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AVOIDING COMPLEX BEHAVIOR

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Abstract

The general problem of guaranteeing the absence of unwanted complex behavior in the context of circuit design is discussed. More specifically, the methods available for proving unique asymptotic behavior are reviewed. A new method is presented and compared, for a circuit example, with the conventional approach by quadratic Liapunov functions.

1. Introduction

Nonlinear electrical circuits may have very complicated dynamics. In particular, a circuit which is driven by a sinusoidal source of frequency f may have voltages and currents whose asymptotic (large time) behavior is periodic with frequency f/n (**subharmonics**), or even **chaotic**. A series of review articles on chaotic phenomena in circuits can be found in the special issue of the IEEE Proceedings [1].

Subharmonic solutions of the circuit equations are undesirable in many applications (e.g. amplifiers, AC power transmission) but necessary in others (e.g. frequency dividers). Chaotic solutions are never acceptable. The only application that has been proposed so far for circuits with chaotic solutions are noise generators. Their efficiency still has to be proved. Therefore, in the context of electrical engineering, methods have to be developed, which permit to exclude any undesirable complex behavior of circuit dynamics.

In this paper we will concentrate on **unique asymptotic behavior**. This excludes of course chaotic solutions, but also the presence of multiple equilibria in the case of autonomous circuits and subharmonics in the case of periodically forced circuits. Nevertheless, for a large class of devices this condition is a basic requirement. A precise definition will be given in section 2.

to the nature of the connections. The nature of the elements is always known, whereas the condition on the element connections can be checked by inspection, at least for moderately sized circuits.

Such a theorem is readily accessible to the average electrical engineer. Unfortunately, there are not enough such theorems to cover the current circuit technology. To be specific, theorem 1.1 is unable to give any information on circuits with transistors. In fact, there are almost no theorems on the dynamics of transistor circuits at present.

1.3. Criteria that need some computation

It is likely that the qualitative aspects of circuit dynamics cannnot in general be captured by theorems whose conditions are as simple as those of theorem 1.1. Therefore, it is reasonable to involve computation in some well controlled way. More precisely, criteria for qualitative features of the circuit dynamics are sought which cannot be decided upon by simple inspection, but which neccessitate a **finite number of numerical computations of finite precision**. The engineer then uses the corresponding software to investigate the circuit behavior. This procedure is quite different from brute force numerical simulation. The result is just as reliable as the paper-and-pencil application of a theorem.

In this paper we will present a new approach to the uniqueness of the asymptotic behior in the case of a system of two nonlinear nonautonomous differential equations. It belongs to the third category. Even though our examples are circuits, there is nothing which refers to the special structure of the circuit equations.

Before introducing the method, we will discuss the traditional approach by Liapunov functions. Finally, the two approaches are compared and possible generalizations to dimension n are discussed.

2. Unique asymptotic behavior and Liapunov functions.

We consider in this paper a system of two ordinary differential equations of the form

$$dx_1/dt = f_1(x_1, x_2, t) \quad (1)$$

$$dx_2/dt = f_2(x_1, x_2, t) \quad (2)$$

where x_1, x_2 and t are scalars and where the functions f_1 and f_2 are supposed to be lipshitz continuous in $\xi = (x_1, x_2)$ and piecewise continuous in t.

The practicing engineer needs tools that are readily applicable, without much knowledge of the theory of nonlinear ordinary differential equations or dynamical systems. To prepare these tools is the basic task of circuit theory. We can roughly classify the different methods that are available for the engineer as follows:

- Numerical simulation
- Theorems of circuit theory
- Criteria that need some computation

1.1. Numerical simulation

This is the only method that is always applicable. Computing power and general purpose circuit simulation software is almost universally available. However, numerical simulation is time consuming and unreliable. In order to establish qualitative properties such as unique asymptotic behavior, in principle an infinite number of infinitely long time domain solutions have to be computed. In practice, only a finite number of time limited simulation runs can be carried out. Usually this number is even quite small. Consequently, the conclusions that can be drawn from such a numerical investigation are of questionable value.

1.2. Theorems of circuit theory

Such theorems express the properties of the circuit solutions in terms of the nature of the circuit components and the nature of their connections. The following example may illustrate this.

Theorem 1.1:

Let a circuit be composed of

- positive linear capacitors
- positive linear inductors
- linear and nonlinear resistors with a strictly increasing characteristic in the voltage-current plane. For sufficiently large positive (negative) voltages, the corresponding currents are also positive (negative).
- voltage and current sources. If they are time dependent, their amplitude is bounded.

Suppose that

- there is no capacitor-inductor-voltage source loop nor a capacitor-inductor-current source cutset.

Then the circuit has a unique asymptotic behavior.

For the proof of this theorem we refer to [2]. Note that the first four conditions refer to the nature of the circuit elements and the last condition

Lipshitz continuity in ξ and continuity in t is required for the existence and uniqueness of the solution starting from given initial conditions $\xi(t_0)$ [3]. At the (isolated) discontinuities in t we use the past solutions as initial conditions for the future solutions. In terms of circuits, these discontinuities correspond to the action of switches. The existence of the solution cannot be guaranteed for all times $t \in [t_0, \infty)$, but if we can prove that the solution is bounded wherever it exists in this interval, it must exist in the whole interval.

Henceforth, we shall assume the boundedness of the solutions in $[t_0, \infty)$ and consequently the existence question will not be addressed any more. In circuit examples, this condition is usually not difficult to establish, using the energy stored in the capacitors and the inductors as a Liapunov function [2].

Definition 2.1:

The *asymtotic behavior* of the system (1,2) is *unique*, if the following two conditions are satisfied:

a) A solution starting at an initial time t_0 is bounded on the whole interval $[t_0, \infty)$.

b) For any two solutions $\xi_1(t)$ and $\xi_2(t)$

$$||\xi_1(t) - \xi_2(t)|| \rightarrow 0 \quad \text{as} \quad t \rightarrow +\infty \tag{3}$$

Note that this definition does not specify in any way the nature of the asymptotic behavior. How can (2) be proved for specific examples? The only non-perturbative method we know of is by inventing a Liapunov function. This function must have two arguments, since we are considering simultaneously two solutions.

Definition 2.2:

An *incremental Liapunov function* for the system (1,2) is a C^1 function $W: R^4 \rightarrow R$ with the following properties:

$$\text{a)} \quad W(\xi_1,\xi_2) > 0 \text{ for } \xi_1 \neq \xi_2 \quad \text{and} \quad W(\xi_1,\xi_2) = 0 \text{ for } \xi_1 = \xi_2 \tag{4}$$

$$\text{b)} \quad \frac{d}{dt} W(\xi_1(t),\xi_2(t)) < 0 \text{ for } \xi_1 \neq \xi_2 \quad \text{and} \quad = 0 \text{ for } \xi_1 = \xi_2 \tag{5}$$

If the solutions of (1,2) are bounded as required by definition 2.1 and if there exists an incremental Liapunov function, then the asymptotic behavior is unique.

Definition 2.2 gives no clue how the Liapunov functions, if they exist, are to be deduced from the system (1,2). The following methods to define Liapunov functions can be distinguished:

- Energy as a Liapunov function: If (1,2) models a dissipative physical system, the most obvious choice is to use some form of energy as a candidate for a Liapunov function. In the context of electrical circuits, it is the energy stored in the capacitors and the inductors. This leads to the kind of theorems mentioned in 1.2. In the case of linear capacitors and inductors, an *incremental stored energy* can be defined [2]. It is used in particular to prove theorem 1.1.
- Guessed Liapunov functions: By some sort of intuition, a certain function is taken as a candidate for a Liapunov function. Often, free parameters are introduced and their values are adjusted until the requirements for a Liapunov function are fulfilled. This leads to some kind of optimization procedure. The simplest and most popular choice are quadratic functions. We shall briefly discuss this approach in section 3 for a circuit example.
- Constructed Liapunov functions: Sometimes Liapunov functions are constructed through some other concept. This approach will be pursued later in this paper. Computer generation of Liapunov functions has been reported [4].

3. Quadratic incremental Liapunov functions for a circuit example.

Consider the circuit of fig. 1. It is driven by a sinusoidal voltage

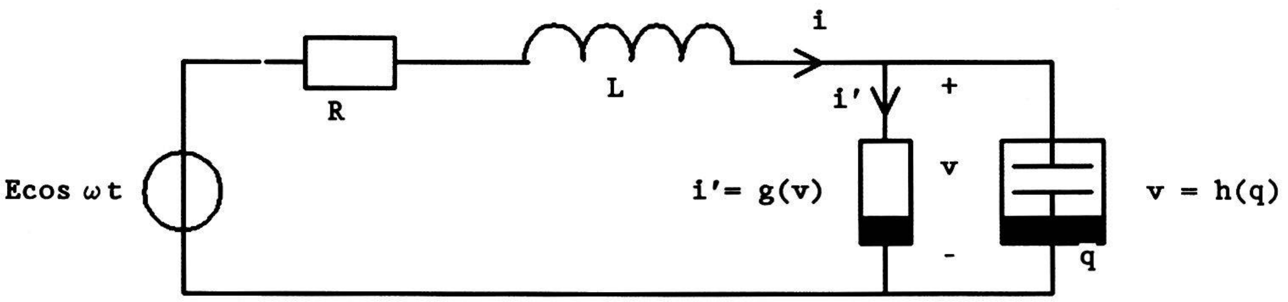

Fig.1

source. The series resistor and inductor are linear, whereas the shunt resistor and capacitor are nonlinear. At this point, we do not make any assumtions about the characteristics of the nonlinear elements, except that the resistor is voltage controlled, i.e. its constitutive relation can be written in the form

$$i = g(v) \tag{6}$$

and that the capacitor is charge controlled, i.e. its constitutive relation has the form

$$v = h(q) \tag{7}$$

where i, v, q are the current, the voltage and the charge, respectively.

The two nonlinear elements together model a diode. The circuit of fig.1 with the diode is known to exhibit a period doubling route to chaos in certain parameter ranges [5]. The experimental results of [5] have been confirmed by numerical simulation using the standard circuit analysis program SPICE [6]. This program uses an equivalent circuit as in fig.1 to model the diode. In SPICE g is an exponential function and the inverse of h is a sum of an exponential and a fractional power. Again by numerical simulation, it has been observed that the same qualitative behavior is obtained by reducing the model to its bare essentials, namely a piecewise linear functions g and h, with just two linear regions, the blocked and the conducting region of the diode (fig.2,3) [7]. Note that a nonlinear capacitor has to be included in the model. Otherwise theorem 1.1 can be applied which guarantees unique asymptotic behavior, and thus the model does not reproduce the qualitative behavior of the physical circuit.

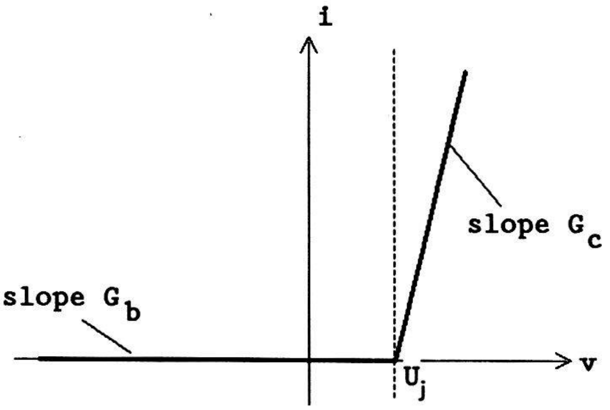

Fig.2

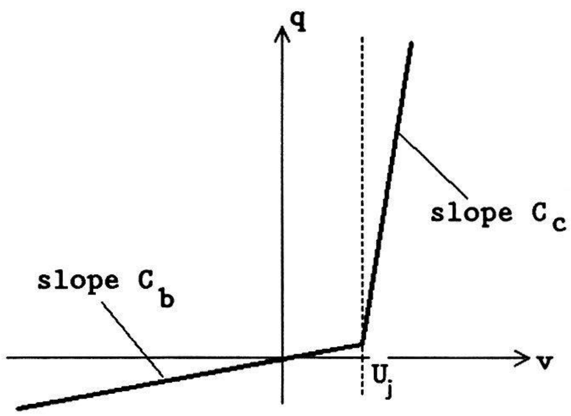

Fig.3

While the circuit of fig.1 exhibits complex dynamics in some parameter ranges, There are large regions in the parameter space, where the asymptotic behavior is unique. We now try to find these regions, or at least some subregion, by using a quadratic incremental Liapunov function.

The dynamics of the circuit of fig.1 is determined by the state equations

$$dq/dt = - g(h(q) + i \tag{8}$$
$$di/dt = - h(q)/L - iR/L + E\cos\omega t \tag{9}$$

where q is the capacitor charge and i the inductor current.

Consider two solutions $(q_1(t), i_1(t))$ and $(q_2(t), i_2(t))$ of (8,9) and define the increments

$$\Delta q(t) = q_1(t) - q_2(t) \tag{10}$$
$$\Delta i(t) = i_1(t) - i_2(t) \tag{11}$$

As a candidate for an incremental Liapunov function we use the quadratic functon in the increments

$$W(q_1,i_1,q_2,i_2) = (\Delta q)^2/2C + (\Delta i)^2L/2 \tag{12}$$

where C is an arbitrary positive constant which will be chosen later in an optimal way. Clearly, condition 2.2.a for an incremental Liapunov function is satisfied. In order to prove condition 2.2.b, we calculate the derivative of (12) along two solutions $(q_1(t), i_1(t))$ and $(q_2(t), i_2(t))$ of (8,9):

$$dW/dt = (\Delta q/C)d(\Delta q)/dt + (L\Delta i)d(\Delta i)/dt \tag{13}$$

The derivatives of the increments are calculated by rewriting (8,9) as follows:

$$d\Delta q/dt = - [g(h(q_1(t))) - g(h(q_2(t)))] + \Delta i(t) \tag{14}$$
$$d\Delta i/dt = - [h(q_1(t)) - h(q_2(t))]/L - \Delta iR/L \tag{15}$$

Combining (13) with (14,15) yields

$$\begin{aligned} dW/dt = & - (\Delta q/C)[g(h(q_1(t))) - g(h(q_2(t)))] + \Delta q\Delta i/C \\ & - (\Delta i/L)[h(q_1(t)) - h(q_2(t))] -(\Delta i)^2R/L \end{aligned} \tag{16}$$

Without any assumption on g and h, we cannot prove that dW/dt is negative. Assume that the slopes of g and h are bounded by positive finite constants above and below. More generally, we assume that there are positive finite constants C_{min} and C_{max} such that for any two charges q_1 and q_2, we have

$$(\Delta q)/C_{max} \leq h(q_1) - h(q_2) \leq (\Delta q)/C_{min} \tag{17}$$

Similarly, we assume that there are positive finite constants ω_{min} and ω_{max} such that for any pair q_1 and q_2

$$(\Delta q)\omega_{min} \leq g(h(q_1)) - g(h(q_2)) \leq (\Delta q)\omega_{max} \tag{18}$$

In the special case of the characteristics of figs.2,3 we have $C_{max} = C_c$, $C_{min} = C_b$, $\omega_{max} = G_c/C_c$ and $\omega_{min} = 0$.

Introducing (17,18) into (16), we get

$$dW/dt = -(\Delta q)^2 a/C + \Delta q\Delta i/C + \Delta q\Delta i.b - (\Delta i)^2 R \tag{19}$$

where

$$\omega_{min} \leq a(t) \leq \omega_{max} \tag{20}$$
$$1/C_{max} \leq b(t) \leq 1/C_{min} \tag{21}$$

We now look for the region in the space of the circuit parameters R, L, C_{min}, C_{max}, ω_{max}, ω_{min} such that the RHS of (19) is negative definite for all a, b satisfying (20), (21). The RHS of (19) is negative definite if

$$(1/C + b)^2 - 4aR/C < 0 \tag{22}$$

According to (20), (21), the LHS of (22) is bounded above by

$$(1/C + 1/C_{min})2 - 4\omega_{min}R/C \tag{23}$$

Remember that C is a free positive finite constant. We choose it in such a way that (23) is minimal. This is our optimization of the Liapunov function candidate W.

If $2\omega_{min}R > 1/C_{min}$, the (23) reaches its minimum at a positive finite value and the minimum is

$$\frac{4\omega_{min}R}{C_{min}} - 4\omega_{min}{}^2R^2 \tag{24}$$

which is negative if $\omega_{min}R > 1/C_{min}$. If on the other hand $2\omega_{min}R \leq 1/C_{min}$, (23) can never be negative. The following theorem summarizes these results.

Theorem 3.1:

The circuit of fig. 1 has unique asymptotic behavior if its elements satisfy the following inequality

$$\omega_{min} > 1/RC_{min} \tag{25}$$

where R is the series resistance and ω_{min}, C_{min} are the parameters of the nonlinear elements defined by (17) and (18).

Theorem 3.1 is at the same time a satisfactory and an unsatisfactory result. Satisfactory, because it gives a very explicit parameter region, where the circuit has a unique asymptotic behavior. Unsatisfactory, because this region is much too small. In fact, extensive numerical simulations have shown that in a far larger region the asymptotic behavior is still unique. It is generally believed that Liapunov's method will always yield too conservative results to be useful for engineering applications. We think that Liapunov's method still has a large unexplored potential.

An obvious method to improve on inequality (25) is to introduce a cross term in the definition (12) of W:

$$W(q_1,i_1,q_2,i_2) = (\Delta q)^2/2C + A\Delta q\Delta i + (\Delta i)^2 L/2 \tag{26}$$

This approach has been pursued in [8]. It needs some computation at the end and as a consequence, the resulting region in parameter space where the circuits have unique asymptotic behavior cannot be described as explicitly as in (25). However, the region is indeed considerably larger than (25), but still far too small if compared with numerical simulations. We shall present some numerical results in section 5.

4. A new approach for establishing unique asymptotic behavior.

4.1. Relation between nonlinear systems and linear time-dependent systems.

The idea is to consider the variational equations [3] corresponding to the system (1), (2). If the solution $\xi(t)$ of this system is expressed as a function of the initial condition ξ_0 at time t_0: $\xi(t,\xi_0)$, then the variational equation is

$$\frac{d}{dt}\frac{\partial\xi(t,\xi_0)}{\partial\xi_0} = \frac{\partial f(\xi(t,\xi_0),t)}{\partial\xi}\cdot\frac{\partial\xi(t,\xi_0)}{\partial\xi_0} \tag{27}$$

If we fix now an initial condition ξ_0, then (27) has the form

$$dy/dt = M(t)y \tag{28}$$

where

$$y(t) = \frac{\partial\xi(t,\xi_0)}{\partial\xi_0}, \qquad M(t) = \frac{\partial f(\xi(t,\xi_0),t)}{\partial\xi} \tag{29}$$

Note that (28) is a **linear homogeneous time-dependent** differential equation for the matrix function y(t). Unfortunately, the time-dependence of M(t) is not known, because it depends on the solution $\xi(t,\xi_0)$.

However, we are not directly interested in (27), but rather in the distance between two solutions of (1), (2) that start from different initial conditions ξ_0 and ξ_1. Using the linear interpolation

$$\xi_\lambda = (1-\lambda)\xi_0 + \lambda\xi_1 \tag{30}$$

we can write

$$\xi(t,\xi_0) - \xi(t,\xi_1) = \int_0^1 \frac{d\xi(t,\xi\lambda)}{d\lambda}\, d\lambda \tag{31}$$

$$= \int_0^1 \frac{\partial\xi(t,\xi_\lambda)}{\partial\xi_0} (\xi_0 - \xi_1)\, d\lambda \tag{32}$$

It follows that if we can show that for $t \to \infty$ $\partial\xi(t,\xi_\lambda)/\partial\xi_0 \to 0$, uniformly in λ, then $||\xi(t,\xi_0) - \xi(t,\xi_1)|| \to 0$. Again, $\partial\xi(t,\xi_\lambda)/\partial\xi_0$ is the solution of an equation of the form (28) with unknown M(t).

While the matrix-valued function M(t) in (28) is unknown, its set of possible values at time t can be deduced from the original system (1), (2). Indeed, it is the set of Jacobian matrices $(\partial f/\partial\xi)(\xi,t)$, where ξ varies over the whole R^2. Therefore, if we can prove that all solutions of (28) converge to 0 for all functions M(t) with this specification on their range, then it follows that the original system (1), (2) has unique asymptotic behavior. Note that it is sufficient to consider this problem for vector-valued solutions y(t).

In the next paragraph, we will discuss families of equations of the form (28) with functions M(t) having a given range in general.

4.2. Linear time-dependent homogeneous systems

Motivatad by the results of the last section, we consider the following general problem

Problem 4.2.1:

Let S be a set of 2x2 matrices. When do **all solutions** $x : [t_0, \infty) \to R^2$ of **all equations**

$$dx/dt = M(t)x \tag{33}$$

where $M : [t_0, \infty) \to S$ is piecewise continuous, converge to zero as $t \to \infty$?

The answer to this problem depends evidently on the set S. In the trivial case where S contains a single matrix A, the function A(t) must be a constant, A. In this case, all solutions of (33) converge to zero, iff both eigenvalues of A are in the complex open left half-plane.

On the other hand, if S contains any number of matrices, but at least one of them, say A, has an eigenvalue in the right half-plane, including the imaginary axis, then not all solutions of all equations (33) converge to zero. Indeed, take $M(t) \equiv A$ and as an initial condition an eigenvector with an eigenvalue in the right half-plane on the imaginary axis. Then the corresponding solution of (33) diverges to infinity.

This suggests that we exclude from S matrices that have eigenvalues in the closed right half-plane. Note that this exludes in particular singular matrices. For convenience, one is tempted to require in addition that the matrices of S be uniformly bounded. It turns out, that this condition is at the same time too weak and too strong. The condition adapted to the problem is point b) of the following general hypothesis on S that will be adopted henceforth.

Hypothesis 4.2.2:

The matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{34}$$

of S satisfy the following two conditions:

a) The eigenvalues are in the open left half-plane, i.e.

$$a + d < 0 \tag{35}$$

$$ad - bc > 0 \tag{36}$$

b) The expression

$$C = \frac{a^2 + b^2 + c^2 + d^2}{ad - bc} \tag{37}$$

is uniformly bounded on S.

In numerical analysis, C is known as the condition number of matrix A.

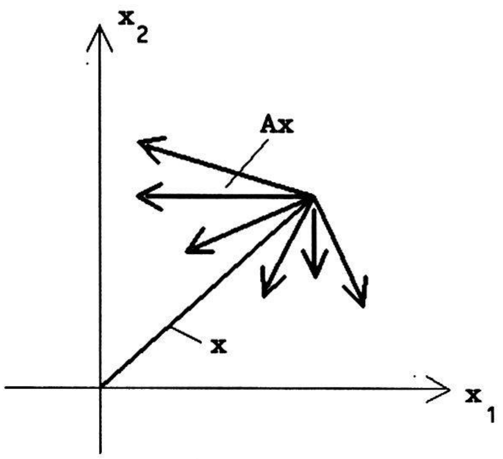

Fig.4

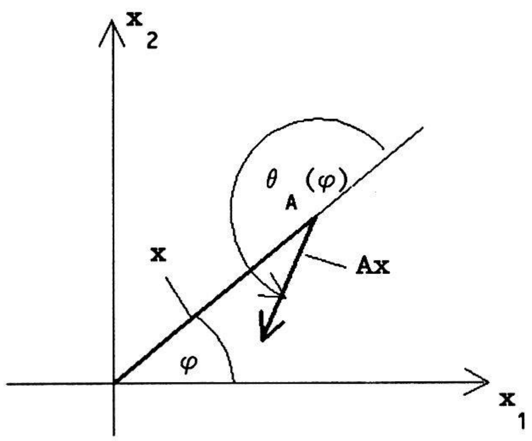

Fig.5

The idea how to solve problem 4.2.1 is as follows. Consider, for fixed $x \in R^2$, all vectors Ax, $A \in S$ (fig.4) and, in particular, their angle θ_A with x. Note that this angle is the same on the whole ray spanned by x. Therefore, this angle can be considered a function of the argument φ of x, $\theta_A(\varphi)$ (fig.5). Since we want the solutions of (33) to converge to 0, the "worst" vectors Ax are those which point "farthest away from 0", i.e. whose angle with x is extremal. Thus we define the two functions

$$\theta_{min}(\varphi) = \inf\{\theta_A(\varphi) \mid A \in S\} \tag{38}$$
$$\theta_{max}(\varphi) = \sup\{\theta_A(\varphi) \mid A \in S\} \tag{39}$$

In order to avoid ambiguities, the determination $[0, 2\pi)$ is taken for $\theta_A(\varphi)$. It follows that

$$0 \le \theta_{min}(\varphi) \le \theta_{max}(\varphi) \le \infty \tag{40}$$

It can be shown that due to hypothesis 4.2.2, the lower and upper limit in (40) are never reached [9].

Normally, $\theta_{min}(\varphi) < \pi$ and $\theta_{min}(\varphi)$ expresses the "worst counter-clockwise direction". However, $\theta_{min}(\varphi) > \pi$ is also possible. In this case $\theta_{min}(\varphi)$ points clockwise, but $\theta_{man}(\varphi)$ is a "less favorable clockwise direction". Hence we loose nothing by limiting the range of $\theta_{min}(\varphi)$ to $[0,\pi]$ and, analogously, the range of $\theta_{man}(\varphi)$ to $[\pi,2\pi]$. Thus we define

$$\theta_+(\varphi) = \min\{\theta_{min}(\varphi), \pi\} \tag{41}$$
$$\theta_-(\varphi) = \max\{\theta_{max}(\varphi), \pi\} \tag{42}$$

Now the interpretation that $\theta_+(\varphi)$ is the "worst counter-clockwise direction" and $\theta_-(\varphi)$ the "worst clockwise direction" is appropriate. The subscript "+" stands for "direction of increasing φ" and "-" for "direction of decreasing φ".

If for an angle φ, $\theta_+(\varphi) - \theta_-(\varphi) > \pi$ then it easy to construct a function M(t) and an initial condition ξ_0 such that the corresponding solution of (33) diverges to ∞. One chooses ξ_0 with an argument close to φ and switches with M(t) back and forth such that $M(t)\xi(t)$ points alternatively close to the directions $\theta_+(\varphi)$ and $\theta_-(\varphi)$, while the solution $\xi(t)$ remains in a sector close to φ (fig.6). This leads to the following proposition. For a complete proof, cf. [9].

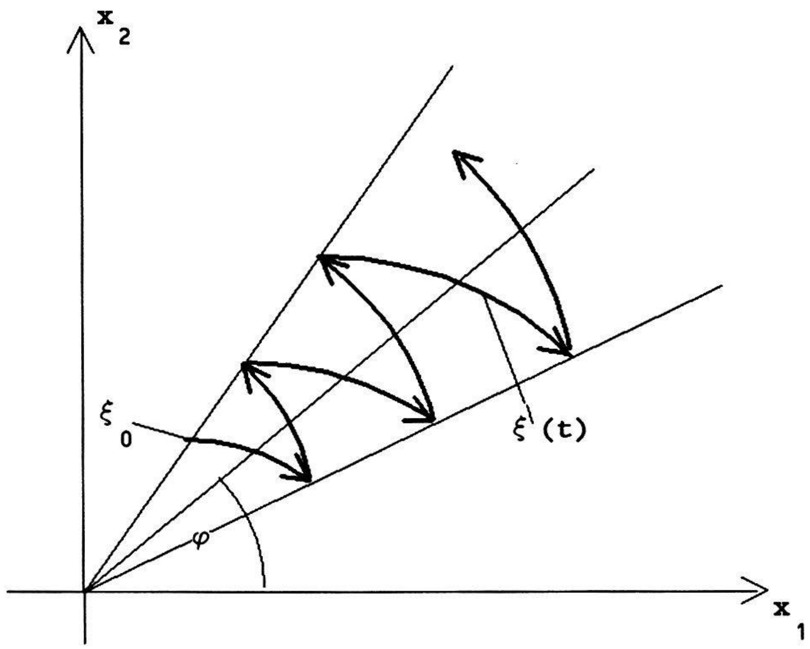

Fig.6

Proposition 4.2.3:

Suppose that the set of matrices S satisfies hypothesis 4.2.2. If for an angle φ

$$\theta_+(\varphi) - \theta_-(\varphi) > \pi \tag{43}$$

then there is a piecewise continuous S-valued function M(t) and an initial condition ξ_0 such that the solution $\xi(t)$ of (33) with $\xi(0) = \xi_0$ satisfies $\xi(t) \to \infty$ as $t \to \infty$.

Consider now a solution of (33) and express it in polar coordinates: $\xi(t) = (r(t), \varphi(t))$. Then from (33) the following equation can be deduced.

$$\frac{1}{r}\frac{dr/dt}{d\varphi/dt} = \cot \theta_{M(t)}(\varphi) \tag{44}$$

Note that in general r need not be a function of φ, since $d\varphi/dt$ may change sign.

We shall compare the solutions of (44) with the functions $r_+(\varphi)$ a and $r_-(\varphi)$ defined by

$$\frac{1}{r_+}\frac{dr_+}{d\varphi} = \cot \theta_+(\varphi) \tag{45}$$

$$\frac{1}{r_-}\frac{dr_-}{d\varphi} = \cot \theta_-(\varphi) \tag{46}$$

Since $\theta_+(\varphi)$ is the "worst counter-clockwise direction", it is expected that whenever for a solution of (33) one has $d\varphi/dt > 0$, then the orbit of this solution approaches zero faster than $r_+(\varphi)$. Indeed, if at time t a solution of (33) is at the point (r_0, φ_0), then the solution of (45) starting at the angle φ_0 with the radius r_0 satisfies

$$\frac{1}{r_0}\frac{dr_+(\varphi_0)}{d\varphi} = \cot \theta_+(\varphi_0) \geq \cot \theta_{M(t)}(\varphi_0) = \frac{1}{r_0}\frac{dr/dt(t)}{d\varphi/dt} \tag{47}$$

An analogous inequality involves r_-.

If we can prove that $r_+(\varphi) \to 0$ as $\varphi \to +\infty$ and $r_-(\varphi) \to 0$ as $\varphi \to -\infty$, then $\xi(t) \to 0$ as $t \to +\infty$, since the solution of (33) approaches 0 faster than $r_+(\varphi)$ and $r_-(\varphi)$. The question whether or not $r_+(\varphi)$ converges to zero as $\varphi \to +\infty$ is easily settled. In fact, it is sufficient to integrate $(1/r_+)dr_+/d\varphi$, which is given by the RHS of (45), from 0 to 2π. If this integral is negative, then $\ln(r_+(0)) > \ln(r_+(\pi))$ and thus $r_+(0) > r_+(\pi)$. Due to the fact that (33) is linear homogeneous, the solution from π to 2π is proportional to the solution from 0 to π, with the proportionality factor $r_+(\pi)/r_+(0) < 1$. Continuing this way, it is clear that $r_+(\varphi)$ spirals in to 0. A similar argument can be given for $r_-(\varphi)$.

This reasoning leads to the theorem 4.2.4 below. Again, the detailed proof can be found in [9]. One of the technical details is that the RHS of (45) and (46) is infinite when $\theta_+(\varphi)$ or $\theta_-(\varphi)$ take the value π. The way out of this difficulty is to limit the maximum of $\theta_+(\varphi)$ in (41) to $\pi-\epsilon$ and the minimum of $\theta_-(\varphi)$ in (41) to $\pi+\epsilon$, rather than to π, where ϵ is chosen sufficiently small.

Theorem 4.2.4:

Suppose that a set of matrices S satisfies hypothesis 4.2.2 and that the following three conditions hold:

a) for all φ

$$\theta_+(\varphi) - \theta_-(\varphi) < \pi \tag{48}$$

b)

$$\int_0^{2\pi} \cot \theta_+(\varphi)\, d\varphi \;< \pi \tag{49}$$

c)

$$\int_0^{2\pi} \cot \theta_-(\varphi)\, d\varphi \;> \pi \tag{50}$$

In (49), cot π has to be set to $-\infty$ and in (50) to $+\infty$.
Then **all solutions of all equations (33)** with piecewise continuous S-valued functions M(t) whose values are in S, **converge to zero**.

While Proposition 4.2.3 shows that condition 4.2.4.a) is necessary, no information has been given so far what happens if conditions 4.2.4.b) and 4.2.4.c) are not satisfied. Suppose that the LHS of (49) is strictly greater than π. Then the $r_+(\varphi)$ spirals out to infinity. It is possible to construct an S-valued function M(t) and to give and initial condition such that th orbit of the corresponding solution of (33) follows exactly, or at least closely, $r_+(\varphi)$ [9]. This implies the following proposition:

Proposition 4.2.5:

Suppose that the set of matrices S satisfies hypothesis 4.2.2 and that one of the following two conditions are satisfied:

a)

$$\int_0^{2\pi} \cot \theta_+(\varphi)\, d\varphi \;> \pi \tag{51}$$

b)

$$\int_0^{2\pi} \cot \theta_-(\varphi)\, d\varphi \;< \pi \tag{52}$$

Then there is a piecewise continuous S-valued function M(t) and an initial condition such that the corresponding solution of (33) diverges to infinity as $t \to \infty$.

Propositions 4.2.3, theorem 4.2.4 and proposition 4.2.5 give a complete solution to problem 4.2.1, except for the special cases where there is an equality in (48), (49), or (50). Problem 4.2.1 has been formulated because of its connection to the problem of unique asymptotic behavior of nonlinear systems. We shall come back to this point in the next section. In addition, however, problem 4.2.1 is of interest in its own right for switched linear systems. The following simple example shows the kind of result that can be obtained.

Example 4.2.6:

Let S be composed of the two matrices

$$A = \begin{bmatrix} -\alpha & +1 \\ -1 & -\alpha \end{bmatrix} \qquad B = \begin{bmatrix} -1/5 & +1 \\ 0 & -1 \end{bmatrix} \tag{53}$$

The eigenvalues of A are $-\alpha \pm j$ and those of B $-1/5$ and -1. Hence hypothesis 4.2.2 is satisfied for any $\alpha > 0$. It is not difficult to show [9] that the conditions of theorem 4.2.4 are equivalent to the positivity of the function $I(\alpha)$:

$$\begin{aligned} I(\alpha) = {} & \alpha(\pi - \tan^{-1}z_1 + \tan^{-1}z_2) + (1/4)\ln(z_1/z_2) \\ & - (5/4)\ln[(z_1 + 4/5)/(z_2 + 4/5)] \\ & + (1/2)\ln[(z_1^2 + 1)/(z_2^2 + 1)] \end{aligned} \tag{54}$$

where

$$z_{1,2} = [1 + 4\alpha/5 \pm (1 + 12\alpha + 16\alpha^2/5)^{1/2}/\surd 5]/(2(1-\alpha)) \tag{55}$$

It turns out that $I(\alpha) < 0$ for $\alpha < \alpha_0$ and $I(\alpha) >$ for $\alpha > \alpha_0$, where $\alpha \simeq 0.0075$. Hence, if $\alpha > \alpha_0$, all solutions of all equations (33) converge to zero, where M(t) is piecewise constant, switching at arbitrary times back and forth between A and B. On the other hand, if $\alpha < \alpha_0$, a certain switching M(t) between A and B can be found, and an initial condition, such that the corresponding solution of (33) diverges to infinity.

Note that the criterion for convergence to zero obtained in example 4.2.6 is almost completely explicit. Only at the very end, the zero of the function $I(\alpha)$ has to be determined numerically. By a finite number of computations, the monotonicity of $I(\alpha)$ can be established and arbitrarily close lower and upper bounds for α_0 can be obtained. Thus, for any $\alpha \neq \alpha_0$, a finite number of computations permits to decide whether or not all solutions of all equations (33), switching back and forth between A and B at arbitrary times, converge to zero. This information is infinitely more precise than direct numerical simulation, whereby only a finite number of finitely many equations (33) can be calculated.

4.3 Application to nonlinear systems.

The results of sections 4.1 and 4.2 can now be combined to the following theorem:

Theorem 4.3.1:

Consider the system of equations $d\xi/dt = f(\xi,t)$, where $\xi \in R^2$ and $f: R^3 \to R^2$ is continuously differentiable in ξ and piecewise continuous in t, i.e. there may be isolated discontinuity points, which divide the time axis into intervals within which f is continuous. Suppose that the solutions are bounded as $t \to \infty$. If the set of Jacobian matrices

$$S = \{(\partial f/\partial \xi)(\xi,t) \mid \xi \in R^2 \text{ and } t_0 \le t < \infty\} \tag{56}$$

satisfies the conditions of theorem 4.2.4, then the solutions defined in $t_0 \le t < \infty$ have unique asymptotic behavior.

The continuous differentiability of f implies the variational equations (27) [3]. The proof of theorem 4.3.1 follows the argments of section 4.1. For details cf. [9]. If f is only lipshitz continuous, some generalized partial derivative would have to be used [10]. Of practical interest is the case of piecewise linear f. In this case the variational equations remain applicable [11]. Indeed, it can be shown that the following theorem holds [9]:

Theorem 4.3.2:

Consider the system of equations $d\xi/dt = f(\xi,t)$, where $\xi \in R^2$ and $f: R^3 \to R^2$ is piecewise linear and continuous, i.e. the plane R^2 is divided by straight lines into N regions and the equation for region i is

$$d\xi/dt = A_i\xi + b_i(t) \tag{57}$$

where $b_i(t)$ is piecewise continuous and at each instant t, the RHS of (57) of the neighboring regions coincide on the dividing lines. If the set of matrices

$$S = \{A_i \mid i = 1,\ldots,N\} \tag{58}$$

satisfies the hypotheses of theorem 4.2.4 then the asymptotic behavior of the solutions of $d\xi/dt = f(\xi,t)$ is unique.

Note that for piecewise linear systems only a finite number of matrices has to be considered. The procedure for checking the conditions of theorem 4.2.4 can be automated, relying on explicit expressions for the integrals in (49) and (50), as in example 4.2.6.

On the other hand, the application of theorem 4.3.1 requires the determination of $\inf\{\theta_A(\varphi) \mid A \in S\}$ and $\sup\{\theta_A(\varphi) \mid A \in S\}$ where S is an infinite set and, in the worst case, the numerical computation of the integrals in (49) and (50). However, results as explicit as in example 4.2.6 are also possible, as shows the following example.

Example 4.3.3:

Consider the circuit of fig.1. Suppose that the nonlinear characteristics (6), (7) are continuously differentiable and satisfy the inequalities (17) and (18), i.e.

$$0 \le 1/C_{max} \le 1/C(q) = dh/dq \le 1/C_{min} \le \infty \qquad (59)$$

$$0 \le \omega_{min} \le \omega(q) = d(g\circ h)/dq \le \omega_{max} \le \infty \qquad (60)$$

With this notation, the Jacobian of the RHS of (8), (9) is

$$(\partial f/\partial \xi)(q) = \begin{bmatrix} -\omega(q) & 1 \\ -1/LC(q) & R/L \end{bmatrix} \qquad (61)$$

and S is the set of all matrices (61), when q varies over the real line. It is not difficult to show that due to the positivity of R, L and the constraints (59), (60), hypothesis 4.2.2 is satisfied. Furthermore, if A(q) is the matrix (61), then

$$\cot \theta_{A(q)}(\varphi) = \zeta(\tan \varphi,\ C(q),\ \omega(q)) \qquad (62)$$

where

$$\zeta(z,C,\omega) = \frac{-\omega + (1 - 1/LC)z - z^2R/L}{-1/LC + (\omega - R/L)z - z^2} \qquad (63)$$

It follows that

$$\partial\zeta/\partial C = - \frac{(z - \omega)(z^2 + 1)/LC^2}{(-1/LC + (\omega - R/L)z - z^2)^2} \qquad (64)$$

$$\partial\zeta/\partial\omega = \frac{zR/L + 1/LC)(z^2 + 1)}{(-1/LC + (\omega - R/L)z - z^2)^2} \qquad (65)$$

At the points where the denominator vanishes, $\theta_{A(q)} = \pi$ and $\cot \theta_{A(q)} = \infty$. The values of $\cot \theta_{min}(\varphi)$ and $\cot \theta_{max}(\varphi)$ coincide with the finite extremal values of $\cot \theta_{A(q)}(\varphi)$. According to (64) and (65), the finite extremal values of $\zeta(z,C,\omega)$ for fixed z and C, ω varying in the intervals (58), (59) are reached by replacing C by C_{max} or C_{min} and ω by ω_{max} or ω_{min}. The choice depends on z. Hence, the integrals (49) and (50) can be evaluated explicitly and the results are similar to (54), (55).

5. Comparison with quadratic Liapunov functions.

In order to compare the new method with the more traditional approach by quadratic Liapunov functions on a specific example, we consider again the circuit of fig. 1, with the piecewise linear characteristics of fig. 2 and 3 for the nonlinear resistor and capacitor. We start from the referenc values R = 25mΩ, L = 919mH, $G_b = 0$, $G_c = 500\Omega^{-1}$, $C_b = 27.57$mF, $C_c = 183.78$F, $U_j = 0.75$V. This is the normalized version of the parameters used in [7] for the numerical simulations which have shown the period-doubling route to chaos when the amplitude of the source is increased from 0 to 6V. All methods to establish unique asymptotic behavior mentioned so far rely on incremental variables and their differential equations (14), (15), or the Jacobian matrices of (8), (9), where the source amplitude and frequency play no role. Therefore, all criteria, when applicable, guarantee **unique asmptotic behavior for all source amplitudes and frequencies**. This property does not hold for the circuit with the reference values.

In order to assess the power of the two approaches, we vary the parameters G_b, G_c, C_b, C_c, one at the time, and determine for both criteria the interval of values where it guarantees unique asymptotic steady state. On the other hand, we determine by numerical simulation, how far from the reference value we still find, for some source amplitude and frequency, multiple asymptotic behaviors. This gives an idea, how close to the actual limit of unique asymptotic behavior for all source amplitudes and frequencies the two criteria get. Of course, the actual limit can only be determined by an infinite number of numerical simulations. However, the finite number of simulations show how good the criteria are **at least**. Additional simulation can only diminish the gap between established multiple asymtotic behavior and guaranteed unique asymptotic behavior.

The first criterion we apply is criterion 2.2 of [8]. This is the best result we have obtained from quadratic Liapunov functions. Theorem 3.1 gives no result for the sets of parameters we consider. The second criterion is deduced from theorem 4.3.2. The corresponding figures are reported in table 1.

Parameter varied	Unique asymptotic behavior guaranteed for by criterion 2.2 of [8]	by theorem 4.3.2	Multiple as. behavior found for
G_b	$G_b > 0.364/\Omega$	$G_b > 0.138/\Omega$	$G_b = 0.138/\Omega$
G_c	$G_c > 67/\Omega$	$G_c > 8.7/\Omega$	$G_c = 5.6/\Omega$
C_b	$C_b > 3.52$F	$C_b > 1.0$F	$C_b = 0.60$F
C_c	$C_c < 1.45$F	$C_c < 10.6$F	$C_c = 16$F

Table 1

Table 1 shows that the results obtained by the new method are considerably better than those produced by quadratic Liapunov functions. The guaranteed limit for unique asymptotic behavior is sufficiently close to the actual limit to be useful for technical applications. We have not explored the whole parameter space in order to give a global view of the capabilities of the new method. Finally, other examples should be considered.

6. Generalization to dimension n.

As it stands, the new method is only capable to handle systems of dimension 2. The connection between the nonlinear problem and the linear problem as described in section 4.1 is not limited to dimension 2. The notion of "worst directions" can also be generalized to higher dimensions. For each x, a cone would be obtained. However, it is not evident, how to define a hypersurface that encloses 0 such that all cones lie inside.

A more crude way is to consider the projections of the solutions of the linear equations to the planes spanned by two coordinates. If all these projections converge to zero then the solution itself converges to zero. Consider equation (33) for $x \in R^n$ and $M: [t_0, \infty) \to \{n \times n \text{ matrices}\}$. Let P be any projection to a linear subspace of dimension 2. Then

$$dPx/dt = PM(t)x \tag{66}$$

We can, for each x, determine the "worst directions" among the vectors PAx, where $A \in S$ and apply the results of section 4. We have not tried this approach so far.

7. Conclusions.

We have presented a new method to prove the unique asymptotic behavior of nonlinear nonautonomous ordinary differential equations of dimension 2. It has been illustrated by circuit examples that the results are better than those obtained by quadratic Liapunov functions. The question, how far Liapunov's method can go at all, remains of course open.

The method is easy to apply. In many cases, only a zero of an explicitly known function on the real line has to be determined numerically. The few examples that have been worked out indicate that the method could be useful for engineering purposes, which is generally not the case for most criteria derived from Liapunov functions. However, the limitation to dimension 2 has to be overcome.

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