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Quantum statistical mechanics of general mean field systems

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Abstract. We consider mean field models for n identical systems interacting with each other, and with another additional system. Each hamiltonian H_n is taken to be symmetric with respect to permutations of the identical systems, and for large n and arbitrary k, $(n+k)^{-1}H_{n+k}$ is approximately equal to $n^{-1}H_n$, taken as an operator of the larger system, and resymmetrized. We give a complete theory of the equilibrium statistical mechanics of such systems. The validity of the Gibbs Variational Principle is established; firstly, at the level of the states of the infinite system, then secondly at the level of the states of the single system. A generalized gap-equation is obtained at this second level. In some cases, the variational problem reduces further; this leads to a non-commutative version of the large deviation results of Cramér–Varadhan for $\mathbb{R}^d$-valued random variables.

I. Introduction

We define a class of statistical mechanical models of mean field type, and obtain a complete theory for them. The models are specified by a C^*-algebra $\mathscr{B}$ for the single system, and a hamiltonian for the aggregate of n single systems (described by the n-fold tensor product of $\mathscr{B}$) interacting with each other, and with a second system specified by a C^*-algebra $\mathscr{A}$. The precise nature of the allowed hamiltonians is described in Section II. The essential features are that the hamiltonian density H_n is invariant with respect to all permutations of the n single systems, and is asymptotically symmetric in the sense that H_{n+1} is given, up to a small correction, by resymmetrizing H_n considered as element of the $(n+1)$-fold tensor product.

For our general mean field model, we prove the validity of the Gibbs Variational Principle at two levels. Firstly, the thermodynamic limit of the free energy density is obtained by minimizing the free energy density functional over the set of (symmetric) states of the (infinite) system. Secondly, the latter variational problem is reduced to that for a free energy density functional on the states of the single system. At this level, the minimizing states are solutions of a

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gap-equation. Since all limiting states of the model are minimizers, we obtain some detailed information about them as well.

In special cases, the reduction proceeds one step further, and a finite dimensional variational problem is obtained. This corresponds to a 'level-1' large deviation result in the terminology of [5], and extends Varadhan's results [21], on the multidimensional Cramér Theorem, to the non-commutative domain. All three levels were obtained in [14] for the simplest possible case.

From a technical point of view, this paper extends the results of [14], particularly those involved in the estimates of the energy density, thus allowing not only for the inclusion of the additional algebra $\mathcal{A}$, but more importantly, for a larger class of hamiltonians.

The basic definitions and the main results are expounded in Section II. Section III contains their proofs. The energy estimates are based on the contents of Section IV, which is essentially selfcontained and describes a general theory of what we call the C^*-algebra of symmetric tensors. In the concluding Section V, we discuss possible extensions, and problems. Some results on the many-variable functional calculus in C^*-algebras (C^*-functions), are given in an Appendix.

II. Main definitions and statement of main results

Throughout the paper $\mathcal{A}$ and $\mathcal{B}$ will be unital C^*-algebras. We shall be concerned with sequences of models with observable algebras $\mathcal{D}_n = \mathcal{A} \otimes (\otimes_{v=1}^n \mathcal{B}_{(v)})$, where $\otimes$ denotes the minimal, or injective, C^*-tensor product [19] and $\mathcal{B}_{(v)}$ is an isomorphic copy of $\mathcal{B}$. The C^*-inductive limit of the sequence $\mathcal{D}_n$ with the natural injections will be denoted by $\mathcal{D}_\infty$. Whenever convenient, $\mathcal{D}_n$ will be considered as a subalgebra of $\mathcal{D}_\infty$. The symmetrization operator $\mathrm{sym}_n : \mathcal{D}_n \to \mathcal{D}_n$ is the continuous linear extension of $\mathrm{sym}_n (a \otimes x_1 \otimes \cdots \otimes x_n) = (1/n!) \sum_\pi a \otimes x_{\pi 1} \otimes \cdots \otimes x_{\pi n}$, where the sum is over all permutations π of $\{1, \ldots n\}$. The same definitions apply when $\mathcal{A} = \mathbb{C}$; we then write $\mathcal{B}_n$ and $\mathcal{B}_\infty$ for $\mathcal{D}_n$ and $\mathcal{D}_\infty$ respectively.

For any C^*-algebra $\mathcal{F}$, $K(\mathcal{F})$ will denote its state space. A state $\phi \in K(\mathcal{D}_\infty)$ (resp. $K(\mathcal{B}_\infty)$) is called *symmetric*, if for all $n \in \mathbb{N}$, and all $X \in \mathcal{D}_n$ (resp. $\mathcal{B}_n$), $\phi(X) = \phi(\mathrm{sym}_n (X))$. The convex set of symmetric states of $\mathcal{D}_\infty$ will be denoted by $K_s(\mathcal{D}_\infty)$. For $\varphi \in K(\mathcal{B})$, the associated infinite product state on $\mathcal{B}_\infty$ is written Π_φ, and is symmetric.

The models we consider, are specified by a sequence of hamiltonians, given abstractly as follows. Firstly, the non-interacting part is determined via a sequence $\{\omega_n \equiv \rho_0 \otimes (\otimes_{v=1}^n \rho_{(v)})\}$ of product states of $\mathcal{D}_n$, where $\rho_0 \in K(\mathcal{A})$, and $\rho_{(v)} = \rho \in K(\mathcal{B})$ are arbitrary *separating* states (i.e. a state such that the associated GNS-vector is separating for the von Neumann algebra generated by the GNS-representation). The interaction is introduced by perturbing each ω_n in the sense of Araki [1] with a relative hamiltonian $n \cdot H_n \in \mathcal{D}_n$. The perturbed (unnormalized) positive linear functional of $\mathcal{D}_n$ will be written $\omega_n^{nH_n}$. This framework provides a generalization of that special case where the state ω is given by $\omega(\circ) = \mathrm{Tr}\,(D\circ)$ with a (non-singular) density D with respect to a trace Tr; there, the state ω^h has density $\exp(\log D + h)$. The number $\log \omega^h(1)$ can be interpreted as a relative negative free energy [1, 14]. The sequences of relative hamiltonian densities we allow are assumed to be approximately symmetric in the sense of the following definition:

II.1 Definition. A *symmetric sequence* in $\mathscr{D}_\infty$ is a sequence (X_n), defined for n larger than some initial value n_0, such that $X_n \in \mathscr{D}_n$, and for all $k \geq 0$ and $n \geq n_0: X_{n+k} = \mathrm{sym}_{n+k}(X_n)$. The set of symmetric sequences will be denoted by $\mathscr{Y}$, or $\mathscr{Y}(\mathscr{A}, \mathscr{B})$. A sequence $(X_n \in \mathscr{D}_n)_{n\geq n_0}$ is called *approximately symmetric,* if for all $n \geq n_0$, $X_n = \mathrm{sym}_n(X_n)$ and $\forall_{\varepsilon>0}\exists_{Y\in\mathscr{Y}}\exists_m\forall_{n\geq m}\,\|X_n - Y_n\| \leq \varepsilon$. The set of approximately symmetric sequences will be denoted by $\tilde{\mathscr{Y}}$, or $\tilde{\mathscr{Y}}(\mathscr{A}, \mathscr{B})$.

Thus, a *mean field model* is specified by the algebras $\mathscr{A}$, and $\mathscr{B}$, with respective separating states ρ_0 and ρ, and by an approximately symmetric sequence $H = (H_n)_{n\geq n_0}$, of relative hamiltonian densities $H_n = H_n^* \in \mathscr{D}_n$.

The simplest examples of such models are the usual quadratic mean field models with hamiltonians of the form

$$n \cdot H_n = \sum_{i=1}^{n} h_i + (n-1)^{-1} \cdot \sum_{i\neq j}^{n} V_{ij},$$

where h_i is a copy of the single particle hamiltonian $h \in \mathscr{B}$, acting in the ith tensor factor, and V_{ij} is a two-particle interaction $V \in \mathscr{B} \otimes \mathscr{B}$, acting in the ith and jth factors. Note that the first term can be included in the second by setting $V' = V + (h \otimes 1 + 1 \otimes h)/2 = H_2$. Clearly, the above sequence H_n is strictly symmetric, and defined for all $n \geq 2$. It is also the most general sequence of this description. The generalization of the quadratic mean field systems to arbitrary N-particle interactions is straightforward, and leads to symmetric sequences H_n defined for $n \geq N$. As in the quadratic case such a model is completely specified by $H_N \in \mathscr{D}_N$, since the higher terms of a symmetric sequence are given by an explicit formula. Just as the requirement of symmetry fixes the scaling of the N-particle interaction-term in H_n, it fixes the scaling of the interaction between $\mathscr{A}$ and $\mathscr{B}_n$. With $\mathscr{A}$ non-trivial there are also symmetric sequences defined for $n \geq 0$, which are of the form $H_n^0 = a \otimes 1 \cdots \otimes 1 \in \mathscr{D}_n$. The corresponding hamiltonian has a factor n, so the non-interacting $\mathscr{A}$-part of the hamiltonian is scaled to infinity with the number of $\mathscr{B}$-particles. This is necessary for H^0 to contribute non-trivially to the thermodynamic functions of the model.

Consider now the sequence $H_n = (n^{-1} \cdot \sum_{i=1}^n h_i)^2$ of hamiltonian densities. This can be written as $H_n = Y_n + R_n$, where Y_n is symmetric with $Y_2 = h \otimes h$ and $\|R_n\| \leq n^{-1}\,\|h\|^2$. Thus H_n is approximately symmetric. More generally, we can take $H_n = f(n^{-1} \cdot \sum_{i=1}^n h_i)$, where f is any continuous function on the spectrum of h. These are exactly the hamiltonian densities considered in [14]. If f is a polynomial, then the sequence $Y \in \mathscr{Y}$ in Definition II.1 can be taken independently of ε. However, for general f we need the full freedom of the definition.

A further generalization covered by the above definition of mean field systems is to allow the function f in the previous paragraph to depend on several variables, which do not have to commute, and may themselves be arbitrary approximately symmetric sequences. Thus we can have $H = f(X_n^1, X_n^2, \ldots) \in \mathscr{D}_n$, with $X^\nu \in \tilde{\mathscr{Y}}$ for some function f (see Proposition II.2 below). However, in order to make this definition of H_n precise we have to clarify what we mean by 'the same function f' in the different C^*-algebras $\mathscr{D}_n$. This is done in the Appendix by introducing the notion of C^*-functions. Here we only remark that the set of C^*-functions is closed under composition, and includes all polynomials of (finitely many) non-commuting variables, as well as the continuous functions of a single variable.

A crucial rôle in the theory is played by the algebra $\mathscr{C}(K(\mathscr{B}), \mathscr{A})$ of continuous functions on the state space of $\mathscr{B}$ (with the w^*-topology) with values in $\mathscr{A}$ (with the norm topology). This is developed in Section IV. To every $x \in \mathscr{D}_n$ we associate a function $j_n(x) \in \mathscr{C}(K(\mathscr{B}), \mathscr{A})$ such that for every $\varphi \in \mathscr{K}(\mathscr{B})$, $j_n(a \otimes b_1 \otimes \cdots \otimes b_n)(\varphi) = a \cdot \prod_{\nu=1}^n \varphi(b_\nu)$. We show in Lemma IV.6 that for $X = (X_n) \in \tilde{\mathscr{Y}}$, the limit $j(X) = \lim_n j_n(X_n)$ exists uniformly and j maps $\tilde{\mathscr{Y}}$ onto $\mathscr{C}(K(\mathscr{B}), \mathscr{A})$. In fact, we equip $\tilde{\mathscr{Y}}$ with the structure of a seminormed *-algebra, and show that j is a C^*-isomorphism. As an application, we obtain a proof of the non-commutative de Finetti Theorem of Størmer [18] and also its extension [6] (without separability assumptions on the algebra $\mathscr{A}$). Returning to our main concern, the statistical mechanics of mean field models, we can show that for a symmetric state ϕ of $\mathscr{D}_\infty$, $\phi(X_n)$ converges as $n \to \infty$, for each $X = (X_n) \in \tilde{\mathscr{Y}}$; and we obtain a formula for this limit in terms of the map j, and the decomposition of ϕ into extremal symmetric states.

If Y is obtained by operating on some other sequences X^ν elementwise with some C^*-function (see the Appendix), then we have the following convenient formula for $j(Y)$ in terms of the functions $j(X^\nu)$.

II.2 Proposition. *Let f be a C^*-function on some compact convex set $\Gamma \subset \mathbb{R}^\infty$, and let $X^\nu \in \tilde{\mathscr{Y}}$ be an approximately symmetric sequence for each $\nu \in \mathbb{N}$ such that $Y_n = f(X_n^1, X_n^2, \ldots) \in \mathscr{D}_n$ is defined for $n \geq n_0$. Then $Y = (Y_n)_{n \geq n_0}$ is approximately symmetric and*

$$j(Y) = f(j(X^1), j(X^2), \ldots)$$

The treatment of the entropy parallels that of [14]; most of the technical details needed in our more general setting are found in [13]. For states ω and φ of a unital C^*-algebra, $S(\omega, \varphi)$ will denote the *relative entropy* of φ with respect to ω (in the sign-convention of [2]). The non-negative real number $S(\omega, \varphi)$ is defined via the GNS representation associated with ω, and is finite only if φ extends to a normal state of the generated von Neumann algebra; in this case, $S(\omega, \varphi)$ is given by the definition of [2] applied to the normal state extensions. $S(\omega, \circ)$ is convex and lower w^*-semicontinuous (the lower semicontinuity in this general context follows readily from [13, Theorem 9]). In the particular case where both states are given by non-singular densities D with respect to a trace

Tr,

$$S(\omega, \varphi) = \mathrm{Tr}\,(D_\varphi\{\log D_\varphi - \log D_\omega\}).$$

The *mean relative entropy* for $\phi \in K(\mathscr{B}_\infty)$ with respect to $\omega \in K(\mathscr{B}_\infty)$ is defined to be

$$S_M(\omega, \phi) = \limsup_n n^{-1} S(\omega \mid \mathscr{B}_n, \phi \mid \mathscr{B}_n);$$

and is affine in ϕ. When the reference state is a symmetric product-state Π_ρ, and ϕ is symmetric, then by Proposition III.4 the upper limit is in fact a proper limit.

The connection between $\omega_n^{nH_n}(1)$ and thermodynamics is the following. Suppose that the separating states $\rho_0 \in K(\mathscr{A})$ and $\rho \in K(\mathscr{B})$ determining ω_n are given by densities $\exp(-\beta h_0)/\mathrm{Tr}_{\mathscr{A}} \exp(-\beta h_0)$, respectively $\exp(-\beta h)/\mathrm{Tr}_{\mathscr{B}} \exp(-\beta h)$, with $\beta > 0$. The non-interacting system then has $\mathscr{H}_n^0 = h_0 \otimes [h \otimes 1_{n-1} + 1 \otimes h \otimes 1_{n-2} + \cdots + 1_{n-1} \otimes h]$ as its hamiltonian. The corresponding free energy density $F_n^0(\beta)$ is then simply

$$\begin{aligned} F_n^0(\beta) &= (-n\beta)^{-1} \log \mathrm{Tr}_{\mathscr{D}_n} \exp(-\beta \mathscr{H}_n^0) \\ &= (-n\beta)^{-1} \log \mathrm{Tr}_{\mathscr{A}} \exp(-\beta h_0) - \beta^{-1} \log \mathrm{Tr}_{\mathscr{B}} \exp(-\beta h), \end{aligned}$$

and its thermodynamic limit is $-\beta^{-1} \log \mathrm{Tr}_{\beta} \exp(-\beta h)$. The free energy density corresponding to the hamiltonian $\mathscr{H}_n^0 + V_n$, i.e. $F_n(\beta) = (-n\beta)^{-1} \log \mathrm{Tr}_{\mathscr{D}_n} \times \exp(-\beta(\mathscr{H}_n^0 + V_n))$, is then given by

$$\beta[F_n^0(\beta) - F_n(\beta)] = n^{-1} \log \omega_n^{-\beta V_n}(1).$$

The following result gives the existence of the thermodynamic limit of the relative free energy density of any mean field model, and establishes the validity of the Gibbs Variational Principle. Moreover, and as is to be expected due to the mean field nature of the models and the non-commutative de Finetti Theorem, the variational problem contracts to one on the direct product of the state space of $\mathscr{A}$ and the ('single particle') state space of $\mathscr{B}$.

II.3 Theorem. *For every mean field model,*

$$\lim_n n^{-1} \log \omega_n^{nH_n}(1) = \sup_{\phi \in K_s(\mathscr{D}_\infty)} \left\{ \lim_n \phi(H_n) - S_M(\Pi_\rho, \phi \mid \mathscr{B}_\infty) \right\} \qquad (*)$$

$$= \sup_{\substack{\varphi \in K(\mathscr{B}) \\ \varphi_0 \in K(\mathscr{A})}} \{\varphi_0\{j(H)(\varphi)\} - S(\rho, \varphi)\}, \qquad (**)$$

Remark that the separating state ρ_0 of $\mathscr{A}$ does not appear in the functionals to be maximized, and also that the $\mathscr{A}$-system does not contribute at all to the entropic part of these functionals. The only influence of the $\mathscr{A}$-system enters via the limiting interaction energy density.

The basic information on the nature of the equilibrium states is collected in the following result.

II.4 Theorem. *For every mean field model one has:*

(1) *Every w^* cluster point of the sequence* $(\mathrm{Norm}^{-1}\,\omega_n^{nH_n})_{n\geq n_0}$ *maximizes* (*);
(2) *The subset $M_* \subset K_s(\mathscr{B}_\infty)$ of states ϕ maximising* (*) *is convex and compact, and the subset $M_{**} \subset K(\mathscr{A}) \times K(\mathscr{B})$ of pairs (φ_0, φ) maximising* (**) *is non-empty and compact. The extreme points of M_* are the states $\varphi_0 \otimes \Pi_\varphi$ with $(\varphi_0, \varphi) \in M_{**}$ and $\varphi_0 \in K(\mathscr{A})$ pure. Every $\phi \in M_*$ has a w^*-integral decomposition $\phi = \int \mu(d\sigma)\varphi_\sigma \otimes \Pi_\sigma$, where μ is a Baire probability measure on $K(\mathscr{B})$, $\varphi_\sigma \in K(\mathscr{A})$ for all $\sigma \in K(\mathscr{B})$, $\sigma \mapsto \varphi_\sigma(a)$ is measurable for all $a \in \mathscr{A}$, and $(\varphi_\sigma, \sigma) \in M_{**}$* a.e. (μ).
(3) *If $\mathscr{A}$ and $\mathscr{B}$ are separable, then for any extreme point ϕ of M_* there exists an approximately symmetric sequence $(\tilde{H}_n)_{n\geq n_0}$ such that*

$$\lim_n \|H_n - \tilde{H}_n\| = 0,$$

and the sequence $(\mathrm{Norm}^{-1}\,\omega_n^{n\tilde{H}_n})_{n\geq n_0}$ *is w^*-convergent to ϕ.*
(4) *Let $X \in \tilde{\mathscr{Y}}(\mathbb{C}, \mathscr{B})$, and suppose that the sequence* $(\mathrm{Norm}^{-1}\,\omega_n^{nH_n})$ *converges to an extreme point of M_*. Then the sequence $(\mathbb{K}_n)$ of probability measures on $\mathbb{R}$, defined by $\int \mathbb{K}_n(dx) f(x) =$* $(\mathrm{Norm}^{-1}\,\omega_n^{nH_n})(f(X_n))$ *for $f \in \mathscr{C}_0(\mathbb{R})$, is w^*-convergent to a point measure.*
(5) *If ϕ is a maximiser of* (*), *then the restriction of ϕ to $\mathscr{B}_n$ is normal with respect to the restriction of ω_n to $\mathscr{B}_n$ for all n.*

Note that the integral decomposition given in (2) is not a decomposition into pure phases, which would be an integral of the form $\phi = \int \nu(d(\varphi_0, \varphi))\varphi_0 \otimes \Pi_\varphi$ with a probability measure ν on $K(\mathscr{A}) \times K(\mathscr{B})$, supported by the set of $(\varphi_0, \varphi) \in M_{**}$ with φ_0 extremal in $K(\mathscr{A})$. For non-separable $\mathscr{A}$ the set of extreme points of $K(\mathscr{A})$ is not measurable in $K(\mathscr{A})$ and hence M_{**} is in general not measurable. The 'support' of the measure ν on $K(\mathscr{A}) \times K(\mathscr{B})$ thus has to be understood in the weaker sense customary in non-metrizable Choquet-theory. The integral decomposition given in the theorem avoids this difficulty and has the additional virtue of being unique in the sense specified in Proposition IV.5.

One may wonder whether the local normality property (5) holds also for the whole algebras $\mathscr{D}_n$ rather than the tensor factors $\mathscr{B}_n$. That this is not the case is seen in the following example. Let $\mathscr{A} = \mathscr{B}(\mathscr{H})$ be the algebra of bounded operators on $\mathscr{H} = L^2([0, 1], dx)$ and let $\mathscr{B} = \mathbb{C}$ be trivial. Let ρ_0 be any faithful normal state on $\mathscr{A}$, and let $H_n \equiv H \in \mathscr{A}$ be the multiplication operator with x in $\mathscr{H}$. Then by Theorem II.4(1) any cluster point $\varphi_0 \in K(\mathscr{A})$ of the sequence $(\mathrm{Norm}^{-1} \cdot \rho_0^{nx})$ satisfies $\varphi_0(H) = \sup \mathrm{spec}\,(H) = 1$. Hence φ_0 must be purely singular on $\mathscr{B}(\mathscr{H})$.

Under a differentiability condition, the maximizers of (**), i.e. the states in M_{**}, satisfy a generalized gap-equation [7, 14] with a state-dependent *effective hamiltonian* $\hbar$:

II.5 Proposition. *Let $(\varphi_0, \varphi) \in M_{**}$, and suppose that $j(H)$ is differentiable* *at* (φ_0, φ) *in the sense that there is some* $h \in \mathscr{B}$ *such that for all* $\psi \in K(\mathscr{B})$:

$$\varphi_0\{j(H)((1-\lambda)\varphi + \lambda\psi)\} = \varphi_0\{j(H)(\varphi)\} + \lambda(\psi(h) - \varphi(h)) + o(\lambda)$$

as $\lambda \to 0^+$. *Then* $\varphi = (\rho^h(1))^{-1} \cdot \rho^h$.

In the case studied in [14], the variational problem (**) contracts further to one on the real line. This was seen to provide an extension of Varadhan's asymptotic formula [20, 21], based on the large deviation results of Cramér for the distribution of sums of independent, identically distributed random variables. We obtain a further generalization of this, which at the same time reduces the computation of the suprema of Theorem II.3 in a certain subclass of mean field models to a variational problem on $\mathbb{R}^k$. The subclass consists of those models where $\mathscr{A}$ is trivial, and the hamiltonian density is given by $H_n = f(X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(k)})$, for some C^*-function f, and k symmetric sequences $X^{(\nu)} \in \mathscr{Y}$, all beginning at $n_0 = 1$.

Consider k self-adjoint elements $x^{(1)}, x^{(2)}, \ldots, x^{(k)}$ in $\mathscr{B}$. For $t \in \mathbb{R}^k$, let $t \cdot x = t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_k x^{(k)}$, and define $G_\rho^x : \mathbb{R}^k \to \mathbb{R}$, $I_\rho^x : \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$ by

$$G_\rho^x(t) = \log \rho^{t \cdot x}(1), \qquad t \in \mathbb{R}^k,$$

$$I_\rho^x(u) = \sup_{t \in \mathbb{R}^k} \{t \cdot u - G_\rho^x(t)\}, \qquad u \in \mathbb{R}^k,$$

where ρ is any separating state of $\mathscr{B}$. G_ρ^x is then convex and differentiable [3] with

$$\{\nabla G_\rho^x\}_j(t) = \rho^{t \cdot x}(x^{(j)}) / \rho^{t \cdot x}(1), \qquad 1 \le j \le k.$$

Moreover, $G_\rho^x(0) = 0$, and the generalized Peirels–Bogoljubov and Golden–Thompson inequalities of [3] imply $\rho(t \cdot x) \le G_\rho^x(t) \le \log \rho(e^{t \cdot x})$. It follows that I_ρ^x is non-negative, convex, and lower semicontinuous, with $I_\rho^x(\rho(x^{(1)}), \rho(x^{(2)}), \ldots, \rho(x^{(k)})) = 0$. Using [3] one can see that G_ρ^x (and hence I_ρ^x, [17, Theorem 26.5]) is strictly convex if and only if the set $\{1, x^{(1)}, \ldots, x^{(k)}\}$ is linearly independent. We remind the reader that the effective domain, dom (I_ρ^x), of I_ρ^x is the convex set where I_ρ^x is finite.

II.6 Theorem. *Let* $x^{(\nu)}$, $1 \le \nu \le k$, *be self-adjoint elements in* $\mathscr{B}$; *then the closure of* dom (I_ρ^x) *is* $E = \{(\varphi(x^{(1)}), \varphi(x^{(2)}), \ldots, \varphi(x^{(k)}) \mid \varphi \in K(\mathscr{B})\}$. *Let the symmetric sequences* $X^{(\nu)}$ *in* $\mathscr{Y}(\mathbb{C}, \mathscr{B})$, $1 \le \nu \le k$, *be given by* $X_1^{(\nu)} = x^{(\nu)} \in \mathscr{B}$, *and let f by any* C^*-*function on E. Set*

$$H_n = f(X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(k)}) \in \mathscr{B}_n.$$

Then, with $H = (H_n)_{n \ge 1} \in \tilde{\mathscr{Y}}(\mathbb{C}, \mathscr{B})$, *one has*

$$\sup_{\varphi \in K(\mathscr{B})} \{j(H)(\varphi) - S(\rho, \varphi)\} = \sup_{u \in E} \{f(u) - I_\rho^x(u)\}$$

$$= \sup_{t \in \mathbb{R}^k} \{f(\{\nabla G_\rho^x\}(t)) - t \cdot \{\nabla G_\rho^x\}(t) + G_\rho^x(t)\}.$$

Let us illustrate these results in the case where the assumptions of Proposition II.5 and Theorem II.6 are both satisfied, i.e. f is differentiable as a scalar function on a neighbourhood of E. Then

$$h = \sum_{j=1}^{k} x_j \nabla_j f(\varphi(x^{(1)}), \ldots, \varphi(x^{(k)})).$$

Consider the maps U, T, and Φ defined by:

$$K(\mathscr{B}) \ni \varphi \mapsto U(\varphi) = (\varphi(x^{(1)}), \ldots, \varphi(x^{(k)})) \in E,$$

$$\mathbb{R}^k \ni t \mapsto \Phi(t) = (\rho^{t\cdot x}(1))^{-1} \rho^{t\cdot x} \in K(\mathscr{B}).$$

$$E \ni u \mapsto T(u) = \nabla f(u) \in \mathbb{R}^k,$$

If $\varphi \in K(\mathscr{B})$ is a maximizer for $j(H)(\circ) - S(\rho, \circ)$, then $U(\varphi)$ is a maximizer for $f(\circ) - I^x_\rho(\circ)$. Given a maximizer $t \in \mathbb{R}^k$, $\Phi(t)$ maximizes $j(H)(\circ) - S(\rho, \circ)$. Finally, an argument similar to that of the proof of Proposition II.5 shows that given a maximizer $u \in E$, $T(u)$ maximizes $t \to f(\{\nabla G^x_\rho\}(t)) - t \cdot \{\nabla G^x_\rho\}(t) + G^x_\rho(t)$. This sets up bijections between the sets of maximizers of the three expressions of Theorem II.6. The gap-equation becomes

$$\varphi = \Phi \circ T \circ U(\varphi) = \mathrm{Norm}^{-1}\, \rho^{\{\nabla f(\varphi(x^{(1)}), \ldots, \varphi(x^{(k)}))\}\cdot x},$$

or, alternatively,

$$t = T \circ U \circ \Phi(t) = \nabla f(\nabla G^x_\rho(t)).$$

The bijective correspondence between the sets of maximizers of the three variational problems of Theorem II.6 is also guaranteed if $\mathrm{dom}\,(I^x_\rho) = \nabla G^x_\rho(\mathbb{R}^k)$. This last condition does not follow from the differentiability of f. If G^x_ρ is strictly convex, then $\nabla G^x_\rho(\mathbb{R}^k) = \mathrm{int}\,(\mathrm{dom}\,(I^x_\rho))$ [17, Theorem 26.5] is open. On the other hand, if $\mathscr{B}$ is finite-dimensional, then $S(\rho, \circ)$ is bounded above, and one can show that $\mathrm{dom}\,(I^x_\rho) = E$, which is closed.

III. Proof of main results

In this section we give the proofs of all results of the previous section except II.2. This is done in the appendix. The basic idea of the proofs is exactly the same as in the paper [14]. The new ingredients are the inclusion of a non-trivial algebra $\mathscr{A}$, and a much larger class of admissible hamiltonians; this becomes possible due to the theory presented in Sect. IV.

The central idea is to use the following important variational characterization of the relative entropy of states in a general C^*-algebra due to Petz [12, 13]. It can be stated by saying that $h \mapsto \log \omega^h(1)$ is the 'Legendre' transform of $\varphi \mapsto S(\omega, \varphi)$, and conversely.

III.1 Lemma (Petz). *Let ω be a separating state of a unital C^*-algebra $\mathscr{A}$, $h = h^* \in \mathscr{A}$, and φ any state of $\mathscr{A}$. Then*

$$\log \omega^h(1) \geq \varphi(h) - S(\omega, \varphi)$$

and equality holds if and only if $\varphi = \omega^h(1)^{-1}\omega^h$. *Moreover,*

$$S(\omega, \varphi) = \sup_{h^*=h\in\mathscr{A}} \{\varphi(h) - \log \omega^h(1)\}.$$

This lemma is now applied to the algebra $\mathscr{D}_n$, with the reference state $\omega = \omega_n = \rho_0 \otimes (\otimes_{\nu=1}^n \rho_{(\nu)})$, the relative Hamiltonian $h = nH_n$, and a symmetric state $\phi = \phi_n$ of $\mathscr{D}_n$. After dividing the inequality by n, we pass to the limit. Thus one has to control two kinds of terms, namely the interaction energy density $\phi_n(H_n)$, and the relative entropy density $n^{-1}S(\omega_n, \phi_n)$. We shall have to require of the sequence (ϕ_n) only that it converges *-weakly to a limiting state on $\mathscr{D}_\infty$. Since the state space of $\mathscr{D}_\infty$ is w^*-compact, this condition can always be met by passing to a subnet (since we are not assuming $\mathscr{A}$ and $\mathscr{B}$ to be separable, subsequences will not do). We will use the following notations. Let ν be a subnet of $\mathbb{N}$, i.e. a function $\nu:\mathbb{A}\to\mathbb{N}$ on a directed set $(\mathbb{A}, \geq)$ such that for every $n \in \mathbb{N}$ there exists $\alpha_0 \in \mathbb{A}$ such that $\nu(\alpha) \geq n$, whenever $\alpha \geq \alpha_0$. If $(a_n)_{n\in\mathbb{N}}$ is a sequence in a Hausdorff space, we write $\lim_{n\to\nu} a_n$ for $\lim_{\alpha\in\mathbb{A}} a_{\nu(\alpha)}$ if it exists, and employ a similar notation for superior and inferior limits of sequences of extended-real numbers.

III.2 Definition. Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of permutation symmetric states $\phi_n \in K(\mathscr{D}_n)$ (resp. $\phi_n \in K(\mathscr{B}_n)$). We say that (ϕ_n) is *convergent along a subnet* $\nu:\mathbb{A}\to\mathbb{N}$, if for all $m \in \mathbb{N}$ and all $X \in \mathscr{D}_m$ (resp. $\mathscr{B}_m$) the limit $\lim_{n\to\nu} \phi_n(X_n) =: \phi(X)$ exists.

For any sequence convergent along a subnet, the limit-functional extends from $\bigcup_n \mathscr{D}_n$ (resp. $\bigcup_n \mathscr{B}_n$) to a unique symmetric state ϕ of $\mathscr{D}_\infty$ (resp. $\mathscr{B}_\infty$), and we shall write $\phi = \lim_{n\to\nu} \phi_n$. By Proposition IV.5 any symmetric state ϕ has an integral decomposition, $\phi = \int \mu(d\sigma)\varphi_\sigma \otimes \Pi_\sigma$, into product states. This decomposition is used in the following proposition, which summarizes the energy estimates we shall need. It is proven at the end of Section IV.

III.3 Proposition. *Let* $X \in \tilde{\mathscr{Y}}$ *and let* $(\phi_n)_{n\in\mathbb{N}}$ *be a sequence of permutation symmetric states of* $\mathscr{D}_n$ *converging along a subnet* ν *to*

$$\lim_{n\to\nu} \phi_n = \int \mu(d\sigma)\varphi_\sigma \otimes \Pi_\sigma \in K_s(\mathscr{D}_\infty).$$

Then

$$\lim_{n\to\nu} \phi_n(X_n) = \int \mu(d\sigma)\varphi_\sigma\{j(X)(\sigma)\}.$$

Recall that $S_M(\omega, \phi)$ was defined as an upper limit. We shall need to know that this lim sup is in fact a limit, if ω is a product state and ϕ is symmetric. The necessary control of the lower limit is stated in the following proposition.

III.4 Proposition. (1) *Let* $(\phi_n)_{n\in\mathbb{N}}$ *be a sequence of permutation symmetric* *states of $\mathscr{B}_n$ converging along a subnet ν to $\lim_{n\to\nu}\phi_n = \phi \in K_s(\mathscr{B}_\infty)$. Then*

$$\liminf_{n\to\nu} n^{-1}S(\Pi_\rho \mid \mathscr{B}_n, \phi_n) \geq S_M(\Pi_\rho, \phi).$$

(2) *If $\phi = \int \mu(d\sigma)\Pi_\sigma$ is the decomposition of the symmetric state ϕ into product states, then $S_M(\Pi_\rho, \phi) = \int \mu(d\sigma)S(\rho, \sigma)$.*

Proof. We may suppose that $S(\Pi_\rho \mid \mathscr{B}_n, \phi_n)$ is finite for every n. Let τ be the right-shift on $\mathscr{B}_\infty$. $S_M(\Pi_\rho, \circ)$ is lower w^*-semicontinuous on the τ-invariant states of $\mathscr{B}_\infty$ (see the appendix of [14]). One has

$$\liminf_{\alpha\in\mathbb{A}} S_M(\Pi_\rho, \xi_\alpha) \geq S_M(\Pi_\rho, \phi),$$

for any net $\{\xi_\alpha \mid \alpha \in \mathbb{A}\}$ of τ-invariant states of $\mathscr{B}_\infty$ which is w^*-convergent to ϕ. The first claim follows if we construct such a net, with the additional property that

$$S_M(\Pi_\rho, \xi_\alpha) = \nu(\alpha)^{-1}S(\Pi_\rho \mid \mathscr{B}_{\nu(\alpha)}, \phi_{\nu(\alpha)}). \tag{***}$$

Define the state $\zeta_\alpha \in K(\mathscr{B}_\infty)$ by

$$\zeta_\alpha \mid \mathscr{B}_{k\nu(\alpha)} = \phi_{\nu(\alpha)} \otimes \phi_{\nu(\alpha)} \otimes \cdots \otimes \phi_{\nu(\alpha)} \ (k \text{ factors}), \text{ for every } k \geq 1;$$

ζ_α is then $\tau^{\nu(\alpha)}$-invariant. For every $m \in \mathbb{N}$, $\zeta_\alpha(X) = \phi_{\nu(\alpha)}(X)$ for all $X \in \mathscr{B}_m$ whenever $\nu(\alpha) \geq m$. Hence, $w^* - \lim_{\alpha\in\mathbb{A}} \zeta_\alpha = \phi$. Put $\xi_\alpha = \nu(\alpha)^{-1}\sum_{j=1}^{\nu(\alpha)} \zeta_\alpha \circ \tau^{j-1}$; then, ξ_α is τ-invariant and has the same limit as $\{\zeta_\alpha\}$ by [14, Lemma 5]. We can repear the argument of [14, Lemma 7] to show that for any m,

$$\lim_k k^{-1}S(\Pi_\rho \mid \mathscr{B}_k, \zeta_\alpha \circ \tau^m \mid \beta_k) = \nu(\alpha)^{-1}S(\Pi_\rho \mid \beta_{\nu(\alpha)}, \phi_{\nu(\alpha)}).$$

Then the argument of [14, Lemma 8] implies (***). This completes the proof of the first claim. The second claim follows from the lower semicontinuity of $S_M(\Pi_\rho, \circ)$ by a standard result of Choquet Theory, and the fact that $S_M(\Pi_\rho, \Pi_\varphi) = S(\rho, \varphi)$.

Proof of Theorem II.3 and of Theorem II.4(1)

Put $a_n = n^{-1}\log \omega_n^{nH_n}(1)$, $A = \sup\{\lim_n \phi(H_n) - S_M(\Pi_\rho, \phi \mid \mathscr{B}_\infty) \mid \phi \in K_s(\mathscr{D}_\infty)\}$, and $B = \sup\{\varphi_0\{j(H)(\varphi)\} - S(\rho, \varphi) \mid \varphi \in K(\mathscr{B}), \varphi_0 \in K(\mathscr{A})\}$.

We first claim that $A \leq B$. Indeed, by Proposition III.3 and the second part of Proposition III.4, for any $\phi \in K_s(\mathscr{D}_\infty)$,

$$\lim_n \phi(H_n) - S_M(\Pi_\rho, \phi \mid \mathscr{B}_\infty) = \int \mu(d\sigma)[\varphi_\sigma\{j(H)(\sigma)\} - S(\rho, \sigma)] \leq B$$

since μ is a probability measure. The first claim follows by taking the supremum with respect to ϕ.

Now we claim that $\liminf_n a_n \geq B$. By Lemma III.1, for arbitrary $\varphi_0 \in K(\mathscr{A})$, and $\varphi \in K(\mathscr{B})$,

$$\begin{aligned}a_n &\geq (\varphi_0 \otimes \Pi_\rho)(H_n) - n^{-1}S(\rho_0 \otimes (\Pi_\rho \mid \mathscr{B}_n), \varphi_0 \otimes (\Pi_\varphi \mid \mathscr{B}_n))\\ &= (\varphi_0 \otimes \Pi_\rho)(H_n) - n^{-1}(S(\rho_0, \varphi_0) + nS(\rho, \varphi)).\end{aligned}$$

Thus, if $S(\rho_0, \varphi_0)$ is finite,

$$\liminf_n a_n \geq \varphi_0\{j(H)(\varphi)\} - S(\rho, \varphi),$$

by Proposition III.3. This implies that

$$\liminf_n a_n \geq \sup\{\varphi_0\{j(H)(\varphi)\} - S(\rho, \varphi) \mid \varphi \in K(\mathscr{B}), \varphi_0 \in K(\mathscr{A}), S(\rho_0, \varphi_0) < \infty\}.$$

Since ρ_0 is separating, the set of states of $\mathscr{A}$ with finite relative entropy with respect to ρ_0 is w^*-dense in $K(\mathscr{A})$, and the second claim follows.

The third claim is that if the sequence $\{\phi_n \equiv \text{Norm}^{-1}\, \omega_n^{nH_n} \mid n \in \mathbb{N}\}$ converges along a subnet ν to $\phi \in K_s(\mathscr{D}_\infty)$, one has

$$\limsup_{n \to \nu} a_n \leq \lim_n \phi(H_n) - S_M(\Pi_\rho, \phi \mid \mathscr{B}_\infty) \leq A.$$

By Lemma III.1, and monotonicty of the relative entropy, for any $\alpha \in \mathbb{A}$,

$$\begin{aligned}a_{\nu(\alpha)} &= \phi_{\nu(\alpha)}(H_{\nu(\alpha)}) - \nu(\alpha)^{-1}S(\omega_{\nu(\alpha)}, \phi_{\nu(\alpha)})\\ &\leq \phi_{\nu(\alpha)}(H_{\nu(\alpha)}) - \nu(\alpha)^{-1}S(\omega_{\nu(\alpha)} \mid \mathscr{B}_{\nu(\alpha)}, \phi_{\nu(\alpha)} \mid \mathscr{B}_{\nu(\alpha)})\\ &= \phi_{\nu(\alpha)}(H_{\nu(\alpha)}) - \nu(\alpha)^{-1}S(\Pi_\rho \mid \mathscr{B}_{\nu(\alpha)}, \phi_{\nu(\alpha)} \mid \beta_{\nu(\alpha)}).\end{aligned}$$

The third claim follows from Propositions III.3 and III.4.

Theorem II.4(1) follows from the three claims. Suppose that $\limsup_n a_n$ is strictly larger than A. Then there is a subnet ν_0 of $\mathbb{N}$ such that (a_n) converges along ν_0 to this larger value. By w^*-compactness of $K(\mathscr{D}_\infty)$, there is a subnet ν of ν_0 such that $(\text{Norm}^{-1}\, \omega_n^{nH_n})$ converges along ν to a symmetric state in the sense of Definition III.2. This contradicts the third claim, and shows that $\limsup a_n \leq A$, which together with the first and second claims proves Theorem II.3.

Proof of Theorem II.4(2)

Since the functional maximized in (*) is affine in ϕ, M_* is convex. Due to the lower semicontinuity of relative entropies the functionals in (*) and (**) are both upper semicontinuous, and hence assume their supremum on a closed set, which is compact since $K_s(\mathscr{D}_\infty)$ and $K(\mathscr{A}) \times K(\mathscr{B})$ are compact. The integral decomposition exists by Proposition IV.5 for any $\phi \in K_s(\mathscr{D}_\infty)$, and we only have to prove that $(\varphi_\sigma, \sigma) \in M_{**}$ almost everywhere. This is clear from the first inequality in the above proof of II.3, which must be an equality for a maximizer ϕ. Clearly, ϕ cannot be extremal unless μ is a point measure, i.e. $\phi = \varphi_0 \otimes \Pi_\varphi$ for some $(\varphi_0, \varphi) \in M_{**}$, and unless φ_0 is extremal. Conversely, if ϕ has the stated property, then if is extremal in $K_s(\mathscr{D}_\infty)$, hence in M_*.

Proof of Theorem II.4(3)

By II.4(2) every extreme point of M_* is of the form $\phi = \varphi_0 \otimes \Pi_\varphi$ with φ_0 pure and $(\varphi_0, \varphi) \in M_{**}$. We claim that there exists $G \in \tilde{\mathcal{Y}}$ such that $\psi_0(j(G)(\psi)) \leq 0$ for all $\psi_0 \in K(\mathcal{A})$, $\psi \in K(\mathcal{B})$, with equality exactly for the pair (φ_0, φ). We then consider for $\varepsilon > 0$ the mean field models with hamiltonian density $\varepsilon G_n + H_n$, and let $\Psi_n^\varepsilon \in K(\mathcal{D}_n)$ denote the state $\Psi_n^\varepsilon = \mathrm{Norm}^{-1} \cdot \omega_n^{n(\varepsilon G_n + H_n)}$. Then by II.4(1) the sequence Ψ_n^ε converges to the given extreme point ϕ for every $\varepsilon > 0$. We construct a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ converging to zero with the property that $\Psi_n^{\varepsilon_n} \to \phi$ as $n \to \infty$. For this consider a metric d on $K(\mathcal{D}_\infty)$, which exists since each $\mathcal{D}_n$ is separable as the tensor product of separable C^*-algebras. Let $N_k \in \mathbb{N}$ such that $d(\phi, \Psi_n^\varepsilon) \leq 1/k$ for $\varepsilon = 1/k$ and all $n \geq N_k$. N_k can be arranged to be an increasing sequence and we set $\varepsilon_n = 1/k$ for $N_k \leq n < N_{k+1}$. Hence $d(\phi, \Psi_n^{\varepsilon_n}) \leq \varepsilon_n$ for all $n \geq N_1$. Then $\tilde{H}_n = \varepsilon_n G_n + H_n$ has the properties stated in the theorem.

It remains to be proven that $G \in \tilde{\mathcal{Y}}$ with the stated properties exists. By Lemma V.6 it suffices to find $g \in \mathcal{C}(K(\mathcal{B}), \mathcal{A})$ such that $g \leq 0$ and $\psi_0(g(\psi)) = 0$ iff $(\psi_0, \psi) = (\varphi_0, \varphi)$. Note that the pair (φ_0, φ) defines an extremal state ϕ of $\mathcal{C}(K(\mathcal{B}), \mathcal{A})$ via $\phi(f) = \varphi_0(f(\varphi))$, and that we are looking for an element $g \in \mathcal{C}(K(\mathcal{B}), \mathcal{A})$ 'exposing' this state, in the sense that $\Psi \in K(\mathcal{C}(K(\mathcal{B}), \mathcal{A}))$ and $\Psi(g) = 0$ imply $\Psi = \phi$. Now since $\mathcal{B}$ is separable, so is $\mathcal{C}(K(\mathcal{B}))$, being generated by the functions of the form $f(\varphi) = \varphi(b)$ for b in a countable dense subset of $\mathcal{B}$ on account of the Stone–Weierstraß-Theorem. Hence $\mathcal{C}(K(\mathcal{B}), \mathcal{A}) \simeq \mathcal{C}(K(\mathcal{B})) \otimes \mathcal{A}$ is separable as the tensor product of separable algebras. Our claim is thus reduced to the general proposition that any pure state Φ of a separable C^*-algebra $\mathcal{F}$ is exposed. (Counterexamples for non-separable $\mathcal{F}$ are easily constructed). By [11, Theorem 3.10.7] every extremal state Φ is characterized as $\{\Phi\} = \{\Psi \in K(\mathcal{F}) \mid \forall_{f \in \mathcal{L}} \Psi(f) = 0\}$, where $\mathcal{L}$ denotes the left ideal $\mathcal{L} = \{f \in \mathcal{F} \mid \Phi(f^* f) = 0\}$. As a subspace of a separable normed space the ideal $\mathcal{L}$ contains a dense sequence $(f_n)_{n \in \mathbb{N}}$, and $g = -\sum_n 2^{-n} \|f_n\|^{-1} f_n$ is an element exposing Φ.

Proof of Theorem II.4(4)

By Proposition II.2 $Y_n = f(X_n)$ is approximately symmetric and $j(Y) = f(j(X))$. Hence, by Proposition III.3, $\lim_n \int \mathbb{K}_n(dx) f(x) = \int \mu(d\sigma) \varphi_\sigma\{j(Y)(\sigma)\}$ with the integral decomposition II.4(2). Since the limit state is pure, μ is a point measure, say at $\varphi \in K(\mathcal{B})$, and since $X \in \tilde{\mathcal{Y}}(\mathbb{C}, \mathcal{B})$, $j(Y)(\sigma) \in \mathcal{A}$ is a multiple of the identity for all σ. Hence $\lim_n \int \mathbb{K}_n(dx) f(x) = j(Y)(\varphi) = f(j(X))(\varphi) = f(j(X)(\varphi))$, which means that $\mathbb{K}_n$ converges to the point measure at $j(X)(\varphi)$.

Proof of Theorem II.4(5)

Clearly, for ϕ a maximizer $S_M(\Pi_\rho, \phi \mid \mathcal{B}_\infty) = \limsup_n n^{-1} S(\Pi_\rho \mid \mathcal{B}_n, \phi \mid \mathcal{B}_n)$ is finite, and hence $s_n = S(\omega_n \mid \mathcal{B}_n, \phi \mid \mathcal{B}_n)$ must be finite for all sufficiently large

n. By the monotonicity property of S, s_n is an increasing sequence, and is hence finite for all n. By [13] this implies that $\phi \mid \mathscr{B}_n$ extends to a normal state on $\pi_\rho(\mathscr{B}_n)''$, where π_ρ denotes the GNS-representation of $\mathscr{B}_n$ with respect to $\Pi_\rho \mid \mathscr{B}_n$ as claimed.

Proof of Proposition II.5

Let $F(\varphi) = \varphi_0\{j(H)(\varphi)\} - S(\rho, \varphi)$. By convexity of $S(\rho, \circ)$, we have

$$F((1-\lambda)\varphi + \lambda\psi) \geq F(\varphi) + \lambda\{\psi(h) - \varphi(h) + S(\rho, \varphi) - S(\rho, \psi)\} + o(\lambda),$$

for all $\psi \in K(\mathscr{B})$ and all $\lambda \in [0, 1]$. If the expression in braces is strictly positive for some ψ, then the left hand side must be strictly larger than $F(\varphi)$ for some small λ, contradicting the maximality of (φ_0, φ). Hence, for all ψ we have $\psi(h) - S(\rho, \psi) \leq \varphi(h) - S(\rho, \varphi)$. Taking the sup over ψ and using Lemma III.1, we find $\varphi(h) - S(\rho, \varphi) = \log \rho^h(1)$, and hence $\varphi = \mathrm{Norm}^{-1} \cdot \rho^h$.

Proof of Theorem II.6

In what follows, we drop the index ρ and the superscript x from I and G. Notice that E is compact, convex and contained in $\times_{\nu=1}^{k} \operatorname{spec}(x^{(\nu)})$. For $t \in \mathbb{R}^k$, we write $\rho_t = (\rho^{t\cdot x}(1))^{-1}\rho^{t\cdot x} \in K(\mathscr{B})$, and remark that, in an obvious vector-notation, $\rho_t(x) = \nabla G(t)$, and moreover, $S(\rho, \rho_t) = t \cdot \rho_t(x) - G(t)$ due to Lemma III.1.

We first prove that $\nabla G(\mathbb{R}^k)$ and dom (I) have the same closure, which is E. By [17, Corollary 26.4.1], $ri(\mathrm{dom}\,(I)) \subset \nabla G(\mathbb{R}^k) \subset \mathrm{dom}\,(I)$, where ri denotes the relative interior [17, p. 44]. By [17, Theorem 6.3] $ri(\mathrm{dom}\,(I))$ and dom (I) have the same closure, so since $\nabla G(\mathbb{R}^k) \subset E$ and E is closed, $\overline{\nabla G(\mathbb{R}^k)} = \overline{\mathrm{dom}\,(I)} \subset E$. For the converse inclusion, suppose $u \notin \overline{\mathrm{dom}\,(I)}$. There exists [17, Theorem 13.1] $t \in \mathbb{R}^k$ such that $t \cdot u > \sup\{t \cdot v \mid v \in \mathrm{dom}\,(I)\}$. Since I is non-negative, we have for every $n \in \mathbb{N}$

$$n^{-1}G(nt) = \sup_{v\in\mathbb{R}^k} \{v \cdot t - n^{-1}I(v)\} = \sup_{v\in\mathrm{dom}\,(I)} \{v \cdot t - n^{-1}I(v)\} \leq t \cdot u - c,$$

for some $c > 0$. Applying Theorem II.3 in the case $\mathscr{A} = \mathscr{B}$ and $\mathscr{B} = \mathbb{C}$, we have $\lim_n n^{-1}G(nt) = \sup\{\varphi(t \cdot x) = t \cdot \varphi(x) \mid \varphi \in K(\mathscr{B})\}$. Hence $u \notin E$ by [17, Theorem 13.1]. This completes the proof of $\overline{\nabla G(\mathbb{R}^k)} = \overline{\mathrm{dom}\,(I)} = E$.

The second part of Lemma III.1 implies that $S(\rho, \varphi) \geq I(\varphi(x))$ (take $h = t \cdot x$ and vary t). Using this, and Proposition II.2,

$$S_1 := \sup_{\varphi\in K(\mathscr{B})} \{j(H)(\varphi) - S(\rho, \varphi)\} = \sup_{\varphi\in K(\mathscr{B})} \{f(\varphi(x)) - S(\rho, \varphi)\}$$

$$\leq \sup_{\varphi\in K(\mathscr{B})} \{f(\varphi(x)) - I(\varphi(x))\} = \sup_{u\in E} \{f(u) - I(u)\} =: S_2.$$

Since I is $+\infty$ outside dom (I) which has closure E, we may rewrite

$$S_2 = \sup_{u \in \operatorname{dom}(I)} \{f(u) - I(u)\}.$$

On the other hand,

$$S_1 \geq \sup_{t \in \mathbb{R}^k} \{f(\rho_t(x)) - S(\rho, \rho_t)\} = \sup_{t \in \mathbb{R}^k} \{f(\nabla G(t)) - t \cdot \nabla G(t) + G(t)\} =: S_3.$$

If $u \in \nabla G(\mathbb{R}^k)$, then $u = \nabla G(t)$ for some $t \in \mathbb{R}^k$, and [17, Theorem 23.5] $I(u) = t \cdot \nabla G(t) - G(t)$. Thus, $S_3 = \sup \{f(u) - I(u) \mid u \in \nabla G(\mathbb{R}^k)\}$. Since $ri(\operatorname{dom}(I)) \subset \nabla G(\mathbb{R}^k)$, $S_3 \geq \sup \{f(u) - I(u) \mid u \in ri(\operatorname{dom}(I))\}$. We have established that

$$\sup_{u \in ri(\operatorname{dom}(I))} \{f(u) - I(u)\} \leq S_1 \leq \sup_{u \in \operatorname{dom}(I)} \{f(u) - I(u)\}. \tag{***}$$

Due to lower semicontinuity [17, Corollary 7.5.1], $\lim_{\lambda \uparrow 1} I((1-\lambda)v + \lambda u) = I(u)$ for every $u \in \mathbb{R}^k$ and $v \in \operatorname{dom}(I)$. Moreover [17, Theorem 6.1], if $v \in ri(\operatorname{dom}(I))$ and $u \in \operatorname{dom}(I)$ then $(1-\lambda)v + \lambda u \in ri(\operatorname{dom}(I))$ for every $0 \leq \lambda < 1$. This implies that given $u \in \operatorname{dom}(I)$ and $\varepsilon > 0$, there exists $v \in ri(\operatorname{dom}(I))$ such that $|I(u) - I(v)| \leq \varepsilon$. Since f is continuous, it follows that the left and right hand sides of (***) are equal. This completes the proof.

IV. A C^*-algebra of symmetric tensors

In this essentially self-contained section we develop the theory of symmetric and approximately symmetric sequences. This provides a systematic background for the energy estimate III.3, as well as the necessary information for showing the equality of the two variational expressions in Theorem II.3. However, we also prove some results of independent interest. The central idea is to equip the set $\tilde{\mathcal{Y}}$ of approximately symmetric sequences with the structure of a (semi-) normed *-algebra in two prima facie different but equivalent ways.

The first product on $\mathcal{Y}$ (in the case of trivial $\mathcal{A} = \mathbb{C}$) is simply the symmetrized tensor product $\star : \mathcal{D}_n \times \mathcal{D}_m \to \mathcal{D}_n \otimes \mathcal{D}_m = \mathcal{D}_{n+m}$. Clearly, this product is commutative. Any symmetric state $\phi \in K_s(\mathcal{D}_\infty)$ defines a state on the algebra $(\mathcal{Y}, \star)$, and the product states of $\mathcal{D}_\infty$ become homomorphisms,, i.e. pure states on this algebra. This is the basic observation behind Størmer's Theorem [18], which says that any symmetric state has an integral decomposition into product states (compared Proposition IV.5).

The second product on $\tilde{\mathcal{Y}}$ is the elementwise product of sequences. It is not immediately obvious that this operation takes $\tilde{\mathcal{Y}} \times \tilde{\mathcal{Y}}$ into $\tilde{\mathcal{Y}}$. However, the elementwise product turns out to be asymptotically equal to the $\star$-product. This equality will make it possible to treat mean field hamiltonians, which are defined for each n as some arbitrary function of a set of sequences from $\tilde{\mathcal{Y}}$.

We shall continue to use the notation introduced in Sect. II. On the set $\mathcal{Y}$ the operations of scalar multiplication, adjoint, and addition will simply be defined elementwise, e.g. $(X+Y)_n = X_n + Y_n \in \mathcal{D}_n$ for all n such that both X_n and Y_n are defined. We shall set $\|X\| = \lim_n \|X_n\|$. This limit exists since $\|X_{n+k}\| = \|\mathrm{sym}_{n+k}(X_n \otimes 1 \otimes \cdots \otimes 1)\| \leq \|X_n \otimes 1 \otimes \cdots \otimes 1\| = \|X_n\|$, i.e. the sequence of norms is decreasing. It is worthwhile to note that this sequence is in general strictly decreasing (unless X_n is defined for all $n \geq 1$), but that it never decreases to zero for $X \neq 0$. (This can be shown with the help of Lemma IV.4 and the fact that product states on $\mathcal{D}_n$ separate points of $\mathrm{sym}_n(\mathcal{D}_n)$; we shall not use this observation).

The product in $\mathcal{Y}$, which we shall denote by $X, Y \mapsto X \star Y$, will be the symmetrized tensor product in the following sense: for $X = a \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_n \in \mathcal{D}_n$ and $Y = b \otimes y_1 \otimes y_2 \otimes \cdots y_m \in \mathcal{D}_m$ let

$$X \star_{n,m} Y = \mathrm{sym}_{n+m}\{ab \otimes x_1 \otimes x_2 \otimes \cdots x_n \otimes y_1 \otimes y_2 \otimes \cdots y_m)\} \in \mathcal{D}_{n+m}$$

Clearly, this extends by linearity and norm continuity to a bilinear map $\star_{n,m} : \mathcal{D}_n \times \mathcal{D}_m \rightarrow \mathcal{D}_{n+m}$. Moreover, this product is associative, and elements of the form $1 \otimes B_n \in \mathcal{D}_n = \mathcal{A} \otimes (\mathcal{B}^{\otimes n})$ commute with all others. Note that a sequence $X_n \in \mathcal{D}_n$ is symmetric iff for all k, $X_{n+k} = X_n \star_{n,k} 1_k$, where 1_k denotes the unit element in $\mathcal{D}_k$. The product $\star : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathcal{Y}$ is now defined by $(X \star Y)_{n+m} := X_n \star_{n,m} Y_m$ for all n, m such that both X_n and Y_m are defined. Since $X_{n+k} \star_{n+k,m} Y_m = (X_n \star_{n,k} 1_k) \star_{n+k,m} Y_m = X_n \star_{n,k+m} (1_k \star_{k,m} Y_m) = X_n \star_{n,k+m} Y_{k+m}$ the value $(X \star Y)_r$ of the sequence $X \star Y$ does not depend on the representation $r = n + m$, and by a similar argument one finds that indeed $X \star Y \in \mathcal{Y}$. It is easy to verify that with these operations $\mathcal{Y}$ becomes a semi-normed *-algebra with unit, and we shall call $\mathcal{Y}$ the *algebra of symmetric* $\mathcal{A}$-valued *tensors* over $\mathcal{B}$.

It is crucial for our application to relate the algebraic properties of the elements $X_n \in \mathcal{D}_n$ to the properties of the sequence $X \in \mathcal{Y}$. The key to such questions is the following combinatorial lemma, which will allow us to transfer the full 'elementwise functional calculus' from the algebras $\mathcal{D}_n$ to the functional calculus of $\mathcal{Y}$.

IV.1 Lemma. *Let $X, Y \in \mathcal{Y}$ and $k, m \in \mathbb{N}$ such that X_k and Y_m are defined. Then for $n \geq k + m$*

$$\|X_n Y_n - (X \star Y)_n\| \leq \frac{k \cdot m}{n} \|X_k\| \cdot \|Y_m\|.$$

In particular,

$$\lim_{n \rightarrow \infty} \|X_n Y_n - (X \star Y)_n\| = 0.$$

Proof. Let $\pi \mapsto \alpha_\pi \in \mathrm{Aut}(\mathcal{D}_n)$ denote the action of the permutations of $\{1, \ldots n\}$ on $\mathcal{D}_n$. Then $X_n = \mathrm{sym}_n(X_k) = (n!)^{-1} \sum_\pi \alpha_\pi(X_k)$ and $X_n Y_n = (n!)^{-2} \sum_{\pi,\pi'} \alpha_\pi(X_k)\alpha_{\pi'}(Y_m)$. Moreover, $(X \star Y)_n$ is represented by the sum over only those terms in the same sum, for which $\pi(\{1, \ldots k\}) \cap \pi'(\{1, \ldots m\}) = \emptyset$.

Let $w_n(k, m)$ denote the relative weight of these terms in the sum. Then $\|X_nY_n - (X \star Y)_n\| \le |1 - w_n(k, m)| \cdot \|X_k\| \cdot \|Y_m\|$. Thus it remains to be proven that $|1 - w_n(k, m)| \le km/n$.

The number of permutations π such that $\pi(\{1, \ldots k\}) \cap \pi'(\{1, \ldots m\}) = \varnothing$ does not depend on π'. Therefore $n! \cdot w_n(k, m)$ is the number of permutations π with

$$\pi(\{1, \ldots k\}) \cap \{1, \ldots m\} = \varnothing, \quad \text{i.e.} \quad \binom{n-m}{k} \cdot k! \cdot (n-k)!.$$

Hence

$$\begin{aligned} w_n(k, m) &= \frac{(n-k)!\,(n-m)!}{n!\,(n-k-m)!} = \frac{(n-m)(n-m-1)\cdots(n-m-k+1)}{n(n-1)\cdots(n-k+1)} \\ &= \prod_{\alpha=0}^{k-1} \frac{n-m-\alpha}{n-\alpha}. \end{aligned}$$

The bound $w_n(k, m) \ge 1 - km/n$ is obviously true for $k = 0$ or $m = 0$. Therefore we may assume $m \ge 1$ and proceed by induction over k. Using the induction hypothesis we find

$$\begin{aligned} w_n(k+1, m) &= w_n(k, m) \cdot (1 - m/(n-k)) \ge (1 - km/n) \cdot (1 - m/(n-k)) \\ &= 1 - m(k+1)/n + km(m-1) \cdot n^{-1} \cdot (n-k)^{-1} \\ &\ge 1 - m(k+1)/n, \quad \text{as long as } n < k. \end{aligned}$$

IV.2 Corollary. *The seminorm defined on $\mathscr{Y}$ satisfies $\|X^* \star X\| = \|X\|^2$.*

Proof. $\|X^* \star X\| = \lim_n \|(X^* \star X)_n\| = \lim_n \|X_n^* X_n\| = \lim_n \|X_n\|^2 = \|X\|^2$

Our next aim is to show that there is a natural one-to-one correspondence between the symmetric states of $\mathscr{D}_\infty$ and the states of $\mathscr{Y}$. A state on $\mathscr{Y}$ is by definition a linear functional $\phi: \mathscr{Y} \to \mathbb{C}$, such that $\phi(1) = 1$, and $\phi(X^* \star X) \ge 0$ and $|\phi(X)| \le \|X\|$ for all $X \in \mathscr{Y}$. The set of such functionals will be denoted by $K(\mathscr{Y})$, and coincides with the state space of the separated completion of $\mathscr{Y}$. It is useful to introduce the following map $k: \bigcup_n \mathscr{D}_n \to \mathscr{Y}$: for $x \in \mathscr{D}_n$, $k(x)$ will be the sequence $k(x)_m = \mathrm{sym}_m\,(x \otimes 1_{(n+1)} \cdots \otimes 1_{(m)})$. Note that k is compatible with the injections $\mathscr{D}_n \hookrightarrow \mathscr{D}_m$, and maps $\bigcup_n \mathscr{D}_n$ onto $\mathscr{Y}$, because $k(X_n) = X$ for all $X \in \mathscr{Y}$ and $n \in \mathbb{N}$ such that X_n is defined. Due to the estimate $\|k(x)\| \le \|x\|$, k has an adjoint k^*, taking continuous linear functionals on $\mathscr{Y}$ to the dual $(\mathscr{D}_\infty)^*$.

IV.3 Lemma. *$k^*: K(\mathscr{Y}) \to K_s(\mathscr{D}_\infty)$ is an isomorphism of compact convex sets. The inverse is given by $(k^{*-1}\phi)(X) = \phi(X_n)$ for $\phi \in K_s(\mathscr{D}_\infty)$, $X \in \mathscr{Y}$, and n large enough for X_n to be defined.*

Proof. Let $\tilde{k}: K_s(\mathscr{D}_\infty) \to \mathscr{Y}^*$ denote the map described in the lemma, which is well defined since for symmetric ϕ and $X \in \mathscr{Y}$ $\phi(X_{n+k}) = \phi(\mathrm{sym}_{n+k}\,(X_n)) = \phi(X_n)$. The functionals $\tilde{k}\phi$ are indeed continuous on $\mathscr{Y}$, since $|\tilde{k}\phi(X)| =$

$\lim_n |\phi(X_n)| \leq \lim_n \|X_n\| = \|X\|$. If ϕ is positive, then so is $\hat{k}\phi$, since $(\hat{k}\phi)(X^* \star X) = \lim_n \phi((X^* \star X)_n) = \lim_n \phi(X_n^* X_n) \geq 0$ by Lemma IV.1. Since for every $X \in \mathcal{Y}$ the map $\phi \mapsto \phi(X_n)$ is w^*-continuous, $\hat{k}$ is continuous for the w^*-topologies and maps $K_s(\mathcal{D}_\infty)$ into $K(\mathcal{Y})$.

We show next that k^* maps $K(\mathcal{Y})$ into $K_s(\mathcal{D}_\infty)$. It is clear that $k^*\omega$ is always a symmetric functional on $\mathcal{D}_\infty$. Suppose that $\omega \in \mathcal{Y}^*$ is positive. Then $\|k^*\omega\| \leq \|\omega\| = \omega(1) = (k^*\omega)(1)$, which implies that $k^*\omega \geq 0$. From the definition of k and $\hat{k}$ it is clear that for $x \in \bigcup_n \mathcal{D}_n$ and $\phi \in K_s(\mathcal{D}_\infty)$, $\phi(x) = (\hat{k}\phi)(kx)$. Hence $k^* \circ \hat{k}$ is the identity on $K_s(\mathcal{D}_\infty)$. On the other hand, for $X \in \mathcal{Y}$ and $\omega \in K(\mathcal{Y})$ we have $(\hat{k} \circ k^*\omega)(X) = (k^*\omega)(X_n) = \omega(k(X_n)) = \omega(X)$. Hence $\hat{k}$ and k are inverse of each other.

This characterization of the symmetric states of $\mathcal{D}_\infty$ is useful only if we can give a concrete representation of $K(\mathcal{Y})$, or, equivalently, of the completion of the algebra $(\mathcal{Y}, \star)$. The following lemma shows that this completion is canonically isomorphic to $\mathscr{C}(K(\mathcal{B}), \mathcal{A})$, and that the embedding of $\mathcal{Y}$ into its completion is just the map $j: \tilde{\mathcal{Y}} \to \mathscr{C}(K(\mathcal{B}), \mathcal{A})$ introduced in Sect. II. We defined $j_n: \mathcal{D}_n \to \mathscr{C}(K(\mathcal{B}), \mathcal{A})$ by $j_n(a \otimes x_1 \otimes \cdots \otimes x_n)(\varphi) = a\Pi_\nu \varphi(x_\nu)$, and $j(X) = \lim_n j_n(X_n)$. The existence of this limit will be established for general $X \in \tilde{\mathcal{Y}}$ in Lemma IV.6. Here we only need the trivial case $X \in \mathcal{Y}$, in which the sequence $n \mapsto j_n(x_n)$ is constant.

IV.4 Lemma. *$j: \mathcal{Y} \to \mathscr{C}(K(\mathcal{B}), \mathcal{A})$ is an isometric *-homomorphism of $\mathcal{Y}$ onto a dense subalgebra of $\mathscr{C}(K(\mathcal{B}), \mathcal{A})$. The pure states of $\mathscr{C}(K(\mathcal{B}), \mathcal{A})$, which are of the form $f \in \mathscr{C}(K(\mathcal{B}), \mathcal{A}) \mapsto \phi_0(f(\phi))$ for a pure state $\phi_0 \in K(\mathcal{A})$ and an arbitrary state $\phi \in K(\mathcal{B})$, are mapped by $k^* \circ j^*$ to the product states $\phi_0 \otimes \Pi_\phi \in K_s(\mathcal{D}_\infty)$.*

Proof. $\mathcal{Y}$ contains two special subalgebras, namely an isomorphic copy of $\mathcal{A}$ consisting of the sequences $A_n = a \otimes 1_{(1)} \otimes \cdots \otimes 1_{(n)} \in \mathcal{D}_n$ with $a \in \mathcal{A}$, and another algebra, isomorphic to $\mathcal{Y}(\mathbb{C}, \mathcal{B})$, consisting of the elements $k(1 \otimes X_n)$ with $X_n \in \mathcal{B}_n$. The algebra $\mathcal{Y}(\mathbb{C}, \mathcal{B})$ belongs to the center of $\mathcal{Y} = \mathcal{Y}(\mathcal{A}, \mathcal{B})$, and since $\mathcal{D}_n = \mathcal{A} \otimes \mathcal{B}_n$, the finite linear combinations of elements AX with $A \in \mathcal{A}$ and $X \in \mathcal{Y}(\mathbb{C}, \mathcal{B})$ are norm dense in $\mathcal{Y}(\mathcal{A}, \mathcal{B})$. Consequently, $\mathcal{Y}(\mathcal{A}, \mathcal{B}) \cong \mathcal{A} \otimes \mathcal{Y}(\mathbb{C}, \mathcal{B})$. As an abelian unital C^*-algebra, the completion of $\mathcal{Y}(\mathbb{C}, \mathcal{B})$ is isomorphic to $\mathscr{C}(\Gamma)$ for some compact space Γ. Hence the completion of $\mathcal{Y}(\mathcal{A}, \mathcal{B})$ is isomorphic to $\mathcal{A} \otimes \mathscr{C}(\Gamma) \cong \mathscr{C}(\Gamma, \mathcal{A})$ by Proposition IV.7.3. and Theorem IV.4.14. of [19]. It remains to be shown that the space Γ is canonically isomorphic to the state space $K(\mathcal{B})$. Thus in the remainder of this proof we can take $\mathcal{A} = \mathbb{C}$.

For any abelian C^*-algebra $\mathscr{C}$, Γ is the set of pure states of $\mathscr{C}$, or, equivalently, the space of unital *-homomorphisms $\gamma: \mathscr{C} \to \mathbb{C}$, equipped with the w^*-topology. Let γ be a homomorphism of $\mathcal{Y}(\mathbb{C}, \mathcal{B})$. We claim that $k^*\gamma$ is then a product state of $\mathcal{B}_\infty$. For let $x \in \mathcal{B}_n$ and $y \in \mathcal{B}_m$, and $x \otimes y \in \mathcal{B}_{n+m}$. Then $k(xy) = k(x) \star k(y)$, and $(k^*\gamma)(xy) = \gamma(k(x) \star k(y)) = \gamma(k(x))\gamma(k(x))$. Conversely, suppose that $\phi \in K_s(\mathcal{B}_\infty)$ is a product state. Then according to Lemma IV.3 we have for all $X, Y \in \mathcal{Y}$ and $k, n \in \mathbb{N}$ sufficiently large: $(\mathscr{k}^{*-1}\phi)(X \star Y) = \phi(\mathrm{sym}_n(X_k \otimes Y_{n-k})) = \phi(X_k \otimes Y_{n-k}) = \phi(X_k)\phi(Y_{n-k}) = (\mathscr{k}^{*-1}\phi)(X)(\mathscr{k}^{*-1}\phi)(Y)$. Hence $(\mathscr{k}^{*-1}\phi)$ is a product state of $\mathcal{Y}(\mathbb{C}, \mathcal{B})$. Hence the extreme points of $K(\mathcal{Y})$ correspond exactly to the product states Π_φ, and since the map $\varphi \mapsto \Pi_\varphi$ is a homeomorphism for the w^*-topologies, the completion of $\mathcal{Y}(\mathbb{C}, \mathcal{B})$ is isomorphic to $\mathscr{C}(K(\mathcal{B}))$, with $K(\mathcal{B})$ taken in this topology.

Putting together these two lemmas we obtain the following generalization of Størmer's Theorem [18]. Our proof is an expansion of the proof given in [6] for the case of separable $\mathcal{A}$ using the theory of liftings [9].

IV.5 Proposition. *Any state $\phi \in K_s(\mathcal{D}_\infty)$ has a w^*-integral decomposition $\phi = \int \mu(d\sigma)\varphi_\sigma \otimes \Pi_\sigma$, where μ is a probability measure on $K(\mathcal{B})$, $\varphi_\sigma \in K(\mathcal{A})$ for all $\sigma \in K(\mathcal{B})$, and $\sigma \mapsto \varphi_\sigma(a)$ is measurable for all $a \in \mathcal{A}$. Moreover, for each $a \in \mathcal{A}$, $\varphi_\sigma(a)$ is uniquely determined by ϕ almost everywhere with respect to μ. If $\mathcal{A}$ is separable, then φ_σ is uniquely determined by ϕ* a.e..

Proof. By the previous two lemmas we have to show that any state Ψ on $\mathscr{C}(K(\mathcal{B}), \mathcal{A})$ has an integral decomposition $\Psi(f) = \int \mu(d\sigma)\varphi_\sigma(f(\sigma))$, for μ and $\sigma \mapsto \varphi_\sigma$ as specified above. For any $a \in \mathcal{A}$ with $a \geq 0$, consider the functional $f \mapsto \Psi(af)$ on scalar functions $f \in \mathscr{C}(K(\mathcal{B}), \mathbb{C})$. Clearly, this is positive and hence of the form $\Psi(af) = \int \mu_a(d\sigma)\, f(\sigma)$ for a unique probability measure μ_a. Since $\Psi(af) \leq \|a\|\, \Psi(1f)$ for all positive f, we have $\mu_a \leq \|a\|\, \mu_1$. Hence μ_a is absolutely continuous with respect to μ_1, and has a Radon–Nikodym derivative $R(a) \in L^1(K(\mathcal{B}), \mu_1)$, which is essentially bounded by $\|a\|$. Hence $R(a) \in L^\infty$, and $a \mapsto R(a) \in L^\infty(K(\mathcal{B}), \mu_1)$ extends to a positive linear map of norm ≤ 1. Note that $R(a)$ is not a single function, but an equivalence class with respect to a.e. equality. However, there exists a 'lifting' $\rho: L^\infty(K(\mathcal{B}), \mu_1) \mapsto \mathscr{L}^\infty(K(\mathcal{B}), \mu_1)$, which associates in a linear and positive manner a single bounded measurable function to each class [9]. Thus for each $\sigma \in K(\mathcal{B})$ the map $a \mapsto R(a) \mapsto \rho(R(a)) \mapsto (\rho R(a))(\sigma)$ is linear and positive, an takes $1 \in \mathcal{A}$ to $1 \in \mathbb{R}$. That is to say there is a state $\varphi_\sigma \in K(\mathcal{A})$ with $\varphi_\sigma(a) = (\rho R(a))(\sigma)$. For discussing the uniqueness statements, let φ_σ and ψ_σ be families of states satisfying the conclusion of the proposition. Then since $\varphi_{\cdot}(a)$ and $\psi_{\cdot}(a)$ both represent the Radon–Nikodym-derivative $R(a)$ they must be equal a.e. If $\mathcal{A}$ is separable and $(a_n)_{n\in\mathbb{N}}$ is a dense sequence, let $N_n = \{\sigma \mid \varphi_\sigma(a_n) \neq \psi_\sigma(a_n)\}$. This is a null set, hence $\{\sigma \mid \varphi_\sigma \neq \psi_\sigma\} \subset \bigcup_n N_n$ is also a null set.

As a simple example showing that separability is essential for the final uniqueness statement, consider $\mathcal{B}$ two dimensional, so that $K(\mathcal{B}) = [0, 1]$, and $\mathcal{A} = L^\infty([0, 1])$. Let ϕ denote the state on $\mathscr{C}(K(\mathcal{B}), \mathcal{A}) \simeq \mathscr{C}([0, 1]) \otimes L^\infty([0, 1])$ given by $\phi(f \otimes g) = \int d\sigma f(\sigma)g(\sigma)$ for $f \in \mathscr{C}([0, 1])$ and $g \in L^\infty([0, 1])$. Thus the function $\sigma \mapsto \varphi_\sigma \in K(\mathcal{A})$ in Proposition IV.5 must satisfy $\varphi_\sigma(g) = g(\sigma)$ almost everywhere, and hence two such functions, say $\varphi_{\cdot}^+$ and $\varphi_{\cdot}^-$, have to coincide a.e. for every g. However, the exceptional null-set may depend on g, and we shall construct φ^+ and φ^- such that $\varphi_\sigma^+ \neq \varphi_\sigma^-$ for all σ. By [9, Theorem VIII.6] we can find a lifting $\rho^+ : L^\infty([0, 1]) \mapsto \mathscr{L}^\infty([0, 1])$ such that $\rho^+(f) = f$ for all functions f, which are continuous from the right. Set $\varphi_\sigma^+(g) = (\rho^+(g))(\sigma)$. Then if $\chi_\sigma \in L^\infty([0, 1])$ denotes the characteristic function of $[0, \sigma]$, we have $\varphi_\sigma^+(\chi_\sigma) = 0$. If ρ^- is a lifting fixing left-continuous functions, and φ^- is defined similarly, then for all σ, $\varphi_\sigma^-(\chi_\sigma) = 1$, and hence $\varphi_\sigma^+ \neq \varphi_\sigma^-$.

The following result states that $\tilde{\mathscr{Y}}$ modulo the equivalence relation $\lim_n \|X_n - Y_n\| = 0$ for $X, Y \in \tilde{Y}$ is exactly the completion of $\mathscr{Y}$.

IV.6 Lemma. *Let $X \in \tilde{\mathscr{Y}}$. Then the limit $\|X\| := \lim_n \|X_n\|$ and the norm limit $j(X) := \lim_n j_n(X_n)$ in $\mathscr{C}(K(\mathscr{B}), \mathscr{A})$ exist. The map $j : \tilde{\mathscr{Y}} \to \mathscr{C}(K(\mathscr{B}), \mathscr{A})$ thus defined maps $\tilde{\mathscr{Y}}$ isometrically onto $\mathscr{C}(K(\mathscr{B}), \mathscr{A})$.*

Proof. Let $\varepsilon > 0$. Then according to Definition II.1, there is some exactly symmetric $Y \in \mathscr{Y}$, and n_0, such that for $n \geq n_0$, $\|X_n - Y_n\| \leq \varepsilon$. Thus for $n, m \geq n_0$: $| \|X_n\| - \|X_m\| | \leq 2\varepsilon + | \|Y_n\| - \|Y_m\| | \leq 3\varepsilon$ for sufficiently large n_0, since the sequence $\|Y_n\|$ is convergent. Similarly, $\|j_n(X_n) - j_m(X_m)\| \leq 2\varepsilon + \|j_n(Y_n) - j_m(Y_m)\| = 2\varepsilon$, since $n \mapsto j_n(Y_n)$ is constant for $n \geq n_0$.

By Lemma IV.4, $j \mid \mathscr{Y}$ is an isometry, and since $\|j(X)\| = \lim_n \|j_n(X_n)\| \leq \lim_n \|X_n\| = \|X\|$, this property carries over to $\tilde{\mathscr{Y}}$. To show that j is onto, let $\xi \in \mathscr{C}(K(\mathscr{B}), \mathscr{A})$. Then by Lemma IV.4 there is a sequence $(X^\alpha)_{\alpha \in \mathbb{N}} \in \mathscr{Y}$ such that $j(X^\alpha) \to \xi$ as $\alpha \to \infty$. We may assume that $\|X^\alpha - X^{\alpha+1}\| \leq 2^{-\alpha}$, and pick some increasing sequence $\alpha \mapsto m(\alpha)$ such that $\|X_n^\alpha - X_n^{\alpha+1}\| \leq 2^{-\alpha+1}$ for $n \geq m(\alpha)$. Now set $Y_n = X_n^\alpha$ for $m(\alpha) \leq n < m(\alpha + 1)$. Then for $n \geq m(\alpha)$, say $m(\beta) \leq n < m(\beta + 1)$ with $\beta \geq \alpha$, $\|Y_n - X_n^\alpha\| = \|X_n^\beta - X_n^\alpha\| \leq \sum_{\nu=\alpha}^{\beta-1} 2^{-\nu+1} \leq 2^{-\alpha+2}$. Hence $Y \in \tilde{\mathscr{Y}}$, and $\|Y - X^\alpha\| \leq 2^{-\alpha+2}$. This implies $\|j(Y) - \xi\| \leq \|j(Y) - j(X^\alpha)\| + \|j(X^\alpha) - \xi\| \leq \text{const } 2^{-\alpha}$, i.e. $j(Y) = \xi$.

We are now ready to prove the convergence of the mean energy of the models under consideration:

Proof of Proposition III.3

First let $X \in \mathscr{Y}$ be strictly symmetric, i.e. $X_n = \text{sym}_n(X_k)$ for some $X_k \in \mathscr{D}_k$. Then $\phi_{\nu(\alpha)}(X_{\nu(\alpha)}) = \phi_{\nu(\alpha)}(\text{sym}_{\nu(\alpha)}(X_k)) = \phi_{\nu(\alpha)}(X_k) \to \phi(X_k)$. The limit is equal to the right hand side of III.3 due to Proposition IV.5 and the definition of j. Now let $X \in \tilde{\mathscr{Y}}$, and $Y \in \mathscr{Y}$ with $\|X_n - Y_n\| \leq \varepsilon$ for $n \geq m$. For any cluster point ξ of $\{\phi_{\nu(\alpha)}(X_{\nu(\alpha)}) \mid \alpha \in \mathbb{A}\}$, $|\xi - \int \mu(d\sigma)\phi_\sigma(j(X)(\sigma))| \leq |\xi - \lim_{n \to \nu} \phi_n(Y_n)| + |\int \mu(d\sigma)\varphi_\sigma(j(Y - X)(\sigma))| \leq \varepsilon + \|j(Y - X)\| \leq 2\varepsilon$. Since ε is arbitrary by definition of $\tilde{\mathscr{Y}}$, the proof is complete.

V. Discussion

In this paper we have focussed only on those features of mean field systems which are thermodynamically relevant, i.e. have an influence on the free energy in the thermodynamic limit. If two hamiltonian densities H_n and $\tilde{H}_n$ satisfy $\|H_n - \tilde{H}_n\| \to 0$, then they are thermodynamically equivalent. It is clear that in each equivalence class the convergence of the states $\mathrm{Norm}^{-1}\, \omega_n^{nH_n}$ to a limit state can be arbitrarily slow. Hence, the asymptotics of the fluctuations of expectation values around the limiting value cannot be discussed at the thermodynamic level [8].

Another question, which cannot be treated at the purely thermodynamic level is the convergence of effective Hamiltonians. Recall that we defined the effective hamiltonian $\hbar$ in Proposition II.5 as a derivative of the energy density $\varphi_0\{j(H)(\varphi)\}$ with respect to φ. Now for suitable sequences H_n we can express $\hbar = \hbar(\varphi_0, \varphi)$ as $\lim_n J_n(H_n)$, where the (φ_0, φ)-dependent operator $J_n : \mathcal{D}_n \to \mathcal{B}$ is given by

$$J_n(a \otimes x_1 \otimes \cdots \otimes x_n) = \sum_{\nu=1}^{n} (x_\nu - \varphi(x_\nu))\varphi_0(a) \prod_{\mu \neq \nu} \varphi(x_\mu).$$

These maps J_n are compatible with symmetrization and the canonical injections $\mathcal{D}_n \hookrightarrow \mathcal{D}_m$. Hence, for a strictly symmetric sequence $H \in \mathcal{Y}$, $J_n(H_n)$ is eventually constant, and in fact equal to $\hbar$. On the other hand, convergence of $J_n(H_n)$ may fail for other, thermodynamically equivalent hamiltonians.To see this, let $G_n = \varepsilon_n 1 \otimes x_n \otimes \cdots \otimes x_n$ for some sequence ε_n going to zero and some hermitian $x_n \in \mathcal{B}$ with $\|x_n\| = 1$. Then $\|G_n\| \to 0$, and $J_n(G_n) = \varepsilon_n n \varphi(x_n)^{n-1}(x_n - \varphi(x_n))$. Now if we choose x_n so that $\varphi(x_n)$ converges rapidly to 1, without x_n converging to $1 \in \mathcal{B}$, we can construct $G \in \tilde{\mathcal{Y}}$ such that $\|J_n(G_n)\|$ diverges.

On the other hand, the condition $\lim_n J_n(H_n) = \hbar$ for all (φ_0, φ) may be of physical interest. For example, if $\mathcal{A} = \mathbb{C}$, one computes that $\lim_n \Pi_\varphi(A[nH_n, B]C - A\delta_n(B)C) = 0$ for all strictly local $A, B, C \in \bigcup_n \mathcal{B}_n$, where $\delta_n(B)$ denotes the commutator of B with $J_n(H_n) \otimes 1_{n-1} + \cdots + 1_{n-1} \otimes J_n(H_n)$. In other words, if $J_n(H_n) \to \hbar$, then the generators of the time evolution in the system of size n converge to a derivation of $\mathcal{B}_\infty$, which corresponds to a one-particle evolution generated by $\hbar$. Hence in this case $\hbar$ can be given a dynamical meaning. This has been exploited in [7, 15] to characterize the equilibrium states of mean field systems by an energy-entropy inequality.

We would like to point out a characteristic difference between the scope of the above results in the quantum and the classical cases. Consider the two functions $G_n(t) = n^{-1} \log \omega_n^{n(H_n + tX_n)}(1)$, and $C_n(t) = n^{-1} \log \omega_n^{nH_n}(e^{tX_n})$ for $t \in \mathbb{R}$. In the classical case, i.e. when $\mathcal{A}$ and $\mathcal{B}$ are abelian, $\omega^h(1) = \omega(e^h)$ holds for all states ω and all hermitian h, and hence $C_n = G_n$. Thus, in the classical case, the function G_n (respectively the limit) not only contains all the thermodynamics but also – via derivatives with respect to t – information about expectation values of X_n with respect to the state $\phi_n = \mathrm{Norm}^{-1}\, \omega_n^{nH_n}$. Here, convergence of $G_n(t)$ for all t is an asymptotic property of the probability measures $\mathbb{K}_n$ on $\mathbb{R}$, given by $\int \mathbb{K}_n(dx) f(x) = \phi_n(f(X_n))$ for bounded continuous $f : \mathbb{R} \to \mathbb{R}$. In fact, if $G(t) := \lim_n G_n(t)$ is differentiable, then the measures $\mathbb{K}_n$ converge to the point measure at $G'(0)$ exponentially fast in the sense made precise by the Large Deviation Principle [10, Theorem 4]. In the non-commutative case, G_n still encodes all thermodynamics, but no longer contains direct information about expectation values. This is contained in C_n, which acts as the cumulant generating function of the measures $\mathbb{K}_n$. By Theorem II.4(4) the measures $\mathbb{K}_n$ still converge to point measures in any pure phase of the system, but the proof [10] of the Large Deviation Principle for differentiable G carries over to the non-commutative case only if the reference states $\rho_0 \in K(\mathcal{A})$ and $\rho \in (\mathcal{B})$ are traces and X is an approximately symmetric sequence such that $[H_n, X_n] = 0$ for all n. However, in general, the Golden–Thompson inequality $G_n(t) \le C_n(t)$ remains a strict inequality in the limit, even though (for $\mathcal{A} = \mathbb{C}$) $\lim_n \|[H_n, X_n]\| = 0$. It would interesting to find asymptotic properties of ρ, H_n, and X_n that would allow the control of the limit of $C_n(t)$, and the proof of the Large Deviation Principle for the measures $\mathbb{K}_n$. However, such properties will again depend on H_n more sensitively than the thermodynamic properties.

The models we have considered here should perhaps more appropriately be called homogeneous mean field models. Indeed, no local features enter the interaction hamiltonian at all. One can also consider 'heterogeneous mean field models' (e.g. the BCS model treated in [4]), where the interaction between particles may depend on their location in some compact space X, and in which the global scaling behaviour of the interaction is of the mean field nature. For each particle number n the locations of the particles are held fixed, and one is interested in the limit in which their density converges to some given measure on x. Extension of our results to this class of models is presently under consideration [16].

Appendix

In this appendix we collect the results on the calculus of C^*-functions referred to in Sections II and IV. These functions are best seen as a many-variable generalization of the ordinary functional calculus in C^*-algebras. There are two natural ways to define 'the same function' in different algebras. The first is abstract, and requires only some transformation behaviour with respect to C^*-morphisms. The second approach starts directly from the algebraic structure and the evaluation of 'the same polynomial' in different algebras, and extends to all functions, which can be approximated by polynomials in a sufficiently strong sense. We shall start from the abstract definition and show the equivalence to the second approach in Lemma A.2.

A.1 Definition. Let Γ be a compact convex subset of $\mathbb{R}^\infty$, the set of real valued sequences with the product topology. Then a *C^*-function* on Γ is a family of functions $f_{\mathcal{A}}$, for every unital C^*-algebra $\mathcal{A}$, with

$$f_{\mathcal{A}} : \{(A_1, A_2, \dots) \in \mathcal{A}^\infty \mid A_\nu = A_\nu^*, \forall_{\varphi \in K(\mathcal{A})} (\varphi(A_1), \varphi(A_2), \dots) \in \Gamma\} \to \mathcal{A}$$

such that for any unital *-homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ into a unital C^*-algebra $\mathcal{B}$,

$$f_{\mathcal{B}}(\Phi(A_1), \Phi(A_2), \dots) = \Phi(f_{\mathcal{A}}(A_1, A_2, \dots)).$$

A C^*-function is called *hermitian,* if the values of all $f_{\mathscr{A}}$ are hermitian for all arguments in its domain. For notational convenience we shall from now on drop the subscripts $\mathscr{A}$, and will sometimes abbreviate the sequence $(A_1, A_2, \ldots)$ of arguments by $\vec{A}$.

We remark that this definition is strictly speaking not legitimate, since it contains a quantifier over the proper class of C^*-algebras. However, it always suffices to define $f_{\mathscr{A}}$ on the separable C^*-subalgebra generated by its countably many arguments. Since every separable C^*-algebra can be faithfully represented on a separable Hilbert space, it suffices to define $f_{\mathscr{A}}$ on the set of separable C^*-algebras on a fixed Hilbert space.

The C^*-functions depending only on a single variable are just the continuous real valued functions on some interval, evaluated in the functional calculus. The interval on which f is defined in the single variable case becomes the set Γ in the many-variable case. Often one can choose Γ to be an infinite product of compact intervals, which amounts to imposing a constraint on the spectrum of each A_ν separately. The composition of C^*-functions, where it is defined, is again a C^*-function. Hence $f(X, Y, Z) = \exp(|i[X, Y]|)/\cosh(Z)$ is a legitimate C^*-function for any choice of $\Gamma \subset \mathbb{R}^3$. As this example shows, a C^*-function of several arguments is not determined by its values on scalars $A_\nu = \lambda_\nu 1$.

A.2 Lemma. *Let f be a C^*-function on $\Gamma \subset \mathbb{R}^\infty$. Then for any $\varepsilon > 0$ there exists a polynomial g depending only on finitely many of the non-commuting variables $A_1, A_2, \ldots$ such that $\|f(\vec{A}) - g(\vec{A})\| \le \varepsilon$. Moreover, there is a constant c such that $\|f(\vec{A})\| \le c$, and*

$$\forall_{\varepsilon>0} \exists_{\delta>0} \exists_{\mu\in\mathbb{N}} (\forall_{\nu\le\mu} \|A_\nu - A'_\nu\| \le \delta) \Rightarrow \|f(\vec{A}) - f(\vec{A}')\| \le \varepsilon$$

These statements are valid for any C^*-algebra $\mathscr{A}$, any admissible sequences of arguments $\vec{A}$ and $\vec{A}'$, and the choices of g, c, δ, and μ can be made independently of $\mathscr{A}$, $\vec{A}$ and $\vec{A}'$.

Proof. Let $\mathscr{F}$ denote the free unital *-algebra over countably many hermitian symbols $X_1, X_2, \ldots$ i.e. the algebra of polynomials in $X_1, X_2, \ldots$ with complex coefficients. Then any choice of a sequence $\vec{A} = (A_\nu)_{\nu\in\mathbb{N}}$ of hermitian elements in some C^*-algebra $\mathscr{A}$ induces a unique unital *-homomorphism $\Phi_{\vec{A}}: \mathscr{F} \to \mathscr{A}$ such that $\varphi_{\vec{A}}(X_\nu) = A_\nu$ for all $\nu \in \mathbb{N}$. Define on $\mathscr{F}$ the seminorm $\|g\| := \sup\{\|\varphi_{\vec{A}}(g)\|\}$, where the supremum is over all sequences $\vec{A}$ in separable *C^*-algebras* $\mathscr{A}$ such that $\varphi(\vec{A}) := (\varphi(A_1), \varphi(A_2), \ldots) \in \Gamma$ for all $\varphi \in K(\mathscr{A})$. This is clearly a C^*-seminorm, and we shall denote by $\hat{\mathscr{F}}$ the separated completion of $\mathscr{F}$ with respect to this seminorm. By definition of the norm on $\mathscr{F}$, each $\Phi_{\vec{A}}$ is continuous, and hence extends to a unique *-homomorphism $\Phi_{\vec{A}}: \hat{\mathscr{F}} \to \mathscr{A}$.

We prove next that $\vec{X} = (X_1, X_2, \ldots) \in \hat{\mathscr{F}}^\infty$ is an admissible sequence of arguments for f, i.e. for any $\phi \in K(\hat{\mathscr{F}})$ we have $\phi(\vec{X}) \in \Gamma$. For any continuous linear functional ξ on $\mathbb{R}^\infty$, i.e. any functional of the form $\xi(x) = \sum_{n=1}^m \xi_n x_n$ for some finite m, let $M_+(\xi) = \sup \xi(\Gamma)$ and $M_-(\xi) = \inf \xi(\Gamma)$. Since Γ is compact and convex $x \in \Gamma$ is equivalent to $x \in [M_-(\xi), M_+(\xi)]$ for all ξ. For any continuous ξ, let $X^\xi \mathscr{F}$ denote the element $X^\xi = \sum_{n=1}^m \xi_n X_n = (1/2)(M_+(\xi) + M_-(\xi)) \cdot 1$. Then, by definition of $\Phi_{\vec{A}}$ and the norm in $\mathscr{F}$:

$$\|X^\xi\| = \sup \{|\varphi(\Phi_{\vec{A}}(X^\xi))|\} = \sup \left\{ \left| \varphi\left(\sum_{n=1}^m \xi_n A_n \right) - (1/2)(M_+(\xi) + M_-(\xi)) \right| \right\}$$

where the supremum is over all admissible sequences $\vec{A} \in \mathscr{A}^\infty$ and all states $\varphi \in K(\mathscr{A})$. Therefore $\varphi(\vec{A}) \in \Gamma$, so that $\xi(\varphi(\vec{a})) \in [M_-(\xi), M_+(\xi)]$, and $\|X^\xi\| \leq (1/2)(M_+(\xi) - M_-(\xi))$. Hence for any $\phi \in K(\hat{\mathscr{F}})$

$$\xi(\phi(\vec{X})) = \sum_{n=1}^m \xi_n \phi(X_n) = \phi(X^\xi) + (1/2)(M_+(\xi) + M_-(\xi)) \leq M_+(\xi).$$

The lower bound $\xi(\phi(\vec{X})) \geq M_-(\xi)$ follows similarly, so that $\phi(\vec{X}) \in \Gamma$.

Now let f be a C^*-function. Set $\hat{f} := f(\vec{X}) \in \hat{\mathscr{F}}$. Then since $\hat{\mathscr{F}}$ is the completion of $\mathscr{F}$, we can find $g \in \mathscr{F}$ $(i \in \mathbb{N})$ such that $\|\hat{f} - g\| \leq \varepsilon$. Thus $\|f(\vec{A}) - g(\vec{A})\| = \|\Phi_{\vec{A}}(\hat{f}) - \Phi_{\vec{A}}(g)\| \leq \|\hat{f} - g\| \leq \varepsilon$ uniformly in $\vec{A}$. Boundedness and uniform continuity are obvious for the polynomials g and follow for f by straightforward estimates.

The final result of this section is the complete transformation of the elementwise functional calculus of approximately symmetric sequences into the functional calculus of $\mathscr{C}(K(\mathscr{B}), \mathscr{A})$ stated in section II:

Proof of Proposition II.2

Consider first the case $f(X^1, X^2) = X^1 X^2$ and fix $\varepsilon_1, \varepsilon_2 > 0$. Let $Z^1, Z^2 \in \mathscr{Y}$ such that $\|X_n^i - Z_n^i\| \leq \varepsilon_1$ for $i = 1, 2$ and $n \geq m_1$, and set $Z = Z^1 \star Z^2$. Then by Lemma IV.1, there is some $m_2 \in \mathbb{N}$, such that $\|Z_n^1 Z_n^2 - Z_n\| \leq \varepsilon_2$ for $n \geq m_2$. Hence for $n \geq \max(m_1, m_2)$ $\|X_n^1 X_n^2 - Z_n\| \leq \varepsilon_1(\|X_n^2\| + \|Z_n^1\|) + \varepsilon_2$, which can be made arbitrarily small by choice of ε_1 and ε_2. Thus by definition $Y_n = X_n^1 X_n^2$ is approximately symmetric, and $\|j_n(Y_n) - j_n(X_n^1) j_n(X_n^2)\| \leq \|Y_n - Z_n\| + \|j_n((Z^1 \star Z^2)_n) - j_n(Z_n^1) j_n(Z_n^2)\| + \|j_n(Z_n^1) j_n(Z_n^2) - j_n(X_n^1) j_n(X_n^2)\|$. The first and last term on the right hand side are estimated as before, and the middle term vanishes, since for $Z \in \mathscr{Y}$ $j_n(Z_n) \equiv j(Z)$ and j is a homomorphism for the $\star$-product. Hence the left hand side becomes small for sufficiently large n, and we find $j(Y) = j(X^1) j(X^2)$.

The case of a monomial $f(X^1, X^2, \ldots X^\Gamma) = X^1 X^2 \cdots X^r$ now follows by induction over r, and the case of general polynomials by taking linear combinations. Let f be a C^*-function and $\varepsilon > 0$. Then by Lemma A.2 we can find a polynomial g such that $\|f(\vec{A}) - g(\vec{A})\| \leq \varepsilon$ for all admissible arguments $\vec{A}$. Consider the sequence $Z_n = g(X_n^1, X_n^2, \ldots)$. By the above arguments $Z \in \tilde{\mathscr{Y}}$, so that we can find $Z' \in \mathscr{Y}$ and $m \in \mathbb{N}$ such that $\|Z_n - Z_n'\| \leq \varepsilon$ for $n \geq m$. Hence $\|Y_n - Z_n'\| \leq \|f(\vec{X}_n) - g(\vec{X}_n)\| + \|Z_n - Z_n'\| \leq 2\varepsilon$ for $n \geq m$. Thus $Y \in \tilde{\mathscr{Y}}$, and $\|j_n(Y_n) - f(j(\vec{X}))\| \leq \|j_n(Y_n - Z_n)\| + \|j_n(g(\vec{X}_n) - g((j(\vec{X}))\| + \|g((j(\vec{X})) -$

$f((j(\vec{X}))\|$. The first and last term on the right are $\leq \varepsilon$ because g approximates f, and the middle term goes to zero since the proposition is valid for polynomials.

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