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Phase Space Analysis of the Charge Transfer Model

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Abstract. Geometric methods are used to establish completeness of scattering for the time dependent Hamiltonian

$$H(t) = \frac{1}{2}p^2 + \sum_{l=1}^{n} V_l(x - x_l(t))$$

The motions $x_l(t)$ of the centers are asymptotically inertial and diverging from each other; the potentials $V_l(x)$ are short range. No assumptions are made on the spectra of the subsystems. Intermediate results of some interest concern the time boundedness of the energy, a RAGE theorem and the asymptotics of some observable.

1. Assumptions and Results

The Charge Transfer Model has been devised for describing the motion of a light particle in a collision between heavy ones, e.g. an electron in the field of heavy ions. In the model only the light particle is subject to quantum dynamics, while the heavy ones follow assigned classical trajectories, which are asymptotically inertial. This leads to the Hamiltonian

$$H(t) = \frac{1}{2}p^2 + \sum_{l=1}^{n} V_l(x - x_l(t)) \quad . \tag{1.1}$$

The problem of asymptotic completeness of scattering has first been solved by Yajima [16], using Howland's formalism [10] and time independent scattering theory. Subsequently Hagedorn produced a proof based on the study of some Faddeev-type equations [8], and he also suggested an 'Enss type' proof [9]. In fact, as we shall see, the ideas developed by V. Enss in a series of papers (among them [2]-[6]), turn out to be very useful for proving completeness in the present context. During the typesetting of this manuscript, we were informed about an independent proof of Wüller [15], working under weaker assumptions on the potentials. Although both his and our approach are examples of 'phase space analysis', they seem to be quite different; we therefore believe that our work also sheds some light on the problem. For instance, in [15] completeness is established by a detailed analysis of the multiple scattering processes the 'electron' can undergo, whereas our basic dynamical property of the scattering states is an estimate on certain time averages (RAGE theorem).

Assumptions: In the course of this work we will use various assumptions on the classical trajectories and on the potentials. The strongest conditions, which cover all the results are

(T1) For any $l = 1, \dots, n$, $x_l \in C^1(\mathbf{R}, \mathbf{R}^\nu)$ and there are $u_l, a_l \in \mathbf{R}^\nu$ such that $u_l \neq u_k$ for $l \neq k$ and, setting $\Delta x_l(t) = x_l(t) - (u_l t + a_l)$,

$$|\Delta x_l(t)|(1+|t|)^{1+\beta}, \qquad |\Delta \dot{x}_l(t)|(1+|t|)^{1+\beta}$$

are bounded for some $\beta > 0$ and large t.

(P1) $V_l(x), l = 1, \dots, n$ are real valued functions satisfying

$$DV_l(x)(1+|x|)^{1+\varepsilon} \in L^\infty(\mathbf{R}^\nu)$$

for some $\varepsilon > 0$, where D is any zeroth or first order distributional derivative.

However, many intermediate results are derived from less restrictive assumptions. To construct the propagator we use

(T2) For any $l = 1, \dots, n$, $x_l \in C^1(\mathbf{R}, \mathbf{R}^\nu)$

(P2) $V_l(x), l = 1, \dots, n$ are real valued functions satisfying

$$V_l \in L^p(\mathbf{R}^\nu) + C_\infty(\mathbf{R}^\nu) \qquad \textit{with} \qquad p > \nu/2,\ p \geq 2,\ p < \infty$$

(C_∞ denotes the continuous functions vanishing at infinity), and

$$DV_l \in L^{p_1}(\mathbf{R}^\nu) + L^\infty(\mathbf{R}^\nu) \qquad \textit{with} \qquad p_1 > \nu/3,\ p_1 \geq 4/3 \quad ,$$

where D is any first order distributional derivative.

To prove time boundedness of the kinetic energy we assume (T1), (P2) and impose one of the following conditions on the forces and on the trajectories (resp. on the forces only): *For $l = 1, \ldots, n$*

(F1) $x_l \in C^2(\mathbf{R}, \mathbf{R}^\nu)$ with $|\ddot{x}_l(t)|(1+|t|)^{1+\beta}$ bounded for some $\beta > 0$ and large t;

$$DV_l(x)F(|x| > R)(1+|x|)^{1+\epsilon} \in L^p(\mathbf{R}^\nu) + L^\infty(\mathbf{R}^\nu)$$

(F2)
$$DV_l(x)(1+|x|)^{1+\epsilon} \in L^p(\mathbf{R}^\nu) + L^\infty(\mathbf{R}^\nu)$$

for some $\epsilon, R > 0$ and $p > \nu/2$, $p \geq 2$, where D is any first order distributional derivative.

Existence of the wave operators is proved assuming (T1) and

(P3) $V_l(x), l = 1, \ldots, n$ are real valued functions satisfying

$$DV_l(x)(1+|x|)^{1+\epsilon} \in L^p(\mathbf{R}^\nu) + L^\infty(\mathbf{R}^\nu) \qquad \textit{with} \qquad p > \nu/2,\ p \geq 2$$

for some $\epsilon > 0$, where D is any zeroth or first order distributional derivative.

Remarks: 1. The short range assumption on the potentials allows for simple asymptotic dynamics (see [15] for the long range case). Boundedness of both the potentials and the forces (instead of p^2-boundedness) is used in Section 6 (Asymptotics of observables).

2. (P1) or (P3) imply (F2), and (P3) implies (P2). (P2) and (F1) allow for Coulomb singularities in $\nu = 3$ dimensions.

3. As a rule, the hypothesis of the theorems will be stated explicitely; lemmas and corollaries hold under the same assumptions on the trajectories and the potentials as the theorem which is proved in the same section.

The Hamiltonian (1.1) is selfadjoint on $\mathcal{H} = L^2(\mathbf{R}^\nu), \nu \geq 1$ with domain $D(p^2)$. Let $U(t,s)$ be the propagator from s to t for the corresponding Schrödinger equation (see Theorem 2.1). One expects that the large time behaviour of a particle under the evolution $U(t,s)$ can be described as a superposition of the following:

i) the particle is free

ii) the particle is bound to one of the centers, say l, i.e. up to the Galilei transform

$$\begin{aligned} x &\longmapsto x - (u_l t + a_l) \\ p &\longmapsto p - u_l \end{aligned} \tag{1.2}$$

it is a superposition of bound states of

$$H_l = \frac{1}{2}p^2 + V_l(x) \quad . \tag{1.3}$$

To formulate this we introduce the projection P_l^{pp} onto the bound state subspace of (1.3) and the operator implementing (1.2), namely

$$G_l(t) = e^{i\frac{u_l^2}{2}t}e^{ip(u_lt+a_l)}e^{-iu_lx} \quad .$$

Here is our main result:

Theorem 1.1. *Let $s \in \mathbf{R}$. The following assertions hold under the assumptions stated in brackets:*

i) Existence of wave operators: (T1), (P3)
The limits

$$\Omega_0^-(s) = \text{s}-\lim_{t\to+\infty} U(s,t)e^{-i\frac{p^2}{2}(t-s)}$$

$$\Omega_l^-(s) = \text{s}-\lim_{t\to+\infty} U(s,t)G_l(t)^{-1}e^{-iH_l(t-s)}P_l^{pp}G_l(s) \qquad l = 1,\ldots,n$$

exist.

ii) Asymptotic orthogonality: (T1), (P3)
The ranges $\operatorname{Ran}\Omega_l^-(s), l = 0,\ldots,n$ *are closed and orthogonal to each other.*

iii) Asymptotic completeness: (T1), (P1)

$$\mathcal{H} = \bigoplus_{l=0}^{n} \operatorname{Ran}\Omega_l^-(s) \quad .$$

An essential step in the proof is time boundedness of the kinetic energy:

Theorem 1.2. *Assume (T1), (P2) and in addition (F1) or (F2). Then for any $s \in \mathbf{R}$ there is a $C > 0$ such that*

$$\sup_{t\geq s}\,(U(t,s)\psi, p^2U(t,s)\psi) \leq C(\psi,(1+p^2)\psi) \tag{1.4}$$

for all $\psi \in Q(p^2)$, i.e. ψ in the form domain of p^2.

Remarks: 1. Analogous statements hold for the past.

2. Write

$$H(t) = \frac{1}{2}p^2 + \sum_{l=1}^{n}\tilde{V}_l(x - \tilde{x}_l(t)) \quad ,$$

where $\tilde{V}_l(x) = V_l(x-a_l)$ and $\tilde{x}_l(t) = x_l(t) - a_l$. These translated trajectories and potentials satisfy the same hypothesis as the original ones, with $\tilde{u}_l = u_l$, $\tilde{a}_l = 0$. The corresponding Galilei transforms are $\tilde{G}_l(t) = e^{iu_l^2t/2}e^{ipu_lt}e^{-iu_lx}$. Because of

$$\begin{aligned}G_l(t)^{-1}e^{-iH_l(t-s)}P_l^{pp}G_l(s) &= \tilde{G}_l(t)^{-1}e^{-ipa_l}e^{-iH_l(t-s)}P_l^{pp}e^{ipa_l}\tilde{G}_l(s)\\ &= \tilde{G}_l(t)^{-1}e^{-i\tilde{H}_l(t-s)}\tilde{P}_l^{pp}\tilde{G}_l(s) \quad ,\end{aligned}$$

where $\tilde{H}_l = \frac{1}{2}p^2 + \tilde{V}_l(x)$ and $\tilde{P}_l^{pp}$ is its bound state projection, we shall restrict ourselves (dropping the tildes) to the case $a_l = 0$.

2. The Propagator

Sufficient conditions for the existence of the propagator for Schrödinger operators with general time dependent potentials have been given by Yajima [17]. Since in our problem the time dependence is given through the trajectories, we are led by his Theorem 1.3 to state Assumptions (T2) and (P2).

Theorem 2.1. *Under the Assumptions (T2), (P2), there is a family of operators $U(t,s)$ on $\mathcal{H}$, $t,s \in \mathbf{R}$, satisfying*

i) $U(t,s)$ *is unitary*

ii) $U(t,s) = U(t,r)U(r,s)$

iii) $U(s,s) = 1$

iv) $U(t,s)D(p^2) \subset D(p^2)$ *and* $U(t,s)$ *is strongly continuous in* $\mathcal{H}$ *and in* $D(p^2)$ *with respect to* (t,s)

v) For $\psi \in D(p^2)$, $U(t,s)\psi$ *is continuously differentiable in* $\mathcal{H}$ *with respect to* (t,s):

$$i\frac{\partial}{\partial t}U(t,s)\psi = H(t)U(t,s)\psi \tag{2.1}$$

$$-i\frac{\partial}{\partial s}U(t,s)\psi = U(t,s)H(s)\psi \quad . \tag{2.2}$$

The family $U(t,s)$ *is uniquely determined by*

vi) $U(t,s)$ *is bounded*

vii) For $\psi \in D(p^2)$: $U(t,s)\psi \in D(p^2)$, $U(s,s)\psi = \psi$ *and* $U(t,s)\psi$ *is differentiable with respect to* t*, the derivative being given by (2.1)*

As an application, we prove the domain invariance property which is implicit in (1.4):

Corollary 2.2. $U(t,s)Q(p^2) \subset Q(p^2)$ *and* $U(t,s)$ *is strongly continuous in* $Q(p^2)$ *with respect to* (t,s).

Proof: By Theorem 2.1 iv) and by the uniform boundedness principle,

$$\|(p^2+1)U(t,s)\psi\| \leq \text{const}\,\|(p^2+1)\psi\| \,, \qquad \psi \in D(p^2),\ t,s \in I$$

for a compact interval I. Interpolation ([13], Proposition IX.9) between this and $\|U(t,s)\psi\| \leq \text{const}\,\|\psi\|$, $\psi \in \mathcal{H}$, $t,s \in I$ gives $U(t,s)Q(p^2) \subset Q(p^2)$ and

$$\|(p^2+1)^{1/2}U(t,s)\psi\| \leq \text{const}\,\|(p^2+1)^{1/2}\psi\| \,, \qquad \psi \in Q(p^2),\ t,s \in I \quad . \tag{2.3}$$

It is therefore enough to know that strong $Q(p^2)$-continuity holds on $D(p^2)$, which is a form core for p^2. ∎

In Section 6 we will need the following domain invariance property:

Corollary 2.3. *$U(t,s)D(p^2+x^2) \subset D(p^2+x^2)$ and $U(t,s)$ is strongly continuous in $D(p^2+x^2)$ with respect to (t,s).*

Proof: We regularize x by $u^\varepsilon(x) = x/(1+\varepsilon x^2)$. We have $u^\varepsilon_{i,j} \xrightarrow{s} \delta_{ij}$, $u^\varepsilon_{i,jj} \xrightarrow{s} 0$ as $\varepsilon \searrow 0$, where indices following a comma stand for partial derivatives. Let $\varphi, \psi \in D(p^2+x^2)$. Integrating $d(U(t,s)\varphi, u^\varepsilon_i U(t,s)\psi)/dt = (U(t,s)\varphi, (u^\varepsilon_{i,j}p_j - iu^\varepsilon_{i,jj}/2)U(t,s)\psi)$ we get

$$u^\varepsilon_i U(t,s)\psi = U(t,s)u^\varepsilon_i\psi + \int_s^t d\tau\ U(t,\tau)(u^\varepsilon_{i,j}p_j - \frac{i}{2}u^\varepsilon_{i,jj})U(\tau,s)\psi \quad ,$$

since by the continuity of the integrand, the integral can be carried inside the scalar product. Furthermore, the integrand is uniformly bounded in ε and τ. As $\varepsilon \searrow 0$, we obtain that $U(t,s)\psi$ is $Q(x^2)$-continuous in (t,s), and

$$x_i U(t,s)\psi = U(t,s)x_i\psi + \int_s^t d\tau\ U(t,\tau)p_i U(\tau,s)\psi \tag{2.4}$$

by dominated convergence and by the closedness of x_i. Now $p_i U(\tau,s)\psi \in Q(p^2)$ by Theorem 2.1 iv), and $U(t,s)x_i\psi$ as well as the integrand in (2.4) are in $Q(p^2)$ by Corollary 2.2 and by $x_i\psi \in Q(p^2)$. Since everything just mentioned is $Q(p^2)$-continuous in the arguments of the propagator, we have $x_i U(t,s)\psi \in Q(p^2)$, $Q(p^2)$-continuously in (t,s). This immediately implies $Q(x^2)$-continuity of $p_i U(t,s)\psi$. Next, (2.4) extends to $Q(p^2+x^2)$, because $D(p^2+x^2)$ is a form core for p^2+x^2 and because of Corollary 2.2. Thus, still assuming $\psi \in D(p^2+x^2)$, (2.4) applies to $x_i\psi$, $p_i U(\tau,s)\psi \in Q(p^2+x^2)$, proving $Q(x^2)$-continuity of the right of (2.4). We conclude that $U(t,s)\psi \in D(x^2)$, with $D(x^2)$-continuous dependence on (t,s). ■

An immediate consequence of the uniqueness of the propagator is the following: let

$$G_l(t) = e^{i\frac{u_l^2}{2}t} e^{ipu_l t} e^{-iu_l x}$$

be the Galilei transform to the asymptotic rest frame of center l, where the total Hamiltonian reads

$$H^l(t) = \frac{1}{2}p^2 + \sum_{k=1}^n V_k\left(x - (x_k(t) - u_l t)\right) \quad .$$

Let $U^l(t,s)$ be the corresponding propagator given by Theorem 2.1:

Corollary 2.4.

$$G_l(t)U(t,s) = U^l(t,s)G_l(s) \quad .$$

Proof: For $\psi \in D(p^2)$

$$i\frac{d}{dt}G_l(t)\psi = -(\frac{u_l^2}{2} + pu_l)G_l(t)\psi$$

$$G_l(t)H(t)\psi = \left(\frac{1}{2}(p+u_l)^2 + \sum_{k=1}^{n} V_k\left(x - (x_k(t) - u_l t)\right)\right) G_l(t)\psi$$

$$\begin{aligned} i\frac{\partial}{\partial t}G_l(t)U(t,s)\psi &= -(\frac{u_l^2}{2} + pu_l)G_l(t)U(t,s)\psi + G_l(t)H(t)U(t,s)\psi \\ &= H^l(t)G_l(t)U(t,s)\psi \end{aligned}$$

Because of $G_l(t)U(t,s)\psi\big|_{t=s} = G_l(s)\psi$ and since $G_l(s)$ maps $D(p^2)$ onto itself, the claim follows from Theorem 2.1. ∎

3. Time Boundedness of the Energy

As a first step to asymptotic completeness we prove boundedness in time of the kinetic energy.

Idea of the proof of Theorem 1.2: The idea is to look at the expectation values of

$$K(t) = \frac{1}{2}(p - \frac{x}{t})^2 + \sum_{l=1}^{n} V_l(x - x_l(t)) \quad .$$

Classically, $K(t)$ will decrease if the particle is far away from the centers, since $(p - x/t)^2$ decreases for the free motion. On the other hand, if x remains close to $x_l(t)$, then $x/t \approx u_l$ and $K(t)$ is essentially the total energy of the corresponding one-center-problem, which is constant. The quantum analogue should be a negative semidefinite expression $i[H(t), K(t)] + \partial K/\partial t$, apart from 'junk' terms which decay integrably in time. Hence time boundedness should hold for $\langle K(t)\rangle$, but this result does not carry over $\langle p^2\rangle$. We will therefore replace the vector field x/t by $v(x,t)$ differing from it mainly by

i) $v(x,t)$ is modified with respect to x/t outside some big ball $\{x \mid |x| < u_0 t\}$, in order to make it bounded. Then p^2 will be relatively bounded with respect to $K(t)$, uniformly in t.

ii) $v(x,t) = u_l$ in an increasingly big neighbourhood of $x = u_l t$, in order to make the intuitive argument really work.

Proof of Theorem 1.2: By (2.3) it suffices to prove (1.4) for s big enough. Moreover, it is enough to prove (1.4) for $\psi \in D(p^2)$, since by the form closedness of p^2 this extends to $Q(p^2)$.

Consider a smooth vector field $v(x,t) : \mathbf{R}^\nu \times [s, +\infty) \to \mathbf{R}^\nu$ and let

$$K(t) = \frac{1}{2}(p - v(x,t))^2 + \sum_{l=1}^{n} V_l(x - x_l(t)) \quad . \tag{3.1}$$

Formally

$$
\begin{aligned}
i[H(t),K(t)]+\frac{\partial K}{\partial t} = & -(p_i-v_i)\frac{v_{i,j}+v_{j,i}}{2}(p_j-v_j) \\
& -\frac{1}{2}\left(p_i\left(v_{i,j}v_j+\frac{\partial v_i}{\partial t}\right)+\left(v_{i,j}v_j+\frac{\partial v_i}{\partial t}\right)p_i\right) \\
& +v_i\left(v_{i,j}v_j+\frac{\partial v_i}{\partial t}\right)+\frac{1}{4}v_{i,ijj}+\sum_{l=1}^{n}(v_i-\dot{x}_{l;i})V_{l,i} \quad , \qquad (3.2)
\end{aligned}
$$

where

- summation over double indices is understood,
- indices following a comma stand for partial derivatives,
- $\dot{x}_{l;i}$ is the i-th component of $\dot{x}_l$.

Provided the vector field $v(x,t)$ and its derivatives in (3.2) are bounded in x and if the last term is relatively bounded with respect to p^2, then (3.2) holds in form sense on $D(p^2)$. Equation (3.2) illustrates what we meant in the heuristic argument: for $v = x/t$ all terms on the right hand side of (3.2), up to the first and the last one, vanish. The first one is negative definite. However, the last one is not integrable in time, but it will become so if modification ii) is taken into account. Then the middle terms no longer vanish but they will be integrable in time.

We will construct a smooth vector field $v(x,t)$, bounded in (x,t), satisfying

$$
\begin{gathered}
v_{i,j}+v_{j,i} \quad \text{is positive semidefinite} \\
\|v_{i,j}v_j+\frac{\partial v_i}{\partial t}\|_\infty+\|v_{i,ijj}\|_\infty \le \text{const}\, t^{-(1+\gamma)} \qquad (3.3) \\
\sum_{l=1}^{n}\|(v_i-\dot{x}_{l;i})V_{l,i}\|_{p,\infty} \le \text{const}\, t^{-(1+\gamma)} \qquad (3.4)
\end{gathered}
$$

for some $\gamma > 0$, where

$$
\|W\|_{p,\infty}=\inf\left\{\|W_1\|_p+\|W_2\|_\infty \;\middle|\; W=W_1+W_2,\ W_1\in L^p(\mathbf{R}^\nu),\ W_2\in L^\infty(\mathbf{R}^\nu)\right\} \quad .
$$

These estimates, if applied to (3.2) together with

$$
\|p\psi\|^2=(\psi,p^2\psi)\le a(\psi,K(t)\psi)+b(\psi,\psi) \qquad (3.5)
$$

with a,b independent of t, prove that for $\psi\in D(p^2)$

$$
\frac{d}{dt}\langle K(t)\rangle_t \le t^{-(1+\gamma)}\langle C_1K(t)+C_2\rangle_t
$$

where $\langle K(t)\rangle_t=(U(t,s)\psi,K(t)U(t,s)\psi)$. By integration time boundedness of $K(t)$ holds in the sense analogous to (1.4). Using (3.5) and $(\psi,K(s)\psi)\le a'(\psi,p^2\psi)+b'(\psi,\psi)$ we obtain (1.4).

Construction of the vector field $v(x,t)$: It is convenient to work first in the scaled coordinates $y = x/t$. We denote the vector field by w, when expressed as a function of y; it is going to depend also on a parameter $\alpha > 0$ (whose dependence on time $\alpha = t^{-\delta}$, $\delta > 0$ as $t \to +\infty$ will be made explicit only later on), which tunes the size of certain regions shown in Figure 1. In case of Assumption (F1) w will also depend explicitely on time. We set

$$w(y,\alpha,t) = w^{(0)}(y,\alpha) + \sum_{l=1}^{n} w^{(l)}(y,\alpha,t) \tag{3.6}$$

where $w^{(0)}$ (resp. $w^{(l)}$) accounts for modification i) (resp. ii)).

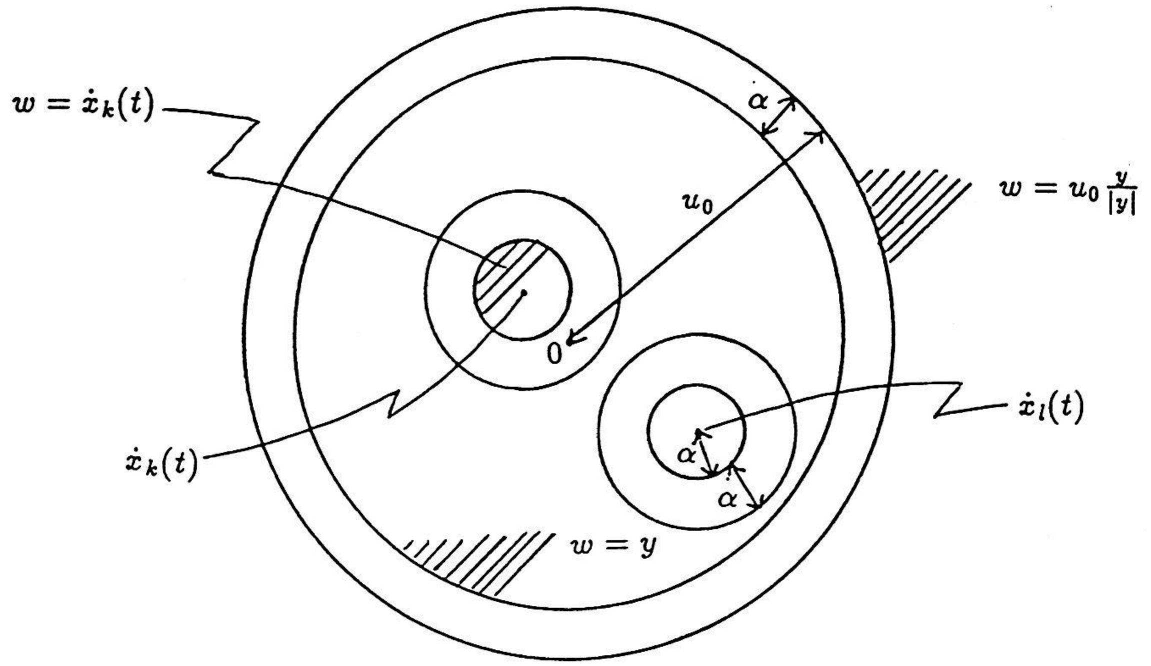

Fig.1 The vector field w in velocity space y. For sake of simplicity only two centers have been drawn.

i) In a first step we build a vector field $w^{(0)}(y,\alpha)$. Let $\varphi \in C^\infty(\mathrm{R})$ with $0 \le \varphi \le 1$, $\varphi' \ge 0$ and

$$\varphi(x) = 0 \quad \text{for } x \le 0, \qquad \varphi(x) = 1 \quad \text{for } x \ge 1 \quad .$$

Let $u_0 = 2\max_{1\le l\le n} |u_l|$ and consider the function

$$\omega(s,\alpha) = s\varphi\left(\frac{u_0 - s}{\alpha}\right) + u_0\left(1 - \varphi\left(\frac{u_0 - s}{\alpha}\right)\right) = -\alpha x\varphi(x)\big|_{x=\frac{u_0-s}{\alpha}} + u_0 \tag{3.7}$$

for $s \ge 0$, $\alpha > 0$ and α small, which is going to be the modulus of $w^{(0)}$. We will check the following bounds:

$$\sup_s |\omega(s,\alpha)| \le \text{const} \qquad (3.8) \qquad\qquad \sup_s \left|(\omega(s,\alpha)-s)\frac{\partial\omega}{\partial s}\right| \le \text{const}\,\alpha \qquad (3.10)$$

$$\omega,\ \frac{\partial\omega}{\partial s} \ge 0 \qquad (3.9) \qquad\qquad \sup_s \left|\frac{\partial^3\omega}{\partial s^3}\right| \le \text{const}\,\alpha^{-2} \qquad (3.11)$$

$$\sup_s \left|\frac{\partial\omega}{\partial\alpha}\right| \le \text{const} \quad , \tag{3.12}$$

where here and elsewhere in this proof, const is independent of all the arguments of the left hand side. The estimate (3.8) follows from (3.7) since $\varphi\left(\frac{u_0-s}{\alpha}\right) = 0$ for $s \ge u_0$. For the same reason, its derivative

$$\frac{\partial\omega}{\partial s} = (x\varphi'(x) + \varphi(x))\big|_{x=\frac{u_0-s}{\alpha}} \tag{3.13}$$

satisfies

$$\operatorname*{supp}_s \frac{\partial\omega}{\partial s} \subset [0, u_0] \quad . \tag{3.14}$$

Moreover, from (3.13) and $x\varphi'(x) \ge 0$ we see that (3.9) and

$$\sup_s \left|\frac{\partial\omega}{\partial s}\right| \le \text{const} \tag{3.15}$$

hold. From $\omega(s,\alpha) - s = (u_0 - s)(1 - \varphi(\frac{u_0-s}{\alpha}))$ we get

$$\sup_{s\le u_0} |\omega(s,\alpha) - s| \le \sup_{u_0-\alpha\le s\le u_0} |u_0 - s| = \alpha$$

since $1 - \varphi\left(\frac{u_0-s}{\alpha}\right) = 0$ for $s \le u_0 - \alpha$. This, together with (3.14), (3.15), proves (3.10). (3.11) follows from (3.13). Last we compute

$$\frac{\partial\omega}{\partial\alpha} = -x\varphi(x)\big|_{x=\frac{u_0-s}{\alpha}} - \alpha(x\varphi'(x) + \varphi(x))\big|_{x=\frac{u_0-s}{\alpha}}\left(-\frac{u_0-s}{\alpha^2}\right) = x^2\varphi'(x)\big|_{x=\frac{u_0-s}{\alpha}} \quad .$$

Since $x^2\varphi'(x)$ has compact support, (3.12) holds. We can now define

$$w^{(0)}(y,\alpha) = \omega(|y|,\alpha)\frac{y}{|y|}$$

and compute

$$w^{(0)}_{i,j} = \frac{\partial\omega}{\partial s}(|y|,\alpha)\frac{y_i y_j}{y^2} + \omega(|y|,\alpha)\frac{\delta_{ij}y^2 - y_i y_j}{|y|^3} \quad .$$

This is symmetric in i, j and positive semidefinite due to (3.9). Moreover, the relations

$$w_{i,j}^{(0)} y_j = \frac{\partial \omega}{\partial s} y_i + \omega \frac{y_i y^2 - y_i y^2}{|y|^3} = \frac{\partial \omega}{\partial s} y_i$$

$$w_{i,j}^{(0)}(w_j^{(0)} - y_j) = w_{i,j}^{(0)}(\omega - |y|)\frac{y_j}{|y|} = (\omega - |y|)\frac{\partial \omega}{\partial s}\frac{y_j}{|y|}$$

$$\begin{aligned} w_{i,i}^{(0)} &= \frac{\partial \omega}{\partial s} + (\nu - 1)\frac{\omega}{|y|} \\ &= \nu \qquad \text{in a neighbourhood of } y = 0 \text{ independent of } \alpha \end{aligned}$$

$$\frac{\partial w^{(0)}}{\partial \alpha} = \frac{\partial \omega}{\partial \alpha}\frac{y}{|y|}$$

carry the bounds (3.8)-(3.12) over to

$$\sup_y |w^{(0)}(y,\alpha)| \leq \text{const} \qquad (3.16)$$

$$\sup_y |w_{i,j}^{(0)}(w_j^{(0)} - y_j)| \leq \text{const}\,\alpha \qquad (3.17)$$

$$\sup_y |w_{i,ijj}^{(0)}| \leq \text{const}\,\alpha^{-2} \qquad (3.18)$$

$$\sup_y |\frac{\partial w^{(0)}}{\partial \alpha}| \leq \text{const} \quad . \qquad (3.19)$$

ii) We first consider case (F1): around the l-th center we now add

$$w^{(l)}(y,\alpha,t) = -(y - \dot{x}_l(t))\varphi\left(2 - \frac{|y - \dot{x}_l(t)|}{\alpha}\right) = -\alpha s \varphi(2 - |s|)\Big|_{s = \frac{y - \dot{x}_l(t)}{\alpha}} \qquad (3.20)$$

according to (3.6). Because of (T1), the sets $K_l = \{y \mid |y - \dot{x}_l(t)| \leq 2\alpha\}$, $l = 1, \ldots, n$ are disjoint and contained in $\{y \mid |y| < u_0 - \alpha\}$, for small α and large t. Since $\operatorname{supp}_y w^{(l)} \subset K_l$ we have

$$w = y + w^{(l)} = y\left(1 - \varphi\left(2 - \frac{|y - \dot{x}_l(t)|}{\alpha}\right)\right) + \dot{x}_l(t)\varphi\left(2 - \frac{|y - \dot{x}_l(t)|}{\alpha}\right)$$

for $y \in K_l$. In particular

$$w = \dot{x}_l(t) \qquad \text{for } |y - \dot{x}_l(t)| \leq \alpha \quad . \qquad (3.21)$$

From

$$w_{i,j} = \left(\delta_{ij}(1 - \varphi(2 - |s|)) + s_i \frac{\varphi'(2 - |s|)}{|s|} s_j\right)\Big|_{s = \frac{y - \dot{x}_l(t)}{\alpha}} \qquad (3.22)$$

we see that $w_{i,j}$ is still symmetric and positive semidefinite, due to $1 - \varphi \geq 0$, $\varphi' \geq 0$. The bounds we will need are

$$\sup_{y \in K_l} |w(y,\alpha,t)| \leq \text{const} \qquad (3.23)$$

$$\sup_{y \in K_l} |w_{i,j}(w_j - y_j)| \leq \text{const}\,\alpha \qquad (3.24)$$

$$\sup_{y \in K_l} |w_{i,ijj}| \leq \text{const}\,\alpha^{-2} \qquad (3.25)$$

$$\sup_{y \in K_l} |\frac{\partial w}{\partial \alpha}| \leq \text{const} \qquad (3.26)$$

$$\sup_{y \in K_l} \left|\frac{\partial w_i}{\partial t}\right| \leq \text{const}\, t^{-(1+\beta)} \quad . \qquad (3.27)$$

Estimate (3.24) follows from $\sup_{y\in K_l}|w_{i,j}| \le \text{const}$, $\sup_{y\in K_l}|w_j - y_j| = \sup_{y\in K_l}|w_j^{(l)}| \le \text{const}\,\alpha$, whereas (3.25) (resp. (3.26)) is immediate from (3.22) (resp. (3.20)). (3.27) holds because of $\partial w_i/\partial t = -w_{i,j}^{(l)}\ddot{x}_{l;j}(t)$ and of (F1). In case (F2) replace $\dot{x}_l(t)$ in (3.20) by u_l. The vector field $v(x,t)$ appearing in (3.1) is defined by

$$v(x,t) = w(\frac{x}{t}, t^{-\delta}, t)$$

where $\delta > 0$ is chosen to satisfy

$$1 + \gamma := \min(1+\delta, 3-2\delta, (1-\delta)(1+\varepsilon), 1+\beta) > 1, \qquad \delta < \beta$$

(see (T1), (F1), (F2) for the definition of β, ε). Most important, $v(x,t)$ is bounded in (x,t) by (3.16), (3.23) and $v_{i,j} + v_{j,i}$ is positive semidefinite.

$$v_{i,j}v_j + \frac{\partial v_i}{\partial t} = \frac{1}{t}w_{i,j}(w_j - y_j) - \delta\frac{\partial w_i}{\partial \alpha}t^{-(1+\delta)} + \frac{\partial w_i}{\partial t} \quad .$$

This has a bound (3.3) by (3.17), (3.19), (3.24), (3.26), (3.27); $v_{i,ijj} = t^{-3}w_{i,ijj}$ shares the same bound by (3.18), (3.25). In case (F1) we have

$$(v_i - \dot{x}_{l;i}(t))V_{l,i}(x - x_l(t)) = 0 \quad \text{for} \quad |x - x_l(t)| \le \frac{1}{2}t^{1-\delta} \tag{3.28}$$

since it follows from (3.21) and

$$\left|\frac{x}{t} - \dot{x}_l(t)\right| \le \frac{1}{t}|x - x_l(t)| + \frac{1}{t}|x_l(t) - t\dot{x}_l(t)| \le \frac{1}{2}t^{-\delta} + \frac{1}{t}\int_0^t dt'|t'\ddot{x}_l(t')|$$
$$\le \frac{1}{2}t^{-\delta} + \frac{1}{t}\text{const}\,t^{1-\beta} \le t^{-\delta}$$

for t large enough. Complementary to (3.28) we have

$$\|F(|x - x_l(t)| \ge \frac{1}{2}t^{1-\delta})(v_i - \dot{x}_{l;i})V_{l,i}\|_{p,\infty} \le \text{const}\,\|F(|x| \ge \frac{1}{2}t^{1-\delta})F(|x| > R)\nabla V_l(x)\|_{p,\infty}$$
$$\le \text{const}\,t^{-(1-\delta)(1+\varepsilon)} \tag{3.29}$$

for $t^{1-\delta} \ge 2R$, proving (3.4). Estimate (3.29) holds also in case (F2), whereas (3.28) fails. We have instead

$$\|F(|x - x_l(t)| \le \frac{1}{2}t^{1-\delta})(v_i - \dot{x}_{l;i})V_{l,i}\|_{p,\infty} \le |\Delta\dot{x}_l(t)|\|\nabla V_l\|_{p,\infty} \le \text{const}\,t^{-(1+\beta)}$$

since for $|x - x_l(t)| \le \frac{1}{2}t^{1-\delta}$, $\quad |\frac{x}{t} - u_l| \le \frac{1}{t}|x - x_l(t)| + \frac{1}{t}|\Delta x_l(t)| \le t^{-\delta}$. ■

There is also a weaker notion of time boundedness of energy:

Corollary 3.1. *For $\psi \in \mathcal{H}$,*

$$\lim_{E\to+\infty}\sup_{t\ge s}\|F(p^2 > E)U(t,s)\psi\| = 0 \quad . \tag{3.30}$$

Proof: It is enough to prove this for $\psi \in Q(p^2)$. Then, by (1.4)

$$\text{const} \ge \sup_{t\ge s}(U(t,s)\psi, p^2U(t,s)\psi) \ge E\sup_{t\ge s}\|F(p^2 > E)U(t,s)\psi\|^2 \quad .$$ ■

4. The Wave Operators

In this section we derive propagation estimates for the free and on the one-center dynamics, from which the existence of the wave operators easily follows. These estimates will be useful once more in Section 7, when proving asymptotic completeness. The standing assumptions of this section are (T1) and (P3).

We will derive the propagation estimates using

$$\|F(|x| > R)V_l(x)(p^2+1)^{-1}\| \leq \text{const}\, R^{-(1+\epsilon)} \quad ,$$

which follows from the part of (P3) concerning the potentials. The assumption on the forces has also been strengthened with respect to (P2), with the only purpose of excluding positive eigenvalues of H_l (see the proof of Theorem 1.1 i)).

Let us start with the free dynamics:

Lemma 4.1. *Let $g \in C_0^\infty(\mathbf{R}^\nu)$ and $v > 0$. Suppose*

i) $g(p) = 0$ for $|p| \geq v$ and fix $\alpha > 1$. Then for $R > 0$, $t \geq 0$ and any $N > 0$

$$\left\|F(|x| > \alpha(R+vt))e^{-i\frac{p^2}{2}t}g(p)F(|x| < R)\right\| \leq C_N(R+vt)^{-N} \quad . \tag{4.1}$$

ii) $g(p) = 0$ for $|p| \leq v$ and fix $v_0 > 0$, $0 < \alpha < 1$. Then for $t > 0$ and any $N > 0$

$$\left\|F(|x| < \alpha(v-v_0)t)e^{-i\frac{p^2}{2}t}g(p)F(|x| < v_0 t)\right\| \leq C_N t^{-N} \quad . \tag{4.2}$$

Since estimates like these are fairly common (e.g.[4], Lemma 6.3), we omit the proof. The next lemma represents to some extent the counterpart of Lemma 4.1 for the one-center dynamics.

Lemma 4.2. *Let $g \in C_0^\infty(\mathbf{R})$ and $v > 0$. Suppose $g(e) = 0$ for $e \geq v^2/2$ and fix $\alpha > 1$. Then for $R > 0$ and $t \geq 0$*

$$\|F(|x| > \alpha(R+vt))e^{-iH_l t}g(H_l)F(|x| < R)\| \leq C(R+vt)^{-\epsilon} \quad . \tag{4.3}$$

Proof: By Lemma 2 in [3] we know that $\|F(|x| > R)(g(H_l) - g(p^2/2))\| \leq \text{const}\, R^{-(1+\epsilon)}$. Thus it is sufficient to estimate $\|F(|x| > \alpha(R+vt))g(p^2/2)e^{-iH_l t}F(|x| < R)\|$. This expression with $e^{-iH_l t}$ replaced by $e^{-i\frac{p^2}{2}t}$ is of order $O((R+vt)^{-N})$ by (4.1). Hence we are left with

$$\|F(|x| > \alpha(R+vt))g(p^2/2)(e^{-iH_l t} - e^{-i\frac{p^2}{2}t})F(|x| < R)\| \quad .$$

Write $\alpha = \alpha_1\alpha_2$ with $\alpha_1, \alpha_2 > 1$, let $f \in C_0^\infty(\mathbf{R}^\nu)$ with $f(y) = 0$ for $|y| \geq 1$, and set $R(t,s) = \alpha_1(R+vt) - vs$. Then for any $N > 0$

$$\sup_{0\leq s\leq t} \|F(|x| > \alpha(R+vt))g(p^2/2)e^{-i\frac{p^2}{2}s}f(x/R(t,s))\| \leq C_N(R+vt)^{-N} \quad . \tag{4.4}$$

This follows from (4.1) because $\alpha_2(R(t,s)+vs) = \alpha(R+vt)$, and $R(t,s) \geq \alpha_1 R > 0$. Using (4.4) with $s = 0, t$ we are allowed to modify once more the quantity to be estimated:

$$\begin{aligned}
&\left\|F(|x| > \alpha(R+vt))g(\frac{p^2}{2})\left((1-f(\frac{x}{R(t,0)}))e^{-iH_lt} - e^{-i\frac{p^2}{2}t}(1-f(\frac{x}{R(t,t)}))\right)F(|x| < R)\right\| \\
&\leq \int_0^t ds \left\|F(|x| > \alpha(R+vt))g(\frac{p^2}{2})e^{-i\frac{p^2}{2}s} \cdot \right. \\
&\qquad \left. \cdot\left[-i\frac{p^2}{2}(1-f(\frac{x}{R(t,s)})) + i(1-f(\frac{x}{R(t,s)}))H_l - \frac{\partial}{\partial s}f(\frac{x}{R(t,s)})\right]\right\| \quad .
\end{aligned} \tag{4.5}$$

The expression in square brackets splits into (a)-(c) below:

(a) $(1 - f(x/R(t,s)))V_l(x) = V_l(x)F(|x| > R(t,s)/2)(1 - f(x/R(t,s)))$

since we may take f to satisfy also $f(y) = 1$ for $|y| \leq 1/2$. Its contribution to the integral in (4.5) is bounded by

$$\begin{aligned}
&\int_0^t ds\ \|g(\frac{p^2}{2})(p^2+1)\|\ \|(p^2+1)^{-1}V_l(x)F(|x| > R(t,s)/2)\| \\
&\qquad\qquad \leq \text{const} \int_0^t ds\ (\alpha_1(R+vt) - vs)^{-(1+\varepsilon)} \\
&\qquad\qquad \leq \text{const}\,(\alpha_1R + (\alpha_1 - 1)vt)^{-\varepsilon} \leq \text{const}\,(R+vt)^{-\varepsilon} \quad .
\end{aligned}$$

(b) $\left[\frac{p^2}{2}, f(x/R(t,s))\right] = (\Delta f)(x/R(t,s)) \cdot \frac{R(t,s)^{-2}}{2} - ip(\nabla f)(x/R(t,s)) \cdot R(t,s)^{-1}$

This gives rise to a contribution $O((R+vt)^{-N})$ to the integrand in (4.5), since (4.4) holds with $g(p^2/2)$ replaced by $g(p^2/2)(p^2+1)$. The corresponding integral is then $O((R+vt)^{-(N-1)})$.

(c) $\frac{\partial}{\partial s}f(x/R(t,s)) = (x/R(t,s))(\nabla f)(x/R(t,s)) \cdot vR(t,s)^{-1}$,

which is treated like (b). ■

Lemma 4.3.

i) Let $0 < v_0 < v$ and $g \in C_0^\infty(\mathbf{R}^\nu)$ with $g(p) = 0$ for $|p - u_l| \leq v$, $l = 1,\dots,n$. Then for any $s \in \mathbf{R}$

$$\lim_{t_1\to+\infty}\sup_{t_2\geq t_1}\left\|\left(U(t_2,t_1) - e^{-i\frac{p^2}{2}(t_2-t_1)}\right)e^{-i\frac{p^2}{2}(t_1-s)}g(p)\prod_{l=1}^n F(|x - u_ls| < v_0(t_1-s))\right\| = 0\ . \tag{4.6}$$

ii) Let $v, v_0 > 0$ with $v+v_0 < \min_{k\neq l}|u_k - u_l|$ and $g \in C_0^\infty(\mathbf{R})$ with $g(e) = 0$ for $e \geq v^2/2$. Then

$$\lim_{t_1\to+\infty}\sup_{t_2\geq t_1}\left\|\left(U^l(t_2,t_1) - e^{-iH_l(t_2-t_1)}\right)g(H_l)F(|x| < v_0t_1)\right\| = 0 \quad . \tag{4.7}$$

Proof: i) Take $\alpha < \alpha_1 < 1$ and let $f \in C_0^\infty(\mathbf{R}^\nu)$ with $f(y) = 0$ if $|y - u_l| \geq \alpha(v - v_0)$ for all $l = 1, \ldots, n$. Then $|f(x/t)| \leq |f(x/t)|\sum_{l=1}^n F(|x - u_lt| < \alpha_1(v - v_0)(t - s))$ for t large enough, since $\alpha t \leq \alpha_1(t - s)$.

$$\begin{aligned}
&\left\|f(x/t)e^{-i\frac{p^2}{2}(t-s)}g(p)\prod_{l=1}^n F(|x - u_ls| < v_0(t - s))\right\| \\
&\quad\leq \|f\|_\infty\sum_{l=1}^n\left\|F(|x - u_lt| < \alpha_1(v - v_0)(t - s))e^{-i\frac{p^2}{2}(t-s)}g(p)F(|x - u_ls| < v_0(t - s))\right\| \\
&\quad\leq \|f\|_\infty\sum_{l=1}^n\left\|F(|x| < \alpha_1(v - v_0)(t - s))e^{-i\frac{p^2}{2}(t-s)}g(p + u_l)F(|x| < v_0(t - s))\right\| \\
&\quad\leq \text{const}\,(t - s)^{-N} \quad .
\end{aligned} \tag{4.8}$$

This follows by applying $G_l(t)$ to the second expression within the bars, by commuting it through, and by (4.2), since $g(p + u_l)$ satisfies its hypothesis. By (4.8) it is enough to estimate

$$\begin{aligned}
&\sup_{t_2\geq t_1}\left\|\left(U(t_2,t_1)(1 - f(x/t_1)) - (1 - f(x/t_2))e^{-i\frac{p^2}{2}(t_1-s)}\right)g(p)\prod_{l=1}^n F(|x - u_ls| < v_0(t_1 - s))\right\| \\
&\leq \int_{t_1}^{+\infty} dt\,\left\|\left[iH(t)(1 - f(x/t)) - i(1 - f(x/t))\frac{p^2}{2} - \frac{\partial}{\partial t}f(x/t)\right]\cdot\right. \\
&\qquad\qquad \left.\cdot e^{-i\frac{p^2}{2}(t-s)}g(p)\prod_{l=1}^n F(|x - u_ls| < v_0(t_1 - s))\right\| \quad .
\end{aligned} \tag{4.9}$$

The expression within square brackets consists of (a)-(c) below:

(a) $V_l(x - x_l(t))(1 - f(x/t)) = (1 - f(x/t))F(|x - u_lt| > \alpha(v - v_0)t/2)V_l(x - x_l(t))$

for $l = 1, \ldots, n$, where we took f to satisfy also $f(y) = 1$ if $|y - u_l| \leq \alpha(v - v_0)/2$. Its contribution to the integrand in (4.9) is bounded by

$$\|F(|x| > \alpha(v - v_0)t/2)V_l(x - \Delta x_l(t))(p^2 + 1)^{-1}\|\|(p^2 + 1)g(p)\| \leq \text{const}\,t^{-(1+\varepsilon)} \quad .$$

(b) $\frac{p^2}{2}f(x/t) - f(x/t)\frac{p^2}{2} = -\frac{t^{-2}}{2}(\Delta f)(x/t) - it^{-1}(\nabla f)(x/t)p$

and

(c) $\frac{\partial}{\partial t}f(x/t) = -t^{-1}(x/t)(\nabla f)(x/t)$

are treated using (4.8).

ii) Choose $\alpha > 1$ and v_1 with $\alpha(v + v_0) < v_1 < \min_{k\neq l} |u_k - u_l|$ and let $f \in C_0^\infty(\mathbf{R}^\nu)$ with $f(y) = 1$ for $|y| \leq \alpha(v + v_0)$ and $f(y) = 0$ for $|y| > v_1$. We first claim that

$$\lim_{t_1\to+\infty} \sup_{t\geq t_1} \left\| \left(1 - e^{-ip\Delta x_l(t)} f(x/t)\right) e^{-iH_l(t-t_1)} g(H_l) F(|x| < v_0 t_1) \right\| = 0 \quad . \tag{4.10}$$

To show this, we write

$$1 - e^{-ip\Delta x_l(t)} f(x/t) = e^{-ip\Delta x_l(t)}(1 - f(x/t)) + (1 - e^{-ip\Delta x_l(t)}) \quad . \tag{4.11}$$

The first term on the right gives rise to a vanishing contribution to (4.10), since $1 - f(x/t)$ is supported in $|x| \geq \alpha(v + v_0)t \geq \alpha(v_0 t_1 + v(t - t_1))$ and

$$\left\|F(|x|>\alpha(v_0t_1+v(t-t_1))e^{-iH_l(t-t_1)}g(H_l)F(|x|<v_0t_1)\right\| \leq \text{const}\,(v_0t_1+v(t-t_1))^{-\varepsilon} \ , \tag{4.12}$$

which follows from (4.3). The contribution related to the second term in (4.11) is bounded by

$$\begin{aligned} \|(1 - e^{-ip\Delta x_l(t)})g(H_l)\| &\leq \|(1 - e^{-ip\Delta x_l(t)})(p^2 + 1)^{-1}\| \|(p^2 + 1)g(H_l)\| \\ &\leq \text{const}\, |\Delta x_l(t)| \| |p|(p^2 + 1)^{-1}\| \xrightarrow[t\to+\infty]{} 0 \quad . \end{aligned}$$

Using (4.10) with $t = t_1$, t_2, the task is now to estimate

$$\begin{aligned} &\sup_{t_2\geq t_1} \left\| \left(U^l(t_2, t_1) e^{-ip\Delta x_l(t_1)} f(x/t_1) - e^{-ip\Delta x_l(t_2)} f(x/t_2) e^{-iH_l(t_2-t_1)}\right) g(H_l) F(|x| < v_0 t_1) \right\| \\ &\leq \int_{t_1}^{+\infty} dt \, \Big\| \Big[iH^l(t) e^{-ip\Delta x_l(t)} f(x/t) - i e^{-ip\Delta x_l(t)} f(x/t) H_l + \frac{\partial}{\partial t}\left(e^{-ip\Delta x_l(t)} f(x/t)\right)\Big] \cdot \\ &\qquad \cdot e^{-iH_l(t-t_1)} g(H_l) F(|x| < v_0 t_1) \Big\| \quad , \end{aligned} \tag{4.13}$$

where the derivative $\partial(e^{-ip\Delta x_l(t)} f(x/t))/\partial t$ is meant in the strong sense and exists on $D(p^2)$. As above, a discussion of terms (a)-(d) now follows:

(a) $V_l(x - \Delta x_l(t)) e^{-ip\Delta x_l(t)} f(x/t) - e^{-ip\Delta x_l(t)} f(x/t) V_l(x) = 0$.

(b) $V_k(x - (x_k(t) - u_l t)) e^{-ip\Delta x_l(t)} f(x/t) = e^{-ip\Delta x_l(t)} f(x/t) V_k(x - (x_k(t) - x_l(t)))$

for $k \neq l$. Notice that $f(x/t) = f(x/t)F(|x - (x_k(t) - x_l(t))| > (v_2 - v_1)t)$ for any v_2 with $v_1 < v_2 < \min_{k\neq l} |u_k - u_l|$, and t large enough. The contribution of (b) to the integrand in (4.13) is bounded by a constant times

$$\|F(|x| > (v_2 - v_1)t) V_k(x)(p^2 + 1)^{-1}\| \|(p^2 + 1) g(H_l)\| \leq \text{const}\, t^{-(1+\varepsilon)} \quad .$$

(c) $\left[\frac{p^2}{2}, e^{-ip\Delta x_l(t)}f(x/t)\right] = e^{-ip\Delta x_l(t)}\left(\frac{t^{-2}}{2}(\Delta f)(x/t) - it^{-1}p(\nabla f)(x/t)\right)$

The first term is integrable by itself, while the contribution related to the second one is bounded by

$$
\begin{aligned}
&t^{-1}\|p(\nabla f)(x/t)e^{-iH_l(t-t_1)}g(H_l)F(|x| < v_0 t_1)\| \\
&\le t^{-1}\|p(H_l+i)^{-1}(\nabla f)(x/t)e^{-iH_l(t-t_1)}(H_l+i)g(H_l)F(|x| < v_0 t_1)\| + \mathrm{O}(t^{-2}) \quad , \qquad (4.14)
\end{aligned}
$$

since $[(\nabla f)(x/t), (H_l+i)^{-1}] = (H_l+i)^{-1}[p^2, (\nabla f)(x/t)](H_l+i)^{-1} = (H_l+i)^{-1}\mathrm{O}(t^{-1})$, where $\mathrm{O}(t^{-1})$ is meant in norm sense. Then (4.14) is integrable, due to $\|p(H_l+i)^{-1}\| < \infty$, to (4.12) (with $g(H_l)$ replaced by $(H_l+i)g(H_l)$), and to the support property of $(\nabla f)(x/t)$.

(d) $\frac{\partial}{\partial t}(e^{-ip\Delta x_l(t)}f(x/t))$

$$= e^{-ip\Delta x_l(t)}\left(f(x/t)(-i\Delta \dot{x}_l(t)p) - (\nabla f)(x/t)\Delta \dot{x}_l(t)t^{-1} + (x/t)(\nabla f)(x/t)t^{-1}\right)$$

The terms which contain $\Delta \dot{x}_l(t)$ are integrable, since $\Delta \dot{x}_l(t)$ is and $\|pg(H_l)\| < \infty$; the one which does not can again be treated by (4.12). ■

Proof of Theorem 1.1 i): It is enough to prove the existence of the strong limit $\Omega_0^-(s)$ on a dense set D: set

$$D = \left\{g(p)f(x)\psi \mid g \in C_0^\infty(\mathbf{R}^\nu \setminus \{u_1, \ldots, u_n\}),\ f \in C_0^\infty(\mathbf{R}^\nu),\ \psi \in \mathcal{H}\right\} \quad .$$

$g(p)$ satisfies the hypothesis of Lemma 4.3 i) with a suitable $v > 0$. Take $0 < v_0 < v$ and note that

$$\left(\prod_{l=1}^{n} F(|x - u_l s| < v_0(t_1 - s))\right) f(x) = f(x)$$

for t_1 big enough. For $t_2 \ge t_1$ we estimate

$$
\begin{aligned}
&\left\|U(s,t_1)e^{-i\frac{p^2}{2}(t_1-s)}g(p)f(x)\psi - U(s,t_2)e^{-i\frac{p^2}{2}(t_2-s)}g(p)f(x)\psi\right\| \\
&\le \left\|\left(U(t_2,t_1) - e^{-i\frac{p^2}{2}(t_2-t_1)}\right)e^{-i\frac{p^2}{2}(t_1-s)}g(p)\prod_{l=1}^{n}F(|x-u_l s| < v_0(t_1-s))\right\| \|f(x)\psi\| \quad .
\end{aligned}
$$

(4.6) now tells us that $U(s,t)e^{-i\frac{p^2}{2}(t-s)}g(p)f(x)\psi$ is Cauchy as $t \to +\infty$.

Now we consider $\Omega_l^-(s)$, $l = 1, \ldots, n$. Because of

$$U(s,t)G_l(t)^{-1}e^{-iH_l(t-s)}P_l^{pp}G_l(s) = G_l(s)^{-1}U^l(s,t)e^{-iH_l(t-s)}P_l^{pp}G_l(s)$$

it suffices to show the existence of

$$\mathrm{s-}\lim_{t\to+\infty} U^l(s,t)e^{-iH_l(t-s)}P_l^{pp} =: \Omega^{l-}(s) \quad . \qquad (4.15)$$

In fact it is enough to prove convergence on eigenstates $H_l\psi = E\psi$, since their finite linear combinations are dense in $\operatorname{Ran} P_l^{pp}$. Due to our assumptions on the potentials, positive eigenvalues are excluded ([7], Corollary 1.4). Thus for any $v > 0$ we can find a g as in Lemma 4.3 ii) with $g(H_l)P_l^{pp} = P_l^{pp}$. To be precise, take $v, v_0 > 0$ with $v + v_0 < \min_{k\neq l}|u_k - u_l|$. For $t_2 \geq t_1$ we can now estimate

$$\begin{aligned}
&\left\|U^l(s,t_1)e^{-iH_l(t_1-s)}P_l^{pp}\psi - U^l(s,t_2)e^{-iH_l(t_2-s)}P_l^{pp}\psi\right\| \\
&\qquad = \left\|U^l(s,t_2)\left(U^l(t_2,t_1) - e^{-iH_l(t_2-t_1)}\right)e^{-iE(t_1-s)}P_l^{pp}\psi\right\| \\
&\qquad \leq \left\|\left(U^l(t_2,t_1) - e^{-iH_l(t_2-t_1)}\right)g(H_l)F(|x| < v_0t_1)\right\| \|\psi\| \\
&\qquad\quad + 2\|g\|_\infty \|F(|x| > v_0t_1)\psi\| \quad .
\end{aligned} \tag{4.16}$$

(4.7), together with the fact that the last term above vanishes as $t_1 \to +\infty$, proves that $U^l(s,t)e^{-iH_l(t-s)}P_l^{pp}\psi$ is Cauchy. ■

The proof above has a

Corollary 4.4.

$$\text{s-}\lim_{s\to+\infty}\left(\Omega^{l-}(s) - 1\right)P_l^{pp} = 0 \quad . \tag{4.17}$$

Proof: Put $t_1 = s$ in (4.16), take the the supremum over $t_2 \geq s$ on the right and the limit $t_2 \to +\infty$ on the left. ■

Proof of Theorem 1.1 ii): Because the wave operators are partial isometries, they have closed ranges. Let $\varphi_0 = \Omega_0^-(s)\psi_0$, $\varphi_l = \Omega_l^-(s)\psi_l$, $l = 1, \ldots, n$ i.e.

$$\left\|U(t,s)\varphi_0 - e^{-i\frac{p^2}{2}(t-s)}\psi_0\right\| \xrightarrow[t\to+\infty]{} 0 \quad ,$$

$$\left\|U(t,s)\varphi_l - G_l(t)^{-1}e^{-iH_l(t-s)}P_l^{pp}G_l(s)\psi_l\right\| \xrightarrow[t\to+\infty]{} 0 \quad .$$

As in the proof of part i), it is enough to consider the case where $G_l(s)\psi_l$ is an eigenstate of H_l : $H_lG_l(s)\psi_l = E_lG_l(s)\psi_l$. Then

$$\begin{aligned}
(\varphi_l, \varphi_0) &= \lim_{t\to+\infty}(U(t,s)\varphi_l, U(t,s)\varphi_0) \\
&= \lim_{t\to+\infty}\left(G_l(t)^{-1}e^{-iH_l(t-s)}P_l^{pp}G_l(s)\psi_l, e^{-i\frac{p^2}{2}(t-s)}\psi_0\right) \\
&= \lim_{t\to+\infty} e^{iE_l(t-s)}\left(G_l(s)\psi_l, e^{-i\frac{p^2}{2}(t-s)}G_l(s)\psi_0\right) \\
&= 0 \quad ,
\end{aligned}$$

since $e^{-i\frac{p^2}{2}t} \xrightarrow[t\to+\infty]{w} 0$. Similarly, for $l \neq k$

$$\begin{aligned}
(\varphi_l, \varphi_k) &= \lim_{t\to+\infty}(U(t,s)\varphi_l, U(t,s)\varphi_k) \\
&= \lim_{t\to+\infty}\left(G_l(t)^{-1}e^{-iH_l(t-s)}P_l^{pp}G_l(s)\psi_l, G_k(t)^{-1}e^{-iH_k(t-s)}P_k^{pp}G_k(s)\psi_k\right) \\
&= \lim_{t\to+\infty} e^{-i(E_k-E_l)(t-s)}\left(G_l(s)\psi_l, G_l(t)G_k(t)^{-1}G_k(s)\psi_k\right) \\
&= 0 \quad ,
\end{aligned}$$

because

$$G_l(t)G_k(t)^{-1} = e^{i\frac{(u_l-u_k)^2}{2}t}e^{ip(u_l-u_k)t}e^{-i(u_l-u_k)x} \xrightarrow[t\to+\infty]{w} 0$$

for $u_l - u_k \neq 0$. ■

5. The RAGE Theorem

The purpose of this section is to give a dynamical characterization of the states ψ orthogonal to the ones which are asymptotically bound to center l, i.e. $\psi \perp \operatorname{Ran}\Omega_l^-(s)$, or $\psi \perp \operatorname{Ran}\Omega^{l-}(s)$ if one is referring, as we will do, to the asymptotic rest frame of center l. We denote by $P_l^{pp}(s)$ the orthogonal projection onto $\operatorname{Ran}\Omega^{l-}(s)$.

Theorem 5.1. *Assume (T1), (P3) and let C be a compact operator. Then for any $\psi \in \mathcal{H}$*

$$\lim_{T\to+\infty}\frac{1}{T}\int_s^{s+T} dt\ \|CU^l(t,s)(1-P_l^{pp}(s))\psi\| = 0 \quad . \tag{5.1}$$

More useful in scattering theory is the following

Corollary 5.2. *Let C be a bounded operator, relatively compact with respect to p^2. Then (5.1) still holds.*

Proof: Given $\varepsilon > 0$, there exists by (3.30) an E such that for $t \geq s$

$$\|F(p^2 > E)U^l(t,s)(1-P_l^{pp}(s))\psi\| \leq \varepsilon \quad .$$

Hence

$$\begin{aligned}\limsup_{T\to+\infty}\frac{1}{T}\int_s^{s+T} dt\ &\|CU^l(t,s)(1-P_l^{pp}(s))\psi\| \\ &\leq \limsup_{T\to+\infty}\frac{1}{T}\int_s^{s+T} dt\ \|CF(p^2\leq E)U^l(t,s)(1-P_l^{pp}(s))\psi\| \\ &\quad + \limsup_{T\to+\infty}\frac{1}{T}\int_s^{s+T} dt\ \|CF(p^2> E)U^l(t,s)(1-P_l^{pp}(s))\psi\| \\ &\leq \|C\|\varepsilon\end{aligned}$$

since $CF(p^2 \leq E)$ is compact. ■

Taking $C = F(|x| < R)$, (5.1) tells us that a state $\psi \perp \operatorname{Ran}\Omega^{l-}(s)$ will leave the ball of radius R in time mean.

As we will see, it would be useful for the proof of Theorem 5.1 if we knew that $P_l^{pp}(t) \xrightarrow[t\to+\infty]{s} P_l^{pp}$. Unfortunately we are not able to prove this, since it does not follow from (4.17). However the weaker Lemma 5.3 also does the job:

Lemma 5.3. *Take an orthonormal basis in* $\operatorname{Ran} P_l^{pp}$ *of eigenstates of* H_l *and let* P_n *be the orthogonal projection on the first* n *of them; denote by* $P_n(s)$ *the orthogonal projection onto* $\operatorname{Ran} \Omega^{l-}(s) P_n$. *Then*

$$P_n \xrightarrow[n \to +\infty]{s} P_l^{pp} \tag{5.2}$$

$$P_n(s) \xrightarrow[n \to +\infty]{s} P_l^{pp}(s) \tag{5.3}$$

$$U^l(t,s) P_n(s) = P_n(t) U^l(t,s) \tag{5.4}$$

$$\|P_n(t) - P_n\| \xrightarrow[t \to +\infty]{} 0 \quad . \tag{5.5}$$

Proof: (5.2) is quite evident.
Since

$$\operatorname{Ran} P_n(s) = \operatorname{Ran} \Omega^{l-}(s) P_n \subset \operatorname{Ran} \Omega^{l-}(s) = \operatorname{Ran} P_l^{pp}(s)$$

we have $P_n(s) P_l^{pp}(s) = P_n(s)$ and therefore

$$P_n(s) - P_l^{pp}(s) = (P_n(s) - 1) P_l^{pp}(s) \quad . \tag{5.6}$$

Given $\varphi \in \mathcal{H}$, there exists by definition of $P_l^{pp}(s)$ a $\psi \in \mathcal{H}$ with $P_l^{pp}(s)\varphi = \Omega^{l-}(s) P_l^{pp} \psi$. Then

$$\begin{aligned} P_l^{pp}(s)\varphi - \Omega^{l-}(s) P_n \psi &= P_l^{pp}(s)\varphi - \Omega^{l-}(s)\left(P_l^{pp}\psi + (P_n - P_l^{pp})\psi\right) \\ &= \Omega^{l-}(s)(P_l^{pp} - P_n)\psi \xrightarrow[n \to +\infty]{} 0 \end{aligned} \tag{5.7}$$

by (5.2). The left hand side is an orthogonal sum

$$P_l^{pp}(s)\varphi - \Omega^{l-}(s) P_n \psi = (1 - P_n(s)) P_l^{pp}(s)\varphi + P_n(s)(P_l^{pp}(s)\varphi - \Omega^{l-}(s) P_n \psi)$$

showing, together with (5.6),(5.7) that $(P_n(s) - P_l^{pp}(s))\varphi \xrightarrow[n \to +\infty]{} 0$.
Given $\varphi \in \mathcal{H}$, there exists by definition of $P_n(s)$ a $\psi \in \mathcal{H}$ with

$$P_n(s)\varphi = \Omega^{l-}(s) P_n \psi \tag{5.8}$$

From the intertwining property $U^l(t,s)\Omega^{l-}(s) = \Omega^{l-}(t) e^{-iH_l(t-s)}$ which follows from the very definition (4.15) of $\Omega^{l-}(s)$, we get

$$U^l(t,s) P_n(s)\varphi = \Omega^{l-}(t) P_n e^{-iH_l(t-s)}\psi \in \operatorname{Ran} P_n(t) \quad ,$$

i.e. $U^l(t,s) P_n(s) = P_n(t) U^l(t,s) P_n(s)$. By taking adjoints and interchanging t and s we also get $P_n(t) U^l(t,s) = P_n(t) U^l(t,s) P_n(s)$, proving (5.4).

$$P_n(t) - P_n = (1 - P_n)P_n(t) - P_n(1 - P_n(t)) \quad .$$

We begin by discussing the first term on the right hand side. Taking ψ as in (5.8) (with t instead of s), we have $\|P_n(t)\varphi\| = \|P_n\psi\|$, since $\Omega^{l-}(t)$ is an isometry on $\operatorname{Ran} P_l^{pp}$. Now

$$(1 - P_n)P_n(t)\varphi = (1 - P_n)(P_n(t)\varphi - P_n\psi) = (1 - P_n)(\Omega^{l-}(t) - 1)P_n\psi$$

and hence $\|(1 - P_n)P_n(t)\| \le \|(\Omega^{l-}(t) - 1)P_n\|\|P_n(t)\|$, which vanishes by (4.17) and by the fact that P_n is of finite rank. The second term is just the adjoint of

$$(1 - P_n(t))P_n = (1 - P_n(t))(1 - \Omega^{l-}(t))P_n$$

and vanishes for the same reason. ■

Lemma 5.4. *For any $\varphi \in P_l^{cont}\mathcal{H} = (1 - P_l^{pp})\mathcal{H}$ and any $\psi \in \mathcal{H}$:*

$$\lim_{T \to +\infty} \frac{1}{T} \int_s^{s+T} dt \; |(\varphi, U^l(t,s)\psi)| = 0 \quad . \tag{5.9}$$

Proof: Let us put $s = 0$ for simplicity.
Step 1: For $\varphi \in P_l^{cont}\mathcal{H}$

$$\frac{1}{\tau} \int_0^\tau dt \; |(e^{iH_l t}\varphi, \psi)| \le c(\tau)\|\psi\| \quad ,$$

where $c(\tau) \to 0$ as $\tau \to +\infty$.
By the Schwarz inequality it suffices to prove

$$\frac{1}{\tau} \int_0^\tau dt \; |(e^{iH_l t}\varphi, \psi)|^2 \le c(\tau)^2\|\psi\|^2 \quad .$$

The left hand side is equal to

$$\frac{1}{\tau} \int_0^\tau dt \; (\psi, e^{iH_l t}\varphi)(e^{iH_l t}\varphi, \psi) = \left(\psi, \frac{1}{\tau} \int_0^\tau dt \; e^{iH_l t} K P_l^{cont} e^{-iH_l t}\psi\right)$$

where $K = (\varphi, \cdot)\varphi = KP_l^{cont}$ is compact. But

$$\|\frac{1}{\tau} \int_0^\tau dt \; e^{iH_l t} K P_l^{cont} e^{-iH_l t}\| \xrightarrow[\tau \to +\infty]{} 0$$

by the usual RAGE Theorem in the form e.g. of [4], Lemma 4.2.
Step 2: For any $\tau > 0$

$$U^l(T, T + t) \xrightarrow[T \to +\infty]{s} e^{iH_l t} \tag{5.10}$$

uniformly in $0 \le t \le \tau$.

It is enough to prove this on $D(p^2)$. It is easy to check that

$$\sum_{k=1}^{n} V_k(x-(x_k(t)-u_l t)) = \sum_{k=1}^{n} V_k(x-(u_k-u_l)t-\Delta x_k(t)) \xrightarrow[t\to+\infty]{s} V_l(x) \tag{5.11}$$

as bounded operators from $D(p^2)$ to $\mathcal{H}$. Moreover, for any $\varphi \in D(p^2)$ the set $\{e^{iH_l t}\varphi \mid 0 \leq t \leq \tau\}$ is compact in $D(p^2)$, hence the convergence (5.11) is uniform on this set. Then (5.10) holds since

$$(U^l(T,T+t)-e^{iH_l t})\varphi = i\int_T^{T+t} dt'\ U^l(T,t')\Big(\sum_{k=1}^{n} V_k(x-(x_k(t')-u_l t'))-V_l(x)\Big)e^{iH_l(T+t-t')}\varphi \quad .$$

Step 3: For any $\varphi \in P_l^{cont}\mathcal{H}$, $\psi \in \mathcal{H}$ and $\varepsilon > 0$ there are a $\tau > 0$ and a $T_0 > 0$ such that for $T \geq T_0$

$$\frac{1}{\tau}\int_T^{T+\tau} dt\ |(\varphi, U^l(t,0)\psi)| \leq \varepsilon \quad . \tag{5.12}$$

This follows from

$$\begin{aligned}
\frac{1}{\tau}\int_T^{T+\tau} dt\ |(\varphi, U^l(t,0)\psi)| &= \frac{1}{\tau}\int_T^{T+\tau} dt\ |(U^l(T,t)\varphi, U^l(T,0)\psi)| \\
&= \frac{1}{\tau}\int_0^{\tau} dt\ |(U^l(T,T+t)\varphi, U^l(T,0)\psi)| \\
&\leq \frac{1}{\tau}\int_0^{\tau} dt\ |(e^{iH_l t}\varphi, U^l(T,0)\psi)| + \frac{1}{\tau}\int_0^{\tau} dt\ \big|\big((U^l(T,T+t)-e^{iH_l t})\varphi, U^l(T,0)\psi\big)\big| \\
&\leq \Big(c(\tau) + \sup_{0\leq t\leq \tau} \|(U^l(T,T+t)-e^{iH_l t})\varphi\|\Big)\|\psi\|
\end{aligned}$$

by the unitarity of $U^l(T,0)$. Choose first τ by Step 1 and then T_0 by Step 2 big enough such that both terms become smaller than $\varepsilon/2$ for $T \geq T_0$.

In order to prove the lemma we set $f(t) = |(\varphi, U^l(t,0)\psi)|$ and write

$$\begin{aligned}
\frac{1}{T}\int_0^T dt\ f(t) &= \frac{1}{T}\left(\int_0^{T_0} dt\ f(t) + \sum_{k=0}^{[\frac{T-T_0}{\tau}]-1}\int_{T_0+k\tau}^{T_0+(k+1)\tau} dt\ f(t) + \int_{T_0+[\frac{T-T_0}{\tau}]\tau}^{T} dt\ f(t)\right) \\
&\leq \frac{1}{T}\left((T_0+\tau)\|\varphi\|\|\psi\| + \varepsilon\tau\frac{T-T_0}{\tau}\right)
\end{aligned}$$

by (5.12), where $[\cdot]$ denotes the integer part. Thus

$$\limsup_{T\to+\infty}\frac{1}{T}\int_0^T dt\ |(\varphi, U^l(t,0)\psi)| \leq \varepsilon \qquad \blacksquare$$

Proof of Theorem 5.1: By the usual argument it is enough to consider the case where C is a rank one operator, i.e. $C = (\varphi, \cdot)\eta$, thus reducing (5.1) to

$$\lim_{T\to+\infty} \frac{1}{T}\int_s^{s+T} dt \; \left|(\varphi, U^l(t,s)(1-P_l^{pp}(s))\psi)\right| = 0 \quad . \tag{5.13}$$

Now take a sequence P_n as in Lemma 5.3; given $\varepsilon > 0$ choose n such that

$$\|\varphi\|\|(P_n(s) - P_l^{pp}(s))\psi\| < \varepsilon \qquad \text{and} \qquad \|\psi\|\|(P_n - P_l^{pp})\varphi\| < \varepsilon \quad ,$$

which is possible by (5.2), (5.3). Hence making also use of (5.4)

$$\begin{aligned}
|(\varphi, U^l(t,s)(1-P_l^{pp}(s))\psi)| &\le |(\varphi, U^l(t,s)(1-P_n(s))\psi)| + \varepsilon \\
&\le |((1-P_n(t))\varphi, U^l(t,s)\psi)| + \varepsilon \\
&\le |((1-P_n)\varphi, U^l(t,s)\psi)| + \|P_n(t) - P_n\|\|\varphi\|\|\psi\| + \varepsilon \\
&\le |((1-P_l^{pp})\varphi, U^l(t,s)\psi)| + \varepsilon + \|P_n(t) - P_n\|\|\varphi\|\|\psi\| + \varepsilon \quad .
\end{aligned}$$

(5.13) now follows by virtue of (5.5) and (5.9). ∎

6. Asymptotics of Observables

In this section we are interested in scattering states

$$\begin{aligned}
\mathcal{H}_{scatt}^-(s) &= \{\psi \in \mathcal{H} \mid \psi \perp \operatorname{Ran} \Omega_l^-(s),\ l = 1, \ldots, n\} \\
&= \{\psi \in \mathcal{H} \mid P_l^{pp}(s) G_l(s)\psi = 0,\ l = 1, \ldots, n\}
\end{aligned}$$

(the second equality follows from $G_l(s)\Omega_l^-(s) = \Omega^{l-}(s)G_l(s)$) and in the asymptotic behaviour of certain observables along scattering trajectories. As a result we shall see that, roughly speaking, x/t tends to p as for a free particle.
We consider once more

$$K(t) = \frac{1}{2}(p - \frac{x}{t})^2 + \sum_{l=1}^{n} V_l(x - x_l(t)) \quad .$$

Theorem 6.1. *Assume (T1), (P1) and let $\psi \in \mathcal{H}_{scatt}^-(s)$. Then*

$$U(s,t) f(K(t)) U(t,s)\psi \xrightarrow[t\to+\infty]{} f(0)\psi \tag{6.1}$$

for any bounded continuous function f on $\mathbf{R}$.

We first pay attention to the 'free' part of $K(t)$:

Lemma 6.2.

i) $\frac{1}{2}(p-\frac{x}{t})^2$*, defined on* $\mathcal{S}(\mathbf{R}^\nu)$ *for* $t \neq 0$*, is essentially selfadjoint; its closure, denoted by* $K_0(t)$*, satisfies*

$$K_0(t) = e^{-i\frac{x^2}{2}t}\frac{1}{2}\left(\frac{x}{t}\right)^2 e^{i\frac{x^2}{2}t} \quad . \tag{6.2}$$

ii) $D(p^2+x^2)$ *is a core for* $K_0(t)$*. For* $\psi \in D(p^2+x^2)$

$$K_0(t)\psi = \left(p^2 - \frac{px+xp}{t} + \frac{x^2}{t^2}\right)\psi \quad . \tag{6.3}$$

iii) $K_0(t)$ *is Galilei invariant:*

$$G(t)K_0(t) = K_0(t)G(t) \tag{6.4}$$

for $G(t) = e^{-i\frac{u^2}{2}t}e^{iput}e^{-iux}$.

Proof: i) (6.2) holds on $e^{-i\frac{x^2}{2}t}\mathcal{S}(\mathbf{R}^\nu) = \mathcal{S}(\mathbf{R}^\nu)$. Hence it has a selfadjoint closure satisfying (6.2).
ii) (6.3) also holds on $\mathcal{S}(\mathbf{R}^\nu)$, together with $\|(px+xp)\psi\| \leq \text{const}\,\|(p^2+x^2)\psi\|$, $\|(p-\frac{x}{t})^2\psi\| \leq \text{const}\,\|(p^2+x^2)\psi\|$. Hence (6.3) follows by taking closures.
iii) It is enough to verify (6.4) on $\mathcal{S}(\mathbf{R}^\nu)$, where it is evident. ■

Lemma 6.3. *For* $\psi \in D(p^2+x^2)$

$$t^2U(s,t)K(t)U(t,s)\psi = s^2K(s)\psi + \sum_{l=1}^{n}\int_s^t d\tau\; \tau G_l(s)^{-1}U^l(s,\tau)W_l(\tau)U^l(\tau,s)G_l(s)\psi \tag{6.5}$$

where $W_l(\tau) = 2V_l(x-\Delta x_l(\tau)) + (x-\tau\Delta\dot{x}_l(\tau))\nabla V_l(x-\Delta x_l(\tau))$.

Proof: Let $\psi_t = U(t,s)\psi$, $\varphi_t = U(t,s)\varphi$ with ψ, $\varphi \in D(p^2+x^2)$. Then $\psi_t, \phi_t \in D(p^2+x^2) \subset D(K(t))$ follows from Corollary 2.3 and from (6.3). The computation which yielded (3.2) is now seen to hold for $v = x/t$ in form sense on $D(p^2+x^2)$. We only remark that its starting point

$$\begin{aligned}&(\varphi_{t+\Delta t}, K_0(t+\Delta t)\psi_{t+\Delta t}) - (\varphi_t, K_0(t)\psi_t) = \\ &= (\varphi_{t+\Delta t},(K_0(t+\Delta t)-K_0(t))\psi_t)+(K_0(t+\Delta t)\varphi_{t+\Delta t},\psi_{t+\Delta t}-\psi_t)+(\varphi_{t+\Delta t}-\varphi_t,K_0(t)\psi_t)\end{aligned}$$

calls for the continuity of $K_0(t)\varphi_t$, which follows from the propositions mentioned above. Hence, from (3.2)

$$\begin{aligned}\frac{d}{dt}(\varphi_t, K(t)\psi_t) &= -\frac{2}{t}(\varphi_t, K_0(t)\psi_t) + \sum_{l=1}^{n}(\varphi_t, (\frac{x}{t}-\dot{x}_l(t))\nabla V_l(x-x_l(t))\psi_t) \\ &= -\frac{2}{t}(\varphi_t, K(t)\psi_t) + \frac{1}{t}\sum_{l=1}^{n}(\varphi_t, (2V_l(x-x_l(t)) + (x-t\dot{x}_l(t))\nabla V_l(x-x_l(t)))\psi_t) \quad .\end{aligned}$$

Alternatively, this can be written as

$$\begin{aligned}\frac{d}{dt}(\varphi_t, t^2K(t)\psi_t) &= t^2\frac{d}{dt}(\varphi_t, K(t)\psi_t) + 2t(\varphi_t, K(t)\psi_t)\\ &= t\sum_{l=1}^{n}(\varphi_t, (2V_l(x-x_l(t)) + (x - t\dot{x}_l(t))\nabla V_l(x - x_l(t)))\psi_t) \quad .\end{aligned}$$

Due to (P1), $W_l(\tau)$ is a continuous function of $\tau \in \mathbf{R}$ to the bounded operators from $D(p^2)$ to $\mathcal{H}$. Then, integration gives the weak form of (6.5) up to a trivial rearrangement by means of Galilei transforms, since by the continuity of the integrand in (6.5) the integration can be carried inside the scalar product. (6.5) now follows, since φ was arbitrary in a dense set. ■

Proof of Theorem 6.1: In order to shorten notation write $\tilde{K}(t) = U(s,t)K(t)U(t,s)$, so that (6.1) reads $f(\tilde{K}(t))\psi \xrightarrow[t\to+\infty]{} f(0)\psi$. By the so-called Stone-Weierstrass gavotte ([1], Appendix to Chapter 3) this holds for $f \in C_\infty(\mathbf{R})$ if it holds for resolvents $f(x) = (x-z)^{-1}$. The argument allowing to extend this to all bounded continuous functions can be found in [12], proof of Theorem VIII.20. We stress that these implications hold on individual states ψ. Furthermore, by the first resolvent identity, it is enough to prove

$$(\eta, (\tilde{K}(t) - z)^{-1}\psi) \xrightarrow[t\to+\infty]{} (-z)^{-1}(\eta, \psi) \quad , \qquad z \in \mathbf{C}\setminus\mathbf{R}$$

for $\eta \in \mathcal{H}$. Now we take a regularization $\psi^{(t)} \in D(p^2 + x^2)$ of ψ with $\psi^{(t)} \xrightarrow[t\to+\infty]{} \psi$ slow enough that

$$\|K(s)\psi^{(t)}\| \leq \text{const}\, t \tag{6.6}$$

(note that in general $\psi^{(t)} \notin \mathcal{H}^-_{scatt}(s)$). Here we assumed $s \neq 0$ without loss of generality. Then

$$\begin{aligned}(\eta, ((\tilde{K}(t) - z)^{-1} - (-z)^{-1})\psi) &= z^{-1}(\eta, (\tilde{K}(t) - z)^{-1}\tilde{K}(t)\psi)\\ &= z^{-1}((\tilde{K}(t) - z)^{-1}\eta, \tilde{K}(t)\psi^{(t)}) + z^{-1}(\eta, (\tilde{K}(t) - z)^{-1}\tilde{K}(t)(\psi - \psi^{(t)})) \quad .\end{aligned}$$

The second term vanishes as $t \to +\infty$, for $(\tilde{K}(t) - z)^{-1}\tilde{K}(t)$ is uniformly bounded in t. It is therefore enough to estimate $\tilde{K}(t)\psi^{(t)}$ using (6.5):

$$\begin{aligned}\|\tilde{K}(t)\psi^{(t)}\| \leq \frac{s^2}{t^2}\|K(s)\psi^{(t)}\| &+ \sum_{l=1}^{n}\frac{1}{t}\int_s^t d\tau\ \left|\frac{\tau}{t}\right| \|W_l(\tau)U^l(\tau,s)(1 - P_l^{pp}(s))G_l(s)\psi\|\\ &+ \sum_{l=1}^{n}\frac{1}{t}\left(\int_s^t d\tau\ \left|\frac{\tau}{t}\right| \|W_l(\tau)\|\right)\|\psi - \psi^{(t)}\| \quad ,\end{aligned}$$

where we have inserted a factor of $(1 - P_l^{pp}(s))$ due to $\psi \in \mathcal{H}^-_{scatt}(s)$. The first term tends to zero as $t \to +\infty$ because of (6.6), and the third one because $|\tau\Delta\dot{x}_l(\tau) - \Delta x_l(\tau)|$ and hence $\|W_l(\tau)\|$ are uniformly bounded in τ. Since $W_l(\tau)$ is relatively compact with respect to p^2, the second term would vanish by (5.1) as $t \to +\infty$ if $\Delta x_l(\tau) = 0$, i.e. if $W_l(\tau)$ were independent of τ. In order to gain the necessary 'uniform compactness in τ', we look at the decay rate $(1+|x|)^{-\epsilon}$ rather than at the potentials themselves:

$$\begin{aligned}
&\frac{1}{t}\int_s^t d\tau\ \|W_l(\tau)U^l(\tau,s)(1-P_l^{pp}(s))G_l(s)\psi\| \\
&\qquad \le \frac{1}{t}\int_s^t d\tau\ \|W_l(\tau)(1+|x|)^{\epsilon}\|\|(1+|x|)^{-\epsilon}U^l(\tau,s)(1-P_l^{pp}(s))G_l(s)\psi\| \\
&\qquad \le \frac{\text{const}}{t}\int_s^t d\tau\ \|(1+|x|)^{-\epsilon}U^l(\tau,s)(1-P_l^{pp}(s))G_l(s)\psi\| \quad ,
\end{aligned}$$

since

$$\begin{aligned}
&\|W_l(\tau)(1+|x|)^{\epsilon}\| \\
&\qquad \le \left(2\|V_l(x)\|_{\epsilon,\infty} + |\tau\Delta\dot{x}_l(\tau) - \Delta x_l(\tau)|\|\nabla V_l(x)\|_{\epsilon,\infty} + \|x\nabla V_l(x)\|_{\epsilon,\infty}\right)\cdot \\
&\qquad\quad \cdot \sup_x \left(\frac{1+|x|}{1+|x-\Delta x_l(\tau)|}\right)^{\epsilon}
\end{aligned}$$

is uniformly bounded in τ. Here we have set $\|V(x)\|_{\epsilon,\infty} = \|(1+|x|)^{\epsilon}V(x)\|_{\infty}$. ∎

Corollary 6.4. *Let $\psi \in \mathcal{H}^-_{scatt}(s)$. Then there is a sequence $\tau_k \xrightarrow[k\to+\infty]{} +\infty$ such that*

$$U^l(\tau_k,s)G_l(s)\psi \xrightarrow[k\to+\infty]{w} 0 \qquad l = 1,\dots,n \quad , \tag{6.7}$$

and

$$U(s,\tau_k)f(K_0(\tau_k))U(\tau_k,s)\psi \xrightarrow[k\to+\infty]{} f(0)\psi \tag{6.8}$$

for every bounded continuous function f on $\mathbf{R}$.

Proof: By (5.1) we know that for every $k > 0$

$$\sum_{l=1}^{n} \frac{1}{t-s}\int_s^t d\tau\ \|F(|x|<k)U^l(\tau,s)G_l(s)\psi\| \xrightarrow[t\to+\infty]{} 0 \quad .$$

Thus there is a τ_k as big as we like such that

$$\sum_{l=1}^{n} \|F(|x|<k)U^l(\tau_k,s)G_l(s)\psi\| < \frac{1}{k} \quad ,$$

which proves (6.7). Moreover, setting $\psi_k = U(\tau_k, s)\psi$,

$$\begin{aligned}\sum_{l=1}^{n} \|V_l(x - x_l(\tau_k))\psi_k\| &= \sum_{l=1}^{n} \|V_l(x - \Delta x_l(\tau_k))U^l(\tau_k, s)G_l(s)\psi\| \\ &\le \sum_{l=1}^{n} \|V_l\|_\infty \|F(|x| < k)U^l(\tau_k, s)G_l(s)\psi\| \\ &\quad + \sum_{l=1}^{n} \|V_l(x - \Delta x_l(\tau_k))F(|x| > k)\|_\infty \|\psi\| \\ &\xrightarrow[k \to +\infty]{} 0 \quad . \end{aligned} \tag{6.9}$$

The difference of the resolvents of $K(\tau_k)$ and $K_0(\tau_k)$ is

$$\begin{aligned}((K_0(\tau_k) - z)^{-1} - (K(\tau_k) - z)^{-1})\psi_k &= \sum_{l=1}^{n} (K_0(\tau_k) - z)^{-1} V_l(x - x_l(\tau_k))(K(\tau_k) - z)^{-1}\psi_k \\ &= \sum_{l=1}^{n} (K_0(\tau_k) - z)^{-1} V_l(x - x_l(\tau_k))(-z)^{-1}\psi_k \\ &\quad + \sum_{l=1}^{n} (K_0(\tau_k) - z)^{-1} V_l(x - x_l(\tau_k))[(K(\tau_k) - z)^{-1} - (-z)^{-1}]\psi_k \quad . \end{aligned}$$

The first term goes to zero as $k \to +\infty$ by (6.9), while the second does the same by (6.1).

■

7. Asymptotic Completeness

The proof of asymptotic completeness proceeds in two steps. First we prove that on certain subspaces the full dynamics is well approximated by simpler ones (Lemma 7.2). Then we show with the asymptotics of observables that the sum of these subspaces is absorbing with respect to the full dynamics on the scattering states, thus providing an asymptotic description of them (Lemma 7.3). The estimates along the way will be more transparent if one keeps in mind the following criterion for asymptotic completeness:

Lemma 7.1. *Suppose $\psi \in \mathcal{H}$ enjoys the following:*
For every $\varepsilon > 0$ there are a $\tau \in \mathbf{R}$ and $\varphi_l \in \mathcal{H}$, $l = 0, \ldots, n$ such that

$$\|U(\tau, s)\psi - \sum_{l=0}^{n} \varphi_l\| \leq \varepsilon \quad ,$$

and

$$\sup_{t \geq \tau} \|\big(U(t,\tau) - e^{-i\frac{p^2}{2}(t-\tau)}\big)\varphi_0\| \leq \varepsilon \quad ,$$

$$\sup_{t \geq \tau} \|\big(U(t,\tau) - G_l(t)^{-1} e^{-iH_l(t-\tau)} G_l(\tau)\big)\varphi_l\| \leq \varepsilon \quad , \qquad l = 1, \ldots, n \quad .$$

Then $\psi \in \bigoplus_{l=0}^{n} \operatorname{Ran} \Omega_l^-(s)$.

Proof: For $t \geq \tau$ we have

$$\begin{aligned}
&\big\|U(t,s)\psi - \big(e^{-i\frac{p^2}{2}(t-\tau)}\varphi_0 + \sum_{l=1}^{n} G_l(t)^{-1} e^{-iH_l(t-\tau)} G_l(\tau)\varphi_l\big)\big\| \leq \\
&\qquad \leq \big\|U(t,\tau)\big(U(\tau,s)\psi - \sum_{l=0}^{n}\varphi_l\big)\big\| + \big\|\big(U(t,\tau) - e^{-i\frac{p^2}{2}(t-\tau)}\big)\varphi_0\big\| + \\
&\qquad\quad + \sum_{l=1}^{n} \big\|\big(U(t,\tau) - G_l(t)^{-1} e^{-iH_l(t-\tau)} G_l(\tau)\big)\varphi_l\big\| \\
&\qquad \leq (n+2)\varepsilon \quad .
\end{aligned}$$

Moreover, because of the asymptotic completeness of the one-center systems ([4], Theorem 9.1; [14], Theorem XI.112) there is by the Cauchy criterion a $\tilde{\tau} \geq \tau$ such that

$$\sup_{t \geq \tilde{\tau}} \big\|\big(e^{-i\frac{p^2}{2}(t-\tilde{\tau})} - e^{-iH_l(t-\tilde{\tau})}\big) e^{-iH_l(\tilde{\tau}-\tau)} P_l^{cont} G_l(\tau)\varphi_l\big\| \leq \varepsilon$$

Thus, for $t \geq \tilde{\tau}$

$$\begin{aligned}
&\big\|U(t,s)\psi - \big[e^{-i\frac{p^2}{2}(t-\tilde{\tau})}\big(e^{-i\frac{p^2}{2}(\tilde{\tau}-\tau)}\varphi_0 + \sum_{l=1}^{n} G_l(\tilde{\tau})^{-1} e^{-iH_l(\tilde{\tau}-\tau)} P_l^{cont} G_l(\tau)\varphi_l\big) + \\
&\qquad\qquad + \sum_{l=1}^{n} G_l(t)^{-1} e^{-iH_l(t-\tau)} P_l^{pp} G_l(\tau)\varphi_l\big]\big\| \leq
\end{aligned}$$

$$\leq \left\|U(t,s)\psi - \left(e^{-i\frac{p^2}{2}(t-\tau)}\varphi_0 + \sum_{l=1}^{n} G_l(t)^{-1} e^{-iH_l(t-\tau)} G_l(\tau)\varphi_l\right)\right\| +$$
$$+ \left\|\sum_{l=1}^{n}\left(e^{-i\frac{p^2}{2}(t-\bar\tau)} G_l(\bar\tau)^{-1} e^{-iH_l(\bar\tau-\tau)} - G_l(t)^{-1} e^{-iH_l(t-\tau)}\right) P_l^{cont} G_l(\tau)\varphi_l\right\|$$
$$\leq (n+2)\varepsilon + \sum_{l=1}^{n}\left\|G_l(t)^{-1}\left(e^{-i\frac{p^2}{2}(t-\bar\tau)} - e^{-iH_l(t-\bar\tau)}\right) e^{-iH_l(\bar\tau-\tau)} P_l^{cont} G_l(\tau)\varphi_l\right\|$$
$$\leq (2n+2)\varepsilon \quad .$$

This can be written for $t \geq \bar\tau$ as

$$\left\|U(t,s)\psi - \left(e^{-i\frac{p^2}{2}(t-s)}\psi_0 + \sum_{l=1}^{n} G_l(t)^{-1} e^{-iH_l(t-s)} P_l^{pp} G_l(s)\psi_l\right)\right\| \leq (2n+2)\varepsilon \quad ,$$

with $$\psi_0 = e^{-i\frac{p^2}{2}(s-\bar\tau)}\left(e^{-i\frac{p^2}{2}(\bar\tau-\tau)}\varphi_0 + \sum_{l=1}^{n} G_l(\bar\tau)^{-1} e^{-iH_l(\bar\tau-\tau)} P_l^{cont} G_l(\tau)\varphi_l\right)$$

and $$\psi_l = G_l(s)^{-1} e^{-iH_l(s-\tau)} G_l(\tau)\varphi_l \quad , \qquad l = 1,\dots,n \quad .$$

Multiplying by $U(s,t)$ the expression within norm bars, and taking the limit $t \to +\infty$, this shows that

$$\psi \in \overline{\bigoplus_{l=0}^{n} \operatorname{Ran}\Omega_l^-(s)} = \bigoplus_{l=0}^{n} \operatorname{Ran}\Omega_l^-(s)$$

because of the closedness of these ranges. ■

Lemma 7.2.

i) Let $0 < v_0 < v$ and $g \in C_0^\infty(\mathbf{R}^\nu)$ with $g(p) = 0$ for $|p - u_l| \leq v$, $l = 1,\dots,n$. Then

$$\lim_{t_1\to+\infty} \sup_{t_2 \geq t_1} \left\|\left(U(t_2,t_1) - e^{-i\frac{p^2}{2}(t_2-t_1)}\right) g(p) F(K_0(t_1) < \frac{v_0^2}{2})\right\| = 0 \quad . \tag{7.1}$$

ii) Let $v, v_0 > 0$ with $2v + v_0 < \min_{k\neq l}|u_k - u_l|$, $g_1 \in C_0^\infty(\mathbf{R})$ with $g_1(e) = 0$ for $e \geq v^2/2$, and $g_2 \in C_0^\infty(\mathbf{R}^\nu)$ with $g_2(p) = 0$ for $|p| \geq v$. Then

$$\lim_{t_1\to+\infty} \sup_{t_2 \geq t_1} \left\|\left(U^l(t_2,t_1) - e^{-iH_l(t_2-t_1)}\right) g_1(H_l) g_2(p) F(K_0(t_1) < \frac{v_0^2}{2})\right\| = 0 \quad . \tag{7.2}$$

Proof: i) By (6.2)

$$\left(U(t_2,t_1) - e^{-i\frac{p^2}{2}(t_2-t_1)}\right) g(p) F(K_0(t_1) < \frac{v_0^2}{2}) =$$
$$= \left(U(t_2,t_1) - e^{-i\frac{p^2}{2}(t_2-t_1)}\right) e^{-i\frac{p^2}{2}t_1} g(p) F(\frac{1}{2}\frac{x^2}{t_1^2} < \frac{v_0^2}{2}) e^{i\frac{p^2}{2}t_1} \quad ,$$

hence the claim follows by (4.6) with $s = 0$.
ii) We take $\alpha > 1$ with $v + \alpha(v + v_0) < \min_{k\neq l} |u_k - u_l|$. By (4.1) and (6.2)

$$\|F(|x| > \alpha(v+v_0)t_1)g_2(p)F(K_0(t_1) < \frac{v_0^2}{2})\|$$
$$= \|F(|x| > \alpha(v+v_0)t_1)e^{-i\frac{x^2}{2}t_1}g_2(p)F(\frac{1}{2}\frac{x^2}{t_1^2} < \frac{v_0^2}{2})\| \leq \text{const}\, t_1^{-N}$$

It is therefore enough to show

$$\lim_{t_1\to+\infty} \sup_{t_2\geq t_1} \left\|\left(U^l(t_2,t_1) - e^{-iH_l(t_2-t_1)}\right) g_1(H_l)F(|x| < \alpha(v + v_0)t_1)\right\| = 0 \quad ,$$

which is precisely what (4.7) does. ∎

Lemma 7.3. *Take* $0 < 2v < \min_{k\neq l} |u_k - u_l|$ *and* $g, h \in C_0^\infty(\mathbf{R})$ *with* g *satisfying*

$$g(e) = 1 \quad \textit{for } e \leq \frac{1}{2}(\frac{v}{2})^2 \qquad \textit{and} \qquad g(e) = 0 \quad \textit{for } e \geq \frac{1}{2}v^2 \quad .$$

Then for $\psi \in \mathcal{H}^-_{scatt}(s)$ *there is a sequence* $\tau_k \xrightarrow[k\to+\infty]{} +\infty$ *such that*

$$\lim_{k\to+\infty} \sup_{t\geq\tau_k} \left\|\left(U(t,\tau_k) - e^{-i\frac{p^2}{2}(t-\tau_k)}\right)\left(1 - \sum_{l=1}^{n} g^2(\frac{1}{2}(p-u_l)^2)\right) h(\frac{p^2}{2})U(\tau_k,s)\psi\right\| = 0 \quad (7.3)$$

$$\lim_{k\to+\infty} \sup_{t\geq\tau_k} \left\|\left(U(t,\tau_k) - G_l(t)^{-1}e^{-iH_l(t-\tau_k)}G_l(\tau_k)\right) g^2(\frac{1}{2}(p-u_l)^2)h(\frac{p^2}{2})U(\tau_k,s)\psi\right\| = 0 \quad (7.4)$$

Proof: Take v_0 with $0 < v_0 < v/2$, $2v + v_0 < \min_{k\neq l} |u_k - u_l|$, and $f \in C_0(\mathbf{R})$ with $f(0) = 1$ and $f(e) = 0$ for $e \geq v_0^2/2$, and let τ_k be the sequence given by Corollary 6.4. Then, by (6.8)

$$(f(K_0(\tau_k)) - 1)U(\tau_k,s)\psi \xrightarrow[k\to+\infty]{} 0 \quad ,$$

and it is enough to prove (7.3) with a factor of $f(K_0(\tau_k)) = F(K_0(\tau_k) < \frac{v_0^2}{2})f(K_0(\tau_k))$ inserted to the left of $U(\tau_k,s)$. Then (7.3) follows from (7.1), since $\left(1-\sum_{l=1}^n g^2((p-u_l)^2/2)\right)$ $h(p^2/2)$ satisfies its hypothesis with $v/2$ instead of v (in particular it has compact support, which is the reason for introducing h).
Applying $G_l(t)$ to the left we see that the norm in (7.4) is equal to

$$\left\|\left(U^l(t,\tau_k) - e^{-iH_l(t-\tau_k)}\right) g^2(\frac{p^2}{2})h(\frac{1}{2}(p+u_l)^2)U^l(\tau_k,s)G_l(s)\psi\right\| \quad . \qquad (7.5)$$

Because of (6.7) and the compactness of $g(H_l) - g(\frac{p^2}{2})$ one can replace in (7.5) one of the factors $g(\frac{p^2}{2})$ by $g(H_l)$. We can then insert as above a factor of $F(K_0(\tau_k) < \frac{v_0^2}{2})f(K_0(\tau_k))$ since, due to (6.4)

$$(f(K_0(\tau_k)) - 1)U^l(\tau_k,s)G_l(s)\psi = G_l(\tau_k)(f(K_0(\tau_k)) - 1)U(\tau_k,s)\psi \xrightarrow[k\to+\infty]{} 0 \quad ,$$

and then apply (7.2). ■

Proof of Theorem 1.1 iii): It is enough to show $\psi \in \bigoplus_{l=0}^{n} \operatorname{Ran} \Omega_l^-(s)$ for $\psi \in \mathcal{H}_{scatt}^-(s)$. Given $\varepsilon > 0$, we can take by (3.30) a function $h \in C_0^\infty(\mathbf{R})$ such that

$$\sup_{t \geq s} \|(1 - h(\frac{p^2}{2}))U(t,s)\psi\| \leq \varepsilon \quad .$$

Now take g as in Lemma 7.3 and $\tau = \tau_k$ so that the suprema in (7.3), (7.4) are smaller than ε. Then the hypothesis of Lemma 7.1 are satisfied with

$$\varphi_0 = \left(1 - \sum_{l=1}^{n} g^2(\frac{1}{2}(p - u_l)^2)\right) h(\frac{p^2}{2})U(\tau_k, s)\psi \quad ,$$

$$\varphi_l = g^2(\frac{1}{2}(p - u_l)^2)h(\frac{p^2}{2})U(\tau_k, s)\psi \quad , \qquad l = 1, \dots, n \quad .$$ ■

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