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Time interval statistics of a symmetric two-peaked spectrum

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Abstract. A complete and an exact analysis of the time interval statistics of *pure* Brillouin spectrum (a symmetric two-peaked spectrum) is provided here, by utilizing the *unique* analytic-form of the generating-function recently given by the present author [1]. Unfortunately, the earlier approximate results due to Blake and Barakat [3] are found to violate even the well known equation (28) for the time-interval-distribution for the conditional photocount.

Introduction

One of the long standing problems in the field of photon-counting-statistics (PCS) of chaotic light is the exact analysis of the time-interval-statistics (TIS) of the *pure* Brillouin spectrum (symmetric two-peaked spectrum). This had been mainly due to the non-availability of the required *exact* one-fold generating-function (g.f.) characterising this multiple-peaked spectrum. However, this task has recently been accomplished by the present author in Ref. 1, wherein the usage of the Hadamard's Theorem for the infinite-products [2] coupled with the following boundary conditions, help us in obtaining the unique analytic form of the g.f. for the symmetric two-peaked spectrum:

$$0 \leq Q(s, T) \leq 1 \tag{1}$$

$$\left\{Q(s, T) = \sum_{n=0}^{\infty} (1-s)^n P(n, T)\right\}\Bigg|_{s=0} = \sum_{n=0}^{\infty} P(n, T) = 1 \tag{2}$$

$$\{(-1)\, \partial Q(s, T)/\partial s\}|_{s=0} = \langle n \rangle \tag{3}$$

The central idea behind such a formalism being that if any useful spectral information is to be obtained via the one-fold photon counting analysis like the TIS here, the relevant g.f., *first* ought to satisfy the boundary conditions stated in equations (1)–(3).

We show in this paper (Section 3) how the mathematical rigor of [1] now helps us in capturing the exact details of the various Time-Interval-Distributions (TIDs). Unfortunately, the earlier analysis due to Blake and Barakat [3] based on the numerical scheme suggested in Ref. 4, has completely failed on this account as far as the TID of registering a conditional photo-count is concerned (see Fig. 5).

The Time-Interval (TI) analysis offers us a simple and direct way of studying the spectral features of an optical beam, as the experimental set-up here, requires a combination of Time-to-Amplitude-Converter (TAC) and Pulse-Height-Analyser (PHA). Sensitive TI measurements with the resolution $\sim 10^{-6}$–10^{-7} s have been reported in literature (see Refs. 5, 6 for example). Reference is also made to 'Time-of Arrival-Correlator' recently constructed by Dhadwal et al. [7].

For a theoretical study of the TIDs, we require the g.f. characterizing the spectrum under study, as shown by Glauber in Ref. 8. A simple analysis of the TIDs is then offered via the one-fold g.f. where the underlying assumption is [8],

$$\int_t^{t+\delta t} I(t')\,dt' \sim I(t)\,\delta t \tag{4}$$

It must be mentioned here that for a rigorous analysis of the TIDs (where the field strength is arbitrary), we require a third-order g.f. as shown by Barakat and Blake in their review article [9]. This task has been accomplished for the case of Gaussian–Lorentzian (G–L) light in [10], by employing the exact third-order g.f. given in Ref. 11.

2. The generating-functions

The desired one-fold analysis of the present symmetric two-peaked spectrum is facilitated by solving the following Fredholm integral equation of the second-kind:

$$s\int_0^T g\,|t-t'|\,\phi_k(t')\,dt' = \lambda_k \phi_k(t) \tag{5}$$

where $g\,|t-t'|$ is the field-correlation of the spectrum under study and $\phi_k s$ and $\lambda_k s$ are the eigen-functions and eigen-values respectively. The field-correlation $g(\tau)$ for this symmetric two-peaked spectrum is given by [3]:

$$\begin{aligned} g(\tau) &= \alpha e^{-\delta\,|\tau|}\cos\Delta t \\ &= \frac{\alpha}{2}\left(e^{-\beta\,|\tau|} + e^{-\beta^*\,|\tau|}\right) \quad (\alpha = 1 \text{ here}) \end{aligned} \tag{6}$$

where 'δ' represents the half-width of the Brillouin lines, 'Δ' the frequency-shift and $\beta,\ \beta^* = \delta - i\Delta,\ \delta + i\Delta$. As fair estimates of the line-width 'δ' of the Brillouin spectrum are available in literature (see Mountain [12]), the quantity of main interest in such a study is the frequency-shift 'Δ' – for it is directly related to the speed of sound in a fluid or the condensed-matter [12]. Multipass Fabry–Pérot interferometer with contrast of the order of 10^{12} have been found extremely useful to study these Brillouin frequencies [13]. Interesting details of the various experimental techniques employed to study the Brillouin spectrum are provided in the review article of Borovick-Romanov and Kreines [14] and the recent papers [15, 16]

It is well known that the one-fold g.f. for the Gaussian light can be expressed as follows:

$$Q(s) = \prod_k (1 + s\lambda_k \langle I \rangle)^{-1} \tag{7}$$

where $\langle I \rangle$ represents the mean count rate and $\lambda_k s$ are the eigen-values of equation (5). Blake and Barakat [3] had obtained the following expression for the one-fold g.f. for this two-peaked spectrum by utilizing the numerical-scheme suggested in [4]:

$$Q(s, T) = \exp \sum_{r=1}^{\infty} \frac{[(-\langle I \rangle s)^r]}{r} I_r(T) \tag{8}$$

where

$$I_r(t) = \int_0^T g_r(t', t')\, dt' \tag{9}$$

and $g_r(t', t')$ is the rth iterated kernel of the integral equation (5), defined recursively through the following relations:

$$g_1(t', t'') \equiv g(t', t'')$$

and

$$g_r(t', t'') = \int_0^T g(t', t) g_{r-1}(t, t'')\, dt \quad (r \geqslant 2) \tag{10}$$

Obviously, no functional form for the g.f. $Q(s, T)$ is possible in this formalism, unlike the g.f. given in [1], where we can clearly see its exact relation with the parameters 'δ and Δ' characterising this two-peaked spectrum under study.

We had obtained the *unique* analytic form of the g.f. for this spectrum [1] by first rendering the Fredholm-determinant (obtained on solving equation (5)) analytic in nature to obtain an entire-function $P(\xi)$ ($\xi = \lambda^{-1}$) and then by applying Hadamard's Theorem [2], we could compare the zeros of $P(\xi)$ with those of $Q(s, T)$ in equation (7), so as to finally arrive at the following important relation:

$$Q(s, T) = P(0)/P(-s\langle I \rangle) \tag{11}$$

where $P(0)$ is a constant given by:

$$P(0) = \beta^* \beta(\beta^{*2} - \beta^2)^2 \tag{12}$$

and,

$$P(-s\langle I\rangle) = e^{-(\beta^*+\beta)T}\{\beta\beta^*[Z^{*2}(p_1)Z^{*2}(p_2)F(p_1)F^*(p_2) + Z^2(p_1)Z^{*2}(p_2)F^*(p_1)F(p_2)] - 2[\delta^2 G(p_1)G(p_2) + \Delta^2]Z(p_1)Z^*(p_1)Z(p_2)Z^*(p_2)\}(p_1^2 - p_2^2)^{-2} \tag{13}$$

where

$$Z(p) = (\beta^2 - p^2), \qquad Z^*(p) = (\beta^{*2} - p^2),$$
$$F(p) = [\cosh pT + \tfrac{1}{2}(\beta/p + p/\beta)\sinh pT],$$
$$F^*(p) = [\cosh pT + \tfrac{1}{2}(\beta^*/p + p/\beta^*)\sinh pT],$$
$$G(p) = \left[\cosh pT + \left(\frac{\beta^*\beta + p^2}{(\beta^* + \beta)p}\right)\sinh pT\right] \tag{14}$$

and the values of 'p' are determined from the following equation:

$$p = \pm\sqrt{A \pm \sqrt{A^2 - 4b}/\sqrt{2}}$$

where

$$A = [(\beta^* + \beta) + s\langle I\rangle(\beta^* + \beta)],$$

and

$$B = \beta\beta^*[\beta^*\beta + s\langle I\rangle(\beta^* + \beta)] \tag{15}$$

The probability of zero-counts defined below gives us a measure of this g.f.:

$$P(0, T) = Q(s, T)|_{s=1} \tag{16}$$

3. Time interval statistics

3.1. *Definitions and general features*

Several important and interesting details of the TIS are given in the review article of Barakat and Blake [9]. We briefly recapitulate the following important details required for discussing the TIS of the chaotic light having *any spectral shape*.

Recollecting that the g.f. can be written as,

$$Q(s, T) = \langle \exp[-sE(T)]\rangle \tag{17}$$

where

$$E(T) = \int_t^{t+T} I(t')\, dt' \tag{18}$$

and $I(t)$ is the instraneous intensity of the scattered-field, the TIDs of registering the first photocount, $V(T)$, and the conditional photocount, $P(T)$, are respectively given by:

$$V(T) = \langle I(T) \exp[-sE(T)] \rangle, \tag{19}$$

$$P(T) = \langle I(0) I(T) \exp[-sE(T)] \rangle, \tag{20}$$

if we take $t = 0$ in equation (18). From equation (17), it is easy to see the following relations of these TIDs with the one-fold g.f.,

$$V(T) = -\partial Q(s, T)/\partial T|_{s=1} \tag{21}$$

and

$$P(T) = \frac{1}{\langle I \rangle} [\partial^2 Q(s, T)/\partial T^2]|_{s=1} \tag{22}$$

For a completely polarized light, Barakat and Blake [9] have derived the following interesting expressions for $V(T)$ and $P(T)$, when the interval times are of short duration but the mean count rate $\langle I \rangle$ can be anything,

$$V(T) = \langle I \rangle / (1 + \tfrac{1}{2}\langle I \rangle T)^2 \tag{23}$$

and

$$P(T) = 2\langle I \rangle / (1 + \langle I \rangle T)^3 \tag{24}$$

However, the above expressions for $V(T)$ and $P(T)$ are obviously insensitive to the spectral-shape of the polarized chaotic light.
For $T = 0$, we have from equation (23),

$$V(0) = \langle I \rangle \tag{25}$$

and from equation (24), we get,

$$P(0) = 2\langle I \rangle \quad [\text{or } 2V(0)] \tag{26}$$

Thus, at the origin, the conditional TID is always twice the value of the TID for registering the first photo-count, *irrespective of the spectral* shape of the chaotic light.

On the other hand, when $\langle I \rangle T \gg 1$, we have,

$$P(T)/V(T) \approx (2\langle I \rangle T)^{-1} \tag{27}$$

which implies that $P(T)$ *decays-off faster than* $V(T)$ *for large count rates* and the crossing point is at about $T = 0.088$ for $\langle I \rangle = 5.0$ as can be easily seen from eqs. (23)–(24). It is also instructive to note that for $\langle I \rangle T \ll 1$, the conditional TID, $P(T)$, behaves like the correlation function $C(T)$ (bunching of photons in a chaotic light!):

$$P(T) = \langle I \rangle [1 + g^2(T)] = \langle I \rangle C(T) \tag{28}$$

This important fact has recently been used by Dhadwal et al. [7], while constructing the 'Time-of-Arrival-Correlator' – operative in the μs regime.

At $T = 0$, equation (28) also gives $P(0) = 2\langle I \rangle$ as $g(0) = 1$.

It is interesting to observe that whereas Barakat and Blake in [9] confirm equation (25) (Fig. 6.3) and equation (26) (Fig. 6.4) for the G–L light, the same authors report values of $P(T)$ for the symmetric two-peaked spectrum in [3] (or see Figs. 6.7 and 6.8 in ref. [9]), which neither follow equation (26) (for any $\langle I \rangle$) not the well-known equation (28) for $\langle I \rangle T \ll 1$ (see Fig. 5).

Thus, to give a direct feeling of how the exact TIDs behave for the present case, we now present their explicit functional form (Section 3.2). The direction dependence of these TIDs on the parameters 'δ and Δ' which characterize this spectrum, comes through the g.f. given in equations (11)–(15).

3.2. TIDs *for the symmetric two-peaked spectrum*

Rewriting the g.f. as,

$$Q(s, T) = A(0)e^{(\beta^* + \beta)T} / A(p_1, p_2, T) \tag{29}$$

where

$$A(0) = P(0) = \beta\beta^*(\beta^{*2} - \beta^2)^2 \tag{30}$$

and

$$\begin{aligned} A(p_1, p_2, T) = \{ &\beta\beta^*[Z^{*2}(p_1)Z^2(p_2)F(p_1)F^*(p_2) \\ &+ Z^{*2}(p_2)Z^2(p_1)F^*(p_1)F(p_2)] - 2[\delta^2 G(p_1)G(p_2) + \Delta^2] \\ \times\ &Z(p_1)Z^*(p_1)Z(p_2)Z^*(p_2)\}(p_1^2 - p_2^2)^{-2} \end{aligned} \tag{31}$$

We first determine from equations (29), (30) and (21), the following functional form of the TID for the first photo-count, $V(T)$:

$$\begin{aligned} V &= -\partial Q(s, T)/\partial T|_{s=1} \\ &= P(0, T)\left[A^{-1}(p_1, p_2, T)\frac{\partial A}{\partial T}(p_1, p_2, T) \right]\Bigg|_{s=1} \end{aligned} \tag{32}$$

where $P(0, T)$ is the probability of zero-counts (equation (16)), and

$$\begin{aligned} \frac{\partial A}{\partial T}(p_1, p_2, T) = \{ &\beta\beta^*[Z^2(p_1)Z^2(p_2)(p_1 \not{F}(p_1)F^*(p_2) \\ &+ p_2 F(p_1)\not{F}^*(p_2)) + Z^{*2}(p_2)Z^2(p_1) \\ &\times (p_1 \not{F}^*(p_1)F(p_2) + p_2 F^*(p_1)\not{F}(p_2))] \\ &- 2[\delta^2(p_1 \not{G}(p_1)G(p_2) + p_2(G(p_1)\not{G}(p_2)] \\ &\times Z(p_1)Z^*(p_1)Z(p_2)Z^*(p_2)\}(p_1^2 - p_2^2)^{-2} \end{aligned} \tag{33}$$

and

$$\begin{aligned} \not{F}(p) &= [\sinh pT + \tfrac{1}{2}(\beta/p + p/\beta)\cosh pT], \\ \not{F}^*(p) &= [\sinh pT + \tfrac{1}{2}(\beta^*/p + p/\beta^*)\cosh pT], \\ \not{G}(p) &= \left[\sinh pT + \frac{(\beta^*\beta + p^2)}{(\beta^* + \beta)p}\cosh pT \right] \end{aligned} \tag{34}$$

Similarly, from equations (30)–(33) and (22), we have the following explicit expression for $P(T)$:

$$P(T) = \frac{1}{\langle I\rangle}\left\{\frac{2V^2(T)}{P(0,T)} - P(0,T)\left[A^{-1}(p_1,p_2,T)\frac{\partial^2 A}{\partial T^2}(p_1,p_2,T)\right]\right\}\Bigg|_{s=1} \tag{35}$$

where

$$\frac{\partial^2 A}{\partial T^2}(p_1,p_2,T) = \{(p_1^2+p_2^2)A(p_1,p_2,T) + 2p_1p_2\not{A}(p_1,p_2,T)\} \tag{36}$$

with

$$\begin{aligned}\not{A}(p_1,p_2,T) = \{&\beta\beta^*[Z^{*2}(p_1)Z^2(p_2)\not{F}(p_1)\not{F}^*(p_2)\\ &+ Z^{*2}(p_2)Z^2(p_1)^*\not{F}^*(p_1)\not{F}(p_2)]\\ &- 2[\delta^2\not{G}(p_1)\not{G}(p_2)+\Delta^2]Z(p_1)Z^*(p_1)Z(p_2)Z^*(p_2)\}\\ &\times(p_1^2-p_2^2)^{-2}\end{aligned} \tag{37}$$

From above, it is interesting to observe that the TID for the first count, $V(T)$, is related to the probability of zero counts, $P(0, T)$ (equation 32) and the TID for the conditional photo-count, $P(T)$, is related to both the probability of zero counts $P(0, T)$ as well as the TID for the first photo-count $V(T)$ equation 35).

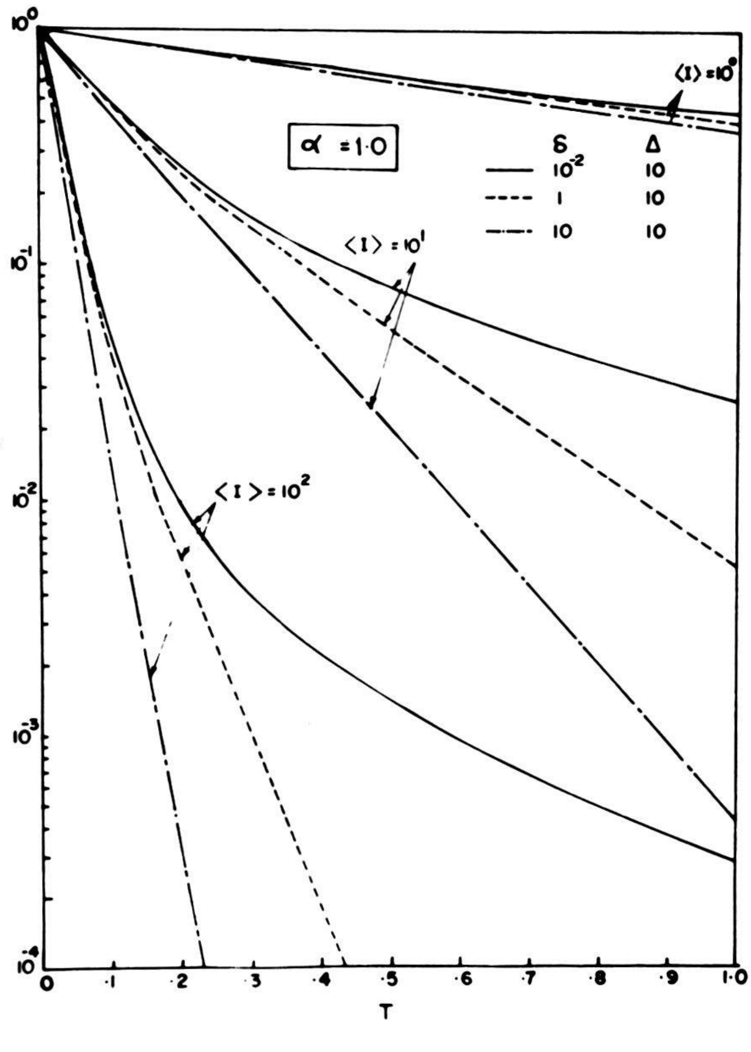

Figure 1
The probability of zero counts.

3.3. *Results and discussion*

It is natural to expect that the probability of zero counts $P(0, T)$ at $T=0$ should be 1 and as T grows, it should fall-off. This is depicted in Fig. 1 and we also see that as $\langle I \rangle$ increases, $P(0, T)$ decreases. In other words, the probability of registering a photocount gets enhanced at a high count rate.

We see in Fig. 2, the behaviour of the TID for the first photo-count $V(T)$. At $T=0$, we see that $V(0)=\langle I \rangle$ for all the values of $\langle I \rangle$, as it ought to be the case (equation (25)). Also we see that for low count rates ($\langle I \rangle = 0.5$, 1.0), $V(T)$ decays-off slowly whereas for relatively high count rates ($\langle I \rangle = 5.0$, 10.0) the decay is more prounced (equation (23)). Unlike the Fig. 1, for $P(0, T)$, we now see the emergence of 'kinks' in this TID. At this juncture, we must mention that the results reported for $V(T)$ for a two-peaked spectrum characterizing the polydisperse medium [17], are in clear distinction with the present case of $V(T)$ for the *pure* Brillouin spectrum, as there were no 'kinks' in it. However, the other feature like $V(0)=\langle I \rangle$ and $V(T)$ decaying faster at higher count rates were also true for the two-peaked spectrum for the polydispersity case [17] (see Fig. 1).

Figure 3 shows the behaviour of the TID for the conditional photocount $P(T)$. At $T=0$, we see that $P(0)=2\langle I \rangle$, as suggested by equation (26) and/or equation (28). We now see that the mere 'kinks' of $V(T)$ have now turned into

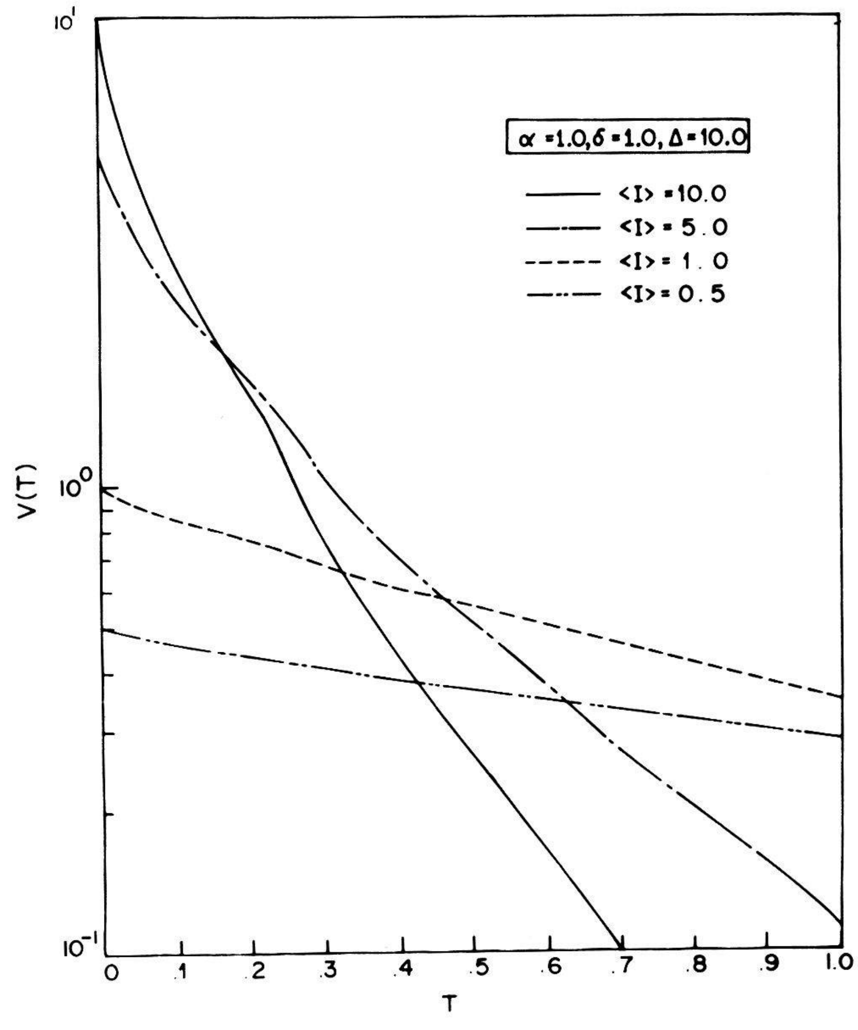

Figure 2
The time-interval-distribution for the first photocount.

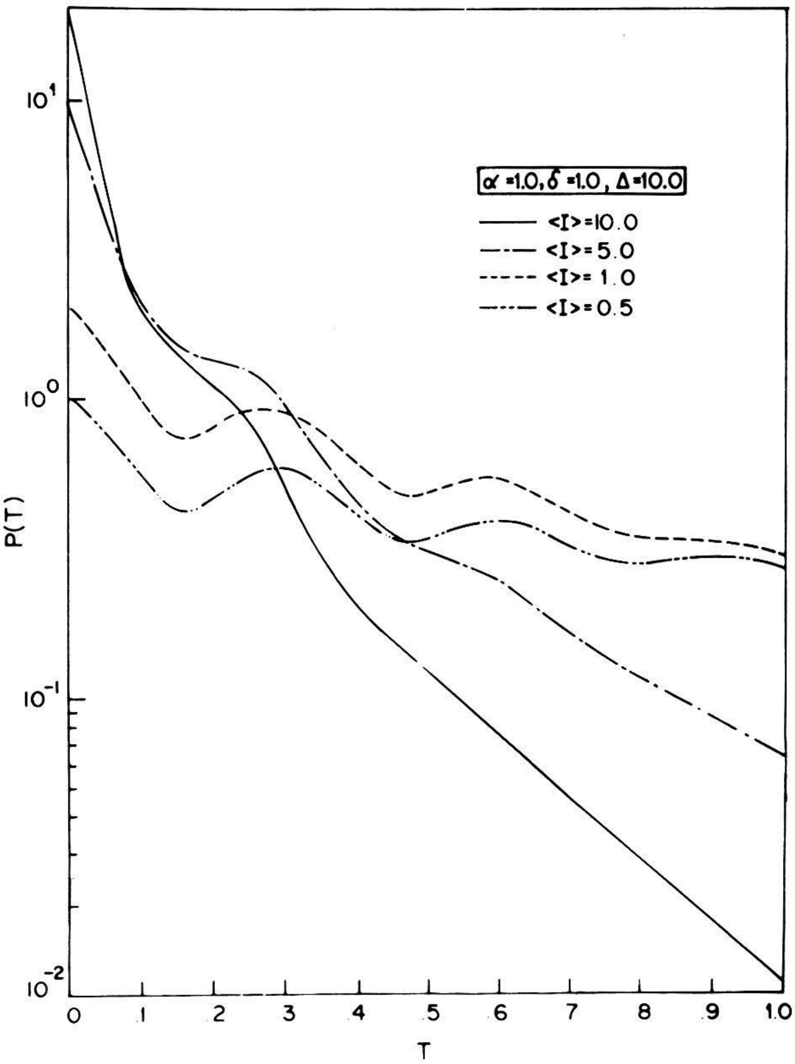

Figure 3
The time-interval-distribution of registering a conditional photocount at time T.

pronounced 'oscillations' for $P(T)$ for the same $\langle I \rangle s$ and Ts. This is to be expected only, as $P(T)$ gives a more sensitive account of the TIS, for its direct dependence on the field-correlation $g(\tau)$ – as evident from the simple equation (28). The general oscillatory behaviour of the TIDs for the present symmetric two-peaked spectrum is due to the '$\cos \Delta T$' factor in $g(T)$ in equation (6). At low count rates, the heterodyning between the Brillouin components produces fluctuations in the intensity on the time scale of the order of Δ^{-1}, and we clearly see the oscillations. But as the mean count rate $\langle I \rangle$ goes higher, these intensity fluctuations get arrested and as a result, the oscillations due to heterodyning also vanish, as made clear in Figs. (2) and (3).

It is instructive to compare the behaviour of the two TIDs, $V(T)$ and $P(T)$, for the same count rates and interval times. This we do in Fig. 4. It is interesting to notice that whereas $P(T)$ oscillates over the $V(T)$ values for a low count rate ($\langle I \rangle = 0.1$), $P(T)$ decays-off faster as compared to $V(T)$ for relatively higher count rate ($\langle I \rangle = 5.0$) (equation (24) or (27)) with $T = 0.08$ as the point of departure-true to the suggestion of equation (27)!

In Fig. 5, we compare the approximate values of $P(T)$ as given by Blake and Barakat in [3], against the exact ones obtained here. First, at the outset, we

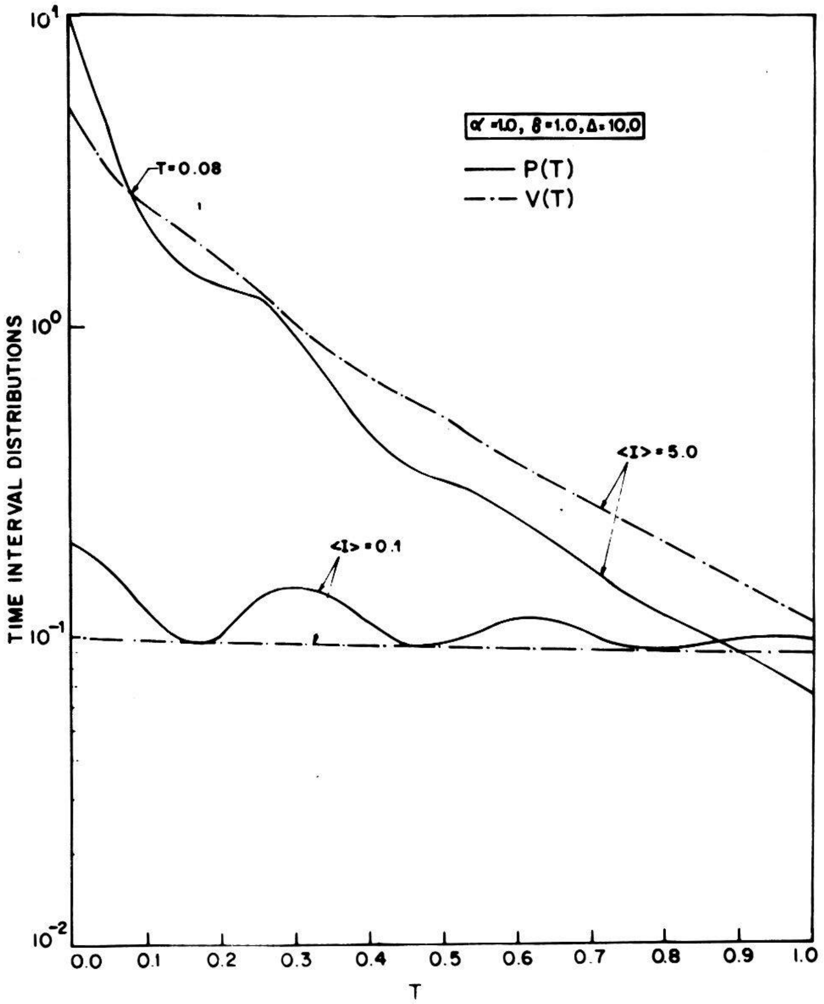

Figure 4
A comparative study of the time-interval-distribution $V(T)$ and $P(T)$, as a function of count rates. For $\langle I \rangle = 5.0$ (moderate count rate $P(T)$ decays off faster than $V(T)$ after $T = 0.08$ but $P(T)$ oscillates above $V(T)$ values throughout, when $\langle I \rangle = 0.1$ (low count rate).

notice that for both the count rates $\langle I \rangle = 0.1$ and $\langle I \rangle = 1.0$, the $P(T)$ values of Blake and Barakat [3] violate $P(0) = 2\langle I \rangle$, as suggested by equations (26) and/or equation (28). For $\langle I \rangle = 0.1$ at $T = 0$, equation (28) suggests $P(0) = 0.2$ but the reported value due to Blake and Barakat [3] is 1.0, thereby suggesting an error of ~400% for $\langle I \rangle = 0.1$. For $\langle I \rangle = 1.0$, equation (28) gives $P(0) = 2$ at $T = 0$, but Blake and Barakat [3] gives this value as 1.25 which now suggests an error of ~38% for $\langle I \rangle = 1.0$. However, the exact and overall estimate of the error in the reported values for $P(T)$ by Blake and Barakat [3],can be determined only from our exact analysis (for arbitrary $\langle I \rangle$ and T) (Fig. 5), *as equation* (28) *is constrained by the condition* $\langle I \rangle T \ll 1$. From Fig. 5, it is evident that as T grows the error also grows for $\langle I \rangle = 1.0$, but the reverse trend is observed for $\langle I \rangle = 0.1$ and an average error of ~77% for $\langle I \rangle = 1.0$ and ~209% for $\langle I \rangle = 0.1$ is found in the numerical-scheme due to Blake and Barakat [3]. However, interestingly the qualitative features are found to be in some agreement with ours, as can be seen from the oscillations in Fig. 5.

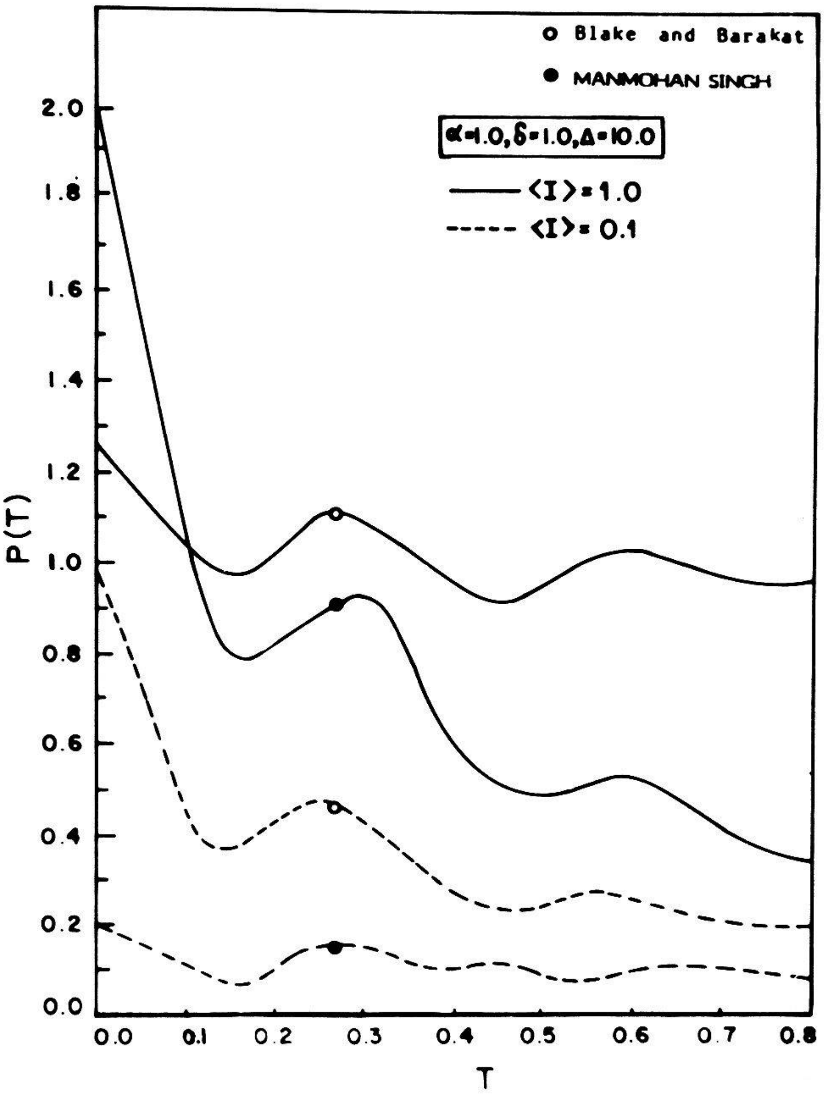

Figure 5
The approximate values of the time-interval-distribution for the conditional photocount reported by Blake and Barakat [3], compared against the exact ones obtained here.

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