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Probability Axioms for Quantum Field Theory.

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Abstract

We give sufficient conditions on a characteristic function of a probability space for its derivatives at the origin to satisfy the Osterwalder-Schrader axioms. These conditions are the *Probability Axioms for Quantum Field Theory.* The weakly coupled $P(\varphi)_2$ models constructed by Glimm, Jaffe and Spencer satisfy these axioms. Some new results about these models are established, such as the strong convergence of measures when the Euclidean-space cutoff is removed, the continuity of some combinations of Schwinger functions [1)], the construction of the Euclidean Wick fields and the generalization of the Feynman-Kac-Nelson formula for the generalized Schwinger distributions.

1) This is a result of the *Thèse de l'Université de Lausanne* of the author.

Introduction

Motivations. In a previous paper [Frochaux, a] the particle structure of a Quantum Field model with weak coupling has been studied using a new method, the *variational perturbation method*, initially proposed by Glimm, Jaffe and Spencer. Some mathematical statements have been used, concerning the existence of some vectors in the domain of the Hamiltonian, and the regularity with respect to the coupling constant of some scalar products. These statements are proved in [Frochaux b]; this last paper use some basic properties of the $P(\varphi)_2$ models, that we establish here.

Contents. We present the $P(\varphi)_2$ models with weak coupling and gives many new results about them. The exposition begins at the axiomatic level, starting from the Osterwalder-Schrader axioms. The First Part gives the conditions on a probability theory to generate an Osterwalder-Schrader model, in a space-time of d dimensions, d≥2. The Second Part gives the example of the weakly coupled $P(\varphi)_2$ models, constructed by Glimm, Jaffe and Spencer. Here d=2. Some new results about these models are established, such as the strong convergence of measures

when the Euclidean-space cutoff is removed, the continuity of some combinations of Schwinger functions, the construction of the Euclidean Wick fields and the generalization of the Feynman-Kac-Nelson formula for the generalized Schwinger distributions.
An effort has been made for the transparency of the mathematical exposition, mostly self-contained, and for the simplicity of the given proofs. A more detailed version of this work is given in [Frochaux, d].

Notations. $\mathbb{N}=\{0, 1, 2, \ldots\}$, $\mathbb{N}^*=\{1, 2, \ldots\}$; $\bar{c} \in \mathbb{C}$ is the complex conjugate of c; the functional spaces $\mathscr{S}(\mathbb{R}^n)$, $\mathscr{D}(\mathbb{R}^n)$, $L^p(\mathbb{R}^n)$ denote spaces of real-valued functions. Let f: $\mathbb{R}^n \to \mathbb{R}$. The translated function $x \cdot f$ is given by $x \cdot f(y)=f(y-x)$ for all $x,y \in \mathbb{R}^n$. The Fourier transform of f is $\tilde{f}(k)=(2\pi)^{-n/2} \int_{\mathbb{R}^n} d^n x\, f(x)\, e^{ikx}$ for all $k \in \mathbb{R}^n$; it may be complex-valued. The dominated convergence theorem is noted as *d.c. theorem*, and the Cauchy-Schwarz inequality as *CS inequality*.

First Part : The Axioms

In the 1950's Wightman wrote down a set of axioms for the Quantum Field Theory, in order to give a precise mathematical framework to this subject. An equivalent formulation was also given, concerning a family of distributions, the *Wightman distributions*, connected in a natural way to the Quantum Field Theory ([Streater, Wightman] or [Jost]). The extension of Wightman distributions to imaginary time leads to analytic functions, the *Schwinger functions.* Osterwalder and Schrader found a set of properties for a family of functions to be the set of Schwinger functions of a Wightman model [Osterwalder, Schrader]. These are the *Axioms for the Euclidean formulation of the Quantum Field Theory.*

The free model (i.e. without interaction) for massive, spinles particles satisfies the Wightman axioms. In order to find more interesting examples (i.e. with interaction) it has been necessary to look at other Euclidean theories constructed on a probability space and connected to the ordinary Quantum Field Theory by the famous Feynman-Kac-Nelson formula [Nelson]. It is possible to write down a set of axioms, the *Probability Axioms for Quantum Field Theory*, concerning a probability space, such that some expectations are the Schwinger functions of a Wightman model [Glimm, Jaffe, §6 and 19]. The proof of this statement, rather difficult, consists in the construction of the basic objects of the Quantum Field Theory, starting from a probability space.
We give here another proof, using slightly different probability axioms (§I), constructing only the Schwinger functions, and verifying that they satisfy the Osterwalder-Schrader axioms (§II). The main interest in doing this is the simplicity of our axioms (we do not need analyticity of the characteristic function) and also of the proof (because we lean on the Osterwalder-Schrader reconstruction theorem). The generality of our axioms make necessary the discussion of which Euclidean Hilbert space must be use for the reconstruction (§III).

I. The Axioms

We restrict ourselves to a world with only with one sort of particles without spin and without electric charge, moving in a d-dimensional space-time, with $d \geq 2$.
Let (Q,Σ,μ) be a probability space, where $Q=\mathcal{S}'(\mathbb{R}^d)$, Σ is the Borelian σ-algebra of Q (given the weak topology), and μ is a probability (i.e. positive and normed) measure on Σ. Let ϕ_f, for $f \in \mathcal{S}(\mathbb{R}^d)$, be the random variable given by $\phi_f(q)=q(f)$ for all $q \in Q$. The characteristic function $\mathscr{C}$ of μ is defined by:

$$\mathscr{C}: \mathcal{S}(\mathbb{R}^d) \to \mathbb{C} \quad , \quad \mathscr{C}(f) = \int_Q e^{i\phi_f(q)}\, d\mu(q)$$

By a Minlos theorem [Minlos], there is a one-to-one correspondence between the probability measures μ and the functions $\mathscr{C}$, provided that the latter verify :

$$\left.\begin{array}{l} \textit{i)}\ \text{Normalisation} : \mathscr{C}(0) = 1 \\ \textit{ii)}\ \text{Continuity} : \mathscr{C} \text{ is continuous} \\ \textit{iii)}\ \text{Positivity} : \text{for all } n \in \mathbb{N}^*,\ f \in \mathcal{S}^n,\ z \in \mathbb{C}^n : \sum_{i,j=1}^{n} \overline{z_i}\, \mathscr{C}(f_i - f_j)\, z_j \geq 0 \end{array}\right\} \text{(M)}$$

We will call these conditions (M).
Let $\mathcal{G}$ be the Euclidean group on $\mathbb{R}^d$ (rotations, translations and reflections), acting on $\mathcal{S}(\mathbb{R}^d)$ in the usual way. We single out a particular direction in $\mathbb{R}^d$ which we call *Euclidean time* ; a point in $\mathbb{R}^d$ will be written as: $x=(\mathring{x},\vec{x})$, where $\mathring{x}$ is an Euclidean time, and $\vec{x} \in \mathbb{R}^{d-1}$. Let $\{T(x),\ x \in \mathbb{R}^d\} \subset \mathcal{G}$ denotes the subgroup of translations, and $\theta \in \mathcal{G}$ the reflection in the $\mathring{x}=0$ hyperplane. We also define:

$$\mathcal{S}^+ = \{f \in \mathcal{S}(\mathbb{R}^d),\ f(\overset{\circ}{x}, \vec{x})=0 \text{ if } \overset{\circ}{x} < 0\}$$

From now on, we write $\mathcal{S}$ instead of $\mathcal{S}(\mathbb{R}^d)$.

The *Probability Axioms for Quantum Field Theory*, denoted by (P), are the following conditions on a function $\mathcal{C}$ satisfying (M) :

i) Euclidean invariance : $\mathcal{C}(\gamma\cdot f) = \mathcal{C}(f)$ for all $\gamma \in \mathcal{G}$, $f \in \mathcal{S}$

ii) Osterwalder-Schrader positivity : $\sum_{i,j=1}^{n} \overline{z}_i\ \mathcal{C}(\theta\cdot f_i - f_j)\ z_j \geq 0$

for all $n \in \mathbb{N}^*$, $f \in (\mathcal{S}^+)^n$, $z \in \mathbb{C}^n$

iii) Cluster property : $\lim_{s\to\infty} [\mathcal{C}(f+T(sx)\cdot g) - \mathcal{C}(f)\mathcal{C}(g)] = 0$

for all $x \in \mathbb{R}^d - \{0\}$ and $f, g \in \mathcal{S}$

iv) Regularity : for all $f \in \mathcal{S}$, $\alpha \to \mathcal{C}(\alpha f)$ is of class C^∞ in an $\mathbb{R}$-neighborhood of $\alpha=0$ and there exist a Schwartz space norm $|...|$ and finite positive numbers a,b,c such that :

$$\left| \partial_\alpha^n \mathcal{C}(af)\Big|_{\alpha=0} \right| \leq a\, b^n\, (n!)^c\, |f|^n \text{ for all } n \in \mathbb{N}^*$$

(P)

The axioms *i), ii)* and *iii)* can be read in [Glimm, Jaffe, §6]. The regularity axioms of these authors requires analyticity of the characteristic function, and other complicated properties that we do not need here.

The interest of (P) appears in the following *reconstruction theorem.*

Theorem. *Let* $\mathcal{C}:\mathcal{S}\to\mathbb{C}$ *satisfy* (M) *and* (P). *Then there exists a Wightman model whose Schwinger distributions* S_n *are the derivatives of* $\mathcal{C}$:

$$S_n(f,\ldots,f) = i^{-n}\ \partial_\alpha^n \mathcal{C}(af)\Big|_{\alpha=0} \tag{1}$$

for all $n \in \mathbb{N}^*$ *and* $f \in \mathcal{S}$.

The demonstration consists of a Proposition (§II) that we will prove and then of the Osterwalder-Schrader reconstruction Theorem [Osterwalder, Schrader].

The Osterwalder-Schrader axioms concern the Schwinger distributions $S_n \in \mathcal{S}'((\mathbb{R}^d)^n)$, $n \in \mathbb{N}$ (we give the case where the S_n are real). We need some definitions; for all $n \in \mathbb{N}^*$, let us denote :

$$\mathcal{S}_{n,c} = \{f \in \mathcal{S}((\mathbb{R}^d)^n),\ f(x_1,\ldots,x_n)=0 \text{ if } x_i=x_j \text{ for some } 1\leq i\neq j\leq n, \text{ and so are all derivatives of } f\}$$

$$\mathcal{S}_{n,+} = \{f \in \mathcal{S}_{n,c},\ f(x_1,\ldots,x_n)=0 \text{ unless } 0 < \overset{\circ}{x}_1 < \ldots < \overset{\circ}{x}_n\}\ .$$

For all $f \in \mathcal{S}((\mathbb{R}^d)^n)$ and $\pi \in \sigma_n$, the set of permutations of $\{1,\ldots,n\}$, we denote by f^* and πf the following functions :

$$f^*(x_1,\ldots,x_n) = f(x_n,\ldots,x_1) \quad \text{and} \quad \pi f(x_1,\ldots,x_n) = f(x_{\pi(1)},\ldots,x_{\pi(n)})$$

$\mathcal{SG}$ will be the subgroup of $\mathcal{G}$ consisting of translations and rotations of $\mathbb{R}^d$ only (the special group of $\mathcal{G}$), acting on $\mathcal{S}((\mathbb{R}^d)^n)$ in the usual way.

We state now the Osterwalder-Schrader axioms. For all n, $m \in \mathbb{N}^*$:

$$\left.\begin{array}{l}
i)\ \text{Euclidean invariance : } S_n(\gamma\cdot f) = S_n(f) \text{ for all } \gamma\in \mathcal{SG} \text{ and } f\in \mathcal{S}_{n,c} \\
ii)\ \text{Osterwalder-Schrader positivity : } \sum_{i,j=0}^{n} S_{i+j}(\theta\cdot f_i{}^*\otimes f_j) \geq 0 \\
\quad \text{for all } f_o\in \mathbb{C} \text{ and } f_i\in \mathcal{S}_{i,+},\ 1\leq i\leq n. \\
iii)\ \text{Cluster property : } \lim_{s\to\infty} \left[S_{n+m}(\theta\cdot f^*\otimes T(sx)\cdot g) - S_n(\theta\cdot f^*)S_m(g) \right] = 0 \\
\quad \text{for all } x=(0,\vec{x}),\ \vec{x}\in \mathbb{R}^{d-1}\text{-}\{0\} \text{ and } f\in \mathcal{S}_{n,+},\ g\in \mathcal{S}_{m,+} \\
iv)\ \text{Temperedness : } S_o=1,\ S_n\in \mathcal{S}'((\mathbb{R}^d)^n) \text{ and there exist a Schwartz} \\
\quad \text{space norm } |...| \text{ and finite positive numbers a,b,c such that :} \\
\quad | S_n(f_1,\dots,f_n) | \leq a\, b^n\, (n!)^c \prod_{i=1}^{n} |f_i| \qquad \text{for all } f_1,\dots,f_n \in \mathcal{S}(\mathbb{R}^d) \\
v)\ \text{Symmetry : } S_n(\pi f)=S_n(f) \text{ for all } \pi\in \sigma_n \text{ and } f\in \mathcal{S}_{n,c}
\end{array}\right\} \quad \text{(OS)}$$

II. The main result

Proposition. *Let* $\mathcal{C}:\mathcal{S}\to\mathbb{C}$ *satisfy* (M) *and* (P). *Then the* S_n *distributions defined by* (1) *verify* (OS) *with the same Schwartz space norm* |...| *and the same numbers* a, b, c *in* (OS) *iv) as in* (P) *iv)*.

Proof of the Proposition. We work in five steps. Step 1 : (P) *iv)* ⇒ (OS) *iv)* ; Step 2 : (P) *iv)* ⇒ (OS) *v)* ; Step 3 : (P) *iv)* and *i)* ⇒ (OS) *i)* ; Step 4 : (P) *iv)*, *i)* and *ii)* ⇒ (OS) *ii)* ; Step 5 : (P) *iv)*, *i)* and *iii)* ⇒ (OS) *iii)*

Step 1 : (P) *iv)* ⇒ (OS) *iv)*.

We consider a more simple probability space: $(\mathbb{R},\mathcal{B},\sigma)$ where σ is a probability measure on the Borel σ-algebra $\mathcal{B}$ of $\mathbb{R}$. Here the characteristic function is now : $\mathcal{C}(\alpha) = \int_{\mathbb{R}} e^{i\alpha x}\, d\sigma(x)$ for all $\alpha\in\mathbb{R}$, and the Bochner theorem takes the place of the Minlos theorem. For all $n\in\mathbb{N}$ we define the numbers, whenever they exist :

$$\mathcal{C}_{2n} = \lim_{\alpha\to 0,\alpha\in\mathbb{R}} \alpha^{-2n} \sum_{-n\leq k\leq n} \binom{2n}{n+k} (-1)^k\, \mathcal{C}(k\alpha)$$

where $\binom{i}{j}$ denotes the binomial coefficients. If $\alpha\to\mathcal{C}(\alpha)$ is of class C^N for some $N\in\mathbb{N}$ in an $\mathbb{R}$-neighborhood of $\alpha=0$, then $\mathcal{C}_{2n}=(-1)^n\, \mathcal{C}^{(2n)}(0)$ for all $n\in\mathbb{N}$ with $2n\leq N$.

Lemma 1. *Let* σ *be a probability measure on* $\mathcal{B}$ *and* $N\in\mathbb{N}$ *such that* $\mathcal{C}_{2n}$ *exist for all* $n\leq N$. *Then : a) the moments* $\int_{\mathbb{R}} x^n\, d\sigma(x)$ *exist for all* $n\leq 2N$

b) $\alpha\to\mathcal{C}(\alpha)$ *belongs to* $C^{2N}(\mathbb{R})$ *and the formulas hold :*

$$\mathcal{C}^{(2n)}(\alpha) = i^n \int_{\mathbb{R}} x^n\, e^{i\alpha x}\, d\sigma(x) \quad \textit{for all } n\leq 2N \textit{ and } \alpha\in\mathbb{R}.$$

Proof of *a)* (from [Loeffel]). The cases N=0 and n=0 are trivial. Fix two integer n, N with $0<n\le N$ and two real numbers Λ,α with $\Lambda>0$ and $|\alpha|\le\pi/\Lambda$. Because $\mathbb{R}\ni x\to$

$\frac{\sin x}{x}$ is even and decreasing on $[0,\pi/2]$: $|x| \le \left| \frac{\alpha\Lambda}{2\sin\frac{\alpha\Lambda}{2}} \; \frac{2\sin\frac{\alpha x}{2}}{\alpha} \right|$ for $|x|\le\Lambda$.

Then : $$\int_{|x|\le\Lambda} x^{2n}\, d\sigma(x) \;\le\; \left(\frac{\alpha\Lambda}{2\sin\frac{\alpha\Lambda}{2}}\right)^{2n} \alpha^{-2n} \int_{|x|\le\Lambda} \left(2\sin\frac{\alpha x}{2}\right)^{2n} d\sigma(x)\,.$$

We use the inequality $\int_{|x|\le\Lambda} (sin\ldots)^{2n}\, d\sigma \le \int_{\mathbb{R}} (sin\ldots)^{2n}\, d\sigma$ and the Euler formula

$$\left(2\sin\frac{\alpha x}{2}\right)^{2n} = \sum_{-n\le k\le n} \binom{2n}{n+k} (-1)^k e^{ik\alpha x}$$

to find : $$\int_{|x|\le\Lambda} x^{2n}\, d\sigma(x) \;\le\; \left(\frac{\alpha\Lambda}{2\sin\frac{\alpha\Lambda}{2}}\right)^{2n} \alpha^{-2n} \sum_{-n\le k\le n} \binom{2n}{n+k} (-1)^k \mathscr{C}(k\alpha)$$

The limit $\alpha\to 0$ gives : $\int_{|x|\le\Lambda} x^{2n}\, d\sigma(x) \le \mathscr{C}_{2n}$ (note that $\mathscr{C}_{2n}$ must be positive!). By a monotonicity argument, the limit $\Lambda\to\infty$ exists, and is bounded by $\mathscr{C}_{2n}$. With the CS inequality the existence of odd moments follows.

Proof of *b)*. Fix $n\le 2N$ and $\alpha,\beta\in\mathbb{R}$ with $\beta\neq 0$. From the Euler formula:

$$\beta^{-n}\sum_{k=0}^{n} \binom{n}{k} (-1)^k \mathscr{C}(\alpha-\beta k+n\tfrac{\beta}{2}) \;=\; i^n \int_{\mathbb{R}} \left(\frac{2\sin\frac{\beta x}{2}}{\beta}\right)^{n} e^{i\alpha x}\, d\sigma(x)$$

The limit $\beta\to 0$ gives the announced formula (for the r.h.s. the existence of the limit follows from *a)* and the d.c. theorem). ♦

Let us return to the probability space (Q,Σ,μ).

Lemma 2. *Let* $\mathscr{C}:\mathscr{S}\to\mathbb{C}$ *satisfy* (M) *and* (P) *iv). Then for all* $n\in\mathbb{N}^*$ *and* $f\in\mathscr{S}^n$:

a) $$\left(\prod_{i=1}^{n}\partial_{\alpha_i}\right)\mathscr{C}\left(\sum_{i=1}^{n}\alpha_i f_i\right)\Bigg|_{\alpha_1=\ldots=\alpha_n=0} = i^n\int_Q \phi_{f_1}(q)\cdots\phi_{f_n}(q)\, d\mu(q)$$

b) $$\left|\int_Q \phi_{f_1}(q)\cdots\phi_{f_n}(q)\, d\mu(q)\right| \;\le\; a\, b^n\, (n!)^c \prod_{i=1}^{n} |f_i|$$

By *a)*, the following definition of the Schwinger distributions agrees with (1) :

$$\left.\begin{aligned} &S_o = 1 \\ &S_n(f_1,\ldots,f_n) = \int_Q \phi_{f_1}(q)\ldots\phi_{f_n}(q)\, d\mu(q) \quad \text{for all } n\in\mathbb{N}^* \text{ and } f\in\mathscr{S}^n \end{aligned}\right\} \tag{2}$$

(OS) *iv)* follows from *b)* and the Schwartz nuclear theorem.

Proof of Lemma 2 *a*). Recall that Σ is generated by the sets $C_f(B)=\{q\in\mathscr{S}', \phi_f(q)\in B\}$ for all $f\in\mathscr{S}$ and $B\in\mathcal{B}$; then for all $f\in\mathscr{S}$, $B\to\mu(C_f(B))$ is a probability measure on $(\mathbb{R},\mathcal{B})$; we call it σ_f. From (P) *iv)* :

$$\mathscr{C}(\alpha f) = \int_Q e^{i\alpha\phi_f(q)}\, d\mu(q) = \int_{\mathbb{R}} e^{i\alpha x}\, d\sigma_f(x)$$

is of class C^∞ in an $\mathbb{R}$-neighborhood of $\alpha=0$; it follows from Lemma 1 that :

$$\partial_\alpha^n \mathscr{C}(\alpha f) = i^n \int_Q (\phi_f)^n\, e^{i\alpha\phi_f(q)}\, d\mu = i^n \int_{\mathbb{R}} x^n\, e^{i\alpha x}\, d\sigma_f$$

exists for all $n\in\mathbb{N}$. $\alpha=0$ gives the announced formula, in the case where all f_i are the same. The general case follows by n-linearity.

Proof of Lemma 2 *b*). From *a)* and (P) *iv)* : $|\int_Q (\phi_f)^n\, d\mu| \le a\, b^n\, (n!)^c\, |f|^n$ for all $f\in\mathscr{S}$. The announced formula follows from Hölder inequality. ♦

Step 2 : (P) *iv)* ⇒ (OS) *v)*.

Since $S_n\in\mathscr{S}'((\mathbb{R}^d)^n)$, $S_n(f)$ can be written as a limit, for all $f\in\mathscr{S}((\mathbb{R}^d)^n)$:

$$S_n(f) = \lim_{\varepsilon\to+0} S_n^\varepsilon(f) \quad \text{where, for all } \varepsilon>0 :$$

$$S_n^\varepsilon(f) = \int_{(\mathbb{R}^d)^n} d^{nd}x\, f(x_1,\dots,x_n)\, S_n\big(T(x_1)\cdot j_\varepsilon, \dots, T(x_n)\cdot j_\varepsilon\big) = S_n\big(f*(j_\varepsilon)^{\otimes n}\big)$$

for all $j\in\mathscr{S}$ with $\int j=1$; j_ε is the function $\mathbb{R}^d\ni x\to\varepsilon^{-d}\, j(x/\varepsilon)$. Note that the limit is independent of j. From its definition (2), $S_n(f_1,\dots,f_n)$ is symmetrical, and so is S_n^ε for all $\varepsilon>0$, and so is its limit when $\varepsilon\to0$, too.

Step 3 : (P) *iv)* and *i)* ⇒ (OS) *i)*.

For all $\gamma\in\mathscr{SG}$ and $f\in\mathscr{S}^n$, we have from (2) and (P) *i)* :

$$S_n(\gamma\cdot f_1,\dots,\gamma\cdot f_n) = (\textstyle\prod\partial_{\alpha_i})\, \mathscr{C}(\sum\alpha_i\gamma\cdot f_i)\,|_{\alpha=0} = (\prod\partial_{\alpha_i})\, \mathscr{C}(\sum\alpha_i f_i)|_{\alpha=0} = S_n(f_1,\dots,f_n)\,.$$

For all $f\in\mathscr{S}_{n,c}\subset\mathscr{S}((\mathbb{R}^d)^n)$, we approach $S_n(f)$ by $S_n^\varepsilon(f)$ with $\varepsilon>0$. With the previous result and some change of variables we obtain :

$$S_n^\varepsilon(\gamma\cdot f) = \int d^{nd}x\, f(x_1,\dots,x_n)\;\; S_n\big(\gamma^{-1}T(\gamma\cdot x_1)\cdot j_\varepsilon, \dots, \gamma^{-1}T(\gamma\cdot x_n)\cdot j_\varepsilon\big)$$

A little effort gives : $\gamma^{-1}T(\gamma\cdot x)\cdot j_\varepsilon = T(x)\cdot k_\varepsilon$ for all $x\in\mathbb{R}^d$
where $k=\gamma'\cdot j$ and $\gamma'\in SO(d)$ given by $\gamma'\cdot(x-y) = \gamma^{-1}\cdot x - \gamma^{-1}\cdot y$ for all $x,y\in\mathbb{R}^d$. We obtain $S_n^\varepsilon(\gamma\cdot f) = S_n\big(f*(k_\varepsilon)^{\otimes n}\big)$. Because $\int k=1$ and $k\in\mathscr{S}$, the limit $\varepsilon\to0$ exists and gives $S_n(f)$.

Step 4 : (P) *iv), i)* and *ii)* ⇒ (OS) *ii)*

The action of $\mathcal{G}$ on $\mathcal{S}$ induces an action on $\mathcal{S}'$ in a natural way, and then also on the random variables $\mathcal{S}' \to \mathbb{C}$. By (P) *i)* this action is realized by unitary operators on $L^2(Q,\mu)$. We introduce a closed subspace of $L^2(Q,\mu)$:

$$L^+ = \text{closure of the span of } \{e^{i\phi f},\ f \in \mathcal{S}^+\}$$

The bilinear form $b : L^2(Q,\mu) \ni \psi, \chi \to b(\psi,\chi) = (\theta\cdot\psi, \chi)_{L^2(Q,\mu)}$ satisfy :

Lemma 3. *Let $\mathscr{C}:\mathcal{S}\to\mathbb{C}$ satisfy* (M) *and* (P) *i) and ii). Then for all* $\psi \in L^+$: $b(\psi,\psi) \geq 0$.

Proof. By (P) *i)*, θ acts as an unitary operator on $L^2(Q,\mu)$. Then the form b is well defined. Let us consider a vector $\psi = \sum_{i\leq j\leq n} c_j\, e^{i\phi f_j}$ with $n \in \mathbb{N}^*$, $c \in \mathbb{C}^n$ and $f \in (\mathcal{S}^+)^n$. Note that $\psi \in L^+$. We have :

$$b(\psi,\psi) = \sum_{j,k=1}^{n} \overline{c_j}\, c_k\, (e^{i\phi_{\theta\cdot f_j}}, e^{i\phi_{f_k}})_{L^2(Q,\mu)} = \sum_{j,k=1}^{n} \overline{c_j}\, c_k\, \mathscr{C}(-\theta\cdot f_j + f_k)$$

which is non negative, by (P) *ii)*. Because all vectors of L^+ can be approached by such a ψ, the conclusion holds. ♦

For all $n \in \mathbb{N}^*$ and $f \in \mathcal{S}((\mathbb{R}^d)^n)$ we introduce the random variables :

$$Q \ni q \to \phi_f^n(q) = \lim_{\varepsilon\to+0} \int_{(\mathbb{R}^d)^n} d^{nd}x\ f(x_1,\ldots,x_n)\ \phi_{T(x_1)\cdot j_\varepsilon}(q) \cdots \phi_{T(x_n)\cdot j_\varepsilon}(q) \qquad (3)$$

where j_ε is an approximation of the Dirac distribution as in Step 2. We will use the following functional spaces, for all $n \in \mathbb{N}^*$:

$$\mathcal{S}_n^+ = \{ f \in \mathcal{S}((\mathbb{R}^d)^n) \,;\, f(x_1,\ldots,x_n) = 0 \text{ if } \overset{o}{x}_i < 0 \text{ for some } i \in \{1,\ldots,n\} \}$$

Lemma 4. *Let $\mathscr{C}:\mathcal{S}\to\mathbb{C}$ satisfy* (M) *and* (P) *iv) and ii). Then for all* $n \in \mathbb{N}^*$ *we have* $\phi_f^n \in L^2(Q,\mu)$ *if* $f \in \mathcal{S}((\mathbb{R}^d)^n)$ *and* $\phi_f^n \in L^+$ *if* $f \in \mathcal{S}_n^+$.

Let us discuss the consequences of Lemmas 3 and 4.

Take $\psi = f_o + \sum_{1\leq n\leq N} \phi_{f_n}^n$ with $N \in \mathbb{N}^*$, $f_o \in \mathbb{C}$ and $f_n \in \mathcal{S}_{n,+} \subset \mathcal{S}_n^+$ for all $1 \leq n \leq N$. By Lemma 4 : $\psi \in L^+$. Then by Lemma 3 : $b(\psi,\psi) = \sum_{n,m\geq 0} S_{n+m}(\theta\cdot f_n \otimes f_m) \geq 0$ which, together with the symmetry of S_{n+m} (Step 2), proves (OS) *ii)*.

Proof of Lemma 4. Let us begin with the case where f is a product of functions of $\mathbb{R}^2$: $f(x_1,\ldots,x_n) = f_1(x_1) \cdots f_n(x_n)$. We define :

$$\phi_{f\varepsilon}^n(q) = i^{-n}\ \frac{1}{\varepsilon}\left(e^{i\varepsilon\phi_{f_1}(q)} - 1\right) \cdots \frac{1}{\varepsilon}\left(e^{i\varepsilon\phi_{f_n}(q)} - 1\right)$$

for all $\varepsilon>0$ and $q\in Q$, $\phi^n_{f\varepsilon}\in L^2(Q,\mu)$ and if $f\in \mathscr{S}^+_n$, $\phi^n_{f\varepsilon}\in L^+$. The lemma holds for such f if the pointwise limit $\phi^n_f(q) = lim_{\varepsilon\to+0}\, \phi^n_{f\varepsilon}(q)$ actually converges in $L^2(Q,\mu)$. Look at

$$\left\|\phi^n_{f\varepsilon} - \phi^n_f\right\|^2_{L^2(Q,\mu)} = \int_Q \left| i^n\phi^n_f(q) - \prod_{j=1}^n \frac{1}{\varepsilon}\left(e^{i\varepsilon\phi_{f_j}(q)} - 1\right)\right|^2 d\mu(q)\ .$$

The convergence to 0 if $\varepsilon\to+0$ is assured by the d.c. theorem (a bound independent of ε follows from : $|\varepsilon^{-1}(exp\, i\varepsilon\phi_f(q)-1)| \le |\phi_f(q)| \in L^p(Q,\mu)$ for all $1\le p<\infty$, Lemma 2). If f cannot be factorized, we use the definition (3). The r. h. s., if $\varepsilon>0$, is treated like the factorized case, and thus is in $L^2(Q,\mu)$ and moreover is in L^+ if $f\in\mathscr{S}^+_n$ and $j\in\mathscr{S}^+$. We have only to control the strong limit in L^2 when $\varepsilon\to+0$. So we compute :

$$\left\| \phi^n_f - \int d^{nd}x\, f(x_1,\dots,x_n)\, \phi_{T(x_1)\cdot j_\varepsilon} \cdots \phi_{T(x_n)\cdot j_\varepsilon}\right\|^2_{L^2(Q,\mu)} =$$

$$= S_{2n}(f\otimes f) - 2\int d^{nd}x\, f(x_1,\dots,x_n)\; S_{2n}\big(f\otimes(T(x_1)\cdot j_\varepsilon\otimes \cdots \otimes T(x_n)\cdot j_\varepsilon)\big) +$$

$$+ \int d^{nd}x\, d^{nd}y\, f(x_1,\dots,x_n)\, f(y_1,\dots,y_n) S_{2n}\big(T(x_1)\cdot j_\varepsilon\otimes \cdots \otimes T(x_n)\cdot j_\varepsilon \otimes T(y_1)\cdot j_\varepsilon\otimes \cdots \otimes T(y_n)\cdot j_\varepsilon\big)$$

which converges to 0 if $\varepsilon\to+0$ because $S_{2n}\in\mathscr{S}'((\mathbb{R}^d)^{2n})$. ♦

Step 5 : (P) *iv), i)* and *iii)* $\Rightarrow$ (OS) *iii).*

Because μ is a finite measure, the constant functions are in $L^2(Q,\mu)$. We call P the orthogonal projector in $L^2(Q,\mu)$ on the one-dimensional subspace $\{Q\ni q\to c,\ c\in\mathbb{C}\}$.

Lemma 5. *Let $\mathscr{C}:\mathscr{S}\to\mathbb{C}$ satisfy* (M) *and* (P) *i) and iii). Then:* $\text{weak-}\lim_{|x|\to\infty} T(x) = P$.

For all $n, m\in\mathbb{N}^*$, $f\in\mathscr{S}_{n,+}\subset\mathscr{S}((\mathbb{R}^d)^n)$ and $g\in\mathscr{S}_{m,+}\subset\mathscr{S}((\mathbb{R}^d)^m)$, we know that $\phi^n_{\theta\cdot f}, \phi^m_g \in L^2(Q,\mu)$ (Lemma 4). For all $x=(0,\vec{x})$, $\vec{x}\in\mathbb{R}^{d-1}\text{-}\{0\}$ it follows from Lemma 5 that

$$0 = \lim_{s\to\infty}\left[(\phi^n_{\theta\cdot f^*}, T(sx)\,\phi^m_g)_{L^2(Q,\mu)} - (\phi^n_{\theta\cdot f^*}, 1)_{L^2(Q,\mu)}(1, \phi^m_g)_{L^2(Q,\mu)}\right]$$

$$= \lim_{s\to\infty}\left[S_{n+m}(\theta\cdot f^*\otimes T(sx)\cdot g) - S_n(\theta\cdot f^*)S_m(g)\right]$$

and then (OS) *iii)* holds.

Proof of Lemma 5. By (P) *i)*, $T(x)$ is a bounded operator on $L^2(Q,\mu)$ for all $x\in\mathbb{R}^d$; it is then enough to verify the weak limit on a dense subspace.

The span of $\{e^{i\phi_f}, f\in\mathcal{S}\}$ is dense (Lemma 6); we take two vectors in it :

$$\psi=\sum_{n=1}^{N} c_n\, e^{-i\phi_{f_n}} \qquad \text{and} \qquad \chi=\sum_{n=1}^{M} d_n\, e^{i\phi_{g_n}}$$

with $N, M\in\mathbb{N}^*$, c_i, $d_j\in\mathbb{C}$, f_i, $g_j\in\mathcal{S}$. For all $x\in\mathbb{R}^d$ we have :

$$(\psi, (T(x)-P)\,\chi)_{L^2(Q,\mu)} = \sum_{n=1}^{N}\sum_{m=1}^{M} \bar{c}_n\, d_m \left[\mathcal{C}(f_n+T(sx)\cdot g_m) - \mathcal{C}(f_n)\mathcal{C}(g_m)\right]$$

which goes to 0 when $|x|\to\infty$, by (P) *iii)* . ♦

III. Choice of the Euclidean Space

Let (Q,Σ,μ) be a probability space, where $Q=\mathcal{S}'(\mathbb{R}^n)$, Σ is the Borelian σ-algebra of Q (given the weak topology), and μ is a probability measure on Σ. We introduce two closed subspaces of $L^2(Q,\mu)$:

$\mathcal{E}$ = closed span of $\{e^{i\phi_f}, f\in\mathcal{S}\}$

$\mathcal{M}$ = closed span of $\{(\phi_f)^n, n\in\mathbb{N}, f\in\mathcal{S}\}$

We give two Lemmas about the connection between $L^2(Q,\mu)$ and these subspaces, and we discuss then which Euclidean Hilbert space must be used for the reconstruction.

Lemma 6. $\mathcal{E}= L^2(Q,\mu)$.

Proof. We establish that all F in $L^2(Q,\mu)$ with : $\tilde{F}(f) =\int_Q e^{i\phi_f(q)}\, F(q)\, d\mu(q) = 0$ for all $f\in\mathcal{S}$ satisfies : $\|F\|_{L^2} = 0$. For all compact set Λ of $\mathbb{R}$, let $\{\chi^{\varepsilon}_{\Lambda}, \varepsilon>0\}$ be a set in $\mathcal{S}$ which converges pointwise when $\varepsilon\to 0$ to the characteristic function χ_Λ of Λ. We suppose that $|\chi^{\varepsilon}_{\Lambda}(x)| < 2$ for all $x\in\mathbb{R}$ and $\varepsilon>0$. For all $\varepsilon>0$ and $f\in\mathcal{S}$ we have now :

$$\int_{\mathbb{R}} d\alpha\ \tilde{\chi}^{\varepsilon}_{\Lambda}(\alpha)\ \tilde{F}(\alpha f) = \int_Q F(q)\ \chi^{\varepsilon}_{\Lambda}(\phi_f(q))\ d\mu(q) \ \overset{\varepsilon\to 0}{\rightarrow}\ \int_{C_f(\Lambda)} F\, d\mu$$

($\tilde{\chi}^{\varepsilon}_{\Lambda}$ is the Fourier transform of $\chi^{\varepsilon}_{\Lambda}$). The equality is due to the Fubini theorem which allows us to permute the integrals over $\mathbb{R}$ and Q when the function is L^1 in both variables $\alpha\in\mathbb{R}$ and $q\in Q$. The convergence when $\varepsilon\to 0$ is assured by the d.c. theorem. Because the sets $C_f(\Lambda) = \{q\in Q, \phi_f(q)\in\Lambda\}$, for all $f\in\mathcal{S}$ and Λ compact set of $\mathbb{R}$, generate Σ, we have proved that $\int_B F\, d\mu = 0$ for all $B\in\Sigma$. ♦

For all $n \in \mathbb{N}^*$ the *n-th moment of* μ is $S_n(f) = \int_Q (\phi_f)^n \, d\mu$ for $f \in \mathcal{S}$, when it exists.

Lemma 7. *Suppose that there exists a Schwartz norm* $|...|$ *and two finite positive numbers* a, b *such that* $|S_n(f)| < a\, b^n\, n!\, |f|^n$ *for all* $n \in \mathbb{N}^*$ *and* $f \in \mathcal{S}$. *Then:* $L^2(Q,\mu) = \mathcal{E} = \mathcal{M}$.

Remark. When the growth in n of $|S_n(f)|$ is stronger then in Lemma 7, $\mathcal{E}$ and $\mathcal{M}$ can be different. For an example (in the case where $Q=\mathbb{R}$) see [Feller, §VII.3].

Euclidean Hilbert space. For the reconstruction of the Quantum Field Theory [Glimm, Jaffe, §6, §19] take $\mathcal{E}$ as Euclidean Hilbert space. The regularity axiom used there requires analyticity of the characteristic function, which imposes a growth in n of $|S_n(f)|$ no stronger then in Lemma 7; thus in this case $\mathcal{E} = \mathcal{M} = L^2$. Our regularity axiom is weaker, and admits stronger growth for $|S_n(f)|$ (that is, in (P) *iv)*, c can be >1). Because $\mathcal{E} = L^2$ may be too large, we must require for the reconstruction the Euclidean Hilbert space to be $\mathcal{M}$.

Proof of Lemma 7. The relation $L^2(Q,\mu)=\mathcal{E}$ is just Lemma 6. To prove $\mathcal{E}=\mathcal{M}$, we must construct a random variable $exp(i\phi_f)$ for each $f \in \mathcal{S}$ as a limit in $\mathcal{M}$. For that, we define, for all $\alpha \in \mathbb{R}$, $|\alpha| < (8b|f|)^{-1}$, the multiplication operator in $L^2(Q,\mu)$:

$$A^N(\alpha) = \sum_{n=0}^{N} \frac{(i\alpha)^n}{n!} (\phi_f)^n$$

with $N \in \mathbb{N}$. $A^N(\alpha)$ is well defined on $L^\infty(Q,\mu)$, which is dense in $L^2(Q,\mu)$ (Lemma 6). Moreover for all $F \in L^\infty(Q,\mu)$ the sequence $\{A^N(\alpha)F,\ N \in \mathbb{N}\}$ converges strongly in $L^2(Q,\mu)$ to $e^{i\alpha\phi_f}F$ when $N \to \infty$ because :

$$\|A^N(\alpha)F - e^{i\alpha\phi_f}F\|^2_{L^2} \le \|F\|^2_{L^\infty} \int_Q \left| \sum_{n=0}^{N} \frac{(i\alpha)^n}{n!} (\phi_f)^n - e^{i\alpha\phi_f} \right|^2 d\mu =$$

$$= \|F\|^2_{L^\infty} \int_Q \left| \sum_{n=N+1}^{\infty} \frac{(i\alpha)^n}{n!} (\phi_f)^n \right| d\mu$$

goes to 0 when $N \to \infty$ (this follows from the hypothesis of the Lemma and the d.c. theorem). Because $(\|e^{i\alpha\phi_f}F\|_{L^2})^2 = \int_Q |e^{i\alpha\phi_f(q)}|^2\, |F(q)|^2\, d\mu(q) = (\|F\|_{L^2})^2$ the multiplication operator $e^{i\alpha\phi_f}$ is of norm 1. So it is defined on all L^2, and it can be iterated. Take $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $n\alpha = 1$ and $\alpha < (8b|f|)^{-1}$; since $1 \in \mathcal{M}$, we have obtain :

$$e^{i\phi_f} = \left(e^{i\alpha\phi_f}\right)^n = \lim_{N_1\to\infty} \cdots \lim_{N_n\to\infty} A^{N_1}(\alpha) \cdots A^{N_n}(\alpha)\ 1$$

that is : $exp(i\phi_f)$ is a strong limit (in L^2) of vectors of $\mathcal{M}$. ♦

Second Part. Examples.

The weakly coupled $P(\varphi)_2$ models associate to any pair (P,λ), where P is a positive polynomial and λ is a non negative, sufficiently small number, a Quantum Field model satisfying the Wightman axioms. They are constructed first in a regularized form, starting from the case $\lambda=0$ (the *free model*, §I) and then the regularization is removed carefully. We describe briefly this construction, without proof (§II), giving only the result of the *cluster expansion* of Glimm, Jaffe and Spencer. We will prove that the limit of involving measure holds in a strong sense because of the good asymptotic behavior of the characteristic function (II.2), which is obtain by a generalization of the *Integration by parts* formulas (Appendix I). These models satisfy the axioms (P) (II.3).
We construct the Euclidean Wick fields (§III) in a simple way (which generalize the results of Klein, Landau and of Glimm, Jaffe), starting with the case $\lambda=0$ (I.2), using that some combinations of Schwinger functions, the *Wick-Schwinger functions*, are bounded and continuous (III.1). This permits us to generalize the Feynman-Kac-Nelson formula for the generalized Schwinger distributions (III.2).
We use the following notations, with now d = space-time dimension = 2 : $\mathscr{S}$, $\mathscr{S}'$, $\mathscr{D}$, L^p and C^N instead of $\mathscr{S}(\mathbb{R}^2)$, $\mathscr{S}'(\mathbb{R}^2)$, $\mathscr{D}(\mathbb{R}^2)$, $L^p(\mathbb{R}^2)$, $C^N(\mathbb{R}^2)$ as spaces of real-valued functions (but their Fourier transform may be complex valued). We will call $\{j_n, n\in\mathbb{N}^*\}$ a *sequence in* $\mathscr{D}$ *which approximates* δ if $j_n(x)=n^2 j(nx)$ for all $x\in\mathbb{R}^2$ and $n\in\mathbb{N}^*$, where j is every function of $\mathscr{D}$ satisfying $\int j=1$.

I. The free model

The first example $\mathscr{C}:\mathscr{S}\to\mathbb{C}$ is a Gaussian function, which generates a physically trivial theory.

I.1 The model

I.1.1 Definition. The characteristic function of the free model is $\mathscr{C}_0:\mathscr{S}\ni f\to exp\left(-\frac{1}{2}C(f,f)\right)$, where $C(f,f)=\left(f,(1-\Delta)^{-1}f\right)_{L^2}$. Δ is the Laplacian on $\mathbb{R}^2$.

I.1.2 Proposition. *$\mathscr{C}_0$ satisfies* (M) *and* (P).

Proof. (M) *i)* and *ii)* are obvious. For (M) *iii)* we must prove that, for all $n\in\mathbb{N}^*$, $f\in\mathscr{S}^n$ and $z\in\mathbb{C}^n$:

$$\sum_{i,j=1}^{n} \overline{z_i}\, e^{-\frac{1}{2}C(f_i-f_j,\, f_i-f_j)}\, z_j = \sum_{i,j=1}^{n} \overline{z'_i}\, e^{C(f_i,f_j)}\, z'_j$$

is not negative, where $z'_i=z_i\,C(f_i,f_i)$ for all $1\le i\le n$. If A is a n×n non-negative matrix, so is B, defined by $B_{i,j}=exp A_{i,j}$ [Horn, Johnson, § 7.5]. So we have only to verify that C is a non-negative quadratic form, which is made obvious by the identity: $C(f,f) = \int_{\mathbb{R}^2} d^2k\, (k^2+1)^{-1}\, |\tilde{f}(k)|^2$ for all $f \in \mathscr{S}(\mathbb{R}^2)$.

(P) *i)* is obvious. For (P) *ii)* it is enough, as above, to see that for all $f \in \mathscr{S}^+$: $C(\theta\cdot f,f) = \int_{\mathbb{R}^2} \overline{(\widetilde{\theta\cdot f})(k)}\, \tilde{f}(k)\, (k^2+1)^{-1} d^2k$ is non-negative. Recall that support property gives analyticity by Fourier transformation. If $f \in \mathscr{S}^+$, then $\tilde{f}(\overset{\circ}{k}, \vec{k})$ is analytic for $\overset{\circ}{k}$ in the half plane $\mathscr{I}m\overset{\circ}{k} \geq 0$, and $|\tilde{f}(\overset{\circ}{k}, \vec{k})|$ decreases if $\mathscr{I}m\overset{\circ}{k} \to +\infty$; and so is $\overline{(\widetilde{\theta\cdot f})(\overset{\circ}{k}, \vec{k})} = \tilde{f}(\overset{\circ}{k}, -\vec{k})$ (recall that f is real-valued). Then by the residues formula

$$C(\theta\cdot f,f) = \pi \int_{\mathbb{R}} d\vec{k}\ \overline{\tilde{f}(i\omega(\vec{k}), -\vec{k})}\ \tilde{f}(i\omega(\vec{k}), \vec{k})\ \omega(\vec{k})^{-1}$$

where $\omega(\vec{k}) = (\vec{k}^2+1)^{1/2}$. Because f is real-valued $\overline{\tilde{f}(ic,-\vec{k})} = \tilde{f}(ic,\vec{k})$ for all c, $k \in \mathbb{R}$, $c \geq 0$. Then $C(\theta\cdot f,f) = \pi \int d\vec{k}\, |\tilde{f}(i\omega(\vec{k}),\vec{k})|^2\, \omega(\vec{k})^{-1}$ which is obviously non-negative.

Let us prove (P) *iii)*. For all f, $g \in \mathscr{S}$, $C(f,x\cdot g) = \int \overline{\tilde{f}(k)}\, \tilde{g}(k)\, (k^2+1)^{-1} e^{ik\cdot x} d^2k$ goes to 0 as $\|x\| \to \infty$ (Riemann Lemma); thus the following expression also tends to 0 :

$$\mathscr{C}_o(f+x\cdot g) - \mathscr{C}_o(f)\mathscr{C}_o(g) = \mathscr{C}_o(f)\mathscr{C}_o(g) \left(e^{-C(f,x\cdot g)} - 1\right).$$

We prove now (P) *iv)*. The computation of the derivatives of $\mathscr{C}_o(\alpha f)$ gives :

$$\partial^n_\alpha \mathscr{C}_o(\alpha f)\Big|_{\alpha=0} = \begin{cases} = \dfrac{(-1)^{n/2}\, n!}{(n/2)!\ 2^{n/2}}\ C(f,f)^{n/2} & \text{for n even} \\ = 0 & \text{for n odd} \end{cases}$$

for all $n \in \mathbb{N}$. We define a Schwartz norm $|f| = sup_{k \in \mathbb{R}^2} \sqrt{k^2+1}\, |\tilde{f}(k)|$ and a finite constant $K = \left[\int d^2k\, (k^2+1)^{-2}\right]^{1/2}$. Then $C(f,f) \leq K^2\, |f|^2$. We have found :

$$\left| \partial^n_\alpha \mathscr{C}_o(\alpha f)\Big|_{\alpha=0} \right| \leq n!\ K^n\ |f|^n \ . \quad \blacklozenge$$

I.1.3 The free model. Let ϕ_f, for $f \in \mathscr{S}$, be as in the First Part. The map $\phi : \mathscr{S} \to (Q,\Sigma)$ is the *Euclidean field.* From the Proposition I.1.2 and the Minlos theorem (see First Part) there exists a unique probability measure μ_o on Σ such that :

$$\mathscr{C}_o(f) = \int_{q \in Q} d\mu_o(q)\ exp\ i\phi_f(q)$$

for all $f \in \mathscr{S}$. From the Proposition I.1.2 and the reconstruction theorem (First Part) there exists a Wightman Quantum Field model, the *free model*, the Schwinger distributions S_n of which are the moments of μ_o :

$$S_n(f_1,\ldots,f_n) = \int_{q \in Q} d\mu_o(q)\ \phi_{f_1}(q) \cdots \phi_{f_n}(q) \quad \text{for all } n \in \mathbb{N}^* \text{ and } f \in \mathscr{S}^n.$$

I.2 Euclidean Wick fields of the free model

We construct here useful random variables on the probability space (Q,Σ,μ_o).

I.2.1 Wick polynomials. We define the polynomials of ϕ_f:

$$:\phi_f^{\,n}: \;=\; i^{-n}\;\partial_\alpha^n\;\frac{e^{i\alpha\phi_f}}{\mathscr{C}_o(\alpha f)}\Bigg|_{\alpha=0}$$

for all $n\in\mathbb{N}^*$, called *Wick polynomials*, closely related to the Hermite polynomials; for n=0, we put $:\phi_f^{\,o}:=1$. We obtain $:\phi_{f_1}\cdots\phi_{f_n}:$ with $f\in\mathscr{S}^n$ by n-linearity. We will use the algebraic notation : $:A+B: = :A:+:B:$. Let $\{j_n\ ,\ n\in\mathbb{N}^*\}$ be a sequence in $\mathscr{D}$ which approximate δ, $f\in L^2$ and $m\in\mathbb{N}^*$. We define a sequence : $\{\psi_n=\int d^2x\ f(x)\ :(\phi_{x\cdot j_n})^m:\ ,\ n\in\mathbb{N}^*\}$. The following result is due to Nelson ([Nelson] or [Dimock]).

I.2.2 Lemma. $\{\psi_n\ ,\ n\in N^*\}$ *is a Cauchy sequence in* $L^2(Q,\mu_o)$.

Proof. The expectation of a product of two Wick polynomials gives :

$$\int_Q :\phi_f^{\,n}:\,:\phi_g^{\,m}:\,d\mu_o \;=\; i^{-(n+m)}\,\partial_\alpha^n\,\partial_\beta^m\,\frac{\mathscr{C}_o(\alpha f+\beta g)}{\mathscr{C}_o(\alpha f)\,\mathscr{C}_o(\beta g)}\Bigg|_{\alpha=\beta=0} \;=\; \delta_{n,m}\,n!\,C(f,g)^n$$

for all $n,m\in\mathbb{N}$ and $f,g\in L^2$; $\delta_{.,.}$ is the Kronecker tensor. After calculations we find

$$\|\psi_n-\psi_{n'}\|^2_{L^2(Q,\mu_o)} \;=\; m!\,(2\pi)^2\int_{\mathbb{R}^{2m}}\left(\prod_{i=1}^m\frac{d^2k_i}{k_i^2+1}\right)\left|\tilde f\left(\sum_{i=1}^m k_i\right)\right|^2\;\left|J_n(k)-J_{n'}(k)\right|^2$$

for all $n,n'\in\mathbb{N}^*$, where $J_n(k)=\tilde j_n(k_1)\cdots\tilde j_n(k_m)$. Now $\tilde j_n(k)=\tilde j(k/n)$ for all $k\in\mathbb{R}^2$, which converges pointwise to $\tilde j(0)$ when $n\to\infty$ (recall that $\tilde j(0)=1/2\pi$ for all $j\in\mathscr{D}$ with $\int j=1$). Moreover $|\tilde j(k/n)|\le sup_{x\in\mathbb{R}^2}|\tilde j(x)| = (\|j\|_{L^1})/2\pi$ for all $k\in\mathbb{R}^2$, $n\in\mathbb{N}^*$ and $\int(\prod d^2k_i/(k_i^2+1))|\tilde f(\sum k_i)|^2$ is well defined because it can be written as $\int d^2k\,|\tilde f(k)|^2\,G_n(k)$, bounded by $(\|F\|_{L^2})^2\,\|G_n\|_{L^\infty}$, where G_n is given in Appendix III. It follows now from the d.c. theorem that $\|\psi_n-\psi_{n'}\|$ goes to 0 when $n,n'\to\infty$. ♦

Remark We have taken $j\in\mathscr{S}$ because of the definition of the Euclidean fields I.1.3, but the above proof uses only that $j\in L^1$.

I.2.3 Definition. Because $L^2(Q,\mu_o)$ is complete $\{\psi_n,\ n\in\mathbb{N}^*\}$ has a limit :

$$:\phi^m:(f) \;=\; \lim_{n\to\infty}\int_{\mathbb{R}^2} d^2x\ f(x)\ :(\phi_{x\cdot j_n})^m:$$

and the maps $:\phi^m:$ from $L^2(\mathbb{R}^2)$ to $L^2(Q,\mu_o)$, for all $m\in\mathbb{N}^*$, are called *Euclidean*

Wick fields. (We don't mention j because the limit depends only on $\tilde{j}(0)=1/2\pi$, see the proof of I.2.2). For m=0 we put $:\phi^0:(f)=1$. For m=1 we have $:\phi^1:(f) = \phi_f$.

I.2.4 Lemma. *For all* m, $p\in\mathbb{N}^*$ *and* $f\in L^2$ *we have* $:\phi^m:(f) \in L^p(Q,\mu_o)$.

This result is also due to Nelson ([Nelson] or [Dimock]). For a more detailed proof, see [Frochaux, d].

II. The weakly coupled $P(\varphi)_2$ models

We resume the main steps of the construction of the weakly coupled $P(\varphi)_2$ models, without giving the proofs. Let $P : \mathbb{R} \to \mathbb{R}$ be a polynomial bounded from below, fixed for all this paper.

II.1 The regularized models

Let Λ be a compact set of $\mathbb{R}^2$ (the *cutoff*), $\{j_n, n\in\mathbb{N}^*\}$ be a sequence in $\mathscr{D}$ which approximate δ, and consider the sequence $\left\{\chi_n=exp\left(-\int_\Lambda d^2x :P(\phi_{x\cdot j_n}):\right) ; n\in\mathbb{N}^*\right\}$.

II.1.1 Theorem. $\{\chi_n, n\in\mathbb{N}^*\}$ *is a Cauchy sequence in* $L^p(Q,\mu_o)$ *for all* $p\in\mathbb{N}^*$.

This Theorem is due to Nelson. He uses a slightly different sequence, but his proof ([Nelson] or [Dimock]) admits the case given here.
From the theorem, $\{\chi_n, n\in\mathbb{N}^*\}$ has a unique limit in all $L^p(Q,\mu_o)$, $p\in\mathbb{N}^*$. We denote it $q\to e^{-V_\Lambda(q)}$. Note the factorization law $e^{-V_{\Lambda\cup\Lambda'}} = e^{-V_\Lambda} e^{-V_{\Lambda'}}$ for all disjoint compact sets Λ,Λ' of $\mathbb{R}^2$. If the polynomial is replaced by λP, $\lambda\geq 0$, the limit is noted $exp\text{-}\lambda V_\Lambda$.

II.1.2 Measure of the regularized models. We introduce a probability measure $\mu_{\lambda,\Lambda}$ on Σ by :

$$d\mu_{\lambda,\Lambda}(q) = \frac{1}{Z_{\lambda,\Lambda}} d\mu_o(q) \; e^{-\lambda V_\Lambda(q)} \qquad \text{for all } q\in Q$$

which is well defined for all $\lambda\geq 0$ and Λ compact set of $\mathbb{R}^2$. $Z_{\lambda,\Lambda}$ is the normalization factor. Note that the constant of the polynomial P disappears in the division by $Z_{\lambda,\Lambda}$. Thus the prescription "P bounded from below" can be replaced by "P positive valued".

II.1.3 Moments of $\mu_{\lambda,\Lambda}$. The measure $\mu_{\lambda,\Lambda}$ is absolutely continuous with respect to μ_o. So are well defined the *generalized moments of* $\mu_{\lambda,\Lambda}$:

$$S^m_{\lambda,\Lambda}(f) = \int_Q d\mu_{\lambda,\Lambda} \prod_{i=1}^n :\phi^{m_i}:(f_i)$$

for all $n\in\mathbb{N}^*$, $m\in(\mathbb{N}^*)^n$ and $f\in\mathscr{S}^n$. The goal now is to remove the cutoff, i. e. to perform the limit $\Lambda\to\mathbb{R}^2$. The first step is to establish bounds on $S^{m}_{\lambda,\Lambda}$ independent of Λ. We consider a Banach space $\mathscr{B}$ of functions from $\mathbb{R}^2$ to $\mathbb{R}$, given in Appendix II. For all $n\in\mathbb{N}^*$ and $f\in\mathscr{B}^n$ we write: $A_n(f)=\prod_{1\le i\le n}\|f_i\|_{\mathscr{B}}$. For all U, $V\subset\mathbb{R}^2$ disjoint, we define: $\mathscr{B}_U=\{f\in\mathscr{B}$, support of $f\subset U\}$ and d(U,V) = smallest distance between U and V (usual distance in $\mathbb{R}^2$).

II.1.4 Theorem. *There exist* K, $\underline{\lambda}$, $\underline{m}\in(0,\infty)$, *depending only on* P, *such that for all* Λ *compact set of* $\mathbb{R}^2$ *and all* $\lambda\in[0,\underline{\lambda}]$, *for all* $n, n'\in\mathbb{N}^*$, $m\in(\mathbb{N}^*)^n$, $m'\in(\mathbb{N}^*)^{n'}$, *(with* $\ell=\sum_{i\le i\le n} m_i$ *and* $\ell'=\sum_{i\le i\le n'} m'_i$*) :*

i) $|S^{m}_{\lambda,\Lambda}(f)| < \ell!\, K^{\ell}\, A_n(f)$ *for all* $f\in\mathscr{B}^n$

ii) for all U, $V\subset\mathbb{R}^2$ *disjoint,* $f\in(\mathscr{B}_U)^n$ *and* $g\in(\mathscr{B}_V)^{n'}$:

$$\left|S^{m+m'}_{\lambda,\Lambda}(f\otimes g) - S^{m}_{\lambda,\Lambda}(f)\, S^{m'}_{\lambda,\Lambda}(g)\right| < \ell!\,\ell'!\, K^{\ell+\ell'} A_n(f)\, A_{n'}(g)\, exp\,(-\underline{m}\, d(U,V))$$

The Theorem, due to Glimm, Jaffe and Spencer, is the main and the most difficult step of the construction of the $P(\varphi)_2$ models; its proof consists of the control of the so-called *cluster expansion* [Glimm, Jaffe, §18]. The Theorem gives an extension of $S^{m}_{\lambda,\Lambda}$, originally in $(\mathscr{S}')^n$, to a continuous n-linear form on $\mathscr{B}$. Note that the bounds are independent of λ in $[0,\underline{\lambda}]$.

II.2. Two convergence theorems

Fix $\lambda\in[0,\underline{\lambda}]$ for all this paper. The limit $\Lambda\to\mathbb{R}^2$ is taken first on the moments $S^{m}_{\lambda,\Lambda}$.

Let B_r be the closed ball in $\mathbb{R}^2$ of centre O and radius r. A sequence $\{\Lambda_j, j\in\mathbb{N}\}$ of compact sets of $\mathbb{R}^2$ is said to be *admissible* if $B_j\subset\Lambda_j$ for all $j\in\mathbb{N}$.

II.2.1 Theorem. *Take* $n\in\mathbb{N}^*$, $m\in(\mathbb{N}^*)^n$ *and* $f\in\mathscr{B}^n$. *For all admissible sequences* $\{\Lambda_j, j\in\mathbb{N}\}$ *of compact sets of* $\mathbb{R}^2$, $S^{m}_{\lambda,\Lambda_j}(f)$ *converges if* $j\to\infty$ *to a unique limit.*

The proof is in [Glimm, Jaffe, §18] or [Dimock].
We are interested now in the convergence of the measure $\mu_{\lambda,\Lambda}$.

II.2.2 Theorem. *Take* $\sigma\in\Sigma$. *For all admissible sequences* $\{\Lambda_j, j\in\mathbb{N}\}$ *of compact sets of* $\mathbb{R}^2$, $\mu_{\lambda,\Lambda_j}(\sigma)$ *converge if* $j\to\infty$ *to a unique limit.*

[Glimm, Jaffe, Corollary 18.1.3] gives the convergence of the measures, but in a weaker sense, as in II.2.5. The proof of the theorem needs three lemmas.

II.2.3 Lemma. *Let* $\{\sigma_j,\ j\in\mathbb{N}\}$ *be a family of Borel probability measures on* $\mathbb{R}$ *such that for all* n, $j\in\mathbb{N}$ *the moment* $F_{n,j}=\int_{\mathbb{R}} x^n\, d\sigma_j(x)$ *satisfies : 1)* $|F_{n,j}| < n!\, K^n$ *for some fixed* $K\in(0,\infty)$, *2)* $lim_{j\to\infty} F_{n,j}$ *exists. Then there exists a unique Borel probability measure* σ *on* $\mathbb{R}$ *such that* :

i) $\int_{\mathbb{R}} x^n\, d\sigma(x) = lim_{j\to\infty} F_{n,j}$ *for all* $n\in\mathbb{N}$

ii) $\int_{\mathbb{R}} e^{ipx}\, d\sigma(x) = lim_{j\to\infty} \int_{\mathbb{R}} e^{ipx}\, d\sigma_j(x)$ *for all* $p\in\mathbb{R}$.

II.2.4 Lemma. *Let* $\{\mu_j,\ j\in\mathbb{N}\}$ *be a family of probability measures on* Σ *such that for all* n, $j\in\mathbb{N}$ *and* $f\in\mathscr{S}$ *the moment* $F_{n,j}(f) = \int_Q (\phi_f)^n\, d\mu_j$ *satisfies: 1)* $|F_{n,j}(f)| < n!\, K^n\, |f|^n$ *for some fixed* $K\in(0,\infty)$ *and Schwartz space norm* $|\ldots|$, *2)* $lim_{j\to\infty} F_{n,j}(f)$ *exists. Then there exists a unique probability measure* μ *on* Σ *such that* :

i) $\int_Q (\phi_f)^n\, d\mu = lim_{j\to\infty} F_{n,j}(f)$ *for all* $n\in\mathbb{N}$ *and* $f\in\mathscr{S}$

ii) $\int_Q e^{i\phi_f}\, d\mu = lim_{j\to\infty} \int_Q e^{i\phi_f}\, d\mu_j$ *for all* $f\in\mathscr{S}$.

The proofs of Lemmas II.2.3 and II.2.4 (standard!) are given in [Frochaux, d].

II.2.5 Weak convergence of the measures μ_{λ,Λ_j}**.** The hypothesis of Lemma II.2.4 are satisfied by the sequence $\{\mu_{\lambda,\Lambda_j},\ j\in\mathbb{N}\}$: *1)* follows from theorem II.1.4 *i)* and Lemma A.II *ii)*; 2) follows from Theorem II.2.1. So there exist a unique probability measure μ_λ on Σ such that :

$$\int_Q e^{i\phi_f}\, d\mu_\lambda = lim_{j\to\infty} \mathscr{C}_{\lambda,\Lambda_j}(f) \qquad \text{where} \qquad \mathscr{C}_{\lambda,\Lambda_j}(f) = \int_Q e^{i\phi_f}\, d\mu_{\lambda,\Lambda_j}$$

$$\int_Q (\phi_f)^n\, d\mu_\lambda = lim_{j\to\infty} \int_Q (\phi_f)^n\, d\mu_{\lambda,\Lambda_j}$$

for all $f\in\mathscr{S}$ and $n\in\mathbb{N}$. We have obtained the weak convergence of the measure μ_{λ,Λ_j} as in [Glimm, Jaffe, Corollary 18.1.3].

II.2.6 Lemma. *For all* $f\in\mathscr{S}$, $f\neq 0$, *there exist* $M\in L^1(\mathbb{R})$, *depending only on* P *and* f, *such that* $|\mathscr{C}_{\lambda,\Lambda}(\alpha f)| \leq M(\alpha)$ *for all* $\alpha\in\mathbb{R}$, $\lambda\in[0,\underline{\lambda}]$ *and* Λ *compact set of* $\mathbb{R}^2$.

Proof This is a consequence of the *Integration by parts formulas* of Appendix I. Take $f\in\mathscr{S}$, $f\neq 0$, and let us compute the derivation :

$$D_{Cf}\, e^{i\alpha\phi_f(q)} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(e^{i\alpha q(f) + i\alpha\varepsilon C(f,f)} - e^{i\alpha q(f)} \right) =$$

$$= e^{i\alpha\phi_f(q)} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(e^{i\alpha\varepsilon C(f,f)} - 1 \right) = e^{i\alpha\phi_f(q)}\, i\alpha C(f,f)$$

for all $q\in Q$. We have found : $D_{Cf}\, \dfrac{e^{i\alpha\phi_f(q)}}{i\alpha C(f,f)} = e^{i\alpha\phi_f(q)}$.

We use the Proposition A.I.2 of Appendix I, with N=1, $F=e^{i\alpha\phi_f}/i\alpha C(f,f)$ and $G=(Z_{\lambda,\Lambda})^{-1}\exp(-\lambda V_\Lambda)$, the derivatives of G being given by Lemma A.I.4, to find :

$$\mathscr{C}_{\lambda,\Lambda}(\alpha f) = \frac{1}{i\alpha C(f,f)} \int_Q d\mu_{\lambda,\Lambda}\, e^{i\alpha\phi_f}\, (P'_{\lambda,\Lambda}(Cf) - \phi_f)$$

(P' is the derivative of P). Note that there is no singularity when $\alpha\to 0$, the integral being $O(\alpha)$. The same formula, with N=2 and $F=-e^{i\alpha\phi_f}/\alpha^2 C(f,f)^2$ gives :

$$\mathscr{C}_{\lambda,\Lambda}(\alpha f) = \\ = -\frac{1}{\alpha^2 C(f,f)^2}\int_Q d\mu_{\lambda,\Lambda}\, e^{i\alpha\phi_f}\left(P''_{\lambda,\Lambda}(Cf,Cf) + (P'_{\lambda,\Lambda}(Cf))^2 - 2\phi_f\, P'_{\lambda,\Lambda}(Cf) + :\phi_f^{\,2}:\right)$$

(P" is the second derivative of P). Again, there is no singularity when $\alpha\to 0$, the integral being $O(\alpha^2)$. For $|\alpha|>1$ we use the theorem II.1.4 to write :

$$|\,\mathscr{C}_{\lambda,\Lambda}(\alpha f)\,| \;\le\; \frac{1}{\alpha^2}\;\frac{K'\,|f|^2}{C(f,f)}$$

for some $K'\in(0,\infty)$. For $|\alpha|\le 1$ we use $|\mathscr{C}_{\lambda,\Lambda}(\alpha f)|\le 1$. We define $M(\alpha)$ as the found bound. ♦

Remark Using the integration by parts formula again and again, we find that $\alpha\to\mathscr{C}_{\lambda,\Lambda}(\alpha f)$ decreases when $|\alpha|\to\infty$ faster than the inverse of any polynomial, uniformly in Λ. And so is its limit when $\Lambda\to\mathbb{R}^2$, in the sense of II.2.5.

II.2.7 Proof of the Theorem. Because the sets $C_f(B)=\{q\in Q,\ \phi_f(q)\in B\}$, with $f\in\mathscr{S}$ and B a Borelian of $\mathbb{R}$, generate the σ-algebra Σ, we restrict ourselves to them. From the Fubini theorem and Lemma II.2.6, we have for all $j\in\mathbb{N}$:

$$\int_{\mathbb{R}} d\alpha\; \mathscr{C}_{\lambda,\Lambda_j}(\alpha f)\; \sqrt{2\pi}\;\tilde{\chi}_B(\alpha) \;=\; \int_Q \chi_B(\phi_f(q))\, d\mu_{\lambda,\Lambda_j}(q) \;=\; \mu_{\lambda,\Lambda_j}(C_f(B))$$

where χ_B is the characteristic function of B. We take the limit $j\to\infty$. Referring to the d.c. theorem and the lemma II.2.6, the limit and the integral can be permuted in the l. h. s. and we obtain :

$$\lim_{j\to\infty} \mu_{\lambda,\Lambda_j}(C_f(B)) \;=\; \int_{\mathbb{R}} d\alpha\; \lim_{j\to\infty} \mathscr{C}_{\lambda,\Lambda_j}(\alpha f)\,\sqrt{2\pi}\;\tilde{\chi}_B(\alpha) \;=\; \mu_\lambda(C_f(B)) \;.\quad ♦$$

II.3 The weakly coupled $P(\varphi)_2$ models

Let $\mathscr{C}_\lambda(f) = \int_Q e^{i\phi_f}\, d\mu_\lambda = \lim_{j\to\infty} \mathscr{C}_{\lambda,\Lambda_j}(f)$, for $f\in\mathscr{S}$, be the characteristic function of the measure $\mu_\lambda = \lim_{j\to\infty}\mu_{\lambda,\Lambda_j}$ announced by the theorem II.2.2.

II.3.1 Theorem. $\mathscr{C}_\lambda$ *satisfies* (P).

Proof. We verify each axioms (P) one after a other.
(P) *i)* : let $\gamma\in\mathscr{G}$, Λ a compact set of $\mathbb{R}^2$ and $f\in\mathscr{S}$. From the definition of $\mu_{\lambda,\Lambda}$ (II.1.2)

and the invariance of μ_o (Proposition I.1.2), we have : $\mathscr{C}_{\lambda,\Lambda}(\gamma\cdot f) = \mathscr{C}_{\lambda,\gamma^{-1}\cdot\Lambda}(f)$.
If $\{\Xi_j , j\in \mathbb{N}\}$ is an admissible sequence of compact sets of $\mathbb{R}^2$, so is $\{\gamma^{-1}\cdot\Xi_j , j\in \mathbb{N}\}$. By the Theorem II.2.2, the limit $\Lambda\to\mathbb{R}^2$ gives on both sides the function $\mathscr{C}_\lambda$.
(P) *ii)* : let us define a closed subspace of $L^2(Q,\mu_o)$: L^+= closure of the span of $\{e^{i\phi_f}, f\in \mathscr{S}^+\}$. Because $\mathscr{C}_o$ satisfies (P) (Proposition I.1.2), L^+ have the property $\int_Q \overline{\psi}\, \theta\cdot\psi\, d\mu_o \geq 0$ for all $\psi\in L^+$ (Lemma 3, First Part). Let Λ be a compact set of $\mathbb{R}^2$ such that $\Lambda=\Lambda^+\cup \Lambda^-$, with $\Lambda^+\cap \Lambda^-=\emptyset$ and $\theta\cdot\Lambda^+=\Lambda^-$. Then $d\mu_{\lambda,\Lambda} = Z_{\lambda,\Lambda}{}^{-1}\, d\mu_o\, E_{\Lambda^+}\, \theta\cdot E_{\Lambda^+}$ where :

$$E_{\Lambda^+} = \lim_{n\to\infty}\ exp\left(-\lambda \int_{\Lambda^+} d^2x\ :P(\phi_{x\cdot j_n}):\right)$$

where $\{j_n , n\in \mathbb{N}\}$ is a sequence which approximates δ, with the special condition: $j\in \mathscr{S}^+$. The existence of this limit is assured by Theorem II.1.1. Thus $E_{\Lambda^+}\in L^+$.
Because μ_o satisfies (P) we have, for all $n\in \mathbb{N}^*$, $f\in (\mathscr{S}^+)^n$ and $z\in \mathbb{C}^n$:

$$\sum_{i,j=1}^{n} \overline{z}_i\ \mathscr{C}_{\lambda,\Lambda}(\theta\cdot f_i-f_j)\ z_j = \frac{1}{Z_{\lambda,\Lambda}} \int_Q \overline{\psi}\ \theta\cdot\psi\ d\mu_o \geq 0$$

where $\psi = E_{\Lambda^+} \sum_{i\leq j\leq n} z_j\, e^{i\phi_{f_j}} \in L^+$. We take an admissible sequence of compact sets of $\mathbb{R}^2$: $\{\Lambda_n , n\in \mathbb{N}\}$ such that $\Lambda_n=\Lambda_n{}^+\cup\Lambda_n{}^-$, with $\Lambda_n{}^+\cap\Lambda_n{}^-=\emptyset$ and $\theta\cdot\Lambda_n{}^+=\Lambda_n{}^-$ for all $n\in \mathbb{N}$. Then we take the limit $n\to\infty$, using Theorem II.2.2. The function $\mathscr{C}_\lambda$ appears in the l. h. s. of the above equation. The non-negativity is preserved during the limit.
(P) *iv)* : the inequalities of Theorem II.1.4 *i)*, uniform in the cutoff Λ, are preserved in the limit $\Lambda\to\mathbb{R}^2$; then together with the Lemma A.II *ii)*, they give :

$$\left|\int_Q (\phi_f)^n\, d\mu_\lambda\right| \leq n!\ K^n\ |f|^n$$

for some Schwartz norm $|...|$, for all $f\in \mathscr{S}$ and $n\in \mathbb{N}$. By Lemma 1, First Part, the function $\mathbb{R}\ni\alpha\to\mathscr{C}_\lambda(\alpha f)$, for fixed $f\in \mathscr{S}$, is of class C^∞, and the following formulas hold

$$d^n\mathscr{C}_\lambda(\alpha f)|_{\alpha=0} = i^n \int_Q (\phi_f)^n\, d\mu_\lambda \quad \text{for all } n\in \mathbb{N}^*.$$

(P) *iii)* : because of the inequalities on the moments stated above, the span of $\{(\phi_f)^n, n\in \mathbb{N}, f\in \mathscr{S}\}$ is dense in $L^2(Q,\mu_\lambda)$ (First Part, Lemma 7). We will prove that we can replace $\mathscr{S}$ by $\mathscr{D}$, i. e. $\mathfrak{J}$ = span of $\{(\phi_f)^n, n\in \mathbb{N}, f\in \mathscr{D}\}$, is also dense. Let ξ , ζ be two vectors in $\mathfrak{J}$. We denote by P the orthogonal projector in $L^2(Q,\mu_\lambda)$ on the constant functions of Q. The inequalities of Theorem II.1.4 *ii)* being uniform in the cutoff Λ, they are preserved in the limit $\Lambda\to\mathbb{R}^2$, so we can write :

$$\lim_{\|x\|\to\infty} (\xi , (T(x) - P)\, \zeta)_{L^2(Q,\mu_\lambda)} = 0$$

Because $T(x)-P$ is bounded uniformly in x, (P) *iii)* holds. We must now verify that $\mathfrak{J}$ is dense. For this, we prove that for all $n\in \mathbb{N}^*$ and $f\in \mathscr{S}$ there is a vector of $\mathfrak{J}$ very near from $(\phi_f)^n$. Take $\varepsilon>0$. Because $\mathscr{D}$ is dense in $\mathscr{S}$ there exists $g\in \mathscr{D}$ with :

$$|f-g| < \varepsilon\left[(4!\ (4n)!)^{1/4}\ K^n \sum_{j=1}^{n-1} |f|^j\ (|f|+1)^{n-j-1}\right]^{-1}$$

We suppose that ε is small enough for the above r. h. s. to be <1, that is $|g|<|f|+1$. With $(\phi_f)^n - (\phi_g)^n = \phi_{f-g} \sum_{i\leq j\leq n-1}(\phi_f)^j (\phi_g)^{n-j-1}$ and with the CS inequality, we have :

$$\| (\phi_f)^n - (\phi_g)^n \|_{L^2} \leq \| \phi_{f-g} \|_{L^4} \left\| \sum_{j=0}^{n-1} (\phi_f)^j (\phi_g)^{n-j-1} \right\|_{L^4}$$

With the bounds of the moments of μ_λ just found we obtain $\|(\phi_f)^n-(\phi_g)^n\|_{L^2} \leq \varepsilon$. ♦

II.3.2 The weakly coupled $P(\varphi)_2$ models. From the Theorem II.3.1 and the reconstruction theorem of the First Part there exists for all positive polynomial P and all $\lambda\in[0,\underline{\lambda}]$ a Wightman Quantum Field model, called a *$P(\varphi)_2$ model*, whose Schwinger distributions $S_{\lambda,n}$ are the moments of μ_λ :

$$S_{\lambda,n}(f_1,\dots,f_n) = \int_Q d\mu_\lambda(q)\, \phi_{f_1}(q) \cdots \phi_{f_n}(q)$$

for all $n\in\mathbb{N}^*$ and $f\in\mathcal{S}^n$. These relations, writing the Schwinger distributions of a model of quantum field theory as moments of a probability space, are known as the *Feynman-Kac-Nelson formula*. These models describe a world in which interaction actually occurs [Osterwalder, Sénéor], [Eckmann, Epstein, Fröhlich].

II.3.3 The generalized Schwinger distributions. The limit announced by the Theorem II.2.1 are called the *generalized Schwinger distributions* denoted as S^m_λ, where : $S^m_\lambda(f) = \lim_{j\to\infty} S^m_{\lambda,\Lambda_j}(f)$

for all $\lambda\in[0,\underline{\lambda}]$, $n\in\mathbb{N}^*$, $m\in(\mathbb{N}^*)^n$ and $f\in\mathcal{B}^n$; for $m=\{1,1,\dots,1\}$ we write simply $S^m_\lambda = S_{\lambda,n}$.

The inequalities of Theorem II.1.4 are preserved in the limit, because of their uniformity. Thus we have:

i) $|S^m_\lambda(f)| < \ell!\, K^\ell A_n(f)$ for all $f\in\mathcal{B}^n$

ii) for all $U,V\subset\mathbb{R}^2$ disjoint, $f\in(\mathcal{B}_U)^n$ and $g\in(\mathcal{B}_V)^{n'}$:

$$| S^{m+m'}_\lambda(f\otimes g) - S^m_\lambda(f)\, S^{m'}_\lambda(g) | < \ell!\, \ell'!\, K^{\ell+\ell'} A_n(f)\, A_{n'}(g)\, exp\,(- \underline{m}\; d(U,V))$$

for all $\lambda\in[0,\underline{\lambda}]$, $n, n'\in\mathbb{N}^*$, $m\in(\mathbb{N}^*)^n$, $m'\in(\mathbb{N}^*)^{n'}$, (with $\ell=\sum_{i\leq i\leq n} m_i$ and $\ell'=\sum_{i\leq i\leq n'} m'_i$). The constants K, $\underline{\lambda}$ and $\underline{m}$ are those of Theorem II.1.4. The goal of §III is to find a representation of S^m_λ as an integral over Q, i.e. to generalize the Feynman-Kac-Nelson formula for the generalized Schwinger distributions.

III Euclidean Wick fields of the weakly coupled $P(\varphi)_2$ models

III.1 The Wick-Schwinger functions

Some combinations of Schwinger distributions are generated by continuous functions.

III.1.1 Definition. The *Wick-Schwinger distributions* $SW_{\lambda,n}$ are defined by :

$$SW_{\lambda,n}(f) = \int_Q d\mu_\lambda \;:\phi_{f_1}\cdots\phi_{f_n}:$$

where $\lambda\in[0,\underline{\lambda}]$ and $n\in\mathbb{N}^*$, for all $f\in\mathscr{S}^n$; for $n=0$ we take $SW_{\lambda,0}=1$. Note that for $\lambda=0$, $SW_{0,n}=0$ for all $n\in\mathbb{N}^*$. The importance of these distributions can be seen in the following formula, called the *Wick decomposition.*

III.1.2 Lemma. *Let* X *be a finite non-empty subset of* $\mathbb{N}$, p *a partition of* X *and* $f\in\mathscr{S}^X$. *Then the following formula holds (Wick decomposition) :*

$$\prod_{J\in p}:\prod_{j\in J}\phi_{f_j}: \;=\; \sum_{\emptyset\subseteq Y\subseteq X}\left(\int_Q d\mu_0 \prod_{J\in p_Y}:\prod_{j\in J}\phi_{f_j}:\right):\prod_{j\in X-Y}\phi_{f_j}:$$

where p_Y *is the restriction of* p *to* Y.

This formula is given in [Dimock, Glimm].

Proof. The definition of the Wick polynomials leads to :

$$\prod_{J\in p}:\prod_{j\in J}\phi_{f_j}: \;=\; i^{-n}\,\partial_{\alpha_1}\cdots\partial_{\alpha_n}\, e^{i\phi_{\alpha\cdot f}}\left(\prod_{J\in p}\mathscr{C}_0\Big(\sum_{j\in J}\alpha_j f_j\Big)\right)^{-1}\Bigg|_{\alpha=0}$$

with $\alpha\cdot f=\sum_{1\le j\le n}\alpha_j f_j$. We multiply by $1=\mathscr{C}_0(\alpha\cdot f)\mathscr{C}_0(\alpha\cdot f)^{-1}$:

$$\prod_{J\in p}:\prod_{j\in J}\phi_{f_j}: \;=\; i^{-n}\,\partial_{\alpha_1}\cdots\partial_{\alpha_n}\left(e^{i\phi_{\alpha\cdot f}}\,\mathscr{C}_0(\alpha\cdot f)^{-1}\right)\left(\mathscr{C}_0(\alpha\cdot f)\prod_{J\in p}\mathscr{C}_0\Big(\sum_{j\in J}\alpha_j f_j\Big)^{-1}\right)\Bigg|_{\alpha=0}$$

With the Leibniz formula, the r.h.s. can be written as a sum of product of derivatives of each of the two factors. The derivatives of the first factor give the Wick polynomials, and the derivatives of the second factor give the integrals as was announced. ♦

By integration the relation of Lemma III.1.2 gives the following result.

III.1.3 Lemma. *Let* X *be a finite non-empty subset of* $\mathbb{N}$, p *a partition of* X, $f\in\mathscr{S}^X$ *and* $\lambda\in[0,\underline{\lambda}]$. *Then the Wick decomposition gives :*

$$\int_Q d\mu_\lambda \prod_{J\in p}:\prod_{j\in J}\phi_{f_j}: \;=\; \sum_{\emptyset\subseteq Y\subseteq X}\left(\int_Q d\mu_0 \prod_{J\in p_Y}:\prod_{j\in J}\phi_{f_j}:\right) SW_{\lambda,|X-Y|}(f_{X-Y})$$

where $f_{X-Y}=\otimes_{j\in X-Y} f_j$.

Thus for all λ each Schwinger distribution is a sum of products of Schwinger distributions of the free theory and Wick-Schwinger distributions.

We establish now the *Integration by part* formulas for the Wick-Schwinger distributions. We need some notations, because of a little algebraic complication. Let Q be a polynomial, X a finite non-empty subset of $\mathbb{N}$ and p a partition of X.

We denote by Q^k the k-th derivative of Q (if k>deg Q, Q^k=0). For all $\lambda\in[0,\underline{\lambda}]$, Λ compact set of $\mathbb{R}^2$ and $f\in\mathscr{S}^p$, we define :

$$S^{Q,p}_{\lambda,\Lambda}(f) = \int_Q d\mu_{\lambda,\Lambda} \prod_{J\in p} :Q^{|J|}(\phi):(f_J)$$

The limit $\Lambda\to\mathbb{R}^2$, in the sense of Theorem II.2.1, will be denoted as :

$$S^{Q,p}_{\lambda}(f) = \lim_{\Lambda\to\mathbb{R}^2} S^{Q,p}_{\lambda,\Lambda}(f)$$

III.1.4 Lemma. *For all* $n\in\mathbb{N}^*$ *and* $f\in\mathscr{S}^n$, *the Integration by parts formula gives :*

$$SW_{\lambda,n}(f) = \sum_{p\in\mathscr{P}_n} S^{P,p}_{\lambda}(f_p)$$

where $f_p\in\mathscr{S}^{|p|}$ *is given by* $f_p = \otimes_{J\in p}(\prod_{j\in J} Cf_j)$.

In the Lemma, C is the operator $(1-\Delta)^{-1}$ and $\mathscr{P}_n$ is the set of all partitions of $\{1,\dots,n\}$. This formula is written in [Eckmann, Epstein, Fröhlich, proof of theorem 2]. Note that the l. h. s. involve Cf_j instead of f_j, which is the key for finding the regularity of the distributions SW.

Proof. The Corollary A.I.3, with $F=e^{-\lambda V_\Lambda}/Z_{\lambda,\Lambda}$ for some compact set Λ of $\mathbb{R}^2$, and then the Lemma A.I.4 lead to :

$$\int_Q d\mu_{\lambda,\Lambda} :\phi_{f_1}\cdots\phi_{f_n}: = \sum_{p\in\mathscr{P}_n} S^{P,p}_{\lambda,\Lambda}(f_{p,\Lambda})$$

where $f_{p,\Lambda}=\otimes_{J\in p}(\chi_\Lambda\prod_{j\in J} Cf_j)$. The limit $\Lambda\to\mathbb{R}^2$ is performed using Theorem II.2.1.♦

III.1.5 Lemma. *For all* $n\in\mathbb{N}^*$, *the distributions* $SW_{n,\lambda}$ *are generated by continuous functions* $sw_{n,\lambda}$. *Moreover for all* $n\in\mathbb{N}^*$ *there exist* $K\in(0,\infty)$ *with* $|sw_{n,\lambda}(x)|<K$ *for all* $x\in(\mathbb{R}^2)^n$ *and* $\lambda\in[0,\underline{\lambda}]$.

The boundedness is known [Eckmann, Epstein, Fröhlich, proof theorem 2].

Proof. We prove first that $t:(\mathbb{R}^2)^n\ni x=(x_1,\dots,x_n)\to \sum_{p\in\mathscr{P}_n} S^{P,p}_{\lambda}(c_{p,x})$

is bounded and continuous, where $c_{p,x}\in\mathscr{S}^{|p|}$ is given by $c_{p,x}=\otimes_{J\in p}(\prod_{j\in J} x_j\cdot c)$, and c is the L^2 function, the Fourier transform of which is $\mathbb{R}^2\ni k\to(2\pi)^{-1}(k^2+1)^{-1}$. t(x) can be written as a finite sum of following terms :

$$S^m_\lambda(\prod_{j\in J_1} x_j\cdot c, \dots, \prod_{j\in J_k} x_j\cdot c)$$

The boundedness follows from the inequalities II.3.3 *i)* (which is independent of λ) and because $\prod_{j\in J} x_j\cdot c$ belong to $\mathscr{B}$ and has a $\mathscr{B}$-norm bounded for all x_j

(Appendix II, Lemma A.II.2 *vii)*). Let x,y be two points in $(\mathbb{R}^2)^n$. t(x)-t(y) can be written as a finite sum of following terms :

$$S_\lambda^m\big(\Pi_{j\in J_1} x_j\cdot c\,, \dots , \Pi_{j\in J_r} x_j\cdot c - \Pi_{j\in J_r} y_j\cdot c\,, \dots , \Pi_{j\in J_k} y_j\cdot c \big)$$

We use again the inequalities II.3.3 *i)* and we take the limit $\|x-y\|\to 0$. From Lemma A.II.2, *viii)* : $\| \Pi_{j\in J} x_j\cdot c - \Pi_{j\in J} y_j\cdot c \|_{\mathscr{B}} \to 0$, and then $t(x)-t(y)\to 0$.
Let T be the distribution generated by t. Because $t(x)=U(x_1\cdot c,\dots,x_n\cdot c)$ with $U\in \mathscr{S}'((\mathbb{R}^2)^n)$, we have $T(f)=U(c*f_1,\dots,c*f_n)$ for all $f\in (\mathscr{S})^n$. But $c*f$ is just Cf. Comparing with the formula of Lemma III.1.4 we obtain $T=SW_{\lambda,n}$. ♦

III.2 Euclidean Wick fields

The continuity and boundedness of the sw functions permits us to construct the Euclidean Wick fields as in I.2, but in the case $\lambda\neq 0$. We take again the sequence $\{\psi_n\,,\, n\in \mathbb{N}^*\}$ of I.2.1.

III.2.1 Theorem. *For all* $p\in \mathbb{N}^*$ *the sequence* $\{\psi_n,\, n\in \mathbb{N}^*\}$ *converge in* $L^p(Q,\mu_\lambda)$ *to the same limit, denoted by* $:\phi_\lambda^m:(f)$. *The distributions* S_λ^k *can be written as integrals*

$$S_\lambda^k(g) \;=\; \int_Q d\mu_\lambda \prod_{i=1}^{n} :\phi_\lambda^{k_i}:(g_i) \qquad \textit{for all } n\in \mathbb{N}^*,\ k\in (\mathbb{N}^*)^n,\ g\in \mathscr{S}^n.$$

The Theorem generalizes the result of [Klein, Landau] (for even P) and of [Glimm, Jaffe, Theorem 12.2.1] (for P of type *even + linear*). The relation in the theorem can be seen as a generalization of the Feynman-Kac-Nelson formula of II.3.2 to the generalized Schwinger distributions.

III.2.2 Lemma. $\{\psi_n\,,\, n\in \mathbb{N}^*\}$ *is a Cauchy sequence in* $L^2(Q,\mu_\lambda)$.

Proof. We will work in 4 steps.
Step 1 : Definition of a number. Let us denote by κ the following real number :

$$\kappa = \sum_{\alpha=0}^{m} \binom{m}{\alpha}^2 \alpha! \int_{\mathbb{R}^4} d^2x\, d^2y\, f(x)\, f(y)\, c(x-y)^\alpha\, sw_{\lambda,\beta}(x,\dots,y,\dots)$$

where $\beta=2m-2\alpha$ (the x and y variables appear both $m-\alpha$ time in $sw_{\lambda,\beta}$; this will be always understood in this proof). Because $sw_{\lambda,\beta}$ is bounded and continuous (Lemma III.1.5) and the function c of the proof of Lemma III.1.5 belongs to L^p for all $1\leq p<\infty$ (see for instance [Frochaux, b, Appendix A]), κ is well defined.
Step 2 : We write an expression for $(\psi_n, \psi_{n'})-\kappa$. The strong convergence is proved if the following statement holds : for all $\varepsilon>0$ there exists $N\in \mathbb{N}$ such that $|(\psi_n, \psi_{n'})-\kappa| < \varepsilon$ for all n, n'>N. Because of the Wick decomposition (Lemma III.1.2), $(\psi_n, \psi_{n'})-\kappa$ is a finite sum of the following terms :

$$\int_{\mathbb{R}^4} d^2x d^2y\ f(x)\ f(y) \left[C(x\cdot j_n, y\cdot j_{n'})^\alpha \int_Q d\mu_\lambda :(\phi_{x\cdot j_n}\phi_{y\cdot j_{n'}})^{m-\alpha}: - c(x,y)^\alpha\ sw_{\lambda,\beta}(x,\ldots,y,\ldots)\right]$$

multiplied by some combinatoric factor, for all $0\leq\alpha\leq m$, with $\beta=2(m-\alpha)$. We look at the limit $n,n'\to\infty$. If $\alpha=m$ the convergence to 0 is assured by Lemma I.2.2 (free model). The convergence towards 0 if $\alpha=0$ follows from the continuity and boundedness of the sw functions (Lemma III.1.5) and the d.c. theorem, because $f\in L^1$. In the intermediate case $0<\alpha<m$ we write what we study as :

$$\int_{\mathbb{R}^4} d^2x\ d^2y\ f(x)\ f(y)\ C(x\cdot j_n\ ,\ y\cdot j_{n'})^\alpha \left[\int_Q d\mu_\lambda :(\phi_{x\cdot j_n}\ \phi_{y\cdot j_{n'}})^{m-\alpha}: - sw_{\lambda\beta}(x,\ldots,y,\ldots)\right]$$

$$+ \quad \int_{\mathbb{R}^4} d^2x\ d^2y\ f(x)\ f(y) \left[C(x\cdot j_n\ ,\ y\cdot j_{n'})^\alpha - C(x,y)^\alpha\right] sw_{\lambda\beta}(x,\ldots,y,\ldots)$$

The first term will be called A and the second B.

Step 3 : Convergence of A. We know that both $C(x\cdot j_n,y\cdot j_{n'})^\alpha$ and the factor [...] converge as $n,n'\to\infty$ in the distribution sense. The problem we have here is that one of the convergence of a product of distributions. Now $C(x\cdot j_n,y\cdot j_{n'})= j_{n'}*\hat{j}_n*c(x-y)$ where $\hat{j}_n(x)=j_n(-x)$ for all $x\in\mathbb{R}^2$. From the Young inequality, $\|j_{n'}*\hat{j}_n*c\|_{L^p} \leq \|j_{n'}\|_{L^1}$ $\|j_n\|_{L^1}\ \|c\|_{L^p} = (\|j\|_{L^1})^2\ \|c\|_{L^p}$ for all $n\in\mathbb{N}^*$, which is well defined for all $1\leq p<\infty$ because of the integrability of c (Step 1). With the CS inequality in the y variable :

$$|A| \leq \int_{\mathbb{R}^2} d^2x\ \ |f(x)| \left[\int_{\mathbb{R}^2} d^2y\ \ f(y)^2\ \xi_{n,n'}(x,y)^2\right]^{1/2} \left[\int_{\mathbb{R}^2} d^2y\ C(x\cdot j_n\ ,\ y\cdot j_{n'})^{2\alpha}\right]^{1/2}$$

where $\xi_{n,n'}(x,y) = \int_Q d\mu_\lambda :(\phi_{x\cdot j_n}\ \phi_{y\cdot j_{n'}})^{m-\alpha}: - sw_{\lambda,\beta}(x,\ldots,y,\ldots)$. By the above discussion, the last factor $[\ldots]^{1/2}$ is bounded by a constant $K\in(0,\infty)$ independent of n and n'. Thus with the CS inequality in the x variable we obtain :

$$|A| \leq K \left[\int_{\mathbb{R}^2} d^2x\ |f(x)|\right]^{1/2} \left[\int_{\mathbb{R}^2} d^2x\ \ |f(x)| \int_{\mathbb{R}^2} d^2y\ f(y)^2\ \xi_{n,n'}(x,y)^2\right]^{1/2}$$

Because of the continuity and boundedness of the sw functions (Lemma III.1.5), the functions $x,y\to\ \xi_{n,n'}(x,y)$ are bounded by a constant uniformly in n, n' and they go pointwise to 0 when $n,n'\to\infty$. The convergence towards 0 follows from the d.c. theorem, because $f\in L^1$.

Step 4 : Convergence of B. Let us denote by F the function $x,y\to\ f(x)\ f(y)\ s(x,y)$ where $s(x,y) = sw_{\lambda,\beta}(x,\ldots,y,\ldots)$. Note that F is in $L^1(\mathbb{R}^4)$ and thus $\tilde{F}$ is a continuous function. B can be written as :

$$B = (2\pi)^2 \int_{\mathbb{R}^{2\alpha}} \left(\prod_{i=1}^{\alpha} \frac{d^2k_i}{k_i^2+1}\right) \tilde{F}\left(\sum_{i=1}^{\alpha} k_i\ ,\ -\sum_{i=1}^{\alpha} k_i\right) \left[\overline{J_{n'}(k)}\ J_n(k) - (2\pi)^{-2\alpha}\right]$$

where $J_n(k) = \prod_{i=1}^{\alpha} \tilde{j}_n(k_i)$. The factor [...] is bounded for all n, n' and k, and goes pointwise to 0 when $n,n'\to\infty$ (for more details, see the proof of Lemma I.2.2). Moreover $\int (\prod d^2k_i/(k_i^2+1))\ |\tilde{F}(\sum k,-\sum k)|$ is well defined ; with Appendix III this can be written as $\int d^2k \, |\tilde{F}(k,-k)| \, G_n(k)$ bounded by $\|G_n\|_{L^2} \left(\int \tilde{F}(k,-k)^2 \right)^{1/2}$; and $\int \tilde{F}(k,-k)^2 = \int d^2x \, d^2y \, d^2z \, f(x) \, f(y) \, f(z) \, f(x+y-z) \, s(x,y) \, s(z,x+y-z)$ is well defined. The convergence towards 0 follows from the d.c. theorem. ♦

III.2.3 Definition. Because $L^2(Q,\mu_\lambda)$ is complete $\{\psi_n \, , \, n \in \mathbb{N}^*\}$ has a limit, noted as

$$:\phi_\lambda^m:(f) \;=\; \lim_{n\to\infty} \int_{\mathbb{R}^2} d^2x \, f(x) \, :(\phi_{x \cdot j_n})^m:$$

and the maps $:\phi_\lambda^m:$ from $\mathscr{S}(\mathbb{R}^2)$ to $L^2(Q,\mu_\lambda)$, for all $m \in \mathbb{N}^*$, are called *Euclidean Wick Fields*. Note that we mention explicitly the λ dependance. For m=1 we have simply $:\phi_\lambda^1:(f) = \phi_f$ (independent of λ).

III.2.4 Lemma. $:\phi_\lambda^m:(f) \in L^p(Q,\mu_\lambda)$ *for all* m, $p \in \mathbb{N}^*$ *and* $f \in \mathscr{S}(\mathbb{R}^2)$.

Proof. We have only to show that the L^p-norms are bounded uniformly in n, for all even $p \in \mathbb{N}^*$ (see [Frochaux, d, proof of I.2.4]). $(\|\psi_n\|_{L^p})^p$ is a sum of following terms :

$$\int_{\mathbb{R}^{2p}} \left(\prod_{i=1}^{p} d^2x_i \, f(x_i) \right) \left[\int_Q d\mu_o \prod_{i=1}^{p} :(\phi_{x_i \cdot j_n})^{\alpha_i}: \right] \int_Q d\mu_\lambda \, :\prod_{i=1}^{p} (\phi_{x_i \cdot j_n})^{m-\alpha_i}:$$

multiplied by some combinatoric factor, with $0 \le \alpha \le m$ (multi-indices). The last integral is bounded in n (Lemma III.1.5). With the Wick theorem (see for instance [Dimock]), the positivity of the function c (see for instance [Frochaux, b, Appendix A]), the absolute value of the integral over μ_o is bounded by the same expression, with j(x) replaced by $|j(x)|$, which is also in L^1. Thus we have only to control an expectation of the free model, which can be done as in [Frochaux, d, proof of I.2.4], the function $x \to |f(x)|$ being in L^2. ♦

III.2.5 Proof of the Theorem. Let $\{\Lambda_n \, , \, n \in \mathbb{N}^*\}$ be an admissible sequence of compact sets of $\mathbb{R}^2$. The definition of the generalized Schwinger distributions II.3.3 uses the definition of the Wick fields of the free model I.2.3 as follows :

$$S_\lambda^{\ell}(f) \;=\; \lim_{j\to\infty} \; \lim_{k\to\infty} \int_Q d\mu_{\lambda,\Lambda_j} \prod_{i=1}^{n} \int_{\mathbb{R}^2} d^2x \, f_i(x) \, :(\phi_{x \cdot j_k})^{\ell_i}:$$

The convergence of these two limits have been proved in I.2 and II.2. It follows from II.2 and Lemma III.2.2 and III.2.3 that if we permute the two limits, the double convergence exists too. We must prove that this gives the same result, which holds if the convergence $k \to \infty$ is uniform in j. This is indeed true, because we can repeat Lemma III.2.2, III.2.3 and their proof, replacing everywhere μ_λ by μ_{λ,Λ_j}. We use Theorem II.1.4 instead of inequalities II.3.3. All estimations are uniform in the cutoff Λ_j, thus on j. ♦

Appendix I. Integration by parts formulas

This is a set of relations between Schwinger distributions [Glimm, Jaffe]. Here we give more general relations, always obtained in the same way.

A.I.1 Definitions. Let $F : Q \to \mathbb{C}$ be a Σ-measurable function. If $q' \neq 0$ belongs to Q, we define $D_{q'} F$, the *derivative of F in the direction q'*, by :

$$D_{q'}F(q) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big(F(q+\varepsilon q') - F(q)\Big)$$

for all $q \in Q$, whenever the limits exist. F is a *cylindrical random variable* if there exist $n \in \mathbb{N}^*$, $f \in \mathscr{S}^n$ and $\hat{F}: \mathbb{R}^n \to \mathbb{C}$ Borel measurable such that $F(q) = \hat{F}(\phi_{f_1}(q),\dots,\phi_{f_n}(q))$ for all $q \in Q$. If $\hat{F}$ is derivable, then the derivative of F is given by the formula :

$$D_{q'}F(q) = \sum_{i=1}^{n} \Big(\partial_i \hat{F}(\phi_{f_1}(q) , \dots , \phi_{f_n}(q))\Big)\, \phi_{f_i}(q')$$

for all $q, q' \in Q$. We will say that F is N-time derivable, with $N \in \mathbb{N}^*$, if $\hat{F}$ admits N-order partial derivatives, all being Borel measurable.

A.I.2 Proposition. (*Integration by parts formulas*) *Let* $N \in \mathbb{N}^*$ *and* F, G *be two cylindrical random variables* N*-time derivable. Suppose that* F, G *and all their derivatives are in* $L^p(Q,\mu_o)$ *for all* $1 \le p < \infty$. *Then for all* $f \in \mathscr{S}$:

$$\int_Q d\mu_o \,(D^N_{Cf} F)\, G = (-1)^N \int_Q d\mu_o \; F \sum_{k=0}^{N} (-1)^k \binom{N}{k} :\phi^k_f:\, D^{N-k}_{Cf} G$$

The *Integration by parts formulas* of the literature [Glimm, Jaffe] contain no derivatives under the integral of the l.h.s.. The interest of the above generalization is in its applications to the characteristic function (proof of II.2.6). Let us write these formulas for G=1.

A.I.3 Corollary. *Let* N *and* F *be as in the proposition, and* $f \in \mathscr{S}$. *Then* :

$$\int_Q d\mu_o \; D^N_{Cf}F = \int_Q d\mu_o \; :\phi^N_f:\, F$$

To apply these formulas to the expectations with respect to μ_λ, we take $G = exp(-\lambda V_\Lambda)$ (given in II.1.2), then we divided by $Z_{\lambda,\Lambda}$ and take the limit $\Lambda \to \mathbb{R}^2$. We have thus only to know the derivatives of G.

A.I.4 Lemma. *For all positive polynomial* P, $\lambda \ge 0$ *and* Λ *compact set of* $\mathbb{R}^2$ *let us denote* $G = exp\ {-\lambda} :P(\phi):(\chi_\Lambda)$. *Then for all* $N \in \mathbb{N}^*$, $f \in L^2(\mathbb{R}^2)$ *and* $F \in L^p(Q,\mu_o)$ *for all* $1 \le p < \infty$:

$$\int_Q d\mu_o\ F\ D_f^N G = \int_Q d\mu_o\ F\ G \sum_{p\in\mathscr{P}_N} (-\lambda)^{|p|} \prod_{J\in p} :P^{(|J|)}(\phi):\left(\chi_\Lambda(Cf)^{|J|}\right)$$

$P^{(n)}$ is the n-th derivative of P. To prove the proposition, we take first N=1.

A.I.5 Lemma. *Let* F, G *be two derivable cylindrical random variables. Suppose that* F, G *and its derivatives are in* $L^p(Q,\mu_o)$ *for all* $1\le p<\infty$. *Then for all* $f\in\mathscr{S}$:

$$\int_Q d\mu_o\ (D_{Cf}F)\ G = -\int_Q d\mu_o\ F\ (D_{Cf}G - \phi_f\ G)$$

Proof of Lemma A.I.5. We follow [Dimock]. There exist $n\in\mathbb{N}^*$ and $f_1,\dots,f_n\in\mathscr{S}$ linearly independent, with $f=f_1$, such that $F(q)=\hat{F}(\phi_{f_1}(q),\dots,\phi_{f_n}(q))$ and $G(q)=\hat{G}(\phi_{f_1}(q),\dots,\phi_{f_n}(q))$ for some functions $\hat{F}, \hat{G}: \mathbb{R}^n\to\mathbb{C}$, for all $q\in Q$. Let $\{e_1,\dots,e_n\}$ be an orthonormal basis of the span of $\{f_1,\dots,f_n\}$ with respect to the scalar product $g,h\to C(g,h)$. We note $\check{F}$ the function $\hat{F}(\phi_{f_1}(q),\dots,\phi_{f_n}(q)) = \check{F}(\phi_{e_1}(q),\dots,\phi_{e_n}(q))$ for all $q\in Q$ and we define $\check{G}$ in the same way. The derivative of F is given by :

$$D_{Cf_1}F(q) = \sum_{i=1}^n f_{1,i}\ D_{Ce_i}F(q) = \sum_{i=1}^n f_{1,i}\ \partial_i\check{F}(\phi_{e_1}(q),\dots,\phi_{e_n}(q))$$

for all $q\in Q$, where $f_{i,j}=C(f_i,e_j)$. On the other hand, the definition of $\mathscr{C}_o$ (I.1.1) gives

$$\mathscr{C}_o(\alpha\cdot e) = e^{-\frac{1}{2}\sum_{j=1}^n \alpha_j^2} = \int_{\mathbb{R}^n} e^{i\alpha\cdot x}\ (2\pi)^{-n/2}\ e^{-\frac{1}{2}\sum_{j=1}^n x_j^2}\ d^n x$$

for all $\alpha\in\mathbb{R}^n$, where $\alpha\cdot e=\sum_{i\le i\le n}\alpha_i e_i$. Note that $(2\pi)^{-n/2}d^n x\ exp-\frac{1}{2}\sum x_j^2$ is the only probability measure on $\mathbb{R}^n$ such that its characteristic function is $\alpha\to\mathscr{C}_o(\alpha\cdot e)$, because of the uniqueness in the Bochner theorem. We have now :

$$\int_Q d\mu_o\ (D_{Cf}F)\ G = \sum_{i=1}^n f_{1,i} \int_Q d\mu_o\ (\partial_i\check{F}(\phi_{e_1}(q),\dots,\phi_{e_n}(q)))\ \check{G}(\phi_{e_1}(q),\dots,\phi_{e_n}(q))$$

$$= \sum_{i=1}^n f_{1,i} \int_{\mathbb{R}^n} d^n x\ (2\pi)^{-n/2}\ e^{-\frac{1}{2}\sum_{j=1}^n x_j^2}\ (\partial_i\check{F}(x_1,\dots,x_n))\ \check{G}(x_1,\dots,x_n)$$

With the ordinary integration by parts formula, this becomes :

$$-\sum_{i=1}^n f_{1,i} \int_{\mathbb{R}^n} d^n x\ (2\pi)^{-n/2}\ e^{-\frac{1}{2}\sum_{j=1}^n x^2}\ \check{F}(x_1,\dots,x_n)\ (\partial_i\check{G}(x_1,\dots,x_n) - x_i\ \check{G}(x_1,\dots,x_n))$$

$$= -\int_Q d\mu_o\ \check{F}(\phi_{e_1},\dots,\phi_{e_n})\left(\sum_{i=1}^n f_{1,i}\ \partial_i\check{G}(\phi_{e_1},\dots,\phi_{e_n}) - \left(\sum_{i=1}^n f_{1,i}\ \phi_{e_i}\right)\check{G}(\phi_{e_1},\dots,\phi_{e_n})\right)$$

$$= -\int_Q d\mu_o \; F\,(D_{Cf_1}G - \phi_{f_1}\, G)\,. \qquad \blacklozenge$$

A.I.6 Proof of the Proposition. We work by induction on N. If N=1 the Proposition is true (Lemma A.I.5). Let us chose $N\geq 1$, such that the Proposition is true for N-1. By Lemma A.I.5 :

$$\int_Q d\mu_o\, (D^N_{Cf}F)\, G \;=\; -\int_Q d\mu_o\;\, (D^{N-1}_{Cf}F)\;(D_{Cf}\,G - \phi_f\; G)\,.$$

By the induction hypothesis this is equal to :

$$-\;(-1)^{N-1}\int_Q d\mu_o\;\; F\sum_{k=0}^{N-1}(-1)^k\binom{N-1}{k}:\phi_f^k:\,D^{N-1-k}_{Cf}\,(D_{Cf}G - \phi_f\; G)\,.$$

The derivation gives :

$$D^{N-1-k}_{Cf}\,(D_{Cf}G - \phi_f\; G) = D^{N-k}_{Cf}G \;-\; \phi_f\, D^{N-k-1}_{Cf}G \;-\; (N\text{-}k\text{-}1)\; C(f,f)\; D^{N-k-2}_{Cf}G.$$

We use the Wick decomposition III.1.2 to write $:\phi_f^k:\,\phi_f = :\phi_f^{k+1}: + \,k\, C(f,f)\, :\phi_f^{k-1}:$ for all $k\in \mathbb{N}^*$, and some simple algebra gives the claimed result ♦

A.I.7 Proof of Lemma A.I.4. We follows [Dimock, Glimm, proof of Theorem 3.2]. We write G as weak limit of cylindrical random variables. By Theorem II.1.1, $G=lim_{n\to\infty}exp\,(-\lambda\int_\Lambda d^2x\; :P(\phi_{x\cdot j_n}):)$ where $\{j_n\;,\; n\in\mathbb{N}^*\}$ is a sequence of $\mathscr{D}$ which approximate δ. The integral is approached by its Riemann sums :

$$\int_\Lambda d^2x\; :P(\phi_{x\cdot j_n}): \;=\; \lim_{a\to+0}\;\; a^2 \sum_{x\in\mathscr{R}_{\Lambda,a}} :P(\phi_{x\cdot j_n}):$$

where $\mathscr{R}_{\Lambda,a}$ is a suitable lattice. We prove now that the limit holds in the strong $L^p(Q,\mu_o)$ sense for all $1\leq p<\infty$. Because $\|:P(\phi_{x\cdot j_n}):\|_{L^p}$ is well defined it is enough to see the L^2 convergence. This follows from the Riemann theorem, because of the equality :

$$\left\|\int_\Lambda d^2x\; :(\phi_{x\cdot j_n})^m: \,-\, a^2\sum_{x\in\mathscr{R}_{\Lambda,a}} :(\phi_{x\cdot j_n})^m:\right\|^2_{L^2} \;=$$

$$=\; m!\left\{\int_{\Lambda^2} d^2x\; d^2y \;-\; 2\int_\Lambda d^2x\; a^2\sum_{x\in\mathscr{R}_{\Lambda,a}} \;+\; a^4\sum_{x,y\in\mathscr{R}_{\Lambda,a}}\right\} C(x\cdot f,y\cdot f)^m$$

for all $n\in\mathbb{N}^*$ and $f\in L^2$, and because the function $\Lambda\times\Lambda \ni (x,y)\to C(x\cdot f,y\cdot f)$ is continuous. We have now to derive the following random variables :

$$G_{n,a} \;=\; exp\left(-\;\lambda\;\; a^2\sum_{x\in\mathscr{R}_{\Lambda,a}} :P(\phi_{x\cdot j_n}):\right)$$

The derivatives of the exponential gives a sum over partitions (see for instance [Frochaux, b, Appendix B]) :

$$D_f^m G_{n,a} = G_{n,a} \sum_{p\in\mathscr{P}_m} \prod_{J\in p} D_f^{|J|}\left(-\lambda\ a^2 \sum_{x\in\mathscr{R}_{\Lambda,a}} :P(\phi_{x\cdot j_n}): \right)$$

and the derivation of Wick monomials are deduced from $D_f :\phi_g^n: = n :\phi_g^{n-1}: C(f,g)$.

Collecting all these results and taking the right limits gives the Lemma. ♦

Remark. Because the limits $lim_{n\to\infty}lim_{a\to+0}\ G_{n,a}(q)$ do not exist for all $q\in Q$, the derivatives of G we have found hold only inside integrals.

Appendix II. The Banach space $\mathscr{B}$

We present the Banach space of II.1.4 and II.3.3, then complete the proof of Lemma III.1.5.

A.II.1 Definitions. For all $j=(j_1,j_2)\in\mathbb{Z}^2$ let us denote by Δ_j the square $\Delta_j = \{x=(x_1,x_2)\in\mathbb{R}^2, |x_i- j_i| < \frac{1}{2}$ for $i=1, 2\}$. For all $1\le p<\infty$ and $f\in L^p(\mathbb{R}^2)$ with compact support the following number $|||\ f\ |||_p = \sum_{j\in\mathbb{Z}^2} \|\ f\ \chi_{\Delta_j}\|_{L^p}$ is well defined (χ_Δ is the characteristic function of Δ). We define $\mathscr{B}_p$ as the Banach space of norm $|||...|||_p$ obtained by completion. The Banach space $\mathscr{B}$ used in II.1.4 and II.3.3 is $\mathscr{B}_2$.
Let T be the translation operator on $\mathscr{B}_p$, that is $(T(x)f)(y)=f(y-x)$ for all $x, y\in\mathbb{R}^2$ and $f\in\mathscr{B}_p$. $\|...\|_p$ is the norm operator $\|Af\|_p=sup\{|||Af|||_p, |||f|||_p\le 1\}$. For all $n\in\mathbb{N}^*$ and $x\in(\mathbb{R}^2)^n$ we define the function $\mathbb{R}^2-\{x_1,\dots,x_n\}\ni z \to c_x(z) = \prod_{1\le i\le n} c(z-x_i)$, where c is the Fourier transform of $\mathbb{R}^2\ni k \to (2\pi)^{-1}(k^2+1)^{-1}$.

A.II.2 Lemma. *For all* $1\le p<\infty$, *we have :*
i) $\mathscr{B}_p = \{f\in L^p(\mathbb{R}^2), |||f|||_p < \infty\}$
ii) there exist a Schwartz space norm $|...|_p$ *and* $K\in(0,\infty)$ *such that* $|||f|||_p \le K\ |f|_p$ *for all* $f\in\mathscr{S}(\mathbb{R}^2)$
iii) for all $f\in\mathscr{B}_p$ *and* $\varepsilon>0$ *there exist* K, *compact set of* $\mathbb{R}^2$ *and* $g\in\mathscr{D}(K)$ *such that* $|||f-g|||_p < \varepsilon$
iv) $\|T(x)\|_p \le 4^{p-1}$ *for all* $x\in\mathbb{R}^2$
v) for all $f\in\mathscr{B}_p$, $T(x)f \to T(y)f$ *in* $\mathscr{B}_p$ *if* $x\to y$ *in* $\mathbb{R}^2$
vi) $|||fg|||_p \le |||f|||_q\ |||g|||_r$ *for all* $q, r\in\mathbb{R}$ *such that* $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$ *and for all* $f\in\mathscr{B}_q$, $g\in\mathscr{B}_r$
vii) for all $n\in\mathbb{N}^*$ *and* $x\in(\mathbb{R}^2)^n$, $c_x\in\mathscr{B}_p$, *and* $x\to|||c_x|||_p$ *is bounded on all* $(\mathbb{R}^2)^n$
viii) for all $n\in\mathbb{N}^*$: $c_x\to c_y$ *in* $\mathscr{B}_p$ *if* $x\to y$ *in* $(\mathbb{R}^2)^n$.

Proof. *i)* Let $f\in\mathscr{B}_p$. Then f is locally L^p. From the inequality $\|f\|_{L^p}\le|||f|||_p$ (because all convergent sum of non-negative real numbers $a=\sum a_n$ satisfy $\sum(a_n)^p=\sum a_n(a_n)^{p-1}\le a(sup\{(a_n)^{p-1}\}) = a(sup\{a_n\})^{p-1} \le a^p$) follows that $f\in L^p$. Inversely let $f\in L^p$ with

$|||f|||_p<\infty$. For all $n\in\mathbb{N}$ let us define $\Lambda_n=\left[-n-\frac{1}{2}, n+\frac{1}{2}\right]\times\left[-n-\frac{1}{2}, n+\frac{1}{2}\right]\subset\mathbb{R}^2$ and $f_n=\chi_{\Lambda_n}f$. Of corse $f_n\to f$ in L^p. But $\{f_n , n\in\mathbb{N}\}$ is a Cauchy sequence in $\mathscr{B}_p$. Let $g\in\mathscr{B}_p$ its limit. From the inequality of the first line of this proof follows that g is also the L^p-limit of f_n. So f=g a.e. .

ii) The function $h : \mathbb{R}^2\ni x\to(1+x^2)^{-3/2}$ belongs to $\mathscr{B}_p$ (because for $\|j\|$ big enough : $\|h\chi_{\Delta_j}\|_{L^p} < K'\,\|j\|^{-3}$ for some $K'\in(0,\infty)$, and $\sum_{j\in\mathbb{Z}^2,\|j\|>1}\|j\|^{-3}$ is well defined). Thus for all $f\in\mathscr{S}(\mathbb{R}^2)$ we have $|||f|||_p\le K|f|_p$ with $|f|_p=sup_{x\in\mathbb{R}^2}(1+x^2)^{-3/2}|f(x)|$, and $K=|||h|||_p$.

iii) There exists J, a finite subset of $\mathbb{Z}^2$, such that $|||f-f_J|||_p<\varepsilon/2$, where $f_J=f\chi_{K'}$, $K'=\bigcup_{j\in J}\Delta_j$. There exists also a sequence $\{f_n, n\in\mathbb{N}\}$ of $\mathscr{D}(K)$, where K is a compact neighborhood of K', such that $f_n\to f_J$ in L^p. But for the measurable functions with support in K, the norms of L^p and of $\mathscr{B}_p$ are equivalent, and thus $f_n\to f_J$ in $\mathscr{B}_p$. Then there exists $N\in\mathbb{N}$ such that $|||f_N-f_J|||_p<\varepsilon/2$. If $g=f_N$, we have found $|||f-g|||_p\le |||f-f_J|||_p + |||f_J-g|||_p < \varepsilon$.

iv) We introduce the notations : $\mathscr{R}=\{\Delta_j , j\in\mathbb{Z}^2\}$, $\mathscr{R}_x=\{\Delta_{j+x}=x\cdot\Delta_j , j\in\mathbb{Z}^2\}$ and $\mathscr{R}'=\{\Delta\cap\Delta_x , \Delta\in\mathscr{R} , \Delta_x\in\mathscr{R}_x\}$. Let $f\in\mathscr{B}_p$. We have immediately :

$$|||\,T(x)\,f\,|||_p = \sum_{\Delta\in\mathscr{R}}\left(\int_\Delta |f(y-x)|^p\,d^2y\right)^{1/p} = \sum_{\Delta\in\mathscr{R}}\left(\int_{x\cdot\Delta}|f|^p\right)^{1/p} =$$

$$= \sum_{\Delta\in\mathscr{R}}\left(\sum_{\Delta'\in\mathscr{R}',\,\Delta'\subset x\cdot\Delta}\int_{\Delta'}|f|^p\right)^{1/p} \le$$

$$\le \sum_{\Delta\in\mathscr{R}}\ \sum_{\Delta'\in\mathscr{R}',\,\Delta'\subset x\cdot\Delta}\left(\int_{\Delta'}|f|^p\right)^{1/p} = \sum_{\Delta'\in\mathscr{R}'}\left(\int_{\Delta'}|f|^p\right)^{1/p}$$

where the inequality follows from $(a_1+\ldots+a_n)^{1/p}\le(a_1)^{1/p}+\ldots+(a_n)^{1/p}$ for all $n\in\mathbb{N}^*$, $a_1,\ldots,a_n>0$ (*). We collect the cells $\Delta'\in\mathscr{R}'$ belonging to the same $\Delta\in\mathscr{R}$:

$$\sum_{\Delta'\in\mathscr{R}'}\left(\int_{\Delta'}|f|^p\right)^{1/p} = \sum_{\Delta\in\mathscr{R}}\ \sum_{\Delta'\in\mathscr{R}',\,\Delta'\subset\Delta}\left(\int_{\Delta'}|f|^p\right)^{1/p} \le$$

$$\le \sum_{\Delta\in\mathscr{R}}4^{p-1}\left(\sum_{\Delta'\in\mathscr{R}',\,\Delta'\subset\Delta}\int_{\Delta'}|f|^p\right)^{1/p} = 4^{p-1}\,|||\,f\,|||_p$$

where we have used the inequality $(a_1)^{1/p}+\ldots+(a_4)^{1/p}\le 4^{p-1}(a_1+\ldots+a_4)^{1/p}$ for all $a_1,\ldots,a_4>0$ (*) and the geometric following fact : each Δ of $\mathscr{R}$ intersects at most 4 elements of $\mathscr{R}_x$.

(*) We have used the inequalities, for all $n\in\mathbb{N}^*$, $b_1,\ldots,b_n>0$:

$$(b_1+\ldots+b_n)^m\le(b_1)^m+\ldots+(b_n)^m \quad \text{for all } m\in(0,1]$$

$$(b_1+\ldots+b_n)^m\le n^{m-1}\left((b_1)^m+\ldots+(b_n)^m\right) \quad \text{for all } m\in[1,\infty)$$

easily obtained by calculating the supremum of the following function :

$$(0,\infty)^{n-1}\ni(x_1,\ldots,x_{n-1})\to(1+x_1+\ldots+x_{n-1})^m\left(1+(x_1)^m+\ldots+(x_{n-1})^m\right)^{-1}$$

v) For all $\varepsilon>0$ there exist K, a compact set of $\mathbb{R}^2$, and $g\in\mathscr{D}(K)$ such that

$|||f-g|||_p < \varepsilon/(2^{4p})$, by *ii)*. g being uniformly continuous, there exists $\delta>0$ such that, for all $x\in \mathbb{R}^2$, $\|x\|<\delta$, we have $|g(z-x)-g(z)| < \varepsilon/(k2^{2p})$ for all $z\in \mathbb{R}^2$, where k is the number of cells of $\mathscr{R}$ which intersect K. Thus $|||T(x)g-g|||_p < \varepsilon/(2^{2p})$.
For all x, $y\in \mathbb{R}^2$, $\|x-y\|<\delta$, we have now, using many times *iv)* :

$$|||\, T(x)\, f - T(y)\, f\, |||_p = |||\, T(x)\, (\, f - T(y-x)\, f\,)\, |||_p = 4^{p-1}\, |||\, f - T(y-x)\, f\, |||_p \leq$$
$$\leq 4^{p-1}\left(\, |||\, f - g\, |||_p + |||\, g - T(y-x)\, g\, |||_p + |||\, T(y-x)\, (f - g)\, |||_p\,\right) \leq$$
$$\leq 4^{p-1}\left(\, (1+4^{p-1})\, |||\, f - g\, |||_p + |||\, g - T(y-x)\, g\, |||_p\,\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\, .$$

vi) By the Hölder inequality, we have :

$$|||\, f\, g\, |||_p = \sum_{\Delta\in\mathscr{R}} \|\, f\, g\, \chi_\Delta\, \|_{L^p} = \sum_{\Delta\in\mathscr{R}} \|\, f\, \chi_\Delta\, \|_{L^q}\, \|\, g\, \chi_\Delta\, \|_{L^r}$$

which contains only a part of the terms of :

$$|||\, f\, |||_q\; |||\, g\, |||_r = \sum_{\Delta\in\mathscr{R}}\; \sum_{\Delta'\in\mathscr{R}} \|\, f\, \chi_\Delta\, \|_{L^q}\, \|\, g\, \chi_{\Delta'}\, \|_{L^r}\, .$$

vii) By *iv)* and *vi)* it is enough to verify that $c\in \mathscr{B}_q$ for all $1\leq q<\infty$. Because $c\in L^2$ (Step 1, III.2.2) it is enough by *i)* to control $|||c|||_q$. From the inequality: $c(x) \leq$ *exp*-$\|x\|$ for $\|x\|$ big enough [Frochaux, b, Appendix A], it follows that c is bounded asymptotically by the function h of the proof of *ii)*. So $|||c|||_q$ is well defined.
viii) We write c_x-c_y as follows :

$$c_x(z) - c_y(z) = \sum_{i=1}^{n} \left(\prod_{j=1}^{i-1} c(x_j-z)\right)\Big(c(x_i-z) - c(y_i-z)\Big)\left(\prod_{j=i+1}^{n} c(y_j-z)\right)$$

for all $z\in \mathbb{R}^2-\{x_1,\ldots,x_n,y_1,\ldots,y_n\}$. Using *vi)* and *iv)* we obtain :

$$|||\, c_x-c_y\, |||_p < |||\, c\, |||_q^{n-1} \sum_{i=1}^{n} |||\, T(x_i-y_i)\, c - c\, |||_q$$

with q=np. The announced result follows from *v)* and *vii)*. ♦

Appendix III. A technical Lemma

Let c be the Fourier transform of $\mathbb{R}^2\ni k \to (2\pi)^{-1}(1+k^2)^{-1}$. Note that $\tilde{c} \in \bigcap_{1<p\leq\infty} L^p$.

A.III.1 Lemma. *For all* $n\in \mathbb{N}$, $n>2$, *the function* G_n *given by :*

$$G_n(k) = \int_{\mathbb{R}^{2n}} \left(\prod_{i=1}^{n} d^2p_i\; \tilde{c}(p_i)\right) \delta^{(2)}\!\left(\sum_{i=1}^{n} p_i - k\right)$$

for all $k\in \mathbb{R}^2$, *belongs to* $\bigcap_{1<p\leq\infty} L^p$.

Proof. Note that $G_n= \tilde{c} * \tilde{c} *..* \tilde{c}$ (n time). By the Young inequality $\|G_n\|_{L^p}\leq (\|\tilde{c}\|_{L^q})^n$ with $q=np/[1+p(n-1)]$. We see easily that $q\in(1,\infty]$ when $p\in(1,\infty]$. ♦

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