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Regularization of the Mandelstam Soliton Operator

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Abstract. We show how Mandelstam's soliton operator can be written in regularized form, and can be used to obtain regularized bosonization formulas. The fermion field properties which we obtain include the anticommutation relations, which we find are not canonical except for free massive fermions. We are also able to perform an explicit nonperturbative mass renormalisation of the massive Thirring model, by direct derivation of the field equations. We find that these field equations, and some of the fermion anticommutator relations, are valid only in a weak sense. We indicate how the soliton operator can be used to demonstrate the equivalence of a more general class of boson and fermion models. The regularized formulas are also used to investigate the $N = 2$ case of non-Abelian bosonization, including the groups generated by the fermion bilinears, and the corresponding currents.

1. Introduction

Bosonization is the means by which a fermion theory can be rewritten in terms of boson fields, and provides a powerful tool for understanding some aspects of fermion models, particularly the nonperturbative features. Examples of bosonization are the equivalence between the sine-Gordon and massive Thirring models described by Coleman [1], and the non-Abelian bosonization of Witten [2]. For the sine-Gordon case Mandelstam [3] exhibited an explicit map from the boson to the fermion fields, and was able to obtain the bosonization formulas in a very direct way. This included the identification of the fermion and boson currents, the equivalence of the mass and potential terms, and also of the field equations for each model.

Because Mandelstam's soliton operator ψ provides a direct link between the boson and fermion models, it not only determines the bosonization formulas correctly but also provides a direct method of obtaining properly regularized expressions; this can be done by using a regularized form of the soliton operator. If one employs ψ literally, various ambiguities and infinities are encountered, which of course merely reflect the renormalisation required in the fermion theory when two fermion operators are multiplied at the same point. Mandelstam [3] explains how to regularize some of the infinities in the standard way, by point splitting (following the well known procedure outlined for example by Klaiber [4]). Additional infinities appear, however, which can be related to mass renormalisation and which are not fully explained in Ref. [3]. An example is in the derivation of the massive Thirring model field equations, where the mass term arises from the commutator

$$\int [:\cos \beta\phi(\xi):, \psi(x)]\, d\xi \tag{1.1}$$

(see equations (4.6), (4.8) in [3]). The integrand of (1.1) is zero everywhere, except at $\xi = x$, where it is undefined. Suitable regularization of the soliton operator leads to a regularized form of (1.1), and thence to mass renormalisation. Similarly, we find that the fermion anticommutation relations, when nonzero, must be evaluated with the regularized soliton operator, and then hold only in a weak sense. Only for a special value of the boson coupling constant ($\beta^2 = 4\pi$, corresponding to free massive fermions), are the anticommutation relations canonical. Whilst this fact is implicit in earlier work (see e.g. Johnson [5]), Mandelstam's soliton operator demonstrates this directly, and indicates in regularized form the appropriate distribution required.

The purpose of this paper is to show how the Mandelstam soliton operator can be regularized, and be used to obtain regularized fermion equations. In Section 2 we outline the steps by which the soliton operator ψ is determined from the current-fermion commutation relations, and how ψ is regularized (we repeat here some calculations of Mandelstam briefly for completeness, and to indicate where regularization is necessary). In Section 3 we consider the fermion anticommutation relations and other boson-fermion equivalences; again, the derivations given by Mandelstam, where satisfactory, are mentioned only briefly, and we describe in detail the cases for which further regularization is necessary. To this point no dynamics are imposed on the fields, but in Section 4 we consider first the sine-Gordon model, and then more general boson models, and derive fermion field equations, together with the appropriate renormalisation. For the general case fermion renormalisation is much more complicated, but can in principle be carried out using the regularized soliton operator. We discuss models of this generality in order to demonstrate that the Mandelstam soliton operator has properties which are not model dependent, but has a wide range of applications.

In Section 5 we consider regularization in the context of the $N = 2$ case of non-Abelian bosonization. Witten [2] has generalized the bosonization formulas to the case in which the symmetry group is non-Abelian, $O(N) \times O(N)$. The Mandelstam soliton operator, although not known for general N, can be used to check the $N = 2$ case of the bosonization formulas. In Ref. [6] it was shown that a formula of Witten's [2] relating fermion bilinears to elements of the symmetry group was not entirely correct, but that a certain linear combination of the bilinears was required. Orthogonality properties of the fermion bilinears are investigated using the regularization determined in Section 3, and the associated currents are investigated using the field equations found in Section 4. We find that, for the massless fermion case, the currents generated by both sets of fermion bilinears are the same. This explains why some checks by Witten did not reveal a discrepancy. We hope that the methods used here will be of use in the general N case of non-Abelian bosonization, where corresponding soliton operators are not yet known.

2. The Mandelstam soliton operator

Without being restricted to any particular fermion model, let us assume that the fermion currents $j_\mu(x)$ satisfy the equal-time commutation relations

$$\begin{aligned}&[j_0(x), j_0(y)] = [j_1(x), j_1(y)] = 0,\\ &[j_0(x), j_1(y)] = i\delta'(x-y),\end{aligned} \tag{2.1}$$

where we have included the Schwinger term. As explained by Dell'Antonio et al [7], it is preferable to introduce the currents via these commutation relations, rather than as fermion bilinears, to avoid problems arising from the multiplication of fermion fields at the same point. Because the current is conserved, we can introduce a boson field $\phi(x)$ according to the formulas [8]

$$j_0(x) = \phi'(x), \qquad j_1(x) = \dot{\phi}(x) \equiv \pi(x). \tag{2.2}$$

The canonical commutation relations for ϕ and π then imply that the boson currents (2.2) satisfy the commutation relations (2.1). The identification of the boson and fermion currents in this way is the basis for the correspondence between boson and fermion theories.

Next, we introduce the fermion field operators $\psi(x)$ by means of the equal time commutation relations, following Johnson [5] and Dell'Antonio at al [7]:

$$[j_0(x), \psi(y)] = -\frac{2\pi}{\beta}\delta(x-y)\psi(y) \tag{2.3}$$

$$[j_1(x), \psi(y)] = -\frac{\beta}{2}\delta(x-y)\gamma_5\psi(y), \tag{2.4}$$

where ψ is a two-component field and β is a real constant. (Our convention for the γ-matrices follows Coleman [1] and Mandelstam [3]). The particular constants on the right hand side of equations (2.3), (2.4) are chosen for later convenience in order to identify the fermion operator with a soliton field. If an integral multiple of the coefficient $-2\pi/\beta$ in (2.3) is chosen instead, the fermion operator will correspond to a multisoliton field. For a chirally invariant theory, equations (2.3), (2.4) express charge conservation for the vector and axial charges, but we postulate these relations quite generally. We wish to regard the fields ψ as functionals of the boson field ϕ, and its conjugate π, with an explicit form to be determined from equations (2.3), (2.4). To determine this form, we first integrate (2.3) with respect to x, to obtain

$$[\phi(x), \psi(y)] = \frac{2\pi}{\beta}\theta(y-x)\psi(y), \tag{2.5}$$

an equation which is the starting point for Mandelstam and has a soliton interpretation. Now, by regarding $\phi(x)$ and $\pi(x)$ as functional derivatives, we integrate (2.5) to obtain the following expressions for $\psi_1(x)$ and $\psi_2(x)$, as given by Mandelstam:

$$\begin{aligned}\psi_1(x) &= N_\epsilon :\exp iA_1(x): \\ \psi_2(x) &= -iN_\epsilon :\exp iA_2(x):,\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}A_1(x) &= -\frac{2\pi}{\beta}\int_{-\infty}^{\infty}\theta(x-\xi)\pi(\xi)\,d\xi - \frac{\beta}{2}\phi(x), \\ A_2(x) &= -\frac{2\pi}{\beta}\int_{-\infty}^{\infty}\theta(x-\xi)\pi(\xi)\,d\xi + \frac{\beta}{2}\phi(x).\end{aligned} \tag{2.7}$$

The ordering of the non-commuting operators in (2.7) is determined by the usual normal ordering prescription, in which we choose the Fock space representation for a boson field of some mass μ. Such normal ordering implies a choice, although somewhat arbitrary, of a representation of the operators as free fields, and to this extent is not perturbation independent. The two-point function in this representation is given by

$$\Delta_+(x-y) = [\phi^+(x), \phi^-(y)] = \frac{i}{4}H_0^{(1)}(i\mu(x^2-t^2)), \tag{2.8}$$

where $x^2 - t^2 > 0$, with the asymptotic expansion

$$\Delta_+(x) \sim -\frac{1}{4\pi}\log c^2\mu^2x^2 + O(x^2). \tag{2.9}$$

The ξ-integration in the expressions (2.7) is understood to be regularized by inserting the factor $\exp(\epsilon\xi)$ in the integrand, and the normalisation N_ϵ in equation (2.6) is chosen to be

$$N_\epsilon^2 = \frac{c\mu}{2\pi}\exp\left(\frac{-\pi^2\mu}{\beta^2\epsilon}\right), \tag{2.10}$$

as determined by the normalisation required below for ψ_1 and ψ_2 (equation (3.13)).

In manipulating the operators A_1, A_2 in (2.7) further regularization is necessary. Firstly, in multiplying two fermion operators we use the point-splitting procedure and $\psi_1(x)\psi_2(x)$, for example, will appear in the form $\varepsilon^\sigma\psi_1(x)\psi_2(x+\varepsilon)$, where the index σ is calculated so as to produce a finite result for $\varepsilon \to 0$. In doing this, we regularize the singularity in $\Delta_+(x)$, for $x \sim 0$, by introducing an imaginary time splitting:

$$\Delta_+(x, \varepsilon) \sim -\frac{1}{4\pi} \log c^2\mu^2(x^2 + \varepsilon^2) + O(x^2). \tag{2.11}$$

In addition, we must also consider the meaning of $\theta(x)$. We can always assume

$$\theta(x) + \theta(-x) = 1, \tag{2.12}$$

but when further regularization is necessary, we replace $\theta(x)$ by $\theta_\epsilon(x)$, defined as follows: let $\theta_1(x)$ be any smooth function interpolating between 0 and 1, and define

$$\theta_\epsilon(x) = \theta_1\left(\frac{x}{\epsilon}\right). \tag{2.13}$$

For small ϵ, $\theta_\epsilon(x)$ approximates $\theta(x)$, and we take the limit $\epsilon \to 0$ when possible. A useful representation of $\theta_\epsilon(x)$ is

$$\theta_\epsilon(x) = \frac{1}{\pi} \arctan\left(\frac{x}{\epsilon}\right) + \tfrac{1}{2}. \tag{2.14}$$

If required, further regularization of equation (2.7) is also possible; for example, we could replace

$$\phi(x) \to \int_{-\infty}^{\infty} \theta_\epsilon(x)(x - \xi)\phi'(\xi)\, d\xi \tag{2.15}$$

as occurs in Section 4 (equation (4.19)).

3. Anticommutation relations

We wish to investigate the anticommutators of $\psi(x)$ and $\psi^\dagger(y)$ for all values of x and y. The formulas we need are

$$:e^A:\, :e^B: = e^{[A,B]}\, :e^B:\, :e^A:\,, \tag{3.1}$$

where $[A, B]$ commutes with A, B, and

$$\begin{aligned} [A_1(x), A_1(y)] &= -[A_2(x), A_2(y)] = -i\pi(\theta(x - y) - \theta(y - x)), \\ [A_1(x), A_2(y)] &= i\pi(\theta(x - y) + \theta(y - x)) = i\pi. \end{aligned} \tag{3.2}$$

With these formulas, it is straightforward following Mandelstam [3] to see that, for different arguments, all anticommutators are zero.

In order to investigate $\{\psi_1(x), \psi_1(y)\}$ for $x = y$, we require the commutator

$$[A_1^+(x), A_1^-(y)] = \frac{-i\pi}{2}(\theta(x-y) - \theta(y-x)) + \left(\frac{4\pi^2}{\beta^2} + \frac{\beta^2}{4}\right)\Delta_+(x-y) + \frac{\pi^2\mu}{\beta^2\epsilon} + \frac{4\pi^2\mu^2}{\beta^2}\int_0^{|x-y|} \Delta_+(\xi)\xi\, d\xi \tag{3.3}$$

where ϵ is the regulator which appears in the integrand of (2.7), and where we have used

$$[\pi^+(x), \pi^-(y)] = \left(\frac{\partial^2}{\partial x\, \partial y} + \mu^2\right)\Delta_+(x-y), \tag{3.4}$$

as well as

$$\int_0^\infty \Delta_+(\xi)\, d\xi = \frac{1}{4\mu}. \tag{3.5}$$

For $x \sim y$ we have therefore

$$[A_1^+(x), A_1^-(y)] = -\frac{i\pi}{2}(\theta(x-y) - \theta(y-x)) - (\sigma+1)\log|c\mu(x-y)| + \frac{\pi^2\mu}{\beta^2\epsilon} + O((x-y)^2 \log|x-y|), \tag{3.6}$$

where

$$\sigma = \frac{2\pi}{\beta^2} + \frac{\beta^2}{8\pi} - 1. \tag{3.7}$$

From the normal ordering formula

$$:e^A: :e^B: = e^{[A^+, B^-]} :e^{A+B}:, \tag{3.8}$$

we find, for $x \sim y$,

$$\begin{aligned}\psi_1(x)\psi_1(y) &= N_\epsilon^2 \exp[A_1^+(x), A_1^-(y)] :\exp i(A_1(x) + A_1(y)): \\ &= \frac{c\mu}{2\pi}\exp\left[\frac{i\pi}{2}(\theta(x-y) - \theta(y-x))\right]|c\mu(x-y)|^{\sigma+1} \\ &\quad \times :\exp i(A_1(x) + A_1(y)):\end{aligned} \tag{3.9}$$

Now we consider this equation in a weak sense, i.e. we take matrix elements, and let $y \to x$; since the matrix elements of the normal ordered operator $:\exp i(A_1(x) + A_1(y)):$ are finite, and other factors are bounded, we obtain, using $\sigma + 1 > 0$,

$$\psi_1(x)^2 = \psi_2(x)^2 = 0. \tag{3.10}$$

We emphasize that these equations hold in a weak sense only, whereas $\{\psi_1(x), \psi_1(y)\}$ is zero in a strong sense for all $x \neq y$.

Next, we investigate the anticommutator $\{\psi_1(x), \psi_1^\dagger(y)\}$. Here we must include the regularized functions $\theta_\varepsilon(x)$ and $\Delta_+(x, \varepsilon)$ in order to smooth the singularity at $x = y$. Using the normal ordering formula (3.8), and the commutator (3.3) we find, for $x \sim y$

$$\begin{aligned}\psi_1^\dagger(x)\psi_1(y) &= N_\varepsilon^2 \exp[A_1^+(x), A_1^-(y)] :\exp - i(A_1(x) - A_1(y)): \\ &= \frac{c\mu}{2\pi}\exp\left[\frac{-i\pi}{2}(\theta_{\varepsilon_1}(x-y) - \theta_{\varepsilon_1}(y-x))\right] \\ &\quad \times \exp\left[\left(\frac{4\pi^2}{\beta^2} + \frac{\beta^2}{4}\right)\Delta_+(x-y, \varepsilon_2)\right] :\exp - i(A_1(x) - A_1(y)): \\ &= \frac{c\mu}{2\pi}\exp\left[\frac{-i\pi}{2}(\theta_{\varepsilon_1}(x-y) - \theta_{\varepsilon_1}(y-x))\right] \\ &\quad \times [c^2\mu^2((x-y)^2 + \varepsilon_2^2)]^{-(\sigma+1)/2}, \end{aligned} \tag{3.11}$$

where we have included only the leading contribution (the identity) from the operator $:\exp - i(A_1(x) - A_1(y)):$. A similar expression for $\psi_1^\dagger(y)\psi_1(x)$ follows by taking the hermitean conjugate. We obtain, therefore, for $x \sim y$

$$\begin{aligned}&\{\psi_1^\dagger(x), \psi_1(y)\} \\ &\quad = \frac{c\mu}{\pi}\cos\left[\frac{\pi}{2}(\theta_{\varepsilon_1}(x-y) - \theta_{\varepsilon_1}(y-x))\right][c^2\mu^2((x-y)^2 + \varepsilon_2^2)]^{-(\sigma+1)/2}.\end{aligned} \tag{3.12}$$

One sees that at $\varepsilon_1 = 0$ the expression $\cos[(\pi/2)(\theta_{\varepsilon_1}(x-y) - \theta_{\varepsilon_1}(y-x))]$ is zero except possibly at $x = y$, i.e. $\{\psi_1^\dagger(x), \psi_1(y)\}$ is zero for $x \neq y$ as was noted already above. Equation (3.12) indicates that it is necessary to regularize both the functions $\theta(x-y)$ and $\Delta_+(x-y)$ in order to evaluate the commutator at $x = y$. The meaning of (3.12) depends on the way in which the limits $\varepsilon_1, \varepsilon_2 \to 0$ are taken; let us choose the regulators $\varepsilon_1, \varepsilon_2$ to be equal, $\varepsilon_1 = \varepsilon_2 = \varepsilon$, with ε small. We can now gain a better understanding of (3.12) by substituting the specific choice (2.14) for $\theta_\varepsilon(x)$. We find

$$\{\psi_1^\dagger(x), \psi_1(y)\} = \{\psi_2^\dagger(x), \psi_2(y)\} = \frac{(c\mu)^{-\sigma}}{\pi}\frac{\varepsilon}{[(x-y)^2 + \varepsilon^2]^{1+\sigma/2}}. \tag{3.13}$$

We recognize that for $\sigma = 0$ (for which $\beta^2 = 4\pi$) we obtain a representation of the δ function, i.e., we recover the canonical anticommutation relations. In general, however, the noncanonical nature of (3.13) can be attributed to the renormalisation required for the interacting fields ψ_1, ψ_2. Our result disagrees with that of Ha [9], who has used a different regularization scheme and obtains the canonical commutation relations for the massless Thirring model, even for $\sigma \neq 0$. The difference appears to be partly due to the failure to regularize step functions of the type $\theta(x)$ (see for example equation (2.22) in [9]).

Next, let us turn to the fermion bilinears and derive expressions for their boson equivalents. The fermion currents can be constructed from ψ_1 and ψ_2, with the point splitting prescription exactly as outlined by Mandelstam [3]; we may therefore set $\varepsilon_1 = \varepsilon_2 = 0$. We obtain

$$j_0(x) = \phi'(x) = \lim_{\varepsilon \to 0} \frac{2\pi}{\beta} (c\mu\varepsilon)^{\sigma} \bar{\psi}(x)\gamma_0\psi(x+\varepsilon),$$
$$j_1(x) = \pi(x) = \lim_{\varepsilon \to 0} \left(\frac{\beta}{2} (c\mu\varepsilon)^{\sigma} \bar{\psi}(x)\gamma_1\psi(x+\varepsilon) - \frac{i\beta}{2\pi\varepsilon}\right), \tag{3.14}$$

which explicitly identifies the corresponding fermion and boson currents, together with additive and multiplicative renormalisation factors.

The remaining fermion bilinears, of the form $\psi_2^{\dagger}(x)\psi_1(y)$, can be equated with terms of the boson potential. As is the case with the currents, point splitting regularization is sufficient, and we obtain

$$:\cos\beta\phi(x): = \lim_{\varepsilon \to 0} \frac{\pi}{c\mu} (c\mu\varepsilon)^{-\delta} \bar{\psi}(x)\psi(x+\varepsilon), \tag{3.15}$$

and

$$:\sin\beta\phi(x): = \lim_{\varepsilon \to 0} -\frac{i\pi}{c\mu} (c\mu\varepsilon)^{-\delta} \bar{\psi}(x)\gamma_5\psi(x+\varepsilon), \tag{3.16}$$

where

$$\delta = \frac{2\pi}{\beta^2} - \frac{\beta^2}{8\pi}. \tag{3.17}$$

4. Field equations

So far we have not imposed any dynamics on the boson field or on the corresponding fermion fields. We consider at first the sine-Gordon model and then indicate how to generalise the results to boson models with arbitrary periodic potentials. Included as a special case is the free boson field, shown to be equivalent to the massless Thirring model. These correspondences have in part been demonstrated by Mandelstam, by deriving fermion field equations from the boson dynamics, but without proper consideration of the regularisation required. In particular, the mass renormalisation in the massive Thirring model is best performed by employing the regularized functions $\Delta_+(x, \varepsilon)$ and $\theta_\varepsilon(x)$. As is the case with some of the fermion anticommutators ((3.10) and (3.13)), the field equations, for the massive case, hold only in a weak sense.

The Hamiltonian for the sine-Gordon model is

$$H = \int \left(\tfrac{1}{2}\phi'(\xi)^2 + \tfrac{1}{2}\pi(\xi)^2 - \frac{\alpha}{\beta^2} :\cos\beta\phi(\xi):\right) d\xi = H_0 + H_I. \tag{4.1}$$

The dynamics of the ψ fields are determined from

$$\dot{\psi} = i[H, \psi],$$
$$\psi' = i[P, \psi], \tag{4.2}$$

where the momentum P is given by

$$P=\frac{1}{2}\int \{\pi(\xi), \phi'(\xi)\}\, d\xi. \tag{4.3}$$

In evaluating (4.2) we encounter the commutator $[:\cos\beta\phi(\xi):, \psi(x)]$ which is zero except possibly at $x=\xi$. This follows from Eq. (2.5), which implies

$$\cos\beta(\xi)\psi(x)=\psi(x)\cos[\beta\phi(\xi)+2\pi\theta(x-\xi)]. \tag{4.4}$$

It is necessary, therefore, to calculate a regularized form of this commutator, which we will be able to identify with the mass term of the massive Thirring equations, as suggested by Mandelstam. By using the normal ordering formula (3.8) and the commutator

$$[\phi^{+}(\xi), A_1^{-}(x)]=-\frac{i\pi}{\beta}\theta_\varepsilon(x-\xi)-\frac{\beta}{2}\Delta_{+}(x-\xi, \varepsilon), \tag{4.5}$$

we obtain

$$\begin{aligned}&[:\exp i\beta\phi(\xi):, \psi_1(x)]\\&\quad=2iN_\varepsilon \sin(\pi\theta_\varepsilon(x-\xi))[c^2\mu^2((\xi-x)^2+\varepsilon^2)]^{-\beta^2/8\pi} :\exp(i\beta\phi(x)+iA_1(x)):.\end{aligned} \tag{4.6}$$

Again, we have chosen equal regulators $\varepsilon_1=\varepsilon_2=\varepsilon$ for the functions $\theta_{\varepsilon_1}(x)$ and $\Delta_{+}(x, \varepsilon_2)$. We observe that $\sin(\pi\theta(x-\xi))$ is zero except at $x=\xi$; for small ε therefore the only contribution to the right hand side of (4.6) occurs for small $|\xi-x|$ and so we have replaced $\Delta_{+}(\xi-x, \varepsilon)$ by its asymptotic form (2.11) and $\phi(\xi)$ by $\phi(x)$. The commutator $[:\exp -i\beta\phi(\xi):, \psi_1(x)]$ has a form similar to (4.6), and in this case we can let $\varepsilon\to 0$ and assert that

$$[:\exp -i\beta\phi(\xi):, \psi_1(x)]=0. \tag{4.7}$$

This is clearly true for $\xi\neq x$ (by putting $\varepsilon=0$), but also for $\xi=x$ because the factor $|\xi-x|$ has for this case a positive exponent $\beta^2/4\pi$, which again gives zero for $\xi\to x$, $\varepsilon\to 0$. The commutator (4.7) is zero in a weak sense because we assume that the normal ordered operator $:\exp[i\beta\phi(x)+iA_1(x)]:$ has finite matrix elements.

From (4.6), therefore, we obtain the commutator

$$[:\cos\beta\phi(\xi):, \psi_1(x)]=-\psi_2(x)\sin(\pi\theta_\varepsilon(x-\xi))[c^2\mu^2((x-\xi)^2+\varepsilon^2)]^{-\beta^2/8\pi}. \tag{4.8}$$

If we choose the representation (2.14) for $\theta_\varepsilon(x)$ we find

$$[:\cos\beta\phi(\xi):, \psi_1(x)]=-\psi_2(x)(c\mu)^{-\beta^2/4\pi}\frac{\varepsilon}{[(x-\xi)^2+\varepsilon^2]^{\beta^2/8\pi+1/2}}. \tag{4.9}$$

Only for $\beta^2=4\pi$ does the right hand side include a regularized δ function.

Now let us return to the evaluation of equations (4.2). The free Hamiltonian

H_0 can be written

$$H_0 = \frac{1}{2} \int (j_0^2 + j_1^2), \tag{4.10}$$

where the boson currents j_0, j_1 are given in (2.2). The commutation relations ((2.3), (2.4)) then imply

$$[H_0, \psi_1] = \frac{1}{2}\left\{-\frac{2\pi}{\beta} j_0(x) - \frac{\beta}{2} j_1(x), \psi_1(x)\right\}. \tag{4.11}$$

Similarly

$$[P, \psi_1] = \frac{1}{2}\left\{-\frac{2\pi}{\beta} j_1(x) - \frac{\beta}{2} j_0(x), \psi_1(x)\right\}. \tag{4.12}$$

From (4.8) we obtain (putting $x - \xi = \varepsilon y$ and using (2.13))

$$\begin{aligned} [H_I, \psi_1(x)] &= \psi_2(x)\varepsilon^{1-(\beta^2/4\pi)} \frac{\alpha}{\beta^2} \int_{-\infty}^{\infty} \sin(\pi\theta_1(y))[c^2\mu^2(y^2+1)]^{-\beta^2/8\pi}\, dy \\ &= m\psi_2(x)\varepsilon^{1-\beta^2/4\pi}, \end{aligned} \tag{4.13}$$

where m can be identified from the right hand side of (4.13) and is finite and independent of ε. However, m does depend on the conventions chosen, namely the normal ordering mass μ, and the regulating function $\theta_1(x)$. It is natural to define the bare mass

$$m_0 = m\varepsilon^{1-\beta^2/4\pi}, \tag{4.14}$$

and regard the factor $\varepsilon^{1-\beta^2/4\pi}$ as a mass renormalisation, which is unity only for $\beta^2 = 4\pi$. For $\beta^2 < 4\pi$ it appears that the bare mass m_0 is zero (letting $\varepsilon \to 0$) but in this case the matrix elements of $\psi_2(x)$, which multiplies m_0 and also depends on ε, will diverge such that the product $m_0\psi_2$ remains finite.

We have now

$$\begin{aligned} \dot{\psi}_1 &= \frac{i}{2}\left\{-\frac{2\pi}{\beta} j_0 - \frac{\beta}{2} j_1, \psi_1\right\} + im_0\psi_2 \\ \psi_1' &= \frac{i}{2}\left\{-\frac{2\pi}{\beta} j_1 - \frac{\beta}{2} j_0, \psi_1\right\} \end{aligned} \tag{4.15}$$

together with similar equations for ψ_2. Combining these, we arrive at the field equations

$$(i\gamma^\mu \partial_\mu + m_0)\psi = \frac{g}{2} \gamma^\mu \left\{\frac{\beta}{2\pi} j_\mu, \psi\right\}, \tag{4.16}$$

where

$$g = \frac{4\pi^2}{\beta^2} - \pi. \tag{4.17}$$

If we identify j_μ with the fermion current as in equations (3.14) then (4.16) can be considered as the quantum field equations for the massive Thirring model, given by the classical Lagrangian

$$\mathscr{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{g}{2}(\bar{\psi}\gamma^\mu\psi)^2 + m_0\bar{\psi}\psi. \tag{4.18}$$

The anticommutator in equation (4.16) is usually expressed via a limiting procedure (see for example Refs. [4] and [5]), in order to be well defined; we can regularize singularities in the most natural way by including the regularized $\theta_\varepsilon(x)$ function in the definition of $\psi(x)$, including the $\phi(x)$ term, as shown in equation (2.15). This amounts to replacing $\delta(x-y)$ on the right hand side of ((2.3), (2.4)) by a regularized function $\delta_\varepsilon(x-y)$. Equation (4.16) can then be written in the smeared form

$$(i\gamma_\mu\partial^\mu + m_0)\psi = \frac{g}{2}\gamma^\mu \int \delta_\varepsilon(x-\xi)\frac{\beta}{2\pi}\{j^\mu(\xi), \psi(x)\}\,d\xi. \tag{4.19}$$

An alternative procedure is to regularize H_0 and P in the following way:

$$\begin{aligned} H_0 &= \lim_{\varepsilon\to 0}\frac{1}{2}\int [j_0(\xi)j_0(\xi-\varepsilon) + j_1(\xi)j_1(\xi-\varepsilon)]\,d\xi, \\ P &= \lim_{\varepsilon\to 0}\frac{1}{2}\int [j_0(\xi)j_1(\xi-\varepsilon) + j_1(\xi)j_0(\xi-\varepsilon)]\,d\xi. \end{aligned} \tag{4.20}$$

In this case the right hand side of (4.16) becomes

$$\lim_{\varepsilon\to 0}\frac{g}{2}\frac{\beta}{2\pi}\gamma^\mu((j_\mu(x+\varepsilon)\psi(x) + \psi(x)j_\mu(x-\varepsilon)), \tag{4.21}$$

which is the point splitting prescription described by Johnson [5].

We remark that the equivalence between the boson and fermion models as revealed by the field equations (4.16) does not extend to an equivalence of the corresponding actions; we can equate the actions of the fermion and boson models only at an extremum of the action. Whereas this equivalence is model dependant, the properties of the fermion field described in Section 3, including relations of the form $\bar{\psi}\psi \sim \cos\beta\phi$, hold regardless of the fermion-boson dynamics. One might expect, therefore, given a boson-fermion equivalence, that a perturbation of the boson model, by adding a term of the form $(\cos\beta\phi)^n$ for some integer n, would correspond to a perturbation of the fermion model of the form $(\bar{\psi}\psi)^n$.

Let us investigate such a possible equivalence, in order to demonstrate that the bosonization formulas of Section 3 apply to a much larger range of models than merely the sine-Gordon and massive Thirring models, and to indicate how the Mandelstam soliton operator can be used to find the fermion model corresponding to a given boson model. The boson models we consider are those with potentials that can be written as a sum of terms of the form $(\cos\beta\phi)^n$, which will correspond to fermion interaction terms of the form $(\bar{\psi}\psi)^n$. Although such terms are not renormalisable in the usual sense, the Mandelstam operator shows how to write down a renormalisable fermion model; the fermion model is necessarily renormalisable since the boson model is renormalisable (by normal ordering) and the Mandelstam operator shows how to rewrite the theory in terms of fermion operators. This is remarkable because it means that although the perturbation series for the fermion field with interaction $(\bar{\psi}\psi)^n$ apparently contains new divergences at every order of approximation, these divergences can all be eliminated by the regrouping arising from the boson-fermion correspondence. Although an infinite set of subtractions appear to be called for, they are not ultimately required. Another way of looking at this problem is to say that the boson-fermion correspondence *defines* the putatively non-renormalisable theory. Furthermore, the redefined theory is unitary.

Let us now investigate therefore, the boson Hamiltonian

$$H = \int \left(\tfrac{1}{2}\phi'(\xi)^2 + \tfrac{1}{2}\pi(\xi)^2 + :V(\phi(\xi)):\right) d\xi. \tag{4.22}$$

where $V(\phi)$ is an even, periodic potential:

$$V(\phi) = V\left(\phi + \frac{2\pi}{\beta}\right). \tag{4.23}$$

We may write $V(\phi)$ as a Fourier cosine series

$$:V(\phi): = \frac{1}{\beta^2} \sum_n a_n :\cos(n\beta\phi):. \tag{4.24}$$

where the coefficients $\{a_n\}$ are finite. In order to calculate the time derivative of ψ from equation (4.2), we require the following commutator (similar to (4.6)):

$$\begin{aligned}&[:\exp in\beta\phi(\xi):, \psi_1(x)]\\ &\quad = 2iN_\epsilon \sin(n\pi\theta_\varepsilon(x-\xi))[c^2\mu^2((\xi-x)^2+\varepsilon^2)]^{-n\beta^2/8\pi} :\exp(in\beta\phi(x) + iA_1(x)):.\end{aligned} \tag{4.25}$$

It is apparent from this expression that only integer values of n are allowed, since otherwise the factor $\sin(n\pi\theta(x-\xi))$ will not have point support, as was required in the previous analysis for $n = 1$. This prevents us from considering arbitrary potentials, expressible as a Fourier transform, instead of the Fourier series (4.24). It follows from (4.25), in the same way as for $n = 1$, that

$$[:\exp(-in\beta\phi(\xi)):, \psi_1(x)] = 0, \tag{4.26}$$

by putting $n \to -n$ and letting $\varepsilon \to 0$. Hence, again as before,

$$\int_{-\infty}^{\infty} [:\cos n\beta\phi(\xi):, \psi_1(x)]\, d\xi = iN_\epsilon M_n \varepsilon^{1-n\beta^2/4\pi} :\exp(in\beta\phi(x) + A_1(x)):. \tag{4.27}$$

where

$$M_n = \int_{-\infty}^{\infty} [c^2\mu^2(y^2+1)]^{-n\beta^2/8\pi} \sin n\pi\theta_1(y)\, dy. \tag{4.28}$$

The coefficients M_n are finite for all $n > 0$, and we can define the 'bare constants' $\tilde{M}_n$ by

$$\tilde{M}_n = M_n \varepsilon^{1-n\beta^2/4\pi}. \tag{4.29}$$

Although this indicates how the fermion theory corresponding to the boson Hamiltonian (4.22) is renormalised, we have yet to express the right hand side of (4.27) in terms of the fermion field $\psi(x)$. To do this, we need to reshuffle the normal ordered operators, introducing more renormalisation factors. The right hand side of (4.27) takes the form $(\psi_1^\dagger\psi_2)^{n-1}\psi_2$, in which the arguments of the fermion operators differ successively by ε, and is also multiplied by a power of ε. Because of the point splitting, repeated products of the form $(\psi_1^\dagger\psi_2)^n$ are not necessarily zero, as might be expected from $\psi_1^2 = 0 = \psi_2^2$. The time derivative of ψ_1 then has the form

$$\dot{\psi}_1 = \frac{i}{2}\left\{-\frac{2\pi}{\beta}j_0 - \frac{\beta}{2}j_1,\ \psi_1\right\} + i\sum_n c_n(\psi_1^\dagger\psi_2)^{n-1}\psi_2, \tag{4.30}$$

for some set of coefficients $\{c_n\}$. The fermion Lagrangian which corresponds to the boson model (4.24) therefore has the form

$$\mathscr{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{g}{2}(\bar{\psi}\gamma^\mu\psi)^2 + f(\bar{\psi}\psi), \tag{4.31}$$

where f is some function that is identified explicitly by tracing back the constant and multiplicative factors introduced in the normal ordering process. Renormalisation of this Lagrangian must be carried out by assigning to expressions like $f(\bar{\psi}\psi)$ the meaning indicated precisely by the bosonization formulas. This demonstrates, in outline, that for boson-fermion equivalent models, a perturbation by $(\cos\beta\phi)^n$ corresponds to a perturbation $(\bar{\psi}\psi)^n$ on the fermion model.

5. Non-Abelian bosonization

Witten's non-Abelian bosonization [2] reduces, for $N = 2$, to the case studied above. Although Witten considered only massless fermions, and the chirally invariant case corresponding to $\beta^2 = 4\pi$, we can investigate the bosonization formulas more generally using Mandelstam's soliton operator. For convenience, we define Majorana fermions:

$$\sqrt{2}\psi_1 = \psi_1^+ - i\psi_2^+, \qquad \sqrt{2}\psi_2 = \psi_1^- - i\psi_2^-, \tag{5.1}$$

giving

$$\begin{aligned} \psi_1^+ &= \sqrt{2}N_\epsilon :\cos A_1:, & \psi_2^+ &= -\sqrt{2}N_\epsilon :\sin A_1:, \\ \psi_1^- &= \sqrt{2}N_\epsilon :\sin A_2:, & \psi_2^- &= \sqrt{2}N_\epsilon :\cos A_2:. \end{aligned} \tag{5.2}$$

Define also the group element

$$g = \begin{pmatrix} \cos\beta\phi & \sin\beta\phi \\ -\sin\beta\phi & \cos\beta\phi \end{pmatrix}, \tag{5.3}$$

and the currents

$$J_{12}^{\pm}(x) = \beta(\pi(x) \pm \phi'(x)). \tag{5.4}$$

Previous formulas can be rewritten in terms of the Majorana fermions and currents; for example ((3.15), (3.16)) become:

$$g_{ij}(x) = i \lim_{\varepsilon \to 0} \frac{\pi}{c\mu} (c\mu\varepsilon)^{\sigma+1} (\varepsilon_{ik} \psi_k^-(x) \psi_j^+(x+\varepsilon) - \psi_i^-(x) \varepsilon_{jk} \psi_k^+(x+\varepsilon)). \tag{5.5}$$

As shown in Ref. [6], the matrix g', defined by

$$Mg'_{ij} = -i\psi_i^- \psi_j^+, \tag{5.6}$$

is not equal to g, as Witten suggested. It is, however, orthogonal in an appropriate sense, provided the normalization M is chosen correctly. To verify this, we require the formula

$$\psi_k^+(x)\psi_k^+(y) = 2N_\varepsilon^2 \exp [A_1^+(x), A_1^-(y)] :\cos (A_1(x) - A_1(y)):. \tag{5.7}$$

Then we find, to leading order in $x - y$,

$$\begin{aligned} g'_{ik}(x)g'_{jk}(y) &= M^{-2}\psi_i^-(x)\psi_j^-(y)\psi_k^+(x+\varepsilon)\psi_k^+(y+\varepsilon) \\ &= M^{-2}2N_\varepsilon^2 \exp [A_1^+(x+\varepsilon), A_1^-(y+\varepsilon)]\psi_i^-(x)\psi_j^-(y). \end{aligned} \tag{5.8}$$

We also have, again to leading order in $x - y$,

$$\psi_1^-(x)\psi_2^-(y) = 2N_\varepsilon^2 \exp (-[A_2^+(x), A_2^-(y)]), \tag{5.9}$$

showing that (in a weak sense)

$$\lim_{y \to x} \psi_1^-(x)\psi_2^-(y) = 0 = \lim_{y \to x} \psi_2^-(x)\psi_1^-(y). \tag{5.10}$$

However, $\psi_1^-(x)\psi_1^-(y)$ and $\psi_2^-(x)\psi_2^-(y)$ are nonzero and equal as $y \to x$. (5.8) shows therefore that $g'_{ik}(x)g'_{jk}(y)$ is nonzero only for $i = k$, and so g' is an orthogonal matrix, in a weak sense, provided M is suitably chosen, i.e.

$$g'_{ij}(x) = -i \lim_{\varepsilon \to 0} \frac{\sqrt{2}\pi}{c\mu} (c\mu\varepsilon)^{\sigma+1} \psi_i^-(x)\psi_j^+(x+\varepsilon). \tag{5.11}$$

What are the currents generated by g'? Witten's calculation indicates that these currents are the same left and right currents $J^{\pm}$ as generated by the group elements g. We can investigate the form of the currents by using the equations (4.15) satisfied by the fermion field ψ. In terms of the Majorana components, these equations can be written

$$\begin{aligned} \partial_\pm \psi_i^+ &= m_0 \varepsilon_{ik} \psi_k^- - \frac{1}{4}\left(1 \pm \frac{4\pi}{\beta^2}\right)\{J_{12}^{\pm}, \varepsilon_{ik}\psi_k^+\} \\ \partial_\pm \psi_i^- &= m_0 \varepsilon_{ik} \psi_k^+ - \frac{1}{4}\left(-1 \pm \frac{4\pi}{\beta^2}\right)\{J_{12}^{\pm}, \varepsilon_{ik}\psi_k^-\} \end{aligned} \tag{5.12}$$

where $\partial_{\pm} = \partial/\partial t \pm \partial/\partial x$. Let us ignore the fact that terms involving m_0 appear in a weak sense only, and interpret the anticommutators on the right hand side of (5.12) to mean the limit as shown in (4.21). Then

$$\begin{aligned}\partial_+(g'_{ij}) &= -iM^{-1}m_0(\varepsilon_{ik}\psi_k^+\psi_j^+ + \varepsilon_{jk}\psi_i^-\psi_k^-)\\ &\quad -\frac{1}{2}\left(-1+\frac{4\pi}{\beta^2}\right)J_{12}^+\varepsilon_{ik}g'_{kj} - \frac{1}{2}\left(1+\frac{4\pi}{\beta^2}\right)J_{12}^+\varepsilon_{jk}g'_{ik}\\ &= -iM^{-1}m_0(\varepsilon_{ik}\psi_k^+\psi_j^+ + \varepsilon_{jk}\psi_i^-\psi_k^-) + J_{12}^+(\varepsilon g')_{ij}\end{aligned} \tag{5.13}$$

where we have used $g'\varepsilon = \varepsilon g'$ for all $g' \in SO(2)$. For $m_0 = 0$ therefore we find that g' satisfies $\partial_+ g' = g'J^+$, and similarly $\partial_- g' = J^- g'$, i.e., the group elements g' and g each generate the same currents $J^{\pm}$, given by (5.4). (This was found by Witten in a different way, see [2], equation (35), but this does not imply $g = g'$.)

Since g is obtained by a linear transformation from g', properties of g follow from those of g'. If we assemble the elements of g and g' into column or row vectors of length 4, we can write

$$g = Tg' \tag{5.14}$$

where, up to a normalisation factor,

$$T = \begin{pmatrix} 0 & -1 & 1 & 0\\ 1 & 0 & 0 & 1\\ -1 & 0 & 0 & -1\\ 0 & -1 & 1 & 0\end{pmatrix}. \tag{5.15}$$

T is not invertible, so we cannot obtain g' from g. However, properties of g follow from those of g' by applying T; for example, the fact that both g and g' generate the same currents $J^{\pm}$ follows from

$$[T, \varepsilon \otimes I] = 0 = [T, I \otimes \varepsilon]. \tag{5.16}$$

This is because we can write $\partial_- g' = (I \otimes \varepsilon)g'J_{12}^-$, from which $\partial_- g = J^- g$ follows using ((5.14), (5.16)).

Whilst Mandelstam's soliton operator is useful in order to establish formulas for the $N = 2$ case of non-Abelian bosonization, it does not appear to easily generalize for larger values of N. It is possible to write down generalizations of (5.5), which involve not just the $SO(N)$ vectors $\psi^{\pm}$, but also the dual vectors as appear in (5.5). For example, one can form the vector

$$V_i = \varepsilon_{ij_2j_3\cdots j_n}\psi_{j_2}\psi_{j_3}\cdots\psi_{j_n}, \tag{5.17}$$

which is not identically zero because the fermion operators ψ anticommute, and then construct a generalization of (5.5), with the correct vector commutation relations with respect to the currents $J^{\pm}$. However, the validity of such an expression cannot be readily checked, because of the lack of a suitable soliton operator.

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