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LOCALIZATION AND CROSS-OVER IN LINEAR MODELS OF RANDOM ARRAYS OF BARRIERS

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Abstract. In this paper, two distinct physical models of random arrays of barriers are considered which lead to the same transfer matrix. The first describes a quantum particle (electron) in a sequence of delta-function potentials with random amplitudes and positions. The second model is an $L - C$ line (cable) made up from a periodic array of inductances L and capacitances C in which at random locations a random value of excess capacitance is inserted which acts as an obstacle. In both models the barriers are assumed to have amplitudes which in the average decrease with distance x as $x^{-\nu/2}$, $\nu > 0$. It is shown that the value $\nu = 1$ marks a cross-over between strong localization defined by an exponential increase of reflectance and no localization or bounded reflectance. For electrons this situation corresponds to an insulator-metal transition while for the $L - C$ line it has the characteristics of a high-pass filter.

1 - Introduction

In the problem of electrons in arrays of barrier potentials, the connection between the residual resistance ρ and the coefficients of transmission T and reflection R was first established by Landauer [1,2]. Making use of the Einstein relation between conductivity and diffusion constant he showed that

$$\rho = \frac{h}{e^2}\frac{R}{T}, \tag{1}$$

where h is Planck's constant and e the electronic charge. As is well known, an array of barriers with constant average amplitude gives rise to an exponential increase of the average resistance $< \rho >$ as function of the number N of barriers, corresponding to a homogeneous wire of uniform cross section. In recent times, however, artificial structures with practically arbitrary properties as function of linear distance have become available [3,4]. The investigation of arrays with non-uniform average amplitude therefore is of practical interest.

One such non-uniform array recently considered by Delyon et al. [5] is a random tight-binding model characterized by an amplitude which in the average decreases with distance x as $x^{-\nu/2}$, $\nu > 0$ [5]. However, these authors were not interested in the resistance but in the spectrum of the model which, for $\nu = 1$ exhibits a transition between a pure point spectrum and a purely continuous one.

In this paper we first consider a model which is similar to the one studied by Delyon et al. It is defined by the one-electron Hamiltonian

$$H = -\frac{d^2}{dx^2} + \sum_{j=1}^{N} \gamma_j v_j \delta(x - x_j), \tag{2}$$

where we have put $h = 2\pi$ and the electron mass equal to $1/2$. The x_j's are the successive positions of the barriers which have the form of a Dirac δ, and $\gamma_j v_j$ are the amplitudes where γ_j is a modulation factor satisfying $0 < \gamma_j \leq 1$. x_j and v_j are random variables defined by the probability distribution

$$P(x_1, v_1, ..., x_N, v_N) = \delta(x_1) \prod_{j=2}^{N} f(x_j - x_{j-1}) \prod_{j'=1}^{N} g(v_{j'}), \tag{3}$$

where f and g are peaked at respective average values $< x_j - x_{j-1} >= 1$ and $< v_j^2 >= \sigma^2$, independent of j. The δ-function potentials in Eq.(2) may be considered as limiting case of a square well barrier.

In the case of the $L - C$ line we start from a periodic array of capacitances C and inductances L as shown in Fig.1. Designating the charge on the

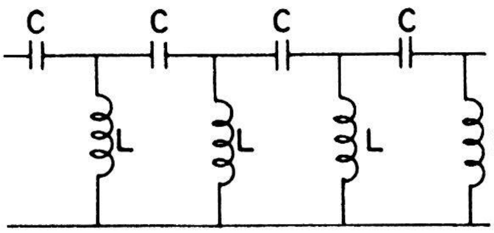

Fig.1: Section of $L - C$-line.

nth cirquit as Q_n it is easy to derive the equation

$$\left(\frac{\omega_o^2}{\omega^2} - 2\right) Q_n + Q_{n+1} + Q_{n-1} = 0, \tag{4}$$

where $\omega_o^2 \equiv 1/CL$. The analogy with the case of electrons comes from the obvious fact that the solution of Eq.(4) has the form of a wave

$$Q_n = Ae^{ikn}, \tag{5}$$

where we chose the cirquit-lenght to be unity and where the wave vector satisfies the dispersion relation

$$4 \sin^2 \frac{k}{2} = \frac{\omega_o^2}{\omega^2}. \tag{6}$$

Randomness is introduced by picking out a random sequence of cirquits n_j, $j = 1, 2, ..., N$, to which a random value C/v_j of excess capacitance is added in series. The high-pass filtering property of the line comes from the fact that for high external frequencies ω the wavelength determined by the dispersion relation (6) is too long for random distructive interference which is the essence of localization.

From this point of view the electron model (2) would be a low-pass filter since in this case Eq.(6) is replaced by

$$k^2 = \omega \tag{7}$$

which here is the Fermi energy so that electrons always have extremely short wavelengths. This is expressed by the condition

$$k(x_j - x_{j-1}) \gg 1, \tag{8}$$

valid for all j, which will also be applied to the $L - C$ line with $x_j = n_j$. Condition (8) combined with the probability distribution (3) is the reason for the random distructive interference mentioned above. Indeed, Felderhof's method [7] requires that $\langle e^{ik\xi} \rangle \simeq 0$. Taking for the position distribution in Eq.(3), e.g. $f(\xi) = (1/\pi)exp(-\xi^2/\pi)$, which satisfies $\langle|\xi|\rangle = 1$, one easily finds $\langle e^{ik\xi} \rangle = exp(-k^2\pi/4)$, which fulfills the mentioned requirement for $k^2 \gg 4/\pi$, in agreement with condition (8).

Choosing the modulation factor in Eq.(2) as [5]

$$\gamma_j = j^{-\nu/2}, \nu > 0 \tag{9}$$

we find three regimes for $< R/T >$ in the limit $N \to \infty$: $< R/T >$ diverges exponentially if $0 \leq \nu < 1$; for $\nu = 1$ the divergence is algebraic while for $\nu > 1$ there is no divergence. More explicitly, with Eq.(9) the third regime corresponding to metallic behaviour is determined by the condition of existence of Riemann's zeta-function,

$$\sum_{j=1}^{\infty} \gamma_j^2 = \zeta(\nu) < \infty, \tag{10}$$

namely $Re\nu > 1$.

Note that, quite independently of the particular modulation (9), $\sum_{j=1}^{\infty} \gamma_j^2 < \infty$ is the general condition for metallic behaviour.Furthermore,the mentioned classification depends on the additional condition that

$$\frac{\gamma_j^2 < v_j^2 >}{k^2} \ll 1 \tag{11}$$

where, in the case of the $L - C$ line, $\gamma_j v_j / k$ is to be replaced by $v_j \omega_o^2/\omega^2 \sin k$. Finally, we note that for a periodic modulation $\gamma_j = \sin \nu j$ [6] it is clear that $\sum_{j=1}^{N} \gamma_j^2$ is proportional to N so that the resistance diverges exponentially with N, and the localization length is a function of ν.

In Section 2 we apply the transfer matrix method as given by Felderhof [7] to the electron barrier model (2). Section 3 is devoted to the model of the $L-C$ line with barriers. In order to show the close analogy between the two models, the transfer matrix for the square well potential is derived in the appendix.

2 - Localization and cross-over in the electron barrier model

We write the wave function between the barriers v_j and v_{j+1} as

$$\psi_j(x) = a_j e^{ikx} + b_j e^{-ikx}, j = o, 1...N \tag{12}$$

where $a_o = 1$ and $b_N = 0$. Then ψ_o describes the incoming and reflected waves and ψ_N the outgoing wave and the reflection and transmission coefficients are

$$R = |b_o|^2,\ T = |a_N|^2. \tag{13}$$

with $R + T = 1$. The transfer matrix equation then reads [7] (see the appendix)

$$\begin{pmatrix} a_j \\ b_j \end{pmatrix} = U_k^+(x_j) M_j U_k(x_j) \begin{pmatrix} a_{j-1} \\ b_{j-1} \end{pmatrix}. \tag{14}$$

Here

$$U_k(x) = \begin{pmatrix} e^{ikx} & 0 \\ 0 & e^{-ikx} \end{pmatrix} \tag{15}$$

and

$$M_j = \begin{pmatrix} \alpha_j & \beta_j \\ \beta_j^* & \alpha_j^* \end{pmatrix} \tag{16}$$

is the transfer matrix whose coefficients are

$$\alpha_j - 1 = \beta_j = \frac{i}{2k} \gamma_j v_j. \tag{17}$$

Making use of Eqs.(12)-(14),(3) and condition (8) one finds [7]

$$< \frac{R}{T} >_N = \frac{1}{2} (< \epsilon >_N - 1) \tag{18}$$

with

$$< \epsilon >_N = \prod_{j=1}^{N} < |\alpha_j|^2 + |\beta_j|^2 > . \tag{19}$$

Inserting Eq.(17) then leads to

$$\log < \epsilon >_N = \sum_{j=1}^{N} \log(1 + \frac{\gamma_j^2 \sigma^2}{2k^2}) \tag{20}$$

where, as mentioned after Eq.(3), $\sigma^2 \equiv < v_j^2 >$ is independent of j.

From Eq.(20) it is evident that when $\gamma_j = 1$, the average of the resistance ρ defined in Eq.(1) diverges exponentially with N. This becomes explicit if, in addition to condition (8) we also require condition (11), in which case Eq.(20) simplifies to

$$\log < \epsilon >_N \cong \frac{\sigma^2}{2k^2} \sum_{j=1}^{N} \gamma_j^2. \tag{21}$$

In particular, with the modulation (9) the following 3 cases of Eq.(21) may be distinguished:

a) $0 \leq \nu < 1$:

$$\log < \epsilon >_N \cong \frac{\sigma^2}{2k^2} \frac{N^{1-\nu}}{1-\nu} \left(1 + O(N^{-1})\right). \tag{22}$$

This implies

$$< \rho >_N \propto \exp\left(\frac{\sigma^2}{2k^2(1-\nu)} N^{1-\nu}\right) \tag{23}$$

which in the limit $N \to \infty$ corresponds to the insulating regime (strong localization).

b) $\nu = 1$:

$$\log < \epsilon >_N \cong \frac{\sigma^2}{2k^2} \left(C + \log N + O(N^{-1})\right), \tag{24}$$

where C is Euler's constant. Hence

$$< \rho >_N \propto N^{\sigma^2/2k^2} \tag{25}$$

which for $N \to \infty$ is algebraically divergent (weak localization).

c) $\nu > 1$: In this case Eq.(10) may be applied in the limit $N \to \infty$ so that

$$\log < \epsilon >_N \cong \frac{\sigma^2}{2k^2} \zeta(\nu) + O(N^{1-\nu}) \tag{26}$$

and $< \rho >_N$ is finite, corresponding to the metallic regime.

As usual, the relative fluctuation $\Delta \equiv ((\langle R^2/T^2 \rangle_N / \langle R/T \rangle_N{}^2) - 1)^{1/2}$ should also be calculated [7]. Making use of Eqs.(9) and (17) one finds that in the insulating case, $\nu \geq 1$, $\Delta \to \infty$ as usual, while in the metallic case, $\nu > 1$, Δ stays finite when $N \to \infty$.

3 - Localization in the $L - C$ line with barriers

Inserting the excess capacitance C/v_j in series at the random position n_j, $j = 1, 2, \ldots N$, but such that $n_j - n_{j-1} \gg 1$, modifies the cirquit equation (4) at the position $l \equiv n_j$ as follows

$$\left(\frac{\omega_j^2}{\omega^2} - 2\right) Q_l + Q_{l+1} + Q_{l-1} = 0, \tag{27}$$

where $\omega_j^2 = (1 + v_j)\omega_o^2$. The solutions of Eq.(27) and of Eq.(4), which is valid for $n \geq l$ and $n \leq l$, may be written in a form analogous to the wave functions (12) and (A1), respectively as

$$Q_n = \alpha e^{i\kappa n} + \beta e^{-i\kappa n}, n = l, l \pm 1,$$

$$Q_n = a_\pm e^{ikn} + b_\pm e^{-ikn}, n = l, l \pm 1. \tag{28}$$

Here the reality conditions $\beta = \alpha^*, b_\pm = a_\pm^*$ obviously hold and κ is defined in analogy with Eq.(6) by

$$4 \sin^2 \frac{\kappa}{2} = \frac{\omega_j^2}{\omega^2}. \tag{29}$$

Eqs.(28) imply two pairs of identities between Q_l and $Q_{l\pm1}$ which are the analogues of the matching conditions (A3) and may be expressed in a matrix form analogous to Eq.(A4),

$$\begin{pmatrix} Q_{l-1} \\ Q_l \end{pmatrix} = m_k U_k(l-1) \begin{pmatrix} a_- \\ b_- \end{pmatrix} = m_\kappa U_\kappa(l-1) \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

$$\begin{pmatrix} Q_l \\ Q_{l+1} \end{pmatrix} = m_k U_k(l) \begin{pmatrix} a_+ \\ b_+ \end{pmatrix} = m_\kappa U_\kappa(l) \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{30}$$

Here

$$m_k = \begin{pmatrix} 1 & 1 \\ e^{ik} & e^{-ik} \end{pmatrix}. \tag{31}$$

Elimination of α and β from Eqs.(30) leads to an equation of the form (14) with a transfer matrix given by Eqs.(A6) and (16) where now, instead of Eq.(17),

$$\alpha_j - 1 = \beta_j = iv_j \frac{1 - \cos k}{\sin k}. \tag{32}$$

Finally, we would like to point out that the $L - C$ line discussed in this section is nothing else than the original Anderson model [8] which here was solved with the method of Felderhof [7].

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Appendix: Transfer matrix for square well potential

Writing the wave function in the interior $0 < x < a$ of the barrier with width a and height V/a as

$$\phi(x) = \alpha e^{i\kappa x} + \beta e^{-i\kappa x}, \tag{A1}$$

where

$$\kappa^2 = k^2 - \frac{V}{a}, \tag{A2}$$

matching onto the exterior solutions (12) implies the conditions

$$\psi_{j-1}(x_j) = \phi(x_j),\ \psi'_{j-1}(x_j) = \phi'(x_j),$$

$$\psi_j(x_j + a) = \phi(x_j + a),\ \psi'_j(x_j + a) = \phi'(x_j + a). \qquad (A3)$$

These conditions may be cast into the matrix form

$$m_k U_k(x_j)\begin{pmatrix} a_{j-1} \\ b_{j-1} \end{pmatrix} = m_\kappa U_\kappa(x_j)\begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

$$m_k U_k(x_j + a)\begin{pmatrix} a_j \\ b_j \end{pmatrix} = m_\kappa U_\kappa(x_j + a)\begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \qquad (A4)$$

Elimination of the amplitudes α and β from the last two equations then leads to the transfer matrix equation analogous to (14)

$$\begin{pmatrix} a_j \\ b_j \end{pmatrix} = U_k^+(x_j + a) M_j U_k(x_j)\begin{pmatrix} a_{j-1} \\ b_{j-1} \end{pmatrix}, \qquad (A5)$$

where

$$M_j = m_k^{-1} m_\kappa U_\kappa(a) m_\kappa^{-1} m_k. \qquad (A6)$$

Evaluation of the last expression yields a transfer matrix which has the general form given by Erdös and Herndon [9],

$$M_j = \begin{pmatrix} \exp(-i\gamma)\cosh\lambda & -i\sinh\lambda \\ i\sinh\lambda & \exp i\gamma\cosh\lambda \end{pmatrix}, \qquad (A7)$$

where

$$\sinh\lambda = \frac{k^2 - \kappa^2}{2k\kappa}\sin\kappa a,$$

$$\tan\gamma = -\frac{k^2 + \kappa^2}{2k\kappa}\tan\kappa a. \qquad (A8)$$

In view of Eq.(A2) the limit $a \to 0$ of Eq.(A7) correctly yields Eqs.(16),(17) if V is replaced by $\gamma_j v_j$.

It is interesting to note that the general form (A7) of the transfer matrix satisfies time-reversal invariance of the ψ_j [9] which is expressed by the form (16). The additional symmetry contained in Eq.(A7) is conservation of the probability current $i(\psi_j \psi_j'^* - \psi_j^* \psi')$ [9] which, combined with time-reversal invariance leads to unimodularity [9],

$$\det M_j = 1. \qquad (A9)$$

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