V}^{h-1}(\Phi^e) = \sum_{n=1}^{\infty} \int d\zeta_1 \ldots d\zeta_q \, G_n^{h-1}(\zeta_1,\ldots,\zeta_q) \; : \Phi^e(\zeta_1)\ldots\Phi^e(\zeta_q) :$$

$$\mathcal{V}^h(\Phi^h+\Phi^e) = \sum_{q+p\geq 1} \int \pi d\xi_j \, \pi d\zeta_k \, G^h_{q,p}(\zeta_1,\ldots,\xi_p) \; : \Phi^e(\zeta_1)\ldots\Phi^e(\zeta_q): \; : \Phi^{(h)}(\xi_1)\ldots\Phi^{(h)}(\xi_p) : .$$

Wick ordering is always done on the natural scale; $\Phi^{(h)}$ is Wick ordered with respect to $d\mu_{C^{(h)}}$ and Φ^e, which is thought of as being $\sum_{-\infty\leq j\leq h-1} \Phi^{(j)}$ is Wick ordered with respect to $\prod_{-\infty<j\leq h-1} d\mu_{C^j}$.

To do this we must rewrite the interaction as a sum of Wick-ordered monomials. Observe that

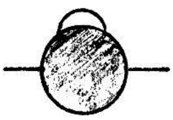
$$\mathcal{V}^o = -\frac{\lambda}{2}\int : \bar{\psi}\psi : v : \bar{\psi}\psi : + \delta\mu \int : \bar{\psi}\psi :$$

where the coefficients of the crossterms in $(\bar{\psi}\psi)(\bar{\psi}\psi) = (: \bar{\psi}\psi : +\text{const})(: \bar{\psi}\psi : +\text{const})$ is absorbed into $\delta\mu$ and the constant is discarded. Discarding another irrelevent constant, $\mathcal{V}^o$ has the form

$$\begin{aligned}\mathcal{V}^o = & -\frac{\lambda}{2}\int d\xi_1\, d\xi_2 \; : \bar{\psi}(\xi_1)\psi(\xi_1)\, v(\xi_1-\xi_2)\bar{\psi}(\xi_2)\psi(\xi_2) : \\ & - \lambda \int d\xi_1\, d\xi_2 \; : \bar{\psi}(\xi_1)\, C^{(\leq 0)}(\xi_1-\xi_2) v(\xi_1-\xi_2)\psi(\xi_2) : \\ & + \delta\mu \int d\xi \; : \bar{\psi}(\xi)\psi(\xi) :\end{aligned}$$

The three Wick-ordered terms are represented by the graphs ,  and  where the line  is the soft covariance $C^{(\leq 0)}(\xi_1-\xi_2)$ of scale zero. Each leaf $\mathcal{G}^o$ at the top our tree is a sum of these three graphs.

When (VI.9) is applied to (VI.8) the $\Phi^e(\zeta_j)$'s of $\mathcal{G}^h(\Phi^{(h)}+\Phi^e)$ become the external fields of $\mathcal{G}^r(\Phi^e)$ and appear as external legs of the graphs in (VI.9). The $\Phi^{(h)}$'s of $\mathcal{G}^h(\Phi^{(h)}+\Phi^e)$ are integrated out to form the internal lines of the graphs of (VI.9). The Wick ordering $:\Phi^{(h)}(\xi_{i,1})\,..\,\Phi^{(h)}(\xi_{i,p_i}):$ graphs that are not only connected, but also have every internal line joining distinct generalized vertices. Hence



M_i

is not allowed.

When (VI.9) is applied to $\mathcal{E}_h^T(\mathcal{M}_1,\ldots,\mathcal{M}_n)$ with

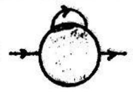
$$\mathcal{M}_i = \int \Pi d\xi_{i,j}\ \Pi d\zeta_{i,k}\ G^h_{q_i,p_i}\ :\prod_{j_i=1}^{q_i} \Phi^e(\zeta_{i,j_i}) : \ :\prod_{k_i=1}^{p_i} \Phi^{(h)}(\xi_{i,k_i}) :$$

the result has external fields $\prod_{i=1}^{n} : \prod_{j_i=1}^{q_i} \Phi^e(\zeta_{i,j_i}) :$. There are terms for which $q_1 = \ldots = q_n = 0$, that is there are no external fields present. We now define Z_r inductively so that the sum of all these terms cancels the constant in (VI.8). Hence, $\mathcal{G}^r$, expressed in terms of Wick order monomials, contains no constant term.

Finally, to write $\mathcal{E}_h^T(\mathcal{M}_1,\ldots,\mathcal{M}_n)$ as a sum of Wick ordered monomials the expression $\prod_{i=1}^{n} \prod_{j_i=1}^{q_i} \Phi^e(\zeta_i,j_i)$: must be re-Wick ordered to obtain the desired form: $\prod_{i=1}^{n} \prod_{j} \phi^e$. This is accomplished by applying the identity

$$\prod_{i=1}^{n} : \prod_{j_i=1}^{q_i} \Phi^e(\zeta_{i,j_i}) :$$

$$= \sum_{G} \pm \prod_{\substack{\text{internal}\\ \text{lines}}} C^{(<h)}(\zeta_{u_\ell}, \zeta_{w_\ell}) \ : \prod_{\substack{\text{external}\\ \text{legs of } G}} \Phi^e(\zeta_\ell) \ :$$

where G runs over all graphs with n generalized vertices (the i^{th} having q_i legs) such that every internal line joins distinct generalized vertices. In other words graphs containing Wick bubbles  are excluded. The sign ± comes from the usual reordering of anticommuting fields. For example

$$\begin{aligned} :\bar\psi(\zeta_1)\psi(\zeta_2): \ :\bar\psi(\zeta_3)\psi(\zeta_4): \ &= \ :\bar\psi(\zeta_1)\psi(\zeta_2)\bar\psi(\zeta_3)\psi(\zeta_4): \\ &+ C(\zeta_2,\zeta_3) \ :\bar\psi(\xi_1)\psi(\zeta_4): \ + C(\zeta_4,\zeta_1) \ :\bar\psi(\zeta_3)\psi(\zeta_2): \\ &- C(\zeta_2,\zeta_3)\, C(\zeta_4,\zeta_1). \end{aligned}$$

Another effect of Wick ordering is

$$\int : \prod_{n=1}^{q} \Phi^j(\xi_n) : d\mu_{C^j} = 0$$

for $q \neq 0$ so that

$$\begin{aligned} &: \prod_{m=1}^{p} (\Phi^{k-1} + \ldots + \Phi^{h+1} + \Phi^e)(\xi_m) : \Big|_{h}^{k} \\ &= \mathcal{E}^T_{h+1}\, \mathcal{E}^T_{h+2} \cdots \mathcal{E}^T_{k-1}\Big(: \prod_{m=1}^{p} (\Phi^{k-1} + \ldots + \Phi^{h+1} + \Phi^e)(\xi_m) :\Big) \\ &= \ : \prod_{m=1}^{p} \Phi^e(\xi_m) : . \end{aligned}$$

We may summarize the discussion which began with (VI.6) in the formula

$$\mathcal{G}^r(\Phi^e) = \sum_{\text{trees } t\in\mathcal{T}} \ \sum_{\text{scales } j\in\mathcal{S}(t)} \ \sum_{\substack{\text{labelled}\\ \text{graphs}}} \ \prod_{f\in t} \frac{1}{p_f!} \operatorname{Val}(G^\gamma). \qquad \text{(VI.10)}$$

Here, $\mathcal{T}$ is the set of all rooted planar trees with each fork f bifurcating upwards into $p_f \geq 2$ branches and $\mathcal{S}(t) = \{\{j_f | f\in t\} \mid r < j_f \leq 0,\ f > f' \to j_f > j_{f'}\}$ is the set of all allowed assignments of scales to the forks of t. The third sum is over all labelled graphs having

a) one vertex , or for each leaf of the tree t

b) each internal line if labelled by a scale j_f, $f \epsilon t$ and is also given a hard/soft label. Interaction lines are always hard and of scale 0.

There are further restrictions on the sum over labelled graphs. They are most easily stated in terms of

$$G_f^{\gamma} := \{\text{lines } \ell \epsilon G^{\gamma} \mid \ell \text{ has scale } j_{f'} \text{ for some } f' \geq f\}$$

and

$$g_f^{\gamma} := G_f^{\gamma} / \{G_{f'}^{\gamma} \mid f' > f\}$$

Here $G/\{G_1,\ldots,G_n\}$ is the graph obtained from G by contracting each subgraph G_i; to a point. The subgraphs and of the leaves are included in the $G_{f'}^{\gamma}$'s and are always collapsed. Hence the lines of g_f^{γ} are precisely the lines of G^{γ} with scale j_f.

c) g_f^{γ} has p_f vertices

d) g_f^{γ} is connected by hard lines

e) g_f^{γ} has no Wick bubbles

The rules defining $\mathrm{Val}(G^{\gamma})(\Phi^e)$ are:

i) each ξ_u ξ_w becomes $-\frac{\lambda}{2}\mathcal{V}(\xi_u-\xi_w)$

each ξ_u ξ_w becomes $-\lambda\,\mathcal{V}(\xi_u-\xi_w)\; C^{\leq 0}(\xi_u-\xi_w)$

each becomes $\delta\mu(\lambda,\mu)$

ii) each hard line $\xi_u \overset{j_f}{\longleftarrow} \xi_w$ becomes $C^{(j_f)}(\xi_u-\xi_w)$

each soft line of scale j_f becomes $C^{(<j_f)}(\xi_u-\xi_w)$

iii) each external leg ξ becomes $\bar{\psi}^e(\xi)$

each external leg ξ becomes $\psi^e(\xi)$

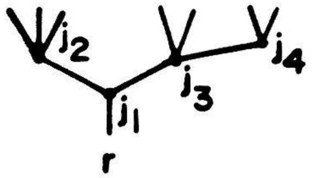
the set of external legs are Wick ordered

iv) each closed fermian loop gets a (-1). There are no other signs arising from the fermion statistics provided the external fields terminating each fermion string $\xi_1 \quad \xi_n$ are written side by side in the order $\bar{\psi}^e(\xi_1)\psi^e(\xi_n)$.

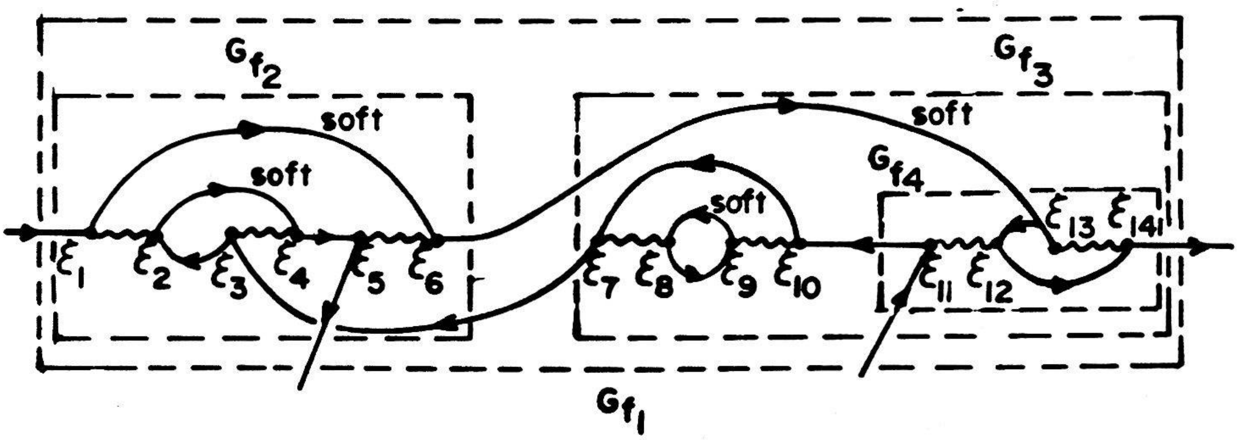
v) the positions of all vertices , are integrated over $\mathbb{R}^{d+1}$

By convention all legs in the $\mathscr{G}^o$'s at the top of the tree are distinguishable. Thus topologically similar graphs are not identified and there are no associated combinatorial factors.

Here is an example of $\mathrm{Val}(G^{\gamma})$:

$t =$ 

$j_2 > j_1 > r$
$j_4 > j_3 > j_1$



$g_{f_4} = G_{f_4}$ $\qquad g_{f_3} =$  $\qquad g_{f_2} = G_{f_2}$

$g_{f_1} =$ 

$$\mathrm{Val}(G^{\gamma}) = (-1)^1\left(-\frac{\lambda}{2}\right)^7 \int d\xi_1 \ldots d\xi_{14} \; : \bar{\psi}^e(\xi_5)\psi^e(\xi_{11})\bar{\psi}^e(\xi_{14})\psi^e(\xi_1)$$

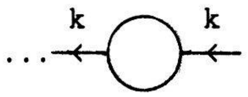
$$\left[\prod_{i=1}^{7} \mathcal{Q}(\xi_{2i-1} - \xi_{2i})\right] C^{<j_2}(\xi_6 - \xi_1) C^{<j_2}(\xi_4 - \xi_2)$$

$$C^{j_2}(\xi_2 - \xi_3)\, C^{j_1}(\xi_3 - \xi_7) C^{j_2}(\xi_5 - \xi_4) C^{<j_1}(\xi_{13} - \xi_6)$$

$$C^{j_3}(\xi_7 - \xi_{10}) C^{<j_3}(\xi_8 - \xi_9) C^{j_3}(\xi_9 - \xi_8) C^{j_3}(\xi_{10} - \xi_{11})$$

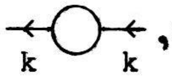
$$C^{j_4}(\xi_{12} - \xi_{13}) C^{j_4}(\xi_{14} - \xi_{12}) .$$

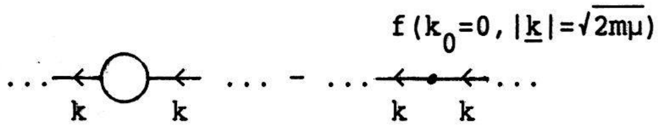
In sections III and V we estimated the values of any graph that does not contain any two-legged subgraph G_f^{γ}. As observed before, graphs containing two-legged subgraphs such as



$$= \int \ldots \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \; \frac{1}{(ik_0 - e(k))^2} \; f(k)$$

diverge because there is a nonintegrable singularity on the Fermi surface.

We now continue the discussion preceeding (VI.2) and construct $\delta\mu(\mu,\lambda)$ perturbatively. The idea underlying the construction is that for every two-legged subgraph , $\delta\mu(\mu,\lambda)$ must generate a counterterm  so that



$$= \int \ldots \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \; \frac{1}{[ik_0 - e(\underline{k})]^2} \; [f(k) - f(k_0 = 0, |\underline{k}| = \sqrt{2m\mu})] \; .$$

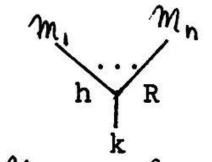
The zero of $f(k) - f(k_0 = 0, |\underline{k}| = \sqrt{2m\mu})$ on the Fermi surface mollifies the singularity in $[ik_0 - e(\underline{k})]^{-2}$ to yield integrability. These conditions will determine $\delta\mu(\mu,\lambda)$ uniquely in perturbation theory.

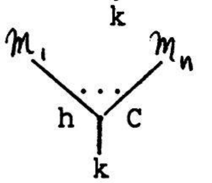
The first step is to introduce a localization operation L which identifies the counterterm corresponding to any given graph. This operator is defined by

$$L\int d\xi_1 d\xi_2 G(\xi_1 - \xi_2) :\bar{\psi}^e(\xi_1)\psi^e(\xi_2):$$
$$= \hat{G}(k_0 = 0, |\underline{k}| = \sqrt{2m\mu}) \int d\xi \ : \bar{\psi}(\xi)^e \psi(\xi)^e: \qquad \text{(VI.11)}$$

$$L\int d\xi_1 \ldots d\xi_n G(\xi_1, \ldots, \xi_n) \ :\overset{(-)}{\psi}{}^e(\xi_1)\ldots\overset{(-)}{\psi}{}^e(\xi_n): \ = 0 \quad \text{for} \quad n \neq 2$$ for Wick monomials and is extended by linearity to all formal power series in $\psi^e, \bar{\psi}^e$. (The fields do carry spin indices, but they have been suppressed.)

The renormalized tree expansion will express the effective potential $\mathcal{G}^r(\Phi^e)$ as a sum of trees similar to (VI.10) but with the difference that each fork will have an additional label taking the values R (for renormalized) and C(for counterterm). The meaning of such a fork is


$$= \chi(k < h \leq 0)\, \mathcal{E}^T_{k+1} \ldots \mathcal{E}^T_{h-1} (1 - L) \frac{1}{n!} \mathcal{E}^T_h(\mathcal{M}_1, \ldots, \mathcal{M}_n)$$


$$= \chi(-\infty < h \leq k)(-L)\frac{1}{n!}\mathcal{E}^T_h(\mathcal{M}_1, \ldots, \mathcal{M}_n)\Big|_{\Phi^{(<h)} \to \Phi^{(\leq k)}}$$

Here $\Phi^{(<h)} \to \Phi^{(\leq k)}$ means that the external fields $\Phi^{(<h)}$ resulting from $(-L)\, \mathcal{E}^T_h$ are replaced by $\Phi^{(\leq k)}$. The output of the R forks automatically pairs each graph with its counterterm. The output of the C forks consists of parts of $\delta\mu(\lambda,\mu) \int :\bar{\psi}(\xi)\psi(\xi): d\xi$ that have not been used up renormalizing the effective potential $\mathcal{G}^k$ of scale k. Note that in contrast to plain and R-forks the scale h of a C-fork is lower than that of its predecessor: $h \leq k$.

We shall use the notation  to stand for the sum of all nontrivial trees (i.e., not $\underset{r}{|}$) with all possible assignments of R and C labels and scales to the forks of the tree. But the root scale r is not summed and the leaves are $-\frac{\lambda}{2}\mathcal{V}(\Phi^e + \sum_{j>r} \Phi^j)$ rather than $\mathcal{G}^0$. Note that $\delta\mu(\lambda,\mu)$ does not appear in the leaves. Furthermore, all vacuum graphs (those with no external legs) are discarded. We shall also use  to denote the sum of all nontrivial trees whose bottom fork is labelled C, and so on.

<u>Theorem VI.1</u> Define

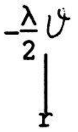
$\delta V =$ 

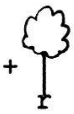
(VI.13)

and

$$\mathcal{G}^r(\Phi^e) = \log\frac{1}{Z_r} \int \exp(-\tfrac{\lambda}{2}\mathcal{V} + \delta\mathcal{V})(\Phi^e + \sum_{j=r+1}^{0} \Phi^j) \prod_{j=r+1}^{0} d\mu_{C^j}(\Phi^j).$$

Then

a) $\mathcal{G}^r =$ 

$+$ 

and (VI.14)

b) if P_n (resp $P_{<n}$) denotes projection onto the n^{th} order (resp. orders less than n) of perturbation theory in λ

$$P_n \delta V(\Phi^e) = -P_n L \log \int \exp(-\tfrac{\lambda}{2}\mathcal{V} + P_{<n}\delta\mathcal{V})(\Phi^e + \Phi)\,d\mu_C(\Phi) \qquad \text{(VI.15)}$$

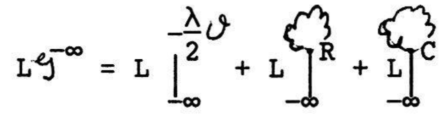
<u>Remarks:</u>

1. The counterterm δV is of the desired form

$$\delta V(\Phi^e) = \delta\mu(\mu,\lambda) \int d\xi :\bar{\psi}^e(\xi)\psi^e(\xi):.$$

Since the interaction is spin independent, it is even diagonal in the (suppressed) spin indices. When the interaction is not spin independent, this will no longer be the case. This does not violate the requirement that the chemical potential be spin independent. This requirement states that the $\int d\xi \bar{\psi}^e(\xi)\psi^e(\xi)$ part of the complete action, including the contribution from the (inverse of) the covariance, be diagonal in the spin indices.

2. The physical effective potential $\mathcal{G}^{-\infty}$ obeys



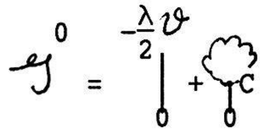
$$= 0$$

To see this observe that the first term is zero because ϑ is a Wick monomial of degree 4; the second is zero because $L(1 - L) = 0$ and the third is zero because of the restriction on scales at C forks.

3. Equation (VI.6) can be regarded as defining a scale dependent map from $\mathcal{G}^h$ to $\mathcal{G}^{h-1}$. Remark 2 may be interpretted as a final value condition at scale $h = -\infty$. This condition determines $\delta\mu(\mu,\lambda)$ and says that the fermi surface occurs (in perturbation theory) at $|\underline{k}| = \sqrt{2m\mu}$ (c.f.(VI.2).

Proof.

a) The proof is by induction on r. When $r = 0$, then a) reduces to

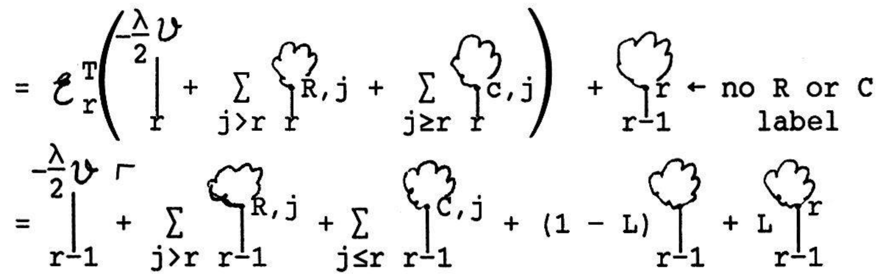


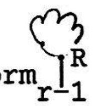
by the restriction on scales at R-forks. This is trivially true.

Assume the claim is true for some given r. Then

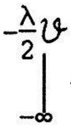
$$\mathcal{G}^{r-1} = \log\int \exp(\mathcal{G}^r)\, d\mu_{C^r} - \text{const}$$

$$= \mathcal{E}^T_r(\mathcal{G}^r) + \sum_{n=2}^{\infty} \frac{1}{n!}\, \mathcal{E}^T_r(\mathcal{G}^r, \ldots, \mathcal{G}^r) - \text{const}$$



The second and fourth terms combine to form  and the third and fifth terms combine to form .

b) Set $r = -\infty$ in (VI.14) and apply $P_n L$.

$$P_n L\,\mathcal{G}^{-\infty} = P_n L$$  $$+ P_n L$$  $$+ P_n L$$ .

$$= 0$$

since ϑ is a wick monomial of degree 4, $L(1 - L) = 0$ and the sum $\sum\limits_{-\infty<h\le-\infty}$ is empty. Thus

$$\begin{aligned} 0 &= P_n L \log\int \exp[-\tfrac{\lambda}{2}\vartheta + P_{\le n}\delta\vartheta]\,d\mu_C \\ &= P_n\delta\vartheta + P_n L \log\int \exp[-\tfrac{\lambda}{2}\vartheta + P_{<n}\delta\vartheta]\,d\mu_C. \end{aligned}$$ ∎

VII Renormalized Bounds

In this section we estimate the coefficients in the formal power series expansion of the 2p-point function generated by the effective potential at scale r, $-\infty \le r \le -1$, and in particular verify that they are finite.

Let

$$G_{2p}^{r}(\bar{\zeta}_1,\xi_1,\bar{\zeta}_2,\xi_2,\ldots,\bar{\zeta}_p,\xi_p)$$

$$= \prod_{j=1}^{p}\left[\frac{\delta}{\delta\bar{\psi}^e(\xi_j)}\,\frac{\delta}{\delta\psi^e(\bar{\zeta}_j)}\right]\mathcal{G}^r(\Phi^e)\Big|_{\Phi^e=0} \qquad \text{(VII.1)}$$

be the 2p-point function generated by the effective potential at scale r. The renormalized tree expansion (V1.14) yields a renormalized perturbation expansion

$$G_{2p}^{r} \sim \sum_{n=1}^{\infty}\lambda^n G_{2p}^{r,n} \qquad \text{(VII.2)}$$

We wish to bound the coefficients $G_{2p}^{r,n}$.

The renormalized tree expansion expresses $G_{2p}^{r,n}$ as a sum over the values of labelled graphs in $\Gamma_{2p}^{r,n}$. Here $\Gamma_{2p}^{r,n}$ is the set of all labelled graphs with scale and hard/soft labels on each line and R or C labels attached to the subgraphs corresponding to forks of the associated tree. In addition, these graphs must satisfy the conditions a)-e) following (VI.10).

The localization operator L introduced in the last section annihilates all but two-legged diagrams. Consequently for a graph without two-legged subgraphs the operators 1-L and -L corresponding to the labels R, C become 1, 0 so that the estimates of sections 3 and 5 apply.

The ultraviolet part of the covariance $U = C^{(0)}$ is integrable but unbounded. So in section 3 it was appropriate to estimate L^1 norms of Schwinger functions. On the other hand, the infrared covariance $\sum_{j=-\infty}^{-1} C^{(j)}$ is bounded but not integrable. So in section 5 L^∞ norms were appropriate. Here, the ultraviolet and infrared regions are treated together. The combined norm

$$\|G_p^r\|_{1,\infty} := \sup\{\int d\xi_1 \ldots d\xi_p |G_p^r(\xi_1,\ldots,\xi_p) f_1(\xi_1)\ldots f_p(\xi_p)| \ : \ \|f_i\|_1, \|f_i\|_\infty \le 1\} \tag{VII.3}$$

is natural.

Directly combining the methods of sections 3 and 5 yields

<u>Lemma V11.1</u> Let the two body potential V either lie in $L^1(\mathbb{R}^d)$, $d \ge 1$ or be the screened Coulomb potential when $d \ge 2$. Let G be in $\Gamma_{2p}^{r,n}$, $r \ge -\infty$, and contain no mass subgraphs that are without external vertices. Then

$$\|\mathrm{Val}(G^{\vartheta})\|_{1,\infty} \le K^{L(G)} \prod_{\substack{f>\phi \\ f \text{ nontrivial} \\ f \text{ not external}}} M^{D_f^1(j_f - j_{\pi(f)})} \prod_{\substack{f>\phi \\ f \text{ nontrivial} \\ f \text{ external}}} M^{\Delta_f^1(j_f - j_{\pi(f)})}$$

$$\prod_{\substack{v\epsilon t(G^{\vartheta}) \\ v \text{ trivial} \\ v \text{ external}}} M^{\Delta_v^1(0 - j_{\pi(v)})} \qquad \text{(VII.4)}$$

The proof of Lemma VII.1 is ommitted.

Now consider two-legged graphs that have no two-legged subgraphs. These graphs arise in three basic ways. The simplest possibility is that the graph appears at scale $r = -\infty$. Its value is a term in the "physical" two point function of the effective potential $\mathcal{Y}^{-\infty}$. In this case the external vertices ξ_1, ξ_2 are integrated against test functions $f_1(\xi_1)$, $f_2(\xi_2)$ lying in $L^1 \cap L^\infty$ and Lemma VII.1 is applicable. However, if the value of G^{ϑ} is a term in $\mathcal{Y}^r$, $r > -\infty$, then there are graphs whose values contribute to $\mathcal{Y}^{r-1}$ that contain G^{ϑ} as a subgraph. When the Feynman rules are used to evaluate the larger graphs the external legs of G^{ϑ} are integrated against expressions that are not necessarily in $L^1 \cap L^\infty$ so Lemma VII.1 cannot be applied.

The graph G^{ϑ} corresponds to a fork in a tree of the renormalized tree expansion. At every fork there is either a C or an R label. If C occurs the operator $-L$ is applied to $\mathrm{Val}(G^{\vartheta})$ and produces a two-legged vertex multiplied by a coupling constant:

$$[-L\ \mathrm{Val}(G^{\vartheta})](\xi_1, \xi_2) = -\ \ell(G^{\vartheta})\delta(\xi_1 - \xi_2)$$
$$\ell(G^{\vartheta}) := \int d^{d+1}\xi\ \mathrm{Val}(G^{\vartheta})(0,\xi)e^{i\underline{\xi}\cdot\underline{k}}\Big|_{|\underline{k}|=\sqrt{2m\mu}} \qquad \text{(VII.5)}$$

(This coupling constant is part of $\delta\mu(\mu,\lambda$, see (VI.13).) If R occurs the operator $1-L$ is applied to $\mathrm{Val}(G^{\vartheta})$ to produce the renormalized value of

$G^{\mathcal{J}}$. (The essential feature of this renormalized value is that its Fourier transform vanishes to first order on the Fermi surface.) Hence the other two ways in which a two-legged graph appear are $(-L)\mathrm{Val}(G^{\mathcal{J}})$ and $(1-L)\mathrm{Val}(G^{\mathcal{J}})$.

We now estimate the result of a C-fork.

<u>Lemma VII.2</u> Let $G^{\mathcal{J}}$ be graph in $\Gamma_2^{r,n}$, $r \geq -\infty$ which contains no proper two-legged subdiagrams and let ϕ be the lowest fork in $t(G^{\mathcal{J}})$. Then

$$\int d^{d+1}\xi\,|\xi|^k|\mathrm{Val}(G^{\mathcal{J}})(0,\xi)|$$

$$\leq K^{L(G)} M^{-kj_\phi} M^{j_\phi \min(2,1/2(d+1))} \prod_{f>\phi} M^{D^1(G_f)(j_f-j_{\pi(f)})} \times \begin{cases} |j_\phi| & \text{if } j_\phi<0,\ d=3 \\ 1 & \text{otherwise} \end{cases}$$

In particular

$$|\ell(G^{\mathcal{J}})| \leq K^{L(G)} M^{j_\phi \min(2,1/2(d+1))} (1+|j_\phi|\delta_{d,3}) \prod_{f>\phi} M^{D^1(G_f)(j_f-j_{\pi(f)})}$$

<u>Proof</u>

Assume for simplicity that all particle lines have scales in the infrared end. It is easy to add in the ultraviolet end using the methods of section 3.

We will be able to follow the argument of Lemma V.3 with one important exception. The vertex ξ is integrated over all $\mathbb{R}^{d+1}$ rather than against an L^1 test function. In other words, it acts as though it were an internal vertex. However we may not apply Lemma V.1 as it stands because ξ is not a dimensionless internal vertex.

Fortunately the assumption that internal vertices are dimensionless was not used until fairly late in the proof of Lemma V.1. Stopping just before that point one obtains

<u>Lemma V.1'</u>. Let G be a general graph as in Lemma V.1 but not necessarily having dimensionless internal vertices. Then, for $2 \leq i \leq n$,

$$\int d\xi_2 \ldots d\xi_n |\mathrm{Val}(G;C_\ell)(0,\xi_2 \ldots,\xi_n)||\xi_i|^k$$
$$\le K^{L(G)} M^{j_\phi D_\phi} \prod_{\substack{f>\phi \\ f \text{ nontrivial}}} M^{D_f(j_f - j_{\pi(f)})} M^{-kj_\phi}$$

The factor $|\xi_i|^k$ did not appear in the statement of Lemma V.1. But this factor may easily be bounded because the vertex 0 must be connected to the vertex ξ_i by hard lines of scale at least j_ϕ. Then we may extract a factor $(1 + M^{j_\phi}|\xi_i|)^{-k}$ from these hard lines and bound

$$|\xi_i|^k[1 + (M^{j_\phi}|\xi_i|)^k]^{-1} \le M^{-kj_\phi}.$$

We sketch the variant of the proof of Lemma V.3 obtained by replacing Lemma V.1 by Lemma V.1'.

Just as before

$$|\xi|^k |\mathrm{Val}(G^{\mathcal{J}})(0,\xi)| \le \sum_{k_\ell = j_\ell}^{-1} \mathrm{Val}(G^{\mathcal{J},\mathcal{K}})(0,\xi) M^{-kj_\phi}$$

where the hard particle lines in $G^{\mathcal{J},\mathcal{K}}$ have covariance

$$M^{-1/2(k_\ell - j_\ell)}[M^{d/2\, k_\ell} g(M^{k_\ell}|\underline{x}|)][M^{1/2 j_\ell} g(M^{j_\ell}|\tau|)],$$

the soft particle lines of $G^{\mathcal{J},\mathcal{K}}$ have covariance

$$M^{-1/2(k_\ell - j_\ell)}[M^{d/2\, k_\ell} g(M^{k_\ell}|\underline{x}|)][M^{1/2\, j_\ell}] \quad \text{if } k_\ell > j_\ell$$

and

$$M^{-1/2(k_\ell - j_\ell)}[M^{d/2\, k_\ell}][M^{1/2\, j_\ell}] \quad \text{if } k_\ell = j_\ell$$

and the interaction lines of $G^{\mathcal{J},\mathcal{K}}$ have covariance

$$M^{-1/2(0-0)}[M^{0d} V(M^0|\underline{x}|)][M^{0\times 1}\delta(M^0\tau)].$$

The integral factorizes

$$\int \mathrm{Val}(G^{\mathcal{J},\mathcal{K}})(0,\xi_2) = \prod_{\ell} M^{-1/2(k_\ell - j_\ell)} \mathrm{Val}^d(\mathcal{K})\mathrm{Val}^1(\mathcal{J})$$

with $\mathrm{Val}^d(\mathcal{K})$ containing the spatial integrals and $\mathrm{Val}^1(\mathcal{J})$ the temporal integrals. Applying Lemma V.1' to the temporal and spatial integrals respectively one obtains

$$\mathrm{Val}^1(\mathcal{J}) \le \mathrm{const}^{L(G)} M^{j_\phi D^1(G)} \prod_{\substack{f \in t(G^{\mathcal{J}}) \\ f > \phi \\ f \text{ nontrivial}}} M^{D^1_f (j_f - j_{\pi(f)})} .$$

and

$$\mathrm{Val}^d(\mathcal{K}) \le \mathrm{const}^{L(G)} M^{k_\phi D^d(G)} \prod_{\substack{f \in t(G^{\mathcal{K}}) \\ f > \phi \\ f \text{ nontrivial}}} M^{D^d_f (k_f - k_{\pi(f)})} .$$

We want to combine the proceeding two estimates. As in the proof of Lemma V.3 this is not quite straightforward. The spatial integrals are done on scales k_ℓ with $k_\ell \ge j_\ell$ while the temporal integrals are performed on the j_ℓ scales. Consequently the forest $F(G^{\mathcal{K}})$ and tree $t(G^{\mathcal{K}})$ can be quite different from $F(G^{\mathcal{J}})$ and $t(G^{\mathcal{J}})$. The way around this difficulty is to observe that $D^d_f \le 0$ for all $f > \phi$ and hence $M^{D^d_f (k_f - k_{\pi(f)})} \le 1$ so that

$$\int d^{d+1}\xi |\xi|^k |\mathrm{Val}(G^{\mathcal{J}})|(0,\xi) \le \mathrm{const}^{L(G)} M^{j_\phi [D^1(G) + D^d(G) - k]} \prod_{\substack{f \in t(G^{\mathcal{J}}) \\ f > \phi \\ f \text{ nontrivial}}}$$

$$M^{D^1(G_f)(j_f - j_{\pi(f)})} \sum_{k_\ell = j_\ell}^{-1} \left[\prod_{\ell} M^{-1/2(k_\ell - j_\ell)} \right] M^{(k_\phi - j_\phi) D^d(G)}$$

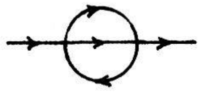
$$\le \mathrm{const}^{L(G)} M^{-kj_\phi} M^{j_\phi \min(2,1/2(d+1))} \prod_{f > \phi} M^{D^1(G_f)(j_f - j_{\pi(f)})}$$

$$\sum_{k_\ell = j_\ell}^{-1} \left[\prod_{\ell} M^{-1/2(k_\ell - j_\ell)} \right] M^{(k_\phi - j_\phi)\min(3/2, d/2)} M^{\max(0,\frac{d-3}{2}) k_\phi} .$$

All that remains is to show that

$$\sum_{k_\ell=j_\ell}^{-1}\left[\prod_\ell M^{-1/2(k_\ell-j_\ell)}\right] M^{(k_\phi-j_\phi)\min(3/2,d/2)}\, M^{\max(0,\frac{d-3}{2}\,k_\phi)}$$

$$\le \mathrm{const}^{L(G)}\begin{cases}|j_\phi| & \text{if } d=3\\ 1 & \text{otherwise}\end{cases}$$

To verify the last estimate we first remark that $G\mathcal{J}$ must have at least three lines whose scales $j_\ell = j_\phi$. To see this observe that every line of $g := G/\{G_f \mid f>\phi\}$ has scale $j_\ell = j_\phi$ and, because of the wick ordering convention, joins distinct vertices of g. Moreover every vertex of g has at least four legs since G has, by assumption, no two-legged subdiagrams. Therefore g has at least $\frac{4V(g)-2}{2} \ge 3$ lines. The smallest possible g is  .

Consequently

$$\sum_{k_\ell=j_\ell}^{-1}\left[\prod_\ell M^{-1/2(k_\ell-j_\ell)}\right] M^{(k_\phi-j_\phi)\min(3/2,d/2)}\, M^{\max(0,\,\frac{d-3}{2})k_\phi}$$

$$= \sum_{k_\phi\ge j_\phi}\ \sum_{k_\ell\ge\max(j_\ell,k_\phi)}\left[\prod_{\ell\notin g} M^{-1/2(k_\ell-j_\ell)}\ \prod_{\ell\in g} M^{-1/2(k_\ell-k_\phi)}\right]$$

$$\times\, M^{-1/2(k_\phi-j_\phi)|g|}\, M^{(k_\phi-j_\phi)\min(3/2,d/2)}\, M^{\max(0,\,\frac{d-3}{2})k_\phi}$$

$$\le \mathrm{const}^{L(G)}\sum_{k_\phi=j_\phi}^{-1} M^{-1/2(k_\phi-j_\phi)(3-d)}\, M^{\max(0,\frac{d-3}{2})k_\phi} \quad \text{since } |g|\ge 3$$

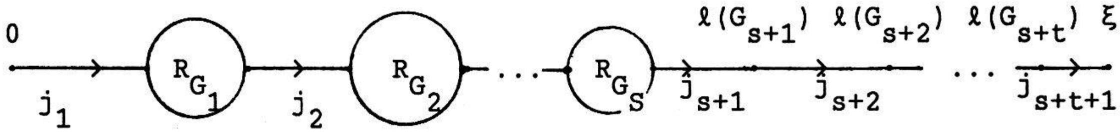
$$\le \mathrm{const}^{L(G)}\begin{cases}|j_\phi| & \text{if } j_\phi<0,\ d=3\\ \mathrm{const} & \text{otherwise}\end{cases}$$

The proof of Lemma VII.2 is complete. ∎

We now estimate the value of an R-fork, namely

$$RG^{\mathcal{J}}(\xi) = \mathrm{Val}(G^{\mathcal{J}})(0,\xi) - \ell(G^{\mathcal{J}})\delta(\xi).$$

If either vertex 0 or ξ is external we may apply Lemma VII.1 Otherwise $RG^{\mathcal{J}}$ occurs in a string:

$S(\xi):=$ 

$$= \int \prod_{i=1}^{s+t} d\zeta_i \prod_{i=1}^{s+t+1} C^{(j_i)}(\xi_{2i}-\xi_{2i-1}) \prod_{i=1}^{s} (RG_i)(\xi_{2i+1}-\xi_{2i})$$
$$\prod_{i=1}^{t} \ell(G_{s+i})\delta(\xi_{2s+2i+1}-\xi_{2s+2i})$$

where $\xi_0 = 0$ and $\xi_{2n+2m+2} = \xi$. Condition d) following (VI.10) forces at least one $C^{(j_i)}$ in the string S to be hard. Furthermore our scale decomposition was constructed so that $\tilde{C}^{(m)}(k)\tilde{C}^{(n)}(k) = 0$ whenever $|m-n| \geq 2$. Hence we may suppose that every $C^{(j_i)}$ is hard and that there is j for which every $j_i = j$ or $(j+1)$. In a typical situation the two-legged diagram G_i of the string S will have scale $h_i > j$, if $1 \leq i \leq s$, and scale j if $(s+1) \leq i \leq s+t$.

We now formulate and prove the Lemma that allows us to estimate strings of renormalized two-legged diagrams.

<u>Lemma VII.3</u> Let G_i, $1 \leq i \leq s+t$, be a two-legged diagram of scale h_i satisfying

$$\||\xi|^k \mathrm{Val}(G_i)(0,\xi)\|_{L^1} \leq K_{i,N'} M^{-kh_i} M^{h_i \min(2,\frac{d+1}{2})}(1+|h_i|\delta_{d,3})$$

for all $1 \leq i \leq s$, $0 \leq k \leq N'$. Then the string of (VII.6) obeys

$$|\mathrm{Val}\ S(\xi)| \le C_1^{s+t} M^j [1+|x|]^{\frac{1-d}{2}} [1+(M^j|\xi|)^{N'}]^{-1}$$
$$\prod_{i=1}^{s} \left[K_{i,N'} M^{h_i \min(2,\frac{d+1}{2})} (1+|h_i|\delta_{d,3}) M^{-(h_i-j)} M^{-j} \right]$$
$$\prod_{i=s+1}^{s+t} \left[|\ell(G_i)| M^{-j} \right] \qquad \text{(VII.7)}$$

Remark (1) It is useful to regard Val $S(\xi)$ as a "covariance." From this point of view Lemma VII.3 is the appropriate analogue of Lemma II.1.

Remark (2) We roughly explain how the different powers of M in (VII.7) arise. Certainly, $|\mathrm{Val}\ S(\xi)|$ includes the convolution of $(s+t+1)$ covariances. By Lemma II.3.1 with $m = n = 0$, $\alpha + \beta = s + t + 1$ the appropriate power of M is

$$M^{j(1-s-t)} = M^j \prod_{i=1}^{s} M^{-j} \prod_{i=s+1}^{s+t} M^{-j}.$$

If G_i, $1 \le i \le s$, were not renormalized it would contribute $M^{h_i \min(1/2(d+1),2)} (1+|h_i|\delta_{d,3})$ by Lemma VII.2. Renormalization replaces $(\widetilde{\mathrm{Val}}\ G_i)(k_0,k)$ by

$$\begin{aligned}
&|(\widetilde{\mathrm{Val}}\ G_i)(k_0,\underline{k}) - (\widetilde{\mathrm{Val}}\ G_i)(0,|\underline{k}|=\sqrt{2m\mu})| \\
&= |k_0 \frac{d}{dk_0} \widetilde{\mathrm{Val}}\ G_i + [|\underline{k}| - \sqrt{2m\mu}] \frac{d}{d|k|} \widetilde{\mathrm{Val}}\ G_i| \\
&\le |k_0|\ \|\frac{d}{dk_0} \widetilde{\mathrm{Val}}\ G_i\|_{L^\infty} + |\ |\underline{k}| - \sqrt{2m\mu}\ |\ \|\frac{d}{d|k|} \widetilde{\mathrm{Val}}\ G_i\|_{L^\infty} \\
&\le \mathrm{const}[|k_0|+||k|-\sqrt{2m\mu}\ |]\ \||\xi|(\mathrm{Val}\ G_i)(0,\xi)\|_{L^1}
\end{aligned}$$

(VII.8)

The support of $C^{(j)}$ forces $[|k_0| + ||\underline{k}| - \sqrt{2m\mu}|] \le \mathrm{const}\ M^j$. As in the proof of Lemma V.1' the $|\xi|$ in $\||\xi|(\mathrm{Val}\ G_i)(0,\xi)\|_{L^1}$ produces a factor M^{-h_i}. Hence the renormalization of G_i gives a $M^{-(h_i-j)}$.

Remark (3)

The expression $M^j[1+|x|]^{\frac{1-d}{2}}[1+(M^j|\xi|)^{N'}]^{-1}$ is our standard bound on a covariance of scale j. Its occurrence on the right hand side of (VII.7) effectively replaces the complicated string S by a single covariance $\underset{0}{\bullet}\overset{j}{\longrightarrow}\underset{\xi}{\bullet}$ of scale j.

Consider the expression

$$M^{h_i \min(2,\frac{d+1}{2})}(1+|h_i|\delta_{d,3})M^{-(h_i-j)}M^{-j} \le \begin{cases} 1 & d=1 \\ M^{h_i/2} & d\ge 2 \end{cases}$$

For d ≥ 2 this expression has two somewhat surprising characteristics. First there is a factor $(1+|h_i|\delta_{d,3})$ which is not a power of M. Second it is bounded by $M^{h_i/2}$ which is summable over $-1 \ge h_i > -\infty$. In the "normal situation" of constructive quantum field theory this would not happen. Rather, as for d = 1, the bound would be essentially independent of h_i, in which case the sums over scales h_i generate powers of j. They in turn generate factorial behaviour for the graph as a whole. However, such factorials are weaker than those of (V.I).

Finally the expression $[|\ell(G_i)M^{-j}|]$ is bounded by a constant since the typical behaviour for $\ell(G_i)$ is (see Lemma VII.2 and (VI.12))

$$|\ell(G_i)| \lesssim \sum_{h\le j} M^{h\min(d,\frac{d+1}{2})}(1+|h|\delta_{d,3})$$

Remark (4)

Two-legged graphs have dimension $\frac{d+1}{2}$. One would normally need an infrared renormalization cancellation of order = (integral part of $\frac{d+1}{2}$). Here, by contrast, a first order renormalization cancellation suffices because of the convolution inequality Lemma II.1.3.

Proof of Lemma VII.3. We simply follow the proof of Lemma II.1. When

$d \geq 2$

$$|(M^j|\xi|)^{N'}S(\xi)| = \text{const}\left|\int_{-\infty}^{\infty} dk_0 \int_0^{\infty} d\rho\ \rho^{d-1} e^{-ik_0\tau} (\rho|\underline{x}|)^{1-d/2} J_{d/2-1}(\rho|\underline{x}|)\right.$$

$$\left.[M^{2j}(-\frac{d^2}{dk_0^2} - \frac{d^2}{d\rho^2} - \frac{d-1}{\rho}\frac{d}{d\rho})]^{N'/2}\ \tilde{S}\ (k_0,|\underline{k}|=\rho)\right|$$

$$\leq \text{const}\ |\underline{x}|^{\frac{1-d}{2}} \int_{-\infty}^{\infty} dk_0 \int_0^{\infty} d\rho\ [M^{2j}(-\frac{d^2}{dk_0^2} - \frac{d^2}{d\rho^2} - \frac{d-1}{\rho}\frac{d}{d\rho})]^{N'/2}|\tilde{S}(k_0,|\underline{k}|=\rho)|$$

The support properties of the $\tilde{C}^{(j_i)}$'s imply that $\tilde{S}$ is supported in a region of (k_0,ρ) space of volume const M^{2j}. The supremum of $\tilde{S}$ is bounded by

$$\text{const}^{s+t+1} M^{-j(s+t+1)} \prod_{i=1}^{s} K_{i,N'} M^{-(h_i-j)} M^{h_i\min(2,\frac{d+1}{2})} (1+|h_i|\delta_{d,3}) \prod_{i=t+1}^{s+t} |\ell(G_i)|.$$

The factor $M^{-j(s+t+1)}$ comes from the $\tilde{C}^{(j_i)}$'s; the product $\prod_{i=1}^{s}$ comes from the $\widetilde{RG}_i$'s via (VII.8) and the observation that on the support of $\tilde{C}^{(j)}$, $[|k_0| + |\underline{k}| - \sqrt{2m\mu}\ ] \leq \text{const}\ M^j$ (c.f. remark 2).

The derivatives $[M^{2j}(-\frac{d^2}{dk_0^2} - \frac{d^2}{d\rho^2} - \frac{d-1}{\rho}\frac{d}{d\rho})]^{N'/2}$ do not materially affect the supremum. This may be seen as follows. As in Lemma II.1 a $\frac{d}{dk_0}$ or $\frac{d}{d\rho}$ acting on a $\tilde{C}^{(j_i)}$ produces, at worst, an M^{-j}. A derivative $\frac{d}{dk_0}$ or $\frac{d}{d\rho}$ acting on $\widetilde{RG}_i$ removes the renormalization, since $\ell(G^{\vartheta})$ is a constant, and hence one factor of $M^{-(h_i-j)}$ from the above supremum but then adds a factor of $M^{-h_i} = M^{-(h_i-j)}M^{-j}$ because of the hypothesis on $\||\xi|^k\text{Val}(G_i)(0,\xi)\|_{L^1}$. Thus the derivative puts back the "renormalization factor" $M^{-(h_i-j)}$ and also gives the usual M^{-j}. As soon as a $\frac{d}{dk_0}$ or $\frac{d}{d\rho}$ acts on $\widetilde{RG}$; the renormalization cancellation disappears. It follows from this observation that at most N' derivatives can act on $\tilde{G}_i$.

Thus, for $d \geq 2$,

$$\left| |\underline{x}|^{\frac{d-1}{2}} (M^j|\xi|^{N'} S(\xi) \right| \leq \text{const}^{s+t+1} M^{j(2-s-t-1)} \prod_{i=s+1}^{s+t} |\ell(G_i)|$$
$$\prod_{i=1}^{s} K_{i,N'} M^{-(h_i-j)} M^{h_i \min(2, \frac{d+1}{2})} (1+|h_i|\delta_{d,3}).$$

To complete the proof we require similar estimates on $|S(\xi)|$ and $|(M^j|\xi|)^{N'} S(\xi)|$. They are derived in the same way. Finally they are added together just as in Lemma II.1 to obtain the stated bound. We omit the argument for d = 1 too. ∎

We now have the essential ingredients necessary to prove the main theorem of this paper.

<u>Theorem VII.4</u> Let $G_{2p}^{r,n}$ $(\xi_1,\ldots,\xi_{2p})$ be the coefficient of λ^n in the expansion of the 2p-point function for the effective potential $\mathcal{G}^r$ at scale r, $-\infty \leq r \leq -1$. Then there exist constants K_p (independent of n and r) and R (independent of p,r and n) such that

$$\|G_{2p}^{r,n}(\xi_1,\ldots,\xi_{2p})\|_{1,\infty} \leq K_p\, R^n\, n!$$

<u>Remark</u> In the introduction we observed that there are approximately $(n!)^2$ graphs contributing to the n^{th} order of perturbation theory, that there is a factor of $\frac{1}{n}!$ arising from the expansion of the exponential and that there are graphs, containing many four legged subgraphs, whose values are of order n!. One of the consequences of Theorem VII.4 is that there must be very few of the latter graphs.

<u>Proof</u> The renormalized tree expansion (VI.14) expresses $G_{2p}^{r,n}$ as a multiple sum. The first sum is over rooted trees having a single branch leaving the root, each fork f bifurcating into $p_f \geq 2$ lines and n endpoints at the top of the tree. The number of such trees is bounded by 2^{4n}. (See, for example, [GJ, page 112].) There is also a sum over the 2^n possible assignments of R/C labels to the forks of a tree τ.

For technical reasons it is convenient to add an internal/external label to each leaf of τ, specifying whether the corresponding vertex is internal or external, and a label E_f to each fork, specifying the number of external lines of G_f that are internal to G. There are at most 2^n possible internal/external labellings. It is enough for the proof to consider a fixed tree and assignment of R/C and internal/external labels since the above powers of two may be absorbed in R^n. Later on it will be necessary to sum over the E_f labellings.

Some forks of a tree require special attention. We now list these forks: (1) C-forks, since they produce a single two legged vertex, multiplied by a scale-dependent coupling constant. (2) R-forks with $E_f=2$ require the convolution identity, see Lemma VII.3 (3) internal R-forks with $E_f=4$ because the factor $M^{D_f^1(j_f-j_{\pi(f)})}$ in (VII.4) fails to provide any decay with which to control the sum over j_f. The strategy will be to first bound $E_f=2,4$ graphs that contain no $E_f=2,4$ subgraphs; then $E_f=2,4$ graphs none of whose $E_f=2,4$ subgraphs contain $E_f=2,4$ subgraphs and so one by induction. Once all $E_f=2,4$ graphs have been treated we can combine the result with a variant of Lemma VII.1 to bound general graphs. In other words we shall proceed by induction on the depth of the tree, which we define to be depth(τ):= max {D| $\exists$ forks $f_1,\dots,f_D$ of τ obeying

(a) $f_1 < f_2 < \dots < f_D$

(b) $E_{f_i} = 2$ or $E_{f_i} = 4$

(c) f_i internal }.

Before formulating the inductive hypothesis we give some motivation. To begin with ignore the number of graphs. Suppose that $f_1,\dots,f_F$ are the forks of τ with $E_f = 4$ (or when d=1, $E_f=2$). The value of a graph arising from τ is of the order of F! To indicate why this happens observe that each scale j_{f_i} must obey $0 \geq j_{f_i} > j_\phi$ and hence can take only $|j_\emptyset|$ values. Furthermore $M^{D^1_{f_i}(j_{f_i}-j_{\pi(f_i)})} = 1$, since $E_{f_1}=4$, and in particular does not decay. Thus, the sum over the scales j_{f_i} is bounded by $|j_\phi|^F$. As we shall see there is a factor $M^{\epsilon j_\phi}$, $\epsilon>0$, associated with the first fork ϕ of τ. For example, when there are no $E_f=2$ subgraphs, one can extract the factor from the estimate of Lemma VII.1. So the product $|j_\phi|^F M^{\epsilon j_\phi} \leq \text{const}^F F!$.

The remark above suffices to prove an n! bound for any single graph. However, to prove that the sum of the values of all n^{th} order graphs is bounded by $\text{const}^n n!$, that is, Theorem VII.4, is much more subtle. Roughly put, either there are many decay factors $M^{D^1_{f_i}(j_{f_i}-j_{\pi(f_i)})}$, $D^1_{f_i} < 0$, controlling the sum over scales and therefore suppressing additional factorials, or most of the forks have $E_{f_i}=4$ generating, as above, an F!, but in this case there are very few graphs contributing to the sum.

We must measure the relative density of four-legged subdiagrams and decay factors. To do this introduce the function

$$\lambda_k(h) := \sum_{i=1}^{\infty} (i+|h|+1)^k M^{-\frac{\alpha}{4}i} .$$

Here i plays the role of $j_f - j_{\pi(f)}$ and h plays the role of $j_{\pi(f)}$. In particular $\sum_{0 \ge j_f > j_{\pi(f)}} (|j_f|+1)^k \, M^{-\frac{\alpha}{4}(j_f - j_{\pi(f)})} \le \lambda_k(j_{\pi(f)})$.

It is proven in [FHRW Lemma 8.2] that

a) $\prod_p \lambda_{n_p}(h) \le \lambda_{\sum n_p}(h)$ if M is large enough

b) $\sum_{h'>h} M^{-\frac{\alpha}{2}(h'-h)} \lambda_n(h') \le \lambda_n(h)$ if M is large enough

c) $\sum_{h'=h+1}^{0} \lambda_n(h') \le \lambda_{n+1}(h)$

d) $\sum_{h \le 0} M^{\frac{\alpha}{2}h} \lambda_n(h) \le C_2^n \, n!$

e) $\sum_{h' \le h} M^{-\frac{\alpha}{2}(h-h')} \lambda_n(h') \le 2\lambda_n(h) \le \lambda_{n+1}(h)$ if M is large enough

f) $n! \le C_3^n \lambda_n(h)$ (VII.9)

We make an inductive hypothesis on

$\mathcal{C}_p^{j,n,D'}(\xi_1,\ldots,\xi_p) = \sup \Big|$  ${}_{j,E_f=p} \Big|$ $\quad p=2,4$

where the sup is over all trees of order at most n, depth at most D' having only internal vertices and over assignments of R and C. The first fork has scale f and E_f=p. The inductive hypothesis is that for all $0 \le k \le d+1$, $2 \le i \le 4$

$$|||\xi|^k \, \mathcal{C}_2^{j,n,D'}(0,\xi)||_1 \le \kappa^{n-1} \lambda_{n-1}(j) M^{-kj} M^{\min(2,\frac{d+1}{2})j} (2+|j|\delta_{d,3}) \quad \text{(VII.10a)}$$

$$|||\xi_i|^k \mathcal{C}_4^{j,n,D'}(0,\xi_2,\xi_3,\xi_4)||_1 \le \kappa_1^{n-1}\lambda_{n-2}(j)M^{-kj} \tag{VII.10b}$$

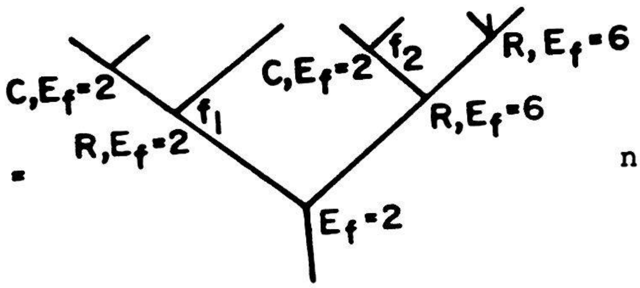
The induction will be on depth. The constant κ is chosen later. Suppose that τ has depth D and that either D=0 or that the inductive hypothesis has been verified for $D' \le D-1$. As above the R/C and E_f labels of τ can be fixed. If the lowest fork has label $E_f=2$ temporarily drop that fork's R/C label since the inductive hypotheses is stated without one.

Decompose the tree τ into a trimmed tree $\tilde{\tau}$ and insertion subtrees $\sigma_1,\ldots,\sigma_m$ by cutting the branches beneath all the minimal internal $E_f=2,4$ forks $f_1,\ldots,f_m$ (i.e. each of the forks $f_1,\ldots,f_m$ is an internal $E_f=2,4$ fork having no internal $E_f=2,4$ fork, except possibly ϕ, below it). If $\tilde{\tau}$ has degree $m+\tilde{n}_0$ and σ_i has degree n_i then

$$n = \tilde{n}_0 + \sum_{i=1}^{m} n_i$$

Further more each σ_i has depth at most D-1. For example, for

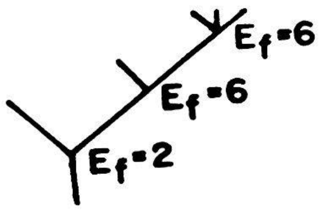
$\tau =$



$n = 7 \quad D = 3$

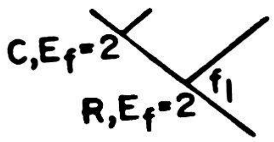
we have

$\tilde{\tau} =$



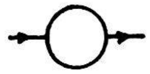
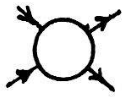
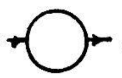
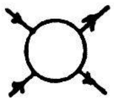
$n_0 = 2$

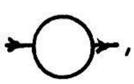
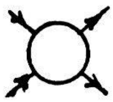
$\sigma_1 =$



$n_1 = 3 \quad$ depth = 2

 $n_2 = 2$ depth = 1

Each graph G contributing to τ may be represented as a graph $\tilde{G}$, of order $\tilde{n}$, associated to $\tilde{\tau}$, having $\tilde{n}_0$ generalized vertices ,  and $\tilde{n}_2$ generalized vertices  and $\tilde{n}_4$ generalized vertices  with the values of the ,  generalized vertices being the values of the insertion subtrees $\sigma_1,\ldots,\sigma_m$. Thus $m = \tilde{n}_2 + \tilde{n}_4$ and $\tilde{n} = \tilde{n}_0 + \tilde{n}_2 + \tilde{n}_4$.

The number of graphs that are represented by each of the generalized vertices ,  in $\tilde{G}$ is built into the inductive hypothesis. In order to continue we must estimate the number of graphs $\tilde{G}$.

The number of unlabelled graphs $\tilde{G}$ is bounded by the standard estimate $2^{7\tilde{n}}(\tilde{n}_0+\tilde{n}_4-p)_+!$ where $(x)_+ = \max(n,0)$. As in [G, Appendix F] the number of labellings of $\tilde{G}$ consistent with the tree expansion and consistent with the E_f labels is bounded, for any $\epsilon>0$, by

$$C_\epsilon^{\tilde{n}} \left[\prod_{f\in\tilde{\tau}} p_f!\right] \exp\left[\epsilon \sum_{f\in\tilde{\tau}} E_f\right]$$

Hence the total number of labelled $\tilde{G}$'s is bounded by

$$C_\epsilon^{\tilde{n}} (\tilde{n}_0 + \tilde{n}_4 - p)_+! \left[\prod_{f\in\tilde{\tau}} p_f!\right] \exp\left[\epsilon \sum_{f\in\tilde{\tau}} E_f\right]. \qquad \text{(VII.11)}$$

We now derive three estimates for the value of an arbitrary $\tilde{G}$, assuming that the generalized vertices corresponding to insertion subtrees satisfy the inductive bounds (VII.10,a,b):

a) If $\tilde{G}$ is two-legged and $0\le k\le d+1$

$$|||\xi|^k \tilde{G}(0,\xi)||_1 \le K^{L(\tilde{G})} M^{-kj_\phi} M^{j_\phi \min(2,\frac{1}{2}(d+1))} (1+|j_\phi|\delta_{d,3}) \prod_{\substack{f\in\tilde{\tau} \\ f>\phi}} M^{D_f^1(j_f - j_{\pi(f)})} Q$$

b) If $\tilde{G}$ is four-legged and $0\le k\le d+1$, $2\le i\le 4$

$$|||\xi|^k \tilde{G}(0,\xi_2,\xi_3,\xi_4)||_1 \le K^{L(\tilde{G})} M^{-kj_\phi} \prod_{\substack{f\in\tilde{\tau} \\ f>\phi}} M^{D_f^1(j_f - j_{\pi(f)})} Q \quad \text{(VII.12)}$$

c) If $\tilde{G}$ has $p\ge 1$ legs

$$||\tilde{G}||_{1,\infty} \le K^{L(\tilde{G})} \prod_{\substack{f\in\tilde{\tau}, f>\phi \\ f \text{ not external} \\ f \text{ not trivial}}} M^{D_f^1(j_f - j_{\pi(f)})} \prod_{\substack{f\in\tilde{\tau}, f>\phi \\ f \text{ external} \\ f \text{ not trivial}}} M^{\Delta_f^1(j_f - j_{\pi(f)})} \times$$

$$\times \prod_{\substack{v \text{ trivial} \\ v \text{ external}}} M^{\Delta_v^1(0-j_{\pi(v)})} Q$$

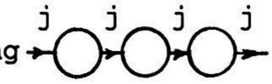
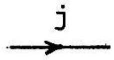
Here

$$Q = \prod_{\substack{\text{insertion tree } \sigma_i \\ \sigma_i \text{ two-legged} \\ \sigma_i \text{ has R label}}} C_1 \kappa^{n_i-1} \lambda_{n_i-1}(h_i) M^{\min(2,\frac{d+1}{2}) h_i} (1+|h_i|\delta_{d,3}) M^{-h_i}$$

$$\prod_{\substack{\sigma_i \\ \sigma_i \text{ two-legged} \\ \sigma_i \text{ has C-label}}} C_1 \kappa^{n_i-1} \lambda_{n_i-1}(h_i) M^{\min(2,\frac{d+1}{2}) h_i} (1+|h_i|\delta_{d,3}) M^{-j_{\pi(\sigma_i)}}$$

$$\prod_{\substack{\sigma_i \\ \sigma_i \text{ four-legged}}} \kappa^{n_i-1} \lambda_{n_i-2}(h_i)$$

where h_i is the scale of the first fork of the insertion subtree σ_i and $j_\pi(\sigma_i)$ is the scale of the first fork below σ_i.

Parts a), b) and c) of (VII.12) follow from Lemmas VII.2, V.1', and VII.1 respectively and two additional observations. First, a key hypothesis is that there are no proper two-legged subgraphs. This may be true for $\tilde{G}$. However Lemma VII.3 eliminates this difficulty. It replaces each string  by a single covariance . Second, only local vertices  were allowed in Lemmas VII.1, VII.2 and V.1'. This was unnecessary. Reviewing the proofs one sees that any graph with four-legged generalized vertices satisfying (VII.10b) obeys the same estimates.

Combining (VII.11) with (VII.12) and estimating the sums over scales and E_f labels allows us to verify the inductive hypotheses at the D^{th} level and finish the proof of the theorem.

First we sum over the scales h_i occuring in Q. We have already observed in Remark (3) following Lemma VII.3 that the factor $M^{h_i \min(2,\frac{d+1}{2})}(1+|h_i|\delta_{d,3})M^{-h_i}$ of the first product in Q is bounded by 1. Hence

$$\prod_{\substack{\sigma_i \text{ two-legged} \\ \sigma_i \text{ has R label}}} \sum_{h_i > j_\pi(\sigma_i)} C_1 \kappa^{n_i-1} \lambda_{n_i-1}(h_i)\, M^{\min(2,\frac{d+1}{2})h_i}(1+|h_i|\delta_{d,3})M^{-h_i}$$

$$\le \prod \sum_{0>h_i>j_\pi(\sigma_i)} C_1\, \kappa^{n_i-1}\, \lambda_{n_i-1}(h_i)$$

$$\le \prod_{\substack{\sigma_i \\ \sigma_i \text{ two legged} \\ \sigma_i \text{ has R label}}} C_1 \, \kappa^{n_i - 1} \, \lambda_{n_i}(j_{\pi(\sigma_i)}) \qquad \text{by (VII.9c)}$$

As for the second and third products we have

$$\prod_{\substack{\sigma_i \text{ 2-legged} \\ \text{C label}}} \sum_{h_i \le j_{\pi(\sigma_i)}} C_1 \kappa^{n_i - 1} \lambda_{n_i - 1}(h_i) M^{\min(2, \frac{d+1}{2}) h_i} (1 + |h_i| \delta_{d,3}) M^{-j_{\pi(\sigma_i)}}$$

$$\le \prod \sum_{h_i \le j_{\pi(\sigma_i)}} \tfrac{1}{2} C_1 \, \kappa^{n_i - 1} \, C_4 \, \lambda_{n_i - 1}(h_i) M^{(h_i - j_{\pi(\sigma_i)})}$$

$$\text{where } C_4 = 2 \max_{h_i \le 0} M^{\min(1, \frac{d-1}{2}) h_i} (1 + |h_i| \delta_{d,3})$$

$$\le \prod_{\substack{\sigma_i \text{ 2-legged} \\ \text{C label}}} C_1 C_4 \, \kappa^{n_i - 1} \, \lambda_{n_i - 1}(j_{\pi(\sigma_i)}) \text{ by (VII.9e)}$$

and

$$\prod_{\sigma_i \text{ 4-legged}} \sum_{0 > h_i > j_{\pi(\sigma_i)}} \kappa^{n_i - 1} \lambda_{n_i - 2}(h_i) \le \prod_{\sigma_i \text{ 4 legged}} \kappa^{n_i - 1} \lambda_{n_i - 1}(j_{\pi(\sigma_i)})$$

Combining these three estimates we have

$$\sum_{\{h_i\}} Q \le (C_1 C_4)^m \, \kappa^{\sum n_i - m} \prod_{\sigma_i \text{ 2 legged}} \lambda_{n_i}(j_{\pi(\sigma_i)}) \prod_{\sigma_i \text{ 4 legged}} \lambda_{n_i - 1}(j_{\pi(\sigma_i)})$$

$$\le (C_1 C_4)^m \, \kappa^{\sum n_i - m} \prod_{\sigma_i \text{ 2 legged}} \lambda_{n_i}(j_\phi) \prod_{\sigma_i \text{ 4 legged}} \lambda_{n_i - 1}(j_\phi)$$

by monotonicity since $|j_\phi| \ge |j_{\pi(\sigma_i)}|$

$$\le (C_1 C_4)^m \, \kappa^{\sum n_i - m} \, \lambda_{\sum n_i - \tilde{n}_4}(j_\phi) \qquad \text{by (VII.9a).}$$

Next we control the factor $\exp[\epsilon \sum_{f\in\tilde\tau} E_f]$ and the sum over E_f's. Note that in (VII.12) every $f\in\tilde\tau$, $f>\phi$ has a factor $M^{D_f^1(j_f-j_{\pi(f)})}$ or $M^{\Delta_f^1(j_f-j_{\pi(f)})}$ with $j_f-j_{\pi(f)} \ge 1$ and

$$D_f^1 \le -\frac{1}{12} E_f \qquad (\text{since } E_f\ge 6)$$

$$\Delta_f^1 \le -\frac{1}{4} E_f \;.$$

We can use half of each of these factors to bound $\exp[\epsilon E_f]$ and to control the sum over E_f. The other half is used to control the sums over j_f, $f>\phi$. All of these sums are bounded by $C_5^{\tilde n_0+m}$.

So far we have shown

a) $$|||\xi|^k \;{}_2^{j_\phi,n,D}(0,\xi)||_1 \le \{\prod_{f\in\tilde\tau} \frac{1}{p_f!}\}\{C_\epsilon^{\tilde n}(\tilde n_0+\tilde n_4-1)_+!\prod_{f\in\tilde\tau} p_f!\}\{K^{2\tilde n} M^{-kj_\phi} M^{j_\phi \min(2,1/2(d+1))}(1+|j_\phi|\delta_{d,3})C_1^m C_4^m C_5^{\tilde n_0+m}\kappa^{\sum n_i-m}\lambda_{\sum n_i-\tilde n_4}(j_\phi)\}.$$

b) $$|||\xi_i|^k \;{}_4^{j_\phi,n,D}(0,\xi_2,\xi_3,\xi_4)||_1 \le \{\prod_{f\in\tilde\tau}\frac{1}{p_f!}\}\{C_\epsilon^{\tilde n}(\tilde n_0+\tilde n_4-2)_+!\prod_{f\in\tilde\tau} p_f!\}\{K^{2\tilde n} M^{-kj_\phi} C_1^m C_4^m C_5^{\tilde n_0+m}\kappa^{\sum n_i-m}\lambda_{\sum n_i-\tilde n_4}(j_\phi)\}$$

c) $$||G_{2p}^{r,n}||_{1,\infty} \le \sum_{j_\phi}\{\prod_{f\in\tilde\tau}\frac{1}{p_f!}\}\{C_\epsilon^{\tilde n}(\tilde n_0+\tilde n_4-p)_+!\prod_{f\in\tilde\tau} p_f!\}\{K^{2\tilde n} M^{-\frac{1}{50}|j_\phi|} C_1^m C_4^m C_5^{\tilde n_0+m}\kappa^{\sum n_i-m}\lambda_{\sum n_i-\tilde n_4}(j_\phi)\} \qquad \text{(VII.13)}$$

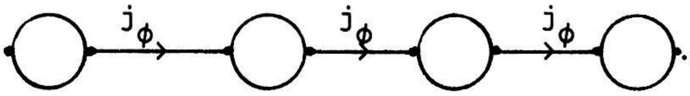
Here the first set of brace brackets { } contains the explicit $\frac{1}{p_f!}$'s that occur in the evaluation of a tree (see (VI.12)), the second set contains our bound on the maximum number of $\tilde{G}$'s and third contains our bound on the maximum value of $\tilde{G}$.

We now observe that the p_f!'s cancel, that

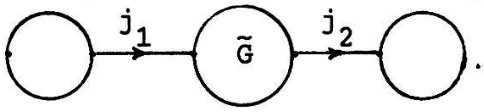
$$(\tilde{n}_0+\tilde{n}_4-p)_+!\lambda_{\sum n_i-\tilde{n}_4}(j_\phi) \le C_3^{\tilde{n}_0+\tilde{n}_4}\lambda_{(\tilde{n}_0+\tilde{n}_4-p)_+}(j_\phi)\lambda_{\sum n_i-\tilde{n}_4}(j_\phi) \quad \text{by (VII.9f)}$$

$$\le C_3^{\tilde{n}_0+\tilde{n}_4}\lambda_{\sum n_i-\tilde{n}_4+(\tilde{n}_0+\tilde{n}_4-p)_+}(j_\phi) \quad \text{by (VII.9a)}$$

and that in cases a) and b) $\tilde{n}_0 + \tilde{n}_4 - p \ge 0$. For example in case a) $p = 1$ so that $\tilde{n}_0 + \tilde{n}_4 - p \ge 0$ unless $\tilde{n}_0 = \tilde{n}_4 = 0$. In other words the only potentially dangerous possibility is that $\tilde{G}$ is of the form



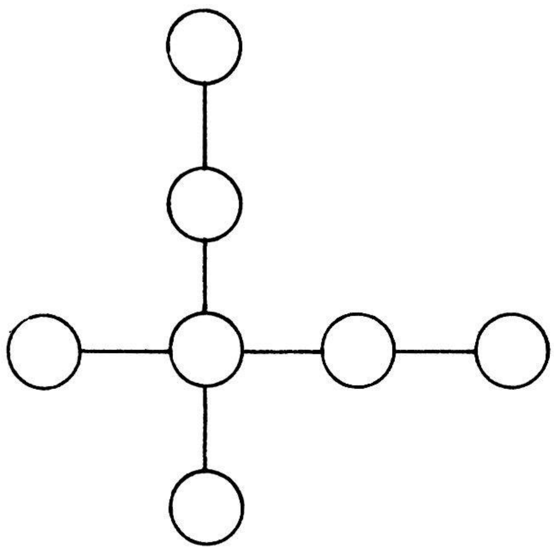
But in case a) $\tilde{G}$ is to be inserted into a larger graph either as a counterterm or with the connecting lines of scale $j_i < j_\phi$



In both cases conservation of momentum implies that the larger graph has value zero. There is another possibility: $j_i = j_\phi - 1$. Then the scale j_ϕ is "tied" to j_i and need not be summed over. It is a dreary technical matter to account for this and would only further obscure the central issues. So, it is ignored. The preceding remarks allow us to conclude that, in case a),

$$\begin{aligned}\sum n_i - \tilde{n}_4 + (\tilde{n}_0+\tilde{n}_4-1)_+ &= \sum n_i - \tilde{n}_4 + \tilde{n}_0 + \tilde{n}_4 - 1\\ &= \tilde{n}_0 + \sum n_i - 1\\ &= n - 1.\end{aligned}$$

Similarly, in case b), $\sum n_i - \tilde{n}_4 + (\tilde{n}_0+\tilde{n}_4-2)_+ = n - 2$. Here the potentially dangerous graph is



Once again conservation of momentum eliminates this possibility. Finally in case c) $\sum n_i - \tilde{n}_4 + (\tilde{n}_0+\tilde{n}_4-p)_+ \le n$.

To verify the inductive hypothesis it must be shown that κ can be chosen so that

$$C_\epsilon^{\tilde{n}} K^{2\tilde{n}} C_1^m C_4^m C_5^{\tilde{n}_0+m} \kappa^{\sum n_i - m} C_3^{\tilde{n}_0+\tilde{n}_4} \le \kappa^{n-1}$$

or, dividing,

$$C_\epsilon^{\tilde{n}} K^{2\tilde{n}} C_1^m C_4^m C_5^{\tilde{n}_0+m} C_3^{\tilde{n}_0+n_4} \le \kappa^{\tilde{n}_0+m-1} .$$

Recall that $\tilde{n} = \tilde{n}_0 + m$ with $m = \tilde{n}_2 + \tilde{n}_4$ and $n = \tilde{n}_0 + \sum n_i$. The left hand side is bounded by $(C_\epsilon K^2 C_1 C_4 C_5 C_3)^{\tilde{n}_0+m}$ so we may choose $\kappa = \max(1, (C_\epsilon K^2 C_1 C_3 C_4 C_5)^2)$ since $\tilde{n}_0 + m \ge 2$.

The proof of Theorem VII.4 is completed by observing

$$\sum_{j_\phi=-\infty}^{0} \lambda_n(j_\phi) M^{-\frac{1}{50}(j_\phi)} \le C_2^n\, n! \quad \text{by (VII.9.d).}$$

sGw

VIII The Interacting Fermi Surface in Perturbation Theory

The average occupation number $n_{\underline{k},\sigma}$ of momentum $\underline{k}$ and spin σ is defined by

$$\begin{aligned} n_{\underline{k},\sigma} &= \int d^3\underline{x}\ \langle\Omega, \bar{\psi}_\sigma(\underline{x},0)\psi_\sigma(\underline{0},0)\Omega\rangle e^{i\underline{k}\cdot\underline{x}} \\ &= \lim_{\tau\to 0+} \int d^3\underline{x}\ S_2(\underline{0},0,\sigma),(\underline{x},\tau,\sigma))e^{i\underline{k}\cdot\underline{x}} \\ &= \hat{S}(\underline{k},\tau=0+,\sigma) \end{aligned} \tag{VIII.1}$$

where $\wedge$ denotes the spatial Fourier transform. In the free model

$$\hat{S}_0(\underline{k},\tau=0+,\sigma) = \chi(\sqrt{2m\mu} - |k|)$$

so that there is a discontinuity at $|k| = \sqrt{2m\mu}$. By definition the interacting Fermi surface is (if it exists) the surface of discontinuity of $\hat{S}(\underline{k},\tau=0+,\sigma)$ where S is the interacting two point Schwinger function.

The inverse of the interacting two point function is the difference of the free one and the proper self-energy Σ (see section VI)

$$S = [S_0^{-1} - \Sigma]^{-1}. \tag{VIII.2}$$

Thus

$$n_{\underline{k},\sigma} = \lim_{\tau\to 0+} \int \frac{d\omega}{(2\pi)} e^{+i\omega\tau} \frac{1}{i\omega - e(\underline{k}) - \tilde{\Sigma}(\underline{k},\omega,\sigma)} \tag{VIII.3}$$

We conclude from (VII.10a) that, in perturbation theory,

$$\begin{aligned} \||\,|\xi|^k\Sigma(\xi)\||_1 &\le O(\lambda) \sum_j M^{-kj} M^{\min(2,\frac{d+1}{2})j}(1+|j|\delta_{d,3}) \\ &\le O(\lambda) \quad \text{when} \begin{cases} k < 1 & \text{if } d = 1 \\ k < 3/2 & \text{if } d = 2 \\ k < 2 & \text{id } d \ge 3 \end{cases} \end{aligned} \tag{VIII.4}$$

Theorem VIII.1

If $\|(1+|\xi|)^{1+\epsilon}\Sigma(\xi)\|_1 \le O(\lambda)$ for some $\epsilon > 0$ and $\Sigma(0,|\underline{k}|=\sqrt{2m\mu}) = 0$ then for λ sufficiently small the limits $\lim_{|k|\to\sqrt{2m\mu}\,\pm} n_{\underline{k},\sigma}$ exist and

$$\lim_{|k|\to\sqrt{2m\mu}\,-} n_{\underline{k},\sigma} - \lim_{|k|\to\sqrt{2m\mu}\,+} n_{\underline{k},\sigma} = \left[1 - \frac{\tilde{\Sigma}_\omega(|\underline{k}|=\sqrt{2m\mu},0)}{i}\right]^{-1}$$

$$\ge 1 - O(\lambda).$$

Remark (1) As we shall see the first hypotheses implies that $\tilde{\Sigma}(\underline{k},\omega)$ and all its first derivatives are continuous and uniformly bounded by $O(\lambda)$. If the free model is a stable fixed point under the renormalization group then, by (VIII.4), the hypothesis ought to be fulfilled in two or more dimensions. This may never be the case. See [KL2] and [LP, page 222].

On the other hand it will fail marginally in one dimension. This observation is consistent with [ML] in which the number density of an exactly soluble one dimensional many fermion system has infinite slope but no jump at $|k| = \sqrt{2m\mu}$. In fact direct calculation to second order in the models considered here yields (for $\sqrt{2m\mu} = 1$)

$$\tilde{\Sigma}(\underline{k},\omega) = \lambda^2 \int_{\substack{e(p)e(\underline{k}+q-p)>0 \\ e(q)[e(\underline{k}+q-p)+e(p)]<0}} \frac{dp}{2\pi}\frac{dq}{2\pi} \frac{\{k_0+(\underline{k}-1)[1+\underline{k}+2q-2p]\}\{\hat{V}(\underline{k}-p)\hat{V}(q-p)-\hat{V}(q-p)^2\}}{\{k_0+i[e(p)-e(q)+e(\underline{k}+q+p)\}[e(p)-e(q)+e(1+q-p)]} + \text{regular}$$

So $\frac{1}{i}\tilde{\Sigma}_\omega(\underline{k},0) \sim \text{const } \ell n\, e(\underline{k})$ and

$$n_{\underline{k}} \sim \text{regular} + \text{const } \Theta(e(\underline{k}))[1+\text{const}|\ell n\, e(\underline{k})|]^{-1}$$

where Θ is the Heavyside step function. In this case $n_{\underline{k}}$ is continuous across the Fermi surface but has infinite slope there.

Remark (2) By (I.2) the free $\hat{S}(\underline{k},\tau,\sigma)$ is real and invariant under the reflection $\underline{k} \to -\underline{k}$. Since the interaction $\hat{V}(k)\delta(\tau)$ is also real and reflection invariant the interacting $\hat{S}(\underline{k},\tau,\sigma)$ and hence $\hat{\Sigma}(\underline{k},\tau,\sigma)$ have these same properties. From this we conclude that

$$\tilde{\Sigma}(\underline{k},\omega=0,\sigma) = \int d\tau\ \hat{\Sigma}(\underline{k},\tau,\sigma)$$

and

$$\frac{1}{i}\frac{\partial}{\partial\omega}\tilde{\Sigma}(\underline{k},\omega=0,\sigma) = \int d\tau\ \tau\hat{\Sigma}(\underline{k},\tau,\sigma)$$

are real.

<u>Proof</u> We decompose

$$\begin{aligned}\hat{S}(\underline{k},\tau,\sigma) &= \int\frac{d\omega}{2\pi}\, e^{i\omega\tau}[i\omega-e(\underline{k}) - \tilde{\Sigma}(\underline{k},\omega,\sigma)]^{-1}\\ &= I_1(\underline{k},\tau) + I_2(\underline{k},\tau) + I_3(\underline{k},\tau)\end{aligned}$$

where

$$I_1(\underline{k},\tau) = \int_{|\omega|<\eta} \frac{d\omega}{2\pi}\, e^{i\omega\tau}[iA\omega-Be(\underline{k})]^{-1}$$

$$I_2(\underline{k},\tau) = \int_{|\omega|<\eta} \frac{d\omega}{2\pi}\, e^{i\omega\tau}\frac{\tilde{R}(k,\omega)}{[iA\omega-Be(\underline{k})][iA\omega-Be(\underline{k})-\tilde{R}(k,\omega)]}$$

$$I_3(\underline{k},\tau) = \int_{|\omega|\geq\eta} \frac{d\omega}{2\pi}\, e^{i\omega\tau}[i\omega-e(\underline{k})]^{-1}$$

$$I_4(\underline{k},\tau) = \int_{|\omega|\geq\eta} \frac{d\omega}{2\pi}\, e^{i\omega\tau}\frac{\tilde{\Sigma}(k,\omega,\sigma)}{[i\omega-e(\underline{k})][i\omega-e(\underline{k})-\tilde{\Sigma}(k,\omega,\sigma)]}$$

$$A = 1 - \frac{1}{i}\frac{\partial}{\partial\omega}\tilde{\Sigma}(|\underline{k}| = \sqrt{2m\mu},0,\sigma)$$

$$B = 1 + \frac{1}{\sqrt{2m\mu}}\frac{\partial}{\partial|\underline{k}|}\tilde{\Sigma}(|k| = \sqrt{2m\mu},0,\sigma)$$

$$\tilde{R} = \tilde{\Sigma} - \frac{\partial}{\partial\omega}\tilde{\Sigma}(|\underline{k}|=\sqrt{2m\mu},0,\sigma)\omega - \frac{\partial}{\partial|\underline{k}|}\tilde{\Sigma}(|\underline{k}| = \sqrt{2m\mu},0,\sigma)\,\frac{e(k)}{\sqrt{2m\mu}}$$

and $\eta > 0$ is a small number to be chosen later. Note that, $\||\xi|\Sigma(\xi)\|_1 < \infty$ implies the continuity of the first derivatives of $\tilde{\Sigma}$ and consequently the existence of A, B and $\tilde{R}$. We still have to show that the integrals converge. Here the extra ϵ is required

For $|k| \neq \sqrt{2m\mu}$

$$\lim_{\tau\to 0+} I_1(\underline{k},\tau) = \lim_{\tau\to 0+} \int_{|\omega|<\eta} \frac{d\omega}{2\pi}\, e^{i\omega\tau}\, [iA\omega-\beta e(\underline{k})]^{-1}$$

$$= \int_{|\omega|<\eta} \frac{d\omega}{2\pi}\, [iA\omega-Be(\underline{k})]^{-1} \quad \text{(since the integrand is } L^1\text{)}$$

$$= \int_{|\omega|<\eta} \frac{d\omega}{2\pi}\, \frac{-iA\omega-Be(k)}{A^2\omega^2+B^2e(\underline{k})^2}$$

$$= -\frac{1}{Be(\underline{k})} \int_{|\omega|<\eta} \frac{d\omega}{2\pi}\, \frac{1}{1+\frac{A^2}{B^2e(\underline{k})^2}\omega^2}$$

$$= -\frac{1}{A}\,\mathrm{sgn}\, e(\underline{k}) \int_{|\omega|<\frac{A}{B|e(\underline{k})|}\eta} \frac{d\omega}{2\pi}\, \frac{1}{1+\omega^2}$$

$$= -\frac{1}{2A}\,\mathrm{sgn}\, e(\underline{k})\, \frac{2}{\pi}\tan^{-1}\left(\frac{A}{B|e(\underline{k})|}\eta\right)$$

As $\underline{k}$ approachs the Fermi surface $\frac{2}{\pi}\tan^{-1}\left(\frac{A}{B|e(\underline{k})|}\eta\right) \to 1$. Hence,

$$\lim_{|\underline{k}|\to\sqrt{2m\mu}\,\mp} I_1(\underline{k},0+) = \pm\frac{1}{2A}\,.$$

So we must show that $I_2(\underline{k},0+)$, $I_3(\underline{k},0+)$ and $I_4(k,0+)$ exist and are continuous across the Fermi surface.

Now consider I_2. It will be shown in Lemma VIII.2 that, for $|\omega|$ and $|e(\underline{k})|$ sufficiently small, $|\tilde{R}(\underline{k},\omega)| \le C[|\omega|^{1+\epsilon} + |e(\underline{k})|^{1+\epsilon}]$.

Consequently

$$|iA\omega-Be(\underline{k})| \ge C'[|\omega|+|e(\underline{k})|)$$

$$|iA\omega-Be(\underline{k})-\tilde{R}(\underline{k},\omega)| \ge C'[|\omega|+|e(\underline{k})|$$

and the integrand

$$\left| \frac{e^{i\omega\tau}}{2\pi} \frac{\tilde{R}(k,\omega)}{[iA-Be(\underline{k})][iA\omega-Be(\underline{k})-\tilde{R}(\underline{k},\omega)]} \right|$$

$$\le \frac{C}{C'^2} \frac{|\omega|^{1+\epsilon}+|e(\underline{k})|^{1+\epsilon}}{[|\omega|+|e(\underline{k})|]^2} .$$

We write

$$\int_{|\omega|<\eta} = \int_{|e(\underline{k})|<|\omega|<\eta} + \int_{|\omega|<\min[\eta,|e(\underline{k})|]} .$$

In the first region, the integrand is bounded by $\frac{2C}{C'^2} \frac{1}{|\omega|^{1-\epsilon}}$, which is L^1. Thus, by the Lebesgue dominated convergence theorem, the first integral is continuous across $e(\underline{k}) = 0$. The second integral is bounded by $2|e(\underline{k})| \frac{C}{C'^2} 2 \frac{|e(\underline{k})|^{1+\epsilon}}{|e(\underline{k})|^2}$ which goes to zero as $e(\underline{k}) \to 0$.

Now consider I_3.

$$I_3(\underline{k},\tau) = \int_{\mathbf{R}^1} \frac{d\omega}{2\pi} e^{i\omega\tau}[i\omega-e(\underline{k})]^{-1} - \int_{|\omega|<\eta} \frac{d\omega}{2\pi} e^{i\omega\tau}[i\omega-e(\underline{k})]^{-1}$$

The second term has already been dealt with (c.f. I_1 with $A = B = 1$) and obeys

$$\lim_{|\underline{k}|\to\sqrt{2m\mu}\,\mp} \ \lim_{\tau\to 0+} \int_{|\omega|<\eta} \frac{d\omega}{2\pi} e^{i\omega\tau}[i\omega-e(\underline{k})]^{-1} = \pm \frac{1}{2}.$$

The first integral is conditionally convergent and may be evaluated explicitly by contour integration. For $\tau > 0$

$$\int_{\mathbf{R}^1} \frac{d\omega}{2\pi} e^{i\omega\tau} [i\omega-e(\underline{k})]^{-1} = e^{-e(\underline{k})\tau} \begin{cases} 1 & e(\underline{k})<0 \\ 0 & e(\underline{k})>0 \end{cases}.$$

The two jumps cancel.

Finally we consider I_4. Observe that

$$|i\omega - e(\underline{k})| \geq |\omega|$$

$$\begin{aligned}|i\omega - e(\underline{k}) - \tilde{\sum}| &\geq |\omega - \text{Im}\ \tilde{\sum}(k,\omega,\sigma)| \\ &\geq |\omega| - |\text{Im}[\tilde{\sum}(k,\omega,\sigma) - \tilde{\sum}(k,0,\sigma)]| \\ &\geq |\omega| - |\omega|\,\|\frac{\partial}{\partial\omega}\tilde{\sum}(k,\omega,\sigma)\|_\infty \\ &\geq \frac{1}{2}|\omega|\end{aligned}$$

if λ is sufficiently small since $\||\tau|\sum(\underline{x},\tau,\sigma)\|_1 \leq 0(\lambda)$. Therefore the integrand is bounded by $\dfrac{0(\lambda)}{\frac{1}{2}|\omega|^2}$. By the Lebesgue dominated convergence theorem I_4 is continuous.

Once Lemma VIII.2 is established the proof of Theorem VIII.1 will be complete.

Lemma VIII.2. Let $f = f(|\underline{x}|,\tau)$. If $\|(1+|\xi|)^{1+\epsilon}f\|_1 \leq C$

$$|\tilde{f}(|\underline{k}|,\omega) - \tilde{f}(k_F,0) - \tilde{f}_\omega(k_F,0)\omega - \tilde{f}_{|\underline{k}|}(k_F,0)[|\underline{k}|-k_F]|$$

$$\leq 2^{1+\epsilon}C[|\omega|^{1+\epsilon} + ||\underline{k}|-k_F|^{1+\epsilon}].$$

Proof

$$\tilde{f}(|\underline{k}|,\omega) - \tilde{f}(k_F,0) - \tilde{f}_\omega(k_F,0)\omega - \tilde{f}_{|\underline{k}|}(k_F,0)[|\underline{k}|-k_F]$$

$$= \int d^d\underline{x}d\tau[e^{i(\tau\omega-|\underline{k}|x_1)} - e^{-ik_Fx_1} - i\tau\omega e^{-ik_Fx_1} + ix_1[|\underline{k}|-k_F]e^{-ik_Fx_1}]f(\underline{x},\tau)$$

But

$$|e^{ia} - 1 - ia| \leq \frac{1}{2}a^2$$

and

$$|e^{ia} - 1 - ia| \leq |e^{ia} - 1| + |a| \leq 2|a|$$

so

$$|e^{ia} - 1 - ia| \leq [\tfrac{1}{2}a^2]^\epsilon[2|a|]^{1-\epsilon} = 2|a|^{1+\epsilon}.$$

Applying this with $a = \tau\omega - [|\underline{k}|-k_F]x_1$ we arrive at

$$|\tilde{f}(|\underline{k}|,\omega) - \tilde{f}(k_F,0) - \tilde{f}_\omega(k_F,0)\omega - \tilde{f}_{|\underline{k}|}(k_F,0)[|\underline{k}|-k_F]|$$

$$\leq \int d^d\underline{x}d\tau\ 2|\tau\omega-[|\underline{k}|-k_F]x_1|^{1+\epsilon}|f(\underline{x},\tau)|$$

$$\leq 2^{1+\epsilon}\int d^d\underline{x}d\tau[|\omega|^{1+\epsilon}|\tau|^{1+\epsilon} + ||\underline{k}|-k_F|^{1+\epsilon}|x_1|^{1+\epsilon}]|f(\underline{x},\tau)|$$

$$\leq 2^{1+\epsilon}C[|\omega|^{1+\epsilon} + ||\underline{k}|-k_F|^{1+\epsilon}]$$

∎

Remark (3) Under the hypotheses of the Theorem the zero set of $i\omega - e(\underline{k}) - \tilde{\sum}(\underline{k},\omega,\sigma)$ is precisely $\omega = 0$, $|\underline{k}| = \sqrt{2m\mu}$ provided λ is sufficiently small. In the course of proving Theorem VIII.1 we showed that for all ω, $\underline{k}$ $|\mathrm{Im}[i\omega-\tilde{\sum}(\underline{k},\omega,\sigma]| \geq \frac{|\omega|}{2}$. Now consider $\omega = 0$. There is precisely one zero for small $|e(\underline{k})|$ because

$$\frac{\partial}{\partial|\underline{k}|}[e(\underline{k})-\sum(\underline{k},0,\sigma)]\Big|_{|\underline{k}|=\sqrt{2m\mu}} \neq 0.$$

There can be no zeroes for $|e(\underline{k})|$ large because $\|\sum(\underline{k},0,\sigma)\|_\infty \leq 0(\lambda)$.

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