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Some Applications of the 3 + 1 Formalism of General Relativity

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Abstract. A variety of basic formulae of general interest are derived in a uniform manner within the 3 + 1 formalism of GR. These include: (i) The 3 + 1 split of electrodynamics in terms of differential forms. (ii) Convenient forms of the Liouville operator for a geodesic spray. (iii) The 3 + 1 split of hydrodynamics. (iv) Applications to spherical collapse and inhomogeneous inflation. (v) Gauge invariant cosmological perturbation theory. Many of the resulting equations are known, but some are new. Numerical solutions for the evolution of cosmological perturbations and inhomogeneous inflationary models are deferred to future publications.

1. Introduction

The importance of the 3 + 1 formulation of general relativity is well known. It plays a crucial role in basic theoretical investigations and is also very useful for more practical purposes. The 3 + 1 or dynamical approach is, for instance, used in discussions of the initial value problem [1], [2] and in numerical relativity [3]. It is directly related to the Hamiltonian formulation of geometrodynamics, which was originally introduced as a tool for quantizing general relativity. (For a description of these roots see Ref. [4].) More recently, the 3 + 1 formalism has become important as a technique for studying astrophysical processes around black holes. This viewpoint is called the 'membrane paradigm' and has been fully described in a recent book [5].

It is the purpose of this paper to systematize and extend the 3 + 1 formalism in various directions. We derive in detail and in a uniform manner a variety of basic formulae of general interest. Many of the resulting equations are known but some are new. Even in cases where we obtain known results, we hope that our derivations will be useful for other workers, since we arrive at them in a smooth manner, making full use of the Cartan calculus. A large portion of the paper is devoted to cosmological perturbation theory. We have solved the resulting equations numerically for various scenarios. This part of our work will be presented in a separate publication [7]. Numerical investigations of inhomogeneous inflationary models are also under way.

In Section 2 we recall the general setup of the 3 + 1 formalism. With the Cartan calculus we obtain in Section 3 the 3 + 1 split of electrodynamics in a very elegant manner for an arbitrary spacetime background. The form of the 3 + 1 split of the Liouville operator for a geodesic spray given in Section 4 is new. Section 5 treats the 3 + 1 split of hydrodynamics and in Section 6 (and Appendix A) we recall for later use the 3 + 1 split of Einstein's field equations.

Specializations to spherically symmetric collapse and inhomogeneous inflation are carried out in Section 7 for a gauge satisfying the maximal slicing condition. Section 8 is devoted to the gauge invariant cosmological perturbation theory. Here, we give first alternative derivations of known results [8], [9]. With the help of the 3 + 1 formalism we arrive much faster at the final equations. We also add some new results. In particular we develop the gauge-invariant perturbation theory of collisionless particles. Furthermore, the entropy production rate of the perturbation is related generally to a certain gauge invariant matter amplitude.

2. 3 + 1 formalism, generalities

We assume that spacetime $(\mathcal{M}, g)$ admits a slicing by slices Σ_t, i.e., there is a diffeomorphism $\phi: \mathcal{M} \to \Sigma \times I$, $I \subset \mathbf{R}$, such that the manifolds $\Sigma_t = \phi^{-1}(\Sigma \times \{t\})$ are spacelike and the curves $\phi^{-1}(\{x\} \times I)$ are timelike. These curves are what we call *preferred* timelike curves. They define a vector field ∂_t, which can be decomposed into normal and parallel components relative to the slicing (Fig. 1):

$$\partial_t = \alpha n + \boldsymbol{\beta}. \tag{2.1}$$

Here n is a unit normal field and $\boldsymbol{\beta}$ is tangent to the slices Σ_t. α is the *lapse function* and $\boldsymbol{\beta}$ the *shift vector* field [1, 2]. A coordinate system $\{x^i\}$ on Σ induces natural coordinates on $\mathcal{M}$: $\phi^{-1}(m, t)$ has coordinates (t, x^i) if $m \in \Sigma$ has coordinates x^i. The preferred timelike curves have constant spatial coordinates. Let us set $\boldsymbol{\beta} = \beta^i\, \partial_i (\partial_i = \partial / \partial x^i)$. From $g(n, \partial_i) = 0$ and (2.1) we find

$$g(\partial_t, \partial_t) = -(\alpha^2 - \beta^i \beta_i),\ g(\partial_t, \partial_i) = \beta_i.$$

Thus in 'comoving coordinates'

$$g = -(\alpha^2 - \beta^i \beta_i)\, dt^2 + 2\beta_i\, dx^i\, dt + g_{ij}\, dx^i\, dx^j \tag{2.2}$$

or

$$g = -\alpha^2\, dt^2 + g_{ij}(dx^i + \beta^i\, dt)(dx^j + \beta^j\, dt). \tag{2.3}$$

This shows that the forms dt and $dx^i + \beta^i\, dt$ are orthogonal.

The tangent and cotangent spaces of $\mathcal{M}$ have two natural decompositions. One is defined by the slicing

$$T_p(\mathcal{M}) = H_p \oplus V_p, \tag{2.4}$$

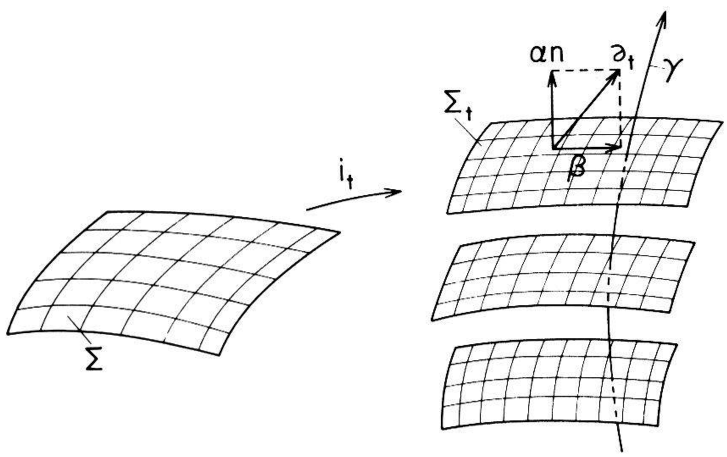

Figure 1.

where the 'horizontal' space H_p consists of the vectors tangent to the slice through p and the 'vertical' subspace is the 1-dimensional space spanned by $(\partial_t)_p$ (preferred direction). The dual decomposition of (2.4) is

$$T_p^*(\mathcal{M}) = H_p^* \oplus V_p^*, \tag{2.5}$$

with $H_p^* = \{\omega \in T_p^*(\mathcal{M}) : \langle \omega, \partial_t \rangle = 0\}$ and $V_p^* = \{\omega \in T_p^*(\mathcal{M}) : \langle \omega, H_p \rangle = 0\}$, which is spanned by $(dt)_p$.

The metric defines – through the normal field n – yet another decomposition

$$T_p(\mathcal{M}) = H_p \oplus H_p^{\perp}, \tag{2.6}$$

where $H_p^{\perp}$ is spanned by n, and dually

$$T_p^*(\mathcal{M}) = (V_p^*)^{\perp} \oplus V_p^*. \tag{2.7}$$

Equation (2.1) reflects the fact that, in general, the two directions V_p and $H_p^{\perp}$ do not agree. Dually this implies that H_p^* and $(V_p^*)^{\perp}$ do not coincide. We have for $\omega^{\perp} \in (V_p^*)^{\perp}$ the following decomposition relative to (2.5)

$$\omega^{\perp} = \mathrm{hor}\,(\omega^{\perp}) + \langle \omega^{\perp}, \boldsymbol{\beta} \rangle\, dt. \tag{2.8}$$

The decompositions (2.4) to (2.7) induce two types of decompositions of arbitrary tensor fields on $\mathcal{M}$. We call a tensor field *horizontal* if it vanishes, whenever at least one argument is ∂_t or dt. Relative to a comoving coordinate system such a tensor has the form

$$\mathbf{S} = S^{i_1 \cdots i_r}_{j_1 \cdots j_s} \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}.$$

This shows that horizontal tensor fields can naturally be identified with families of tensor fields on Σ_t, or with time-dependent tensor fields on Σ ('absolute' space). We shall denote them with boldface letters (except ∂_i and dx^i).

As an often occurring example of a decomposition, we consider a horizontal p-form $\boldsymbol{\omega}$ and its exterior derivative $d\boldsymbol{\omega}$. We have

$$d\boldsymbol{\omega} = \mathbf{d}\boldsymbol{\omega} + dt \wedge \partial_t \boldsymbol{\omega},$$

where $\mathbf{d}\boldsymbol{\omega}$ is again horizontal. In comoving coordinates $\mathbf{d}$ involves only the dx^i ($\mathbf{d} = dx^i \wedge \partial_i$) and $\partial_t \boldsymbol{\omega}$ is the partial time derivative. $\mathbf{d}\boldsymbol{\omega}$ and $\partial_t \boldsymbol{\omega}$ are horizontal and can be interpreted as t-dependent forms on Σ. In this interpretation $\mathbf{d}\boldsymbol{\omega}$ is just the exterior derivative of $\boldsymbol{\omega}$. Similarly, other differential operators (covariant derivative, Lie derivative, etc) can be decomposed. We use two types of bases of vector fields and 1-forms which are adapted to (2.4) and (2.5), respectively (2.6) and (2.7). Obviously, the dual pair $\{\partial_\mu\}$ and $\{dx^\mu\}$ for comoving coordinates $\{x^\mu\}$ are adapted to (2.4) and (2.5). On the other hand, equations (2.1) and (2.3) show that the dual pair

$$\{\partial_i, n\} \quad \text{and} \quad \{dx^i + \beta^i\, dt, \alpha\, dt\} \tag{2.9}$$

is adapted to (2.6) and (2.7).

Instead of $\{\partial_i\}$ we shall also use an orthonormal horizontal basis

$\{\mathbf{e}_i\}(g(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij})$, together with the dual basis $\{\boldsymbol{\vartheta}^i\}$ instead of $\{dx^i\}$. Then we have the following two dual pairs, which will be constantly used:

$$\{\mathbf{e}_i, \partial_t\}, \qquad \{\boldsymbol{\vartheta}^i, dt\} \qquad \text{(adapted to slicing)}, \tag{2.10}$$

$$\{\mathbf{e}_i, e_0 = n\}, \qquad \{\theta^\mu\} \qquad \text{(adapted to (2.6) and (2.7))}, \tag{2.11}$$

where the orthonormal tetrad $\{\theta^\mu\}$ is given by

$$\theta^0 = \alpha\, dt, \qquad \theta^i = \boldsymbol{\vartheta}^i + \beta^i\, dt \tag{2.12}$$

with β^i defined by $\boldsymbol{\beta} = \beta^i \mathbf{e}_i$. We note also the relation

$$e_0 = n = \frac{1}{\alpha}(\partial_t - \beta^i \mathbf{e}_i). \tag{2.13}$$

3. 3 + 1 split of electrodynamics

As a first example of a 3 + 1 split of physical laws we rewrite now Maxwell's equations on $(\mathcal{M}, g)$ in terms of horizontal differential forms for electric and magnetic fields. The resulting equations have been widely used in black-hole electrodynamics [5]. The following derivation is, however, valid for any slicing of an arbitrary spacetime. Using the general setup of Section 2, we first decompose the Maxwell form F as in special relativity with respect to the orthonormal tetrad (2.11):

$$F = E \wedge \theta^0 + B,$$
$$E = E_i \theta^i, \qquad B = \tfrac{1}{2} B_{ij} \theta^i \wedge \theta^j. \tag{3.1}$$

Next we use (2.12) for a further decomposition into horizontal and vertical contributions:

$$E = E_i(\boldsymbol{\vartheta}^i + \beta^i\, dt) = \mathscr{E} + \mathbf{i}_{\boldsymbol{\beta}} \mathscr{E}\, dt, \tag{3.2}$$

where

$$\mathscr{E} = E_i \boldsymbol{\vartheta}^i \tag{3.3}$$

is the horizontal electric 1-form. Similarly, we have

$$B = \mathscr{B} + dt \wedge \mathbf{i}_r \mathscr{B} \tag{3.4}$$

with

$$\mathscr{B} = \tfrac{1}{2} B_{ij} \boldsymbol{\vartheta}^i \wedge \boldsymbol{\vartheta}^j. \tag{3.5}$$

We obtain thus the following decomposition of F into horizontal and vertical components

$$F = \mathscr{B} + (\alpha \mathscr{E} - \mathbf{i}_\beta \mathscr{B}) \wedge dt. \tag{3.6}$$

The homogeneous Maxwell equation $dF = 0$ gives

$$\mathbf{d}\mathcal{B} + dt \wedge \partial_t \mathcal{B} + \mathbf{d}(\alpha\mathcal{E}) \wedge dt - \mathbf{d}(\mathbf{i}_{\boldsymbol{\beta}}\mathcal{B}) \wedge dt = 0$$

and thus

$$\mathbf{d}\mathcal{B} = 0, \qquad \mathbf{d}(\alpha\mathcal{E}) + \partial_t \mathcal{B} = \mathbf{d}(\mathbf{i}_{\boldsymbol{\beta}}\mathcal{B}). \tag{3.7}$$

Here we have everywhere horizontal forms. We reemphasize that they can be interpreted as time-dependent forms on Σ. Using the Cartan formula for the Lie derivative $\mathbf{L}_{\boldsymbol{\beta}} = \mathbf{d}\mathbf{i}_{\boldsymbol{\beta}} + \mathbf{i}_{\boldsymbol{\beta}}\mathbf{d}$, they can also be written as

$$\mathbf{d}\mathcal{B} = 0, \quad \mathbf{d}(\alpha\mathcal{E}) + \partial_t \mathcal{B} = \mathbf{L}_{\boldsymbol{\beta}}\mathcal{B}. \tag{3.8}$$

Next we consider the inhomogeneous Maxwell equation. From (3.6) we find for the Hodge dual $*F$:

$$*F = \mathcal{D} - (\alpha\mathcal{H} + \mathbf{i}_{\boldsymbol{\beta}}\mathcal{D}) \wedge dt \tag{3.9}$$

with

$$\mathcal{H} = \ast\mathcal{B}, \qquad \mathcal{E} = \ast\mathcal{D} \tag{3.10}$$

($\ast$ denotes the 3-dimensional Hodge dual). Now we also decompose the current 3-form $\mathcal{S}$ in $d * F = 4\pi\mathcal{S}$. Its dual 1-form $J = *\mathcal{S}$ can first be decomposed as in special relativity

$$J = \rho_{el}\theta^0 + j_k\theta^k, \tag{3.11}$$

where ρ_{el} is the charge density and j^k the electric current relative to the tetrad $\{\mathbf{e}_k\}$. It is natural to introduce the horizontal forms

$$\boldsymbol{\rho} = \rho_{el}\boldsymbol{\vartheta}^1 \wedge \boldsymbol{\vartheta}^2 \wedge \boldsymbol{\vartheta}^3, \qquad \mathbf{j} = j_k\boldsymbol{\vartheta}^k, \; \mathcal{J} = \ast\mathbf{j}. \tag{3.12}$$

Then one finds with (2.12) easily

$$\mathcal{S} = \boldsymbol{\rho} + (\mathbf{i}_{\boldsymbol{\beta}}\boldsymbol{\rho} - \alpha\mathcal{J}) \wedge dt. \tag{3.13}$$

With this and (3.9) the inhomogeneous Maxwell equation can be written as

$$\begin{aligned} d*F &= \mathbf{d}\mathcal{D} + dt \wedge \partial_t\mathcal{D} - \mathbf{d}(\alpha\mathcal{H}) \wedge dt - \mathbf{d}(\mathbf{i}_{\boldsymbol{\beta}}\mathcal{D}) \wedge dt \\ &= 4\pi\boldsymbol{\rho} + 4\pi(\mathbf{i}_{\boldsymbol{\beta}}\boldsymbol{\rho} - \alpha\mathcal{J}) \wedge dt. \end{aligned}$$

From this we find the following $3+1$ split of the inhomogeneous Maxwell equation

$$\mathbf{d}\mathcal{D} = 4\pi\boldsymbol{\rho}, \; \mathbf{d}(\alpha\mathcal{H}) = (\partial_t - \mathbf{L}_{\boldsymbol{\beta}})\mathcal{D} + 4\pi\alpha\mathcal{J}. \tag{3.14}$$

From (3.14) we obtain immediately the local law of charge conservation in the form

$$(\partial_t - \mathbf{L}_{\boldsymbol{\beta}})\boldsymbol{\rho} + \mathbf{d}(\alpha\mathcal{J}) = 0. \tag{3.15}$$

We write also the Hodge-dual of this equation. Using equation (A.9) of

Appendix A, i.e.,

$$\partial_t \operatorname{vol}(\mathbf{g}) = (\mathbf{div}\, \boldsymbol{\beta} - \alpha \operatorname{tr} \mathbf{K}) \operatorname{vol}(\mathbf{g}), \tag{3.16}$$

where $\mathbf{K}$ denotes the second fundamental form of the slices, we obtain

$$(\partial_t - \mathbf{L}_{\boldsymbol{\beta}})\rho_{el} + \boldsymbol{\nabla} \cdot (\alpha * \mathcal{J}) - \alpha \rho_{el} \operatorname{tr}(\mathbf{K}) = 0. \tag{3.17}$$

4. 3 + 1 split of the Liouville operator for a geodesic spray

In this section we derive a useful form of the Liouville operator for a geodesic spray for an arbitrary 3 + 1 split. In later sections this will be worked out for spherical collapse and in a gauge-invariant manner in cosmological perturbation theory.

We start with some generalities. The metric g of the spacetime manifold $\mathcal{M}$ defines a natural diffeomorphism between the tangent bundle $T\mathcal{M}$ and the cotangent bundle $T^*\mathcal{M}$, which can be used to pull back the natural symplectic form on $T^*\mathcal{M}$. In terms of natural bundle coordinates the diffeomorphism is given by $(x^\mu, p^\mu) \mapsto (x^\mu, p_\mu = g_{\mu\nu} p^\nu)$ and thus the induced symplectic 2-form on $T\mathcal{M}$ is

$$\omega = dx^\mu \wedge d(g_{\mu\nu} p^\nu). \tag{4.1}$$

The Lagrangian $L = \frac{1}{2} g_{\mu\nu} p^\mu p^\nu$ on $T\mathcal{M}$ defines a Hamiltonian vector field X_g on $T\mathcal{M}$, determined by

$$i_{X_g}\omega = dL.$$

In terms of natural bundle coordinates the geodesic spray X_g is given by

$$X_g = (p^\mu, -\Gamma^\mu_{\alpha\beta} p^\alpha p^\beta), \tag{4.2}$$

where $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols for $(\mathcal{M}, g)$. (For further details see [10].)

The one-particle phase space for particles of mass m, i.e., the submanifold $\{v \in T\mathcal{M} : g(v, v) = -m^2\}$, is invariant under the geodesic flow and we denote the restriction of X_g to the one-particle phase space also by X_g.

Let f be a distribution function on the one-particle phase space. The Vlasov and Boltzmann equations for f involve the Lie derivative $L_{X_g} f$. If we consider the spatial components p^i, relative to an orthonormal tetrad $\{e^\mu\}$ as independent variables of f, then the Liouville operator L_{X_g} can be written as

$$L_{X_g} f = p^\mu e_\mu(f) - \omega^i_\alpha(p) p^\alpha \frac{\partial f}{\partial p^i}, \tag{4.3}$$

where ω^μ_ν are the connection forms of $(\mathcal{M}, g)$ relative to the dual basis $\{\theta^\mu\}$.

We derive now a more explicit expression of (4.3) for an arbitrary 3 + 1 slicing. In order to do this, we need the connection forms relative to the basis $\{\theta^\mu\}$ introduced in Section 2. These are derived in detail in Appendix A. They can be expressed in terms of α, $\boldsymbol{\beta}$, $\boldsymbol{\omega}^i_j$, c^i_j, where ω^i_j are the connection forms of the slices Σ_t belonging to the induced metric $\mathbf{g}$ and c^i_j is defined by

$$\partial_t \vartheta^i = c^i_j \vartheta^j. \tag{4.4}$$

Using equations (A.5), (A.3), (A.6) and (A.2) we find ($\mathbf{p} = p^i \mathbf{e}_i$, $E = p^0 = \sqrt{\mathbf{p}^2 + m^2}$):

$$\begin{aligned}
\omega^i_\alpha(p)p^\alpha \frac{\partial}{\partial p^i} &= \omega^i_0(p)p^0 \frac{\partial}{\partial p^i} + \omega^i_j(p)p^j \frac{\partial}{\partial p^i} \\
&= [\omega^i_0(e^0)p^0 + \omega^i_0(\mathbf{p})]p^0 \frac{\partial}{\partial p^i} + [\omega^i_j(e^0)p^0 + \omega^i_j(\mathbf{p})]p^j \frac{\partial}{\partial p^i} \\
&= E^2\alpha^{-1}\alpha^{|i} \frac{\partial}{\partial p^i} - K^i_j E p^j \frac{\partial}{\partial p^i} + \boldsymbol{\omega}^i_j(\mathbf{p})p^j \frac{\partial}{\partial p^i} + \omega^i_j(e^0)Ep^j \frac{\partial}{\partial p^i} \\
&= E^2\alpha^{-1}\alpha^{|i} \frac{\partial}{\partial p^i} + \boldsymbol{\omega}^i_j(\mathbf{p} - \alpha^{-1}\boldsymbol{\beta}E)p^j \frac{\partial}{\partial p^i} - \frac{E}{\alpha}(\beta^{|i}_j - c^i_j)p^j \frac{\partial}{\partial p^i}.
\end{aligned}$$

Here K^i_j are the components of the second fundamental form of Σ_t, for which we also use equation (A.7) of Appendix A. ($_|$ denotes the covariant derivative on $(\Sigma_t, \mathbf{g}_t)$).

This leads to the following useful 3 + 1 split of the Liouville operator:

$$L_{X_g} f = \left[\frac{E}{\alpha}\partial_t + \mathbf{L}_{\mathbf{p} - \frac{E}{\alpha}\boldsymbol{\beta}}\right] f - \left[\boldsymbol{\omega}^i_j\left(\mathbf{p} - \frac{E}{\alpha}\boldsymbol{\beta}\right)p^j + E^2(\ln\alpha)^{|i} - EH^i_j p^j\right]\frac{\partial f}{\partial p^i}, \tag{4.5}$$

where we have introduced the horizontal tensor field

$$H^i_j = \alpha^{-1}(\beta^i_j - c^i_j). \tag{4.6}$$

5. 3 + 1 split of hydrodynamics. Equation of motion for a test particle

Calculations similar to those in the last section lead quite rapidly to a 3 + 1 split of hydrodynamics. The resulting equations have been used, for instance, in black-hole physics [5]. We shall apply them in Section 8 on cosmological perturbation theory.

Let us decompose the energy-momentum tensor into horizontal and vertical components:

$$T = \epsilon e_0 \otimes e_0 + e_0 \otimes \mathbf{S} + \mathbf{S} \otimes e_0 + \mathbf{T}. \tag{5.1}$$

For an ideal fluid with

$$T = (\rho + p)u \otimes u + pg^\# \tag{5.2}$$

we find, setting as in special relativity $u = \gamma(e_0 + \mathbf{v})$, $\gamma = (1 - \mathbf{v}^2)^{-1/2}$,

$$\epsilon = \gamma^2(\rho + p\mathbf{v}^2), \tag{5.3}$$

$$\mathbf{S} = (\rho + p)\gamma^2\mathbf{v}, \tag{5.4}$$

$$\mathbf{T} = (\rho + p)\gamma^2\mathbf{v} \otimes \mathbf{v} + p\mathbf{g}^\#. \tag{5.5}$$

Now we compute $\nabla \cdot T$ for an arbitrary T. From

$$\nabla_{e_0}(\epsilon e_0 \otimes e_0) = L_{e_0}(\epsilon) e_0 \otimes e_0 + \epsilon \omega^i_0(e_0) \mathbf{e}_i \otimes e_0 + \epsilon e_0 \otimes \omega^i_0(e_0) \mathbf{e}_i$$

and

$$\nabla_{\mathbf{e}_k}(\epsilon e_0 \otimes e_0) = L_{\mathbf{e}_k}(\epsilon) e_0 \otimes e_0 + \epsilon \omega^i_0(\mathbf{e}_k) \mathbf{e}_i \otimes e_0 + \epsilon e_0 \otimes \omega^i_0(\mathbf{e}_k) \mathbf{e}_i$$

we obtain

$$\nabla \cdot (\epsilon e_0 \otimes e_0) L_{e_0}(\epsilon) e_0 + \epsilon \omega^i_0(e_0) \mathbf{e}_i + \epsilon \omega^i_0(\mathbf{e}_i) e_0.$$

In the same manner one finds the other contributions with the result:

$$(\nabla \cdot T)^0 = L_{e_0}(\epsilon) + \epsilon \omega^i_0(\mathbf{e}_i) + \omega^0_j(e_0) S^j + S^k_{|k} + \omega^0_j(e_0) S^j + \omega^0_j(\mathbf{e}_i) T^{ij}.$$

Inserting the expressions for the connection forms given in Appendix A, leads to the following form of the energy conservation:

$$\frac{1}{\alpha}(\partial_t - \mathbf{L}_{\boldsymbol{\beta}})\epsilon = -\boldsymbol{\nabla} \cdot \mathbf{S} - 2\,\mathbf{grad}\,(\ln \alpha) \cdot \mathbf{S} + \epsilon\,\mathrm{tr}\,(\mathbf{K}) + \mathrm{tr}\,(\mathbf{K} \cdot \mathbf{T}). \tag{5.6}$$

Similarly one finds

$$(\nabla \cdot T)^i = \omega^i_0(e_0)\epsilon + L_{e_0}(S^i) + [\omega^i_0(\mathbf{e}_j) + \omega^i_j(e_0)] S^j + \omega^j_0(\mathbf{e}_j) S^i + \omega^0_j(e_0) T^{ji} + T^{ij}_{|j}$$

and from this we obtain the momentum conservation equation

$$\frac{1}{\alpha}(\partial_t - \mathbf{L}_{\boldsymbol{\beta}})\mathbf{S} = -\mathbf{grad}\,(\ln \alpha)\epsilon + 2\mathbf{K} \cdot \mathbf{S} + \mathrm{tr}\,(\mathbf{K})\mathbf{S} - \alpha^{-1} \boldsymbol{\nabla} \cdot (\alpha \mathbf{T}). \tag{5.7}$$

This form of the conservation laws will turn out to be very convenient in our treatment of cosmological perturbation theory.

Some of the terms on the right hand side of (5.7) appear also in the equation of motion for a charged test particle. Indeed, the horizontal component of $\nabla_u p$, $p = mu$, is similarly found to be given by

$$\gamma^{-1}(\nabla_u p)^{\mathrm{hor}} = \frac{1}{\alpha}\left(\frac{d}{dt} - \mathbf{L}_{\boldsymbol{\beta}}\right)\mathbf{p} + \boldsymbol{\nabla}_{\mathbf{v}}\mathbf{p} + m\gamma\,\mathbf{grad}\,(\ln \alpha) - 2\mathbf{K} \cdot \mathbf{p}, \tag{5.8}$$

where $\mathbf{p}$ is the horizontal part of p.

In order to work out the Lorentz equation

$$\nabla_u p = eF^{\#} \cdot u, \tag{5.9}$$

we use

$$(F^{\#} \cdot u)^{\flat} = -i_u F, \qquad u = \gamma(e_0 + \mathbf{v}) = \frac{\gamma}{\alpha}(\partial_t + \alpha \mathbf{v} - \boldsymbol{\beta})$$

and (3.6), giving us

$$i_u F = -\frac{\gamma}{\alpha}(\alpha \mathscr{E} - \mathbf{i}_{\boldsymbol{\beta}} \mathscr{B}) + \gamma \mathbf{i}_{\mathbf{v}} \mathscr{B} - \frac{\gamma}{\alpha} \mathbf{i}_{\boldsymbol{\beta}} \mathscr{B} + \text{terms prop. to } dt.$$

Thus

$$-(i_u F)^{\text{hor}} = \gamma(\mathscr{E} - \mathbf{i}_{\mathbf{v}}\mathscr{B}), \tag{5.10}$$

and hence

$$(F^{\#} \cdot u)^{\text{hor}} = \gamma(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B}), \tag{5.11}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic vector fields belonging to $\mathscr{E}$ and $\mathscr{B}$ (via the metric $\mathbf{g}$). Using (5.8) and (5.11) in (5.9) gives us the equation of motion in the form

$$\frac{1}{\alpha}\left(\frac{d}{dt} - \mathbf{L}_{\boldsymbol{\beta}}\right)\mathbf{p} = -m\gamma\, \mathbf{grad}\,(\ln \alpha) - \boldsymbol{\nabla}_{\mathbf{v}}\mathbf{p} + 2\mathbf{K} \cdot \mathbf{p} + e(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B}) \tag{5.12}$$

which shows some similarity of electromagnetic and gravitational 'forces' (relative to 'absolute' space).

Let us finally write the local baryon number conservation law

$$\nabla \cdot (\rho_0 u) = 0 \tag{5.13}$$

in a $3+1$ splitting. Here ρ_0 is the rest mass density of baryons and u the four-velocity of baryon number. With $u = \gamma(e_0 + \mathbf{v})$ and $\tilde{\rho} := \gamma\rho_0$, we find as above

$$\left(\frac{d}{dt} - \mathbf{L}_{\beta}\right)\tilde{\rho} = -\boldsymbol{\nabla} \cdot (\alpha\tilde{\rho}\mathbf{v}) + \alpha\tilde{\rho}\, \text{tr}\,(\mathbf{K}). \tag{5.14}$$

Note the similarity of this equation with (3.17).

6. $3+1$ split of Einstein's field equations

For the sake of completeness and for later applications we discuss also the often-used $3+1$ split of the gravitational field equations. The calculation of the curvature forms relative to the basis (2.12) is presented in Appendix A. The reader will note that Cartan's calculus leads rather quickly to the required results.

We use the notation introduced in the previous section (5.1) for the various projections of the energy-momentum tensor T into normal and horizontal components:

$$T = \epsilon e_0 \otimes e_0 + e_0 \otimes \mathbf{S} + \mathbf{S} \otimes e_0 + \mathbf{T}. \tag{6.1}$$

From equations (A.14), (A.16) and (A.17) of Appendix A for the Einstein and Ricci tensors, Einsteins field equations can be written in the form (recall that boldface letters always refer to the slices Σ_t):

$$\mathbf{R} + (\text{tr}\,\mathbf{K})^2 - \text{tr}\,\mathbf{K}^2 = 16\pi G\epsilon, \tag{6.2}$$

$$\boldsymbol{\nabla} \cdot \mathbf{K} - \boldsymbol{\nabla} \cdot \text{tr}\,(\mathbf{K}) = 8\pi G\mathbf{S}, \tag{6.3}$$

$$\partial_t \mathbf{K} = \mathbf{L}_{\boldsymbol{\beta}}\mathbf{K} - \mathbf{Hess}\,(\alpha) + \alpha[\text{Ric}\,(\mathbf{g}) - 2\mathbf{K} \cdot \mathbf{K} + (\text{tr}\,\mathbf{K})\mathbf{K} - 8\pi G\mathbf{T} - 4\pi G\mathbf{g}(\epsilon - \text{tr}\,\mathbf{T})]. \tag{6.4}$$

In addition to (6.2), (6.3) and (6.4) we have the following relation (Appendix A, equation (A.8)) between **g** *and the second fundamental form* **K**:

$$\partial_t \mathbf{g} = -2\alpha \mathbf{K} + \mathbf{L}_{\boldsymbol{\beta}} \mathbf{g}. \tag{6.5}$$

Note that this decomposition into constraint equations (6.2), (6.3) and dynamical equations (6.4), (6.5) involves only horizontal quantities and thus provides the 3 + 1 split of the gravitational field equations.

Later we shall also use the following consequence of (6.4) and (6.2)

$$\partial_t \operatorname{tr}(\mathbf{K}) = -\Delta\alpha + \mathbf{L}_{\boldsymbol{\beta}} \operatorname{tr}(\mathbf{K}) + \alpha[\operatorname{tr}(\mathbf{K}^2) + 4\pi G(\epsilon + \operatorname{tr}\mathbf{T})]. \tag{6.6}$$

Note that ∂_t and tr do not commute. With (6.5) one shows easily

$$\operatorname{tr}(\partial_t \mathbf{K} - \mathbf{L}_{\boldsymbol{\beta}} \mathbf{K}) = \partial_t \operatorname{tr}(\mathbf{K}) - \mathbf{L}_{\boldsymbol{\beta}} \operatorname{tr}(\mathbf{K}) - 2\alpha \operatorname{tr}(\mathbf{K}^2). \tag{6.7}$$

7. Applications to spherically symmetric spacetimes

We now specialize the previous results to spherically symmetric spacetimes. This will lead to useful expressions for relativistic collapse problems or inhomogeneous inflationary models.

The shift vector $\boldsymbol{\beta}$ has only a radial component: $\boldsymbol{\beta} = \beta\partial_r$ and the metric (2.2) can be put into the form

$$g = -(\alpha^2 - A^2\beta^2)\, dt^2 + 2A^2\beta\, dr\, dt + A^2\, dr^2 + B^2r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{7.1}$$

where A and B are two functions of r and t.

7.1. *Field equations for maximal slicing*

For a spherically symmetric configuration the gravitational field is not dynamical (no radiation). Therefore, it must be possible to find a gauge in which the metric variables are determined *instantaneously* in terms of the matter variables. Such a gauge can be constructed as follows.

First we can arrange β such that $A = B$, which means that the metric **g** on the slices Σ_t is conformally flat. Next we restrict α by imposing the 'maximal slicing' condition [2] $\operatorname{tr}\mathbf{K} = 0$ for all times. It can be shown that this is indeed possible, because the dynamical equations guarantee that this condition propagates. An obvious advantage can be seen from (6.6) which then becomes a time-independent equation:

$$\Delta\alpha = \alpha[\operatorname{tr}(\mathbf{K}^2) + \tfrac{1}{2}(\epsilon + \operatorname{tr}\mathbf{T})]. \tag{7.2}$$

(In this section we set $8\pi G = 1$.) Now we work out the expression (A.7) of Appendix A for the second fundamental form relative to an orthonormal basis, which we repeat here:

$$K_{ij} = \frac{1}{2\alpha}(\beta_{i|j} + \beta_{j|i} - c_{ij} - c_{ji}). \tag{7.3}$$

In the present situation the basis $\{\boldsymbol{\vartheta}^i\}$ of Section 2 is

$$\boldsymbol{\vartheta}^1 = A\,dr, \qquad \boldsymbol{\vartheta}^2 = Ar\,d\theta, \qquad \boldsymbol{\vartheta}^3 = Ar\sin\theta\,d\phi. \tag{7.4}$$

Thus the coefficients c^i_j in (4.4) are

$$c^i_j = \frac{\dot{A}}{A}\,\delta^i_j \qquad \left(\dot{A} = \frac{\partial A}{\partial t}\right). \tag{7.5}$$

The connection forms $\boldsymbol{\omega}^i_j$ of the slices Σ_t are readily found to be (a prime denotes $\partial/\partial r$):

$$\boldsymbol{\omega}^2_1 = -\boldsymbol{\omega}^1_2 = \frac{(rA)'}{rA^2}\,\boldsymbol{\vartheta}^2,\ \boldsymbol{\omega}^3_1 = -\boldsymbol{\omega}^1_3 = \frac{(rA)'}{rA^2}\,\boldsymbol{\vartheta}^3. \tag{7.6}$$

From this one finds for

$$\beta_{j|i} = \mathbf{e}_i(\beta_j) - \boldsymbol{\omega}^k_j(\mathbf{e}_i)\beta_k$$

the expressions

$$\beta_{1|1} = \frac{(A\beta)'}{A}, \qquad \beta_{2|2} = \beta_{3|3} = \frac{(rA)'}{rA^2}\,A\beta, \qquad \beta_{i|j} = 0 \text{ for } i \neq j. \tag{7.7}$$

Thus $\operatorname{tr}\mathbf{K} = 0$ implies

$$(3/A)(A'\beta - \dot{A}) + 2\beta/r + \beta' = 0. \tag{7.8}$$

We note also

$$K^2_2 = K^3_3 = -\tfrac{1}{2}K^1_1, \qquad \operatorname{tr}(\mathbf{K}^2) = \tfrac{3}{2}(K^1_1)^2. \tag{7.9}$$

Furthermore we find

$$\begin{aligned}\alpha K_{11} &= \frac{(A\beta)'}{A} - \frac{\dot{A}}{A}\\ &= \tfrac{2}{3}(\beta' - \beta/r),\end{aligned} \tag{7.10}$$

and thus,

$$\beta = -\tfrac{3}{2}r\int_r^\infty \frac{\alpha K_{11}}{r}\,dr. \tag{7.11}$$

Next, we work out the constraint equations. First we need the Riemann scalar $\mathbf{R}$ for the spatial slices. Since these are conformally flat one can write down the result immediately

$$\mathbf{R} = -8r^{-2}A^{-5/2}\,\partial_r(r^2\,\partial_r A^{1/2}). \tag{7.12}$$

In addition, we need the covariant divergence of K^j_i (with respect to $\mathbf{g}$). The absolute exterior differential [11] is

$$\mathbf{D}K^i_j = \mathbf{d}K^i_j + \boldsymbol{\omega}^i_l K^l_j + \boldsymbol{\omega}^l_j K^i_l.$$

Inserting the connection forms (7.6) gives in particular

$$\nabla_j K^j_1 = r^{-2} A^{-4}\, \partial_r (r^3 A^3 K^1_1). \tag{7.13}$$

Now we write down the constraint equations (6.2) and (6.3):

$$r^{-2}\, \partial_r (r^2\, \partial_r A^{1/2}) = \tfrac{1}{4} A^{-5/2}[\epsilon + \tfrac{3}{4}(K^1_1)^2], \tag{7.14}$$

$$r^{-3} A^{-4}\, \partial_r (r^3 A^3 K^1_1) = S_1. \tag{7.15}$$

The explicit form of (7.2) becomes

$$A^{-3} r^{-2}\, \partial_r (r^2 A\, \partial_r \alpha) = \alpha[\tfrac{1}{2}(\epsilon + \mathrm{tr}\,\mathbf{T}) + \tfrac{3}{2}(K^1_1)^2]. \tag{7.16}$$

The last three equations, together with (7.10), provide four equations for the functions α, β, A, K^1_1 which contain only derivatives with respect to r! Thus the metric coefficients are instantaneously determined by the source terms ϵ, S_1 and $\mathrm{tr}\,\mathbf{T}$ and we have thus found the gauge we were looking for. (See also [13].)

For our choice of gauge ($A = B$, $\mathrm{tr}\,\mathbf{K} = 0$) numerical simulations simplify very much. Numerical studies are underway for spherically symmetric inhomogeneous inflationary models, where the source terms are given in terms of the energy momentum tensor of a Higgs field. The self-consistent set of equations contains then only one partial differential equation. For the Lagrangian

$$L_{\mathrm{Higgs}} = 1/2(d\phi, d\phi) - V(\phi) \tag{7.17}$$

the source terms are

$$\epsilon = 1/2\left[\left(\alpha^{-1}\dot{\phi} - \frac{\beta}{\alpha}\phi'\right)^2 + A^{-2}(\phi')^2\right] + V(\phi),$$

$$S_1 = -A^{-1}\phi'(\alpha^{-1}\dot{\phi} - \beta\phi'),$$

$$\mathrm{tr}\,\mathbf{T} = \tfrac{3}{2}\alpha^{-2}(\dot{\phi} - \beta\phi')^2 - \frac{1}{2A^2}(\phi')^2 - 3V(\phi).$$

The field equation

$$\Box_g \phi - \frac{\partial V}{\partial \phi} = 0$$

reads explicitly

$$-\alpha^{-1}\left(\alpha^{-1}\dot{\phi} - \frac{\beta}{\alpha}\phi'\right)^{\cdot} + \beta\alpha^{-1}\left(\alpha^{-1}\dot{\phi} - \frac{\beta}{\alpha}\phi'\right)'$$
$$+ A^{-1}(A^{-1}\phi')' + A^{-1}\left((\alpha A)^{-1}\phi' + 2\frac{(rA)'}{rA^2}\right)\phi' - \frac{\partial V}{\partial \phi} = 0. \tag{7.18}$$

One of the purposes of a numerical investigation of these coupled equations is to decide whether the chaotic scenario of inflation [14] works for 'generic' initial conditions.

7.2. *Explicit form of the Liouville operator*

It is now very easy to work out the general formula (4.5) for the Liouville operator. Let us parametrize the four momentum p of a massless particle as

$$p = (E, E\mu, E\sqrt{1-\mu^2}\cos\bar{\phi}, E\sqrt{1-\mu^2}\sin\bar{\phi}). \tag{7.19}$$

The distribution function f is considered as function of r, t, E and μ. If we insert the expressions (7.6) in (4.5), we obtain in a first step

$$\begin{aligned} L_{X_g} f = {} & \frac{E}{\alpha}\,\partial_t f + A^{-1}\left(E\mu - \frac{E}{\alpha}A\beta\right)\partial_r f \\ & + \frac{(rA)'}{rA^2}[(p^2)^2 + (p^3)^2]\frac{\partial f}{\partial p^1} \\ & - \left[\frac{(rA)'}{rA^2}p^1p^2 - \frac{1}{rA}\cot\theta(p^3)^2\right]\frac{\partial f}{\partial p^2} \\ & - \left[\frac{(rA)'}{rA^2}p^1p^3 + \frac{1}{rA}\cot\theta p^2p^3\right]\frac{\partial f}{\partial p^3} \\ & - EA^{-1}\alpha^{-1}\alpha'\frac{\partial f}{\partial p^1} + E\alpha^{-1}\left[(\beta_{1|1} - c_{1|1})p^2\frac{\partial}{\partial p^1} + \cdots\right]f. \end{aligned} \tag{7.20}$$

From

$$0 = \frac{\partial f}{\partial\bar{\phi}} = -\frac{\partial f}{\partial p^2}p^3 + \frac{\partial f}{\partial p^3}p^2$$

we see that the terms proportional to $\cot\theta$ in (7.20) cancel. Furthermore, the chain rule gives immediately

$$E(1-\mu^2)\frac{\partial f}{\partial\mu} = [(p^2)^2 + (p^3)^2]\frac{\partial f}{\partial p^1} - p^1p^2\frac{\partial f}{\partial p^2} - p^1p^3\frac{\partial f}{\partial p^3},$$

$$E\frac{\partial f}{\partial E} = p^1\frac{\partial f}{\partial p^1} + p^2\frac{\partial f}{\partial p^2} + p^3\frac{\partial f}{\partial p^3},$$

and thus,

$$E(1-\mu^2)\frac{\partial f}{\partial\mu} + \mu E^2\frac{\partial f}{\partial E} = E^2\frac{\partial f}{\partial p^1}.$$

This enables us to collect terms in (7.20) as follows:

$$\begin{aligned} L_{X_g} f = {} & E\alpha^{-1}\,\partial_t f + A^{-1}(E\mu - E\alpha^{-1}A\beta)\partial_r f + \frac{(rA)'}{rA^2}E(1-\mu^2)\frac{\partial f}{\partial\mu} \\ & - A^{-1}\alpha^{-1}\alpha'\left[E(1-\mu^2)\frac{\partial}{\partial\mu} + \mu E^2\frac{\partial}{\partial E}\right]f \\ & + E\alpha^{-1}\left[(\beta_{1|1} - c_{11})p^1\frac{\partial}{\partial p^1} + \cdots\right]f. \end{aligned}$$

In the last row we insert now (7.7) and (7.5) to obtain

$$(\beta_{1|1}-c_{11})p^1\frac{\partial}{\partial p^1}+\cdots=\left(\frac{A'}{A}\beta+r^{-1}\beta-\frac{\dot{A}}{A}\right)\sum_i p^i\frac{\partial f}{\partial p^i}+(\beta'-r^{-1}\beta)p^1\frac{\partial f}{\partial p^1}$$
$$=\left(\frac{A'}{A}\beta+r^{-1}\beta-\frac{\dot{A}}{A}\right)E\frac{\partial f}{\partial E}$$
$$+(\beta'-r^{-1}\beta)\mu\left[(1-\mu^2)\frac{\partial f}{\partial \mu}+\mu E\frac{\partial f}{\partial E}\right].$$

Finally, we use the maximal slicing condition (7.8) and obtain after some rearrangements

$$E^{-1}L_{X_g}f=\alpha^{-1}\,\partial_t f+A^{-1}(\mu-\alpha^{-1}A\beta)\partial_r f$$
$$+E\frac{\partial f}{\partial E}\left[-\mu A^{-1}\frac{\alpha'}{\alpha}+\alpha^{-1}(r^{-1}\beta-\beta')(\tfrac{1}{3}-\mu^2)\right]$$
$$+(1-\mu^2)\frac{\partial f}{\partial \mu}\left[(1-\alpha^{-1}\mu\beta A)(1/Ar)-\alpha^{-1}\left(\frac{\alpha}{A}\right)'+\frac{\mu}{\alpha}\beta'\right]. \qquad (7.21)$$

This equation can, for example, be used for the Boltzmann equation describing the neutrino transport in a type II supernova explosion.

A similar calculation leads to an explicit expression for $L_{X_g}f$ in the following gauge, which is so far used mostly in numerical simulations of supernovae explosions

$$g=-e^{2\Phi}\,dt^2+e^{2\Lambda}\,dr^2+r^2\,d\Omega. \qquad (7.22)$$

Details can be found in Ref. [12].

8. Gauge-invariant cosmological perturbation theory

The blackbody nature and the astonishing isotropy of the microwave background radiation provide strong evidence that the early universe can be described to a good approximation by a Friedman model. Initial deviations from homogeneity and isotropy must have been very small ($<3\cdot 10^{-5}$, [18]) indeed. Thus there was a long period during which deviations from Friedman models can be studied perturbatively, i.e., by linearizing the Einstein and matter equations around the solutions of the idealized Friedman models.

Until recently this perturbation analysis was always carried out in a coordinate dependent manner [15, 16], by fixing the gauge freedom in some convenient way (synchronous gauge, etc). This leads often to misinterpretations of the growth of density fluctuations – especially on super-horizon scales – and gave rise to incorrect conclusions in the literature.

In recent years, Bardeen [8] and others [9] have developed a manifestly gauge-invariant formalism, which eliminates ambiguities and problems of interpretation. In this section we give simplified derivations of the main equations of this formalism by using the tools developed in this paper. We also present some new results. These include gauge-invariant equations for the entropy production rate and gauge-invariant perturbation equations for collisionless particles (neutrinos, etc). One of us (R.D.) has translated the resulting system of perturbation equations into computer programs. The results of this numerical investigation will be published in an accompanying paper [7].

For the unperturbed Friedman model with metric $g^{(0)}$ we have, if t denotes the *conformal* time,

$$\alpha^{(0)} = a(t), \qquad \beta^{(0)} = 0, \qquad \mathbf{g}^{(0)} = a^2(t)\boldsymbol{\gamma}. \tag{8.1}$$

Here $a(t)$ is the scale factor and $\boldsymbol{\gamma}$ is a metric on Σ with constant curvature $K = 0$, ± 1. In addition, we have matter variables for the various components (baryons, radiation, neutrinos, axions, strings, etc).

In the following we linearize all the basic equations around the unperturbed solution in the usual manner. For the small-amplitude departures in $g = g^{(0)} + \delta g$, etc, we have in the linearized approximation the gauge freedom

$$\delta g \mapsto \delta g + L_\xi g^{(0)}, \text{ etc}, \tag{8.2}$$

where ξ is any vector field. This transformation changes in particular the lapse function and the shift vector field.

8.1. *Harmonic analysis*

The unperturbed spaces $(\Sigma, \boldsymbol{\gamma})$ of constant curvature $K = 0$, ± 1 are highly symmetric which invites us to carry out a harmonic analysis of the perturbations. This is achieved in two steps.

a) *Decomposition into scalar, vector and tensor contributions:*

This decomposition of the quantities $\delta\alpha$, $\delta\boldsymbol{\beta}$, $\delta\mathbf{g}$, $\delta\mathbf{K}$, etc, proceeds as follows. Consider first the set $\mathscr{X}(\Sigma)$ of vector fields on Σ. This module can be decomposed as follows into 'scalar' and 'vector' contributions

$$\mathscr{X}(\Sigma) = \mathscr{X}^S \oplus \mathscr{X}^V,$$

where $\mathscr{X}^S$ consists of all gradients and $\mathscr{X}^V$ of all vector fields with vanishing divergence. (More generally, we have for the p-forms Λ^p on Σ the decomposition $\Lambda^p = d\Lambda^{p-1} \oplus \operatorname{Ker} \delta \mid \Lambda^p$, where δ denotes the codifferential) Similarly, we can decompose a symmetric tensor $\mathbf{t} \in \mathscr{S}(\Sigma)$ (= set of symmetric tensor fields) into 'scalar', 'vector' and 'tensor' contributions:

$$\mathbf{t} = \mathbf{t}^S + \mathbf{t}^V + \mathbf{t}^T,$$

where

$$\begin{aligned} \mathbf{t}^S_{ij} &= \operatorname{tr}(\mathbf{t})\gamma_{ij} + (\boldsymbol{\nabla}_i\boldsymbol{\nabla}_j - \tfrac{1}{3}\,\delta_{ij}\Delta)f, \\ \mathbf{t}^V_{ij} &= \boldsymbol{\nabla}_i\mathbf{X}_j + \boldsymbol{\nabla}_j\mathbf{X}_i, \text{ with } \mathbf{X} \in \mathscr{X}^V(\Sigma) \end{aligned} \tag{8.3}$$

and

$$\mathrm{tr}\,(\mathbf{t}^T) = 0, \ \nabla \cdot \mathbf{t}^T = 0.$$

One can show [9] that these direct decompositions are respected by the covariant derivatives. For example, if $\mathbf{X} \in \mathscr{X}(\Sigma)$, $\mathbf{X} = \mathbf{X}_* + \mathrm{grad}\,(f)$, $\nabla \cdot \mathbf{X}_* = 0$, then one finds easily

$$\Delta \mathbf{X} = \Delta \mathbf{X}_* + \mathbf{grad}\,(\Delta f + 2Kf)$$

and $\nabla \cdot (\Delta \mathbf{X}_*) = 0$. (A group theoretical discussion will be given in [6].)

This observation implies a decoupling of scalar, vector and tensor modes. Only the scalar modes are interesting for the problems of galaxy formation, because only these contribute to density perturbations. In what follows we consider only the scalar modes. (For the other modes we refer to [9] and [6].)

b) *Decomposition into spherical harmonics.* In a second step we decompose the scalar components of all functions, vector fields and symmetric tensor fields on Σ in terms of spherical harmonics on (Σ, γ) and their covariant derivatives. The spherical harmonics Y are eigenfunctions of the Laplace-Beltrami operator on (Σ, γ)

$$(\Delta + k^2)Y = 0. \tag{8.4}$$

The possible eigenvalues depend on the curvature K [19]. Indices, referring to the various modes, are always suppressed. The scalar contributions of the vector and symmetric tensor fields can be expanded in terms of

$$\begin{aligned} &Y_i := -k^{-1}\nabla_i Y, \\ &Y_{ij} := k^{-2}\nabla_i\nabla_j Y + \tfrac{1}{3}\gamma_{ij} Y \quad \text{and} \quad Y\gamma_{ij}. \end{aligned} \tag{8.5}$$

The following properties are easily derived by computing covariant derivatives (using the Riemann tensor for (Σ, γ)):

$$\begin{aligned} \nabla_i Y^i &= kY, \\ \Delta Y_i &= -(k^2 - 2K)Y_i, \\ \nabla_j Y_i &= -k(Y_{ij} - \tfrac{1}{3}\gamma_{ij} Y), \\ \nabla^j Y_{ij} &= \tfrac{2}{3}k^{-1}(k^2 - 3K)Y_i, \\ \nabla_j \nabla^m Y_{im} &= \tfrac{2}{3}(3K - k^2)(Y_{ij} - \tfrac{1}{3}\gamma_{ij} Y), \\ \Delta Y_{ij} &= -(k^2 - 6K)Y_{ij}, \\ \nabla_m Y_{ij} - \nabla_j Y_{im} &= (k/3)(1 - (3K/k^2))(\gamma_{im} Y_j - \gamma_{ij} Y_m). \end{aligned} \tag{8.6}$$

Clearly, different modes in a harmonic expansion do not couple in the linearized approximation. Hence it suffices to consider a generic mode.

For the metric perturbations $\delta\alpha$, $\delta\boldsymbol{\beta}$ and $\delta\mathbf{g}$ we have, using the same notation as [9],

$$\begin{aligned} &\delta\alpha = aA(t)Y, \\ &\delta\boldsymbol{\beta} = -B(t)Y^i\,\partial_i, \\ &\delta\mathbf{g} = a^2[2H_L(t)\gamma_{ij} Y + 2H_T(t)Y_{ij}]\,dx^i\,dx^j. \end{aligned} \tag{8.7}$$

Gauge transformations (8.2) affect scalar modes only for vector fields of the 'scalar' type:

$$\xi = T(t)Y\,\partial_t + L(t)Y^i\,\partial_i. \tag{8.8}$$

Using

$$L_\xi\,dt = d(L_\xi t) = d(TY) = \dot{T}Y\,dt - kTY_i\,dx^i,\ \text{etc}, \tag{8.9}$$

one finds the transformation laws

$$\begin{aligned} &A \to A + \dot{T} + (\dot{a}/a)T,\\ &B \to B - \dot{L} - kT,\\ &H_L \to H_L + (k/3)L + (\dot{a}/a)T,\\ &H_T \to H_T - kL. \end{aligned} \tag{8.10}$$

Next, we consider perturbations of the energy-momentum tensor T. The 4-velocity u is defined to be the normalized timelike eigenvector of T

$$Tu = -\rho u,\ g(u, u) = -1. \tag{8.11}$$

We decompose T as follows

$$T = \rho u \otimes u + \tau, \tag{8.12}$$

where τ is orthogonal to u,

$$\tau_{\mu\nu}u^\nu = 0. \tag{8.13}$$

For the unperturbed model we have

$$u^{(0)} = a^{-1}\,\partial_t, \qquad \tau_\mu^{(0)0} = 0, \qquad \tau_j^{(0)i} = p^{(o)}\delta_j^i. \tag{8.14}$$

For a general mode the perturbations have the form

$$\delta\rho = \rho^{(0)}\delta Y, \tag{8.15}$$

$$\delta u = a^{-1}(-AY\,\partial_t + vY^i\,\partial_i). \tag{8.16}$$

Here the coefficient in the first term for δu is fixed by the normalisation condition in (8.11). For the spatial components of the stress tensor τ we set

$$\delta\tau_j^i = p^{(0)}[\pi_L Y\delta_j^i + \pi_T Y_j^i]. \tag{8.17}$$

The other components are fixed by the condition (8.13):

$$\delta\tau_0^0 = 0,\ \delta\tau_0^i = -(\rho^{(0)} + p^{(0)})vY^i,\ \delta\tau_j^0 = (\rho^{(0)} + p^{(0)})(v - B)Y_j. \tag{8.18}$$

Under a gauge transformation defined by the vector field (8.8) we find

$$\begin{aligned} &\delta \to \delta - 3(1+w)(\dot{a}/a)T,\\ &v \to v - \dot{L},\\ &\pi_L \to \pi_L - 3(c_s^2/w)(1+w)(\dot{a}/a)T,\\ &\pi_T \to \pi_T, \end{aligned} \tag{8.19}$$

where $w = p^{(0)}/\rho^{(0)}$ and $c_s^2 = \dot{p}^{(0)}/\dot{\rho}^{(0)}$ is the velocity of sound. In deriving the substitution rules for δ and π_L use was made of the unperturbed equation

$$\dot{\rho}^{(0)} = -3\rho^{(0)}(1 + w)(\dot{a}/a). \tag{8.20}$$

8.2. *Calculation of geometrical quantities*

For scalar perturbations the 4-velocity field u is hypersurface orthogonal. Indeed, it can easily be seen that u is proportional to the gradient of the function

$$t_m = t + k^{-1}(v - B)Y. \tag{8.21}$$

This function defines another foliation of spacetime, which plays a certain role in what follows. We compute first the second fundamental forms for the two families of slices $\{t = \text{const}\}$ and $\{t_m = \text{const}\}$. This is conveniently done with the use of (6.5). For the original slicing $\{t = \text{const}\}$ we obtain in zeroth order

$$\mathbf{K}^{(0)} = -\dot{a}\gamma_{ij}\, dx^i\, dx^j, \qquad \text{tr}\,(\mathbf{K}^{(0)}) = -3\dot{a}/a^2. \tag{8.22}$$

Using (8.7) and (8.6) one finds in the linearized approximation

$$\begin{aligned} \tilde{\mathbf{K}} &= -ak\sigma_g Y_{ij}\, dx^i\, dx^j, \\ \text{tr}\,(\mathbf{K}) &= -3(\dot{a}/a^2)(1 + \kappa_g Y), \end{aligned} \tag{8.23}$$

where $\tilde{\mathbf{K}}$ is the trace-free part of $\mathbf{K}$ and

$$\kappa_g = -A + \tfrac{1}{3}(\dot{a}/a)^{-1}kB + (\dot{a}/a)^{-1}\dot{H}_L, \tag{8.24}$$

$$\sigma_g = k^{-1}\dot{H}_T - B. \tag{8.25}$$

Similarly, we calculate the second fundamental form $\mathbf{K}^{(m)}$ of the slices $\{t_m = \text{const}\}$. It is easy to see that one obtains the same expression as in (8.23), but with B replaced by v in (8.24) and (8.25). Thus

$$\begin{aligned} \tilde{\mathbf{K}}^{(m)} &= -ak\sigma_m Y_{ij}\, dx^i\, dx^j, \\ \text{tr}\,(\mathbf{K}^{(m)}) &= -3(\dot{a}/a^2)(1 + \kappa_m Y), \end{aligned} \tag{8.26}$$

with

$$\kappa_m = -A + \tfrac{1}{3}(\dot{a}/a)^{-1}kv + (\dot{a}/a)^{-1}\dot{H}_L, \tag{8.27}$$

$$\sigma_m = k^{-1}\dot{H}_T - v. \tag{8.28}$$

We need also the scalar Riemann curvature $\mathbf{R}$ of the slices $\{t = \text{const}\}$. This is derived in Appendix B with the result (B.8)

$$\delta\mathbf{R} = 4a^{-2}(k^2 - 3K)\mathcal{R}Y, \tag{8.29}$$

where

$$\mathcal{R} = H_L + \tfrac{1}{3}H_T. \tag{8.30}$$

8.3. *Gauge-invariant amplitudes*

From the transformation laws already written down we deduce

$$\begin{aligned}&\mathcal{R} \to \mathcal{R} + (\dot{a}/a)T,\\ &\kappa_g \to \kappa_g - (\dot{a}/a)^{-1}[(\dot{a}/a)^2 + (k^2/3) - (\dot{a}/a)^{\cdot}]T,\\ &\sigma_g \to \sigma_g + kT.\end{aligned} \tag{8.31}$$

Therefore, the following combinations, [9], are gauge-invariant:

$$\mathcal{A} = A - a^{-1}\left(\frac{a^2}{\dot{a}}\mathcal{R}\right)^{\cdot}, \tag{8.32}$$

$$\mathcal{B} = k(\dot{a}/a)^{-1}\mathcal{R} - \sigma_g, \tag{8.33}$$

$$\Pi = \pi_T, \tag{8.34}$$

$$\Gamma = \pi_L - (c_s^2/w)\delta, \tag{8.35}$$

$$V = -\sigma_m = v - k^{-1}\dot{H}_T = v - B - \sigma_g, \tag{8.36}$$

$$\Delta_s = \delta + 3(1+w)(a'/a)k^{-1}\sigma_g. \tag{8.37}$$

Bardeen [8] originally introduced also the following combinations, which we have used in our numerical simulations [7]:

$$\Phi = k^{-1}(\dot{a}/a)\mathcal{B} = \mathcal{R} - k^{-1}(\dot{a}/a)\sigma_g, \tag{8.38}$$

$$\Psi = \mathcal{A} + (ka)^{-1}(a\mathcal{B})^{\cdot} = A - (ka)^{-1}(a\sigma_g)^{\cdot}, \tag{8.39}$$

$$\Delta_g = \delta + 3(1+w)\mathcal{R} \tag{8.40}$$

$$= \Delta_s + 3(1+w)\Phi. \tag{8.41}$$

Another gauge-invariant quantity is the acceleration of u

$$\mathbf{a} = \nabla_u u = -kA_m Y^i\,\partial_i, \tag{8.42}$$

with

$$A_m = A - (ka)^{-1}[a(v-B)]^{\cdot} = \Psi - (ka)^{-1}(aV)^{\cdot}. \tag{8.43}$$

This result can be obtained by noting that for an arbitrary slicing one has (see Appendix A, (A.5))

$$\nabla_{e_0} e_0 = \omega^i_0(e_0)\boldsymbol{e}_i = \alpha^{-1}\alpha^{|i}\boldsymbol{e}_i.$$

If this is used for the slicing $\{t_m = \text{const}\}$ one finds the given result.

Summarising, we have shown that each scalar mode is described by 2 gauge invariant metric amplitudes $\mathcal{A}$, $\mathcal{B}$ (or Φ, Ψ) and four gauge-invariant matter amplitudes that describe the source terms in Einstein's field equation. These are Δ_g (density contrast), V (peculiar velocity), Π (anisotropic stress) and Γ, which describes the entropy production, as will be shown in the next subsection.

8.4. *Perturbation Equations*

8.4.1. *Gravitational field equations.* The perturbed field equations can be directly obtained from the general formulae (6.2) to (6.5).

a) Constraint equations. By comparing (8.14) and (8.16) with $n = \alpha^{-1}(\partial_t - \boldsymbol{\beta})$ one sees that u is equal to n up to a first order horizontal contribution. Hence, we obtain to first order from our expression for T in subsection 8.1:

$$T(n, n) = \rho = \rho^{(0)}(1 + \delta Y), \qquad T(n, \partial_i) = -a(\rho^{(0)} + p^{(0)})(v - B)Y_i.$$

Indicating as before the traceless part of a tensor by a twiddle, we have

$$(\operatorname{tr} \mathbf{K})^2 - \operatorname{tr}(\mathbf{K}^2) = \tfrac{2}{3}(\operatorname{tr} \mathbf{K})^2 - \operatorname{tr}(\tilde{\mathbf{K}}^2).$$

Equation (8.23) shows that $\tilde{\mathbf{K}}$ is of first order, and we find for the first order term

$$[(\operatorname{tr} \mathbf{K})^2 - \operatorname{tr}(\mathbf{K}^2)]^{(1)} = 12\left(\frac{\dot{a}}{a^2}\right)^2 \kappa_g Y.$$

Using also (8.29) we can write down the constraint equation (6.2)

$$6(\dot{a}/a)^2\kappa_g + 2a^{-2}(k^2 - 3K)\mathcal{R} = 8\pi G\rho^{(0)}\delta.$$

With (8.40) and using the zeroth order relation

$$4\pi G\rho^{(0)}(1 + w) = a^{-2}[(\dot{a}/a)^2 - (\dot{a}/a)^{\cdot} + K] \tag{8.44}$$

we can write this as

$$4\pi G\rho^{(0)}\Delta_g = 3(\dot{a}/a)^2\kappa_g + a^{-2}[k^2 + 3((\dot{a}/a)^2 - (\dot{a}/a)^{\cdot})]\mathcal{R}.$$

Expressing finally the right hand side in terms of gauge-invariant quantities, one finds

$$4\pi Ga^2\rho^{(0)}\Delta_g = -3(\dot{a}/a)^2\mathcal{A} + k(\dot{a}/a)\mathcal{B}. \tag{8.45}$$

In order to write down (6.3), we note that

$$\begin{aligned}(\operatorname{tr} \mathbf{K})_{|i} - K^j_{i|j} &= \tfrac{2}{3}(\operatorname{tr} \mathbf{K})_{|i} - \tilde{K}^j_{i|j}\\ &= 2k(\dot{a}/a^2)\kappa_g Y_i + \tfrac{2}{3}a^{-1}(k^2 - 3K)\sigma_g Y_i.\end{aligned}$$

Thus (6.3) reads

$$-8\pi G\rho^{(0)}(1 + w)(v - B) = 2k(\dot{a}/a^2)\kappa_g + \tfrac{2}{3}a^{-1}(k^2 - 3K)\sigma_g$$

or, with (8.36) and (8.44)

$$-4\pi Ga^2\rho^{(0)}(1 + w)V = k(\dot{a}/a)\kappa_g + [\tfrac{1}{3}k^2 + (\dot{a}/a)^2 - (\dot{a}/a)^{\cdot}]\sigma_g.$$

Expressing again the right hand side in terms of $\mathcal{A}$ and $\mathcal{B}$ leads to

$$4\pi Ga^2\rho^{(0)}(1 + w)V = k(\dot{a}/a)\mathcal{A} + [(\dot{a}/a)^2 - (\dot{a}/a)^{\cdot}]\mathcal{B}. \tag{8.46}$$

With the two constraint equations (8.45) and (8.46) one can express Φ algebraically in terms of V and Δ_g:

$$\Phi = \frac{4\pi G a^2 \rho^{(0)}}{k^2 - 3K + 12\pi G(1+w)a^2\rho^{(0)}}(\Delta_g + 3(1+w)(\dot{a}/a)k^{-1}V). \tag{8.47}$$

b) *Dynamical equations.* In order to work out the content of the dynamical equation (6.4), it is convenient to start from the tracefree part of this equation. With help of (6.7) we obtain

$$\partial_t \tilde{\mathbf{K}} = \mathbf{L}_{\boldsymbol{\beta}}\tilde{\mathbf{K}} - \widetilde{\mathbf{Hess}}\,\alpha + \alpha[\widetilde{\mathbf{Ric}} - 2\mathbf{K}\cdot\tilde{\mathbf{K}} + \tilde{\mathbf{K}}\,\mathrm{tr}\,(\mathbf{K}) - \tilde{\mathbf{T}}]. \tag{8.48}$$

Since $\tilde{\mathbf{K}}$ is of first order, $\mathbf{L}_{\boldsymbol{\beta}}\tilde{\mathbf{K}}$ is of second order. If we use also $\widetilde{\mathbf{Ric}}_{ij} = -k^2\mathcal{R}Y_{ij}$, (see Appendix B, equation (B.9)) we find immediately

$$8\pi G a^2 p^{(0)}\Pi = k^2[-(A - (ka)^{-1}(a\sigma_g)^{\cdot}) + ((\dot{a}/a)k^{-1}\sigma_g - \mathcal{R})],$$

or with (8.38) and (8.39)

$$-8\pi G a^2 k^{-2} p^{(0)}\Pi = \Phi + \Psi. \tag{8.49}$$

This also enables us to express Ψ algebraically in terms of matter variables.

8.4.2. *Energy-momentum conservation.* These 'conservation' laws are of course a consequence of the field equation. In order to work them out we start from the $3+1$ split of $\nabla \cdot T = 0$ carried out in Section 5.

To first order the quantities in (5.6) and (5.7) are

$$\epsilon = \rho^{(0)}(1 + \delta Y),$$
$$\mathbf{S} = (\rho^{(0)} + p^{(0)})v^i\,\partial_i, \qquad v^i = a^{-1}(v - B)Y^i,$$
$$\mathbf{T} = a^{-2}p^{(0)}[(1 + \pi_L Y)\gamma^{ij} + \pi_T Y^{ij}]\partial_i \otimes \partial_j,$$
$$\alpha = a(1 + AY), \qquad \beta = -BY^i\,\partial_i.$$

Inserting this in (5.6) and using the zeroth order relation

$$\dot{\rho}^{(0)} = -3(\dot{a}/a)(\rho^{(0)} + p^{(0)}),$$

gives

$$(\rho^{(0)} + p^{(0)})3(\dot{a}/a)(\kappa_m + A) + \rho^{(0)}\dot{\delta} - p^{(0)}3(\dot{a}/a)(\delta - \pi_L) = 0.$$

This can be expressed in terms of gauge-invariant quantities. Making use also of the unperturbed relation

$$\dot{w} = -3(c_s^2 - w)(1 + w)\dot{a}/a, \tag{8.50}$$

we find

$$\dot{\Delta}_g + 3(c_s^2 - w)(\dot{a}/a)\Delta_g + (1 + w)kV + 3w(\dot{a}/a)\Gamma = 0. \tag{8.51}$$

Similarly, equation (5.7) gives after a short calculation

$$\dot{V} + (\dot{a}/a)(1 - 3c_s^2)V = k(\Psi - 3c_s^2\Phi) + \frac{c_s^2}{1+w} k\Delta_g + \frac{kw}{1+w}\left(\Gamma - \frac{2}{3}\left(1 - \frac{3K}{k^2}\right)\Pi\right). \tag{8.52}$$

If we substitute the algebraic expressions (8.47), (8.49) for Φ and Ψ in terms of the matter variables Δ_g, V, Π and Γ into the last two equations, we obtain dynamical equations for Δ_g and V, with source terms determined by Π and Γ.

8.4.3. *Entropy production and heat flux.* In the following discussion use is made of Appendix B in [11], where basic aspects of general relativistic thermodynamics are developed. Adopting the fitting procedure of Eckart for small departures from equilibrium, one obtains for the energy-momentum tensor (see equation (B.12) of [11]):

$$T^{\mu\nu} = (\rho + p_{eq})U^\mu U^\nu + p_{eq} g^{\mu\nu} + \delta T^{\mu\nu}, \tag{8.53}$$

where p_{eq} is the pressure of the equilibrium state which is fitted to the actual state, and $\delta T^{\mu\nu}$ is of first order and satisfies

$$\delta T^{\mu\nu} U_\mu U_\nu = 0.$$

We recall that U^μ is the four-velocity of particle transport, i.e., the particle current N^μ is given by $N^\mu = nU^\mu$.

On the other hand, the matter four-velocity u^μ is defined by (8.11) and $T^{\mu\nu}$ can also be written in the form given by (8.12):

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pq^{\mu\nu} + \Pi^{\mu\nu}, \tag{8.54}$$

where we have set

$$\tau^{\mu\nu} = p(u^\mu u^\nu + g^{\mu\nu}) + \Pi^{\mu\nu}, \quad \Pi^\lambda_\lambda = 0. \tag{8.55}$$

The tensor $\Pi^{\mu\nu}$ is orthogonal to u^μ, $\Pi^{\mu\nu}u_\nu = 0$. Defining Q^μ by

$$u^\mu = U^\mu + Q^\mu,$$

we can rewrite (8.54) in the following manner:

$$\begin{aligned} T^{\mu\nu} &= (\rho + p)U^\mu U^\nu + pg^{\mu\nu} + U^\mu q^\nu + U^\nu q^\mu + \Pi^{\mu\nu} \\ &= (\rho + p_{eq})U^\mu U^\nu + p_{eq} g^{\mu\nu} + U^\mu q^\nu + U^\nu q^\mu \\ &\quad + [(p - p_{eq})(U^\mu U^\nu + g^{\nu\mu}) + \Pi^{\mu\nu}], \end{aligned} \tag{8.56}$$

where

$$q^\mu = (\rho + p)Q^\mu. \tag{8.57}$$

Since $u^2 = U^2 = -1$, we have to first order $q \cdot U = 0$, $q \cdot u = 0$. Comparison of (8.56) with (B.16) of [11] shows that q is the heat flux. In addition, we obtain with

(8.53) up to first order

$$\delta T^{\mu\nu} = (p - p_{eq})(u^\mu u^\nu + g^{\mu\nu}) + \Pi^{\mu\nu} + (u^\mu q^\nu + u^\nu q^\mu). \tag{8.58}$$

From $p = p^{(0)}(1 + \pi_L Y)$ and $p_{eq} = p^{(0)}(1 + (c_s^2/w)\delta Y)$ we find with (8.35)

$$p - p_{eq} = p^{(0)}\left(\pi_L + \frac{c_s^2}{w}\delta\right)Y = p^{(0)}\Gamma Y. \tag{8.59}$$

Hence,

$$\delta T^{\mu\nu} = t^{\mu\nu} + u^\mu q + u^\nu q^\mu,$$

with

$$t^{\mu\nu} = p^{(0)}\Gamma Y(u^\mu u^\nu + g^{\mu\nu}) + \Pi^{\mu\nu}, \; t^{\mu\nu}u_\mu = 0. \tag{8.60}$$

Finally we use this expression in (B.29) of [11],

$$S^\mu_{;\mu} = -T^{-2}(T_{,\mu} - Ta_\mu)q^\mu + T^{-1}\theta_{\lambda\mu}t^{\lambda\mu},$$

for the divergence of the entropy current. (T denotes the temperature.) Since $T_{,\mu}$ is in zeroth order proportional to u_μ, we have to first order $T_{,\mu}\, q^\mu = u_\mu q^\mu = 0$. Furthermore, equation (8.42) shows that a_μ is also of first order. Thus to first order

$$S^\mu_{;\mu} = T^{-1}\theta_{\lambda\mu}t^{\lambda\mu}. \tag{8.61}$$

Now $\theta_{\mu\nu} = -K_{\mu\nu}$, since in an orthonormal basis $\{e_\mu\}$ adapted to the slicing $\{t_m = \text{const}\}$ with $u = e_0$ we have

$$\theta_{ij} = \tfrac{1}{2}[(e_i, \nabla_{e_j}e_0) + (e_j, \nabla_{e_i}e_0)] = \tfrac{1}{2}[\omega^i_0(e_j) + \omega^i_0(e_j)] = -K_{ij}.$$

In such a basis (8.61) becomes with (8.22)

$$S^\mu_{;\mu} = -T^{-1}K^{(0)}_{ij}t^{ij} = 3\frac{\dot a}{a^2}T^{-1}p^{(0)}\Gamma Y. \tag{8.62}$$

This demonstrates that the entropy production rate is proportional to Γ.

8.5 *Gauge-invariant perturbation theory for collisionless particles*

In this section we shall derive a gauge-invariant version of the perturbed Liouville equation. For that we first have to find gauge invariant amplitudes which describe the perturbation of the distribution function. For the sake of simplicity, we specialize now to $K = 0$. The scalar harmonics are then given by $Y(\boldsymbol{x}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$. This simplification can always be made in applications, since the deviation of Ω from 1 is very small in the regime where the deviations from homogeneity and isotropy can be treated by linear perturbation theory.

8.5.1. *Gauge-invariant perturbation amplitudes for the distribution function.* We split the perturbed distribution function, f, on the one-particle phase space $P_m = \{(x, p) \in T\mathcal{M} \mid g(x)(p, p) = -m^2\}$ into an unperturbed contribution $f^{(0)}$ and a perturbation $f^{(1)}$ in the following manner:

$$f(p^\mu e_\mu) = f^{(0)}(p^\mu e_\mu^{(0)}) + f^{(1)}(p^\mu e_\mu^{(0)}), \tag{8.63}$$

where $\{e_\mu\}$ is the adapted orthonormal basis (2.11) and $\{e_\mu^{(0)}\}$ is the corresponding basis for the unperturbed metric. Under a gauge transformation given by a vector field ξ, f transforms into $\bar{f} = f + L_{T\xi} f$, where $T\xi$ is the natural lift of ξ to $T\mathcal{M}$. Similarly to (8.63) we set

$$\bar{f}(p^\mu \bar{e}_\mu) = f^{(0)}(p^\mu e_\mu^{(0)}) + \bar{f}^{(1)}(p^\mu e_\mu^{(0)}), \tag{8.64}$$

The subtraction of these two decompositions shows that the change $\delta_\xi f^{(1)}$ of $f^{(1)}$ under a gauge transformation is given by (ignoring higher order terms)

$$\begin{aligned}\delta_\xi f^{(1)}(p^\mu e_\mu^{(0)}) &= \bar{f}(p^\mu \bar{e}_\mu) - f(p^\mu e_\mu)\\ &= (L_{T\xi} - L_{(T\xi)^\perp}) f^{(0)}(p^\mu e_\mu^{(0)})\end{aligned}$$

where

$$(T\xi)^\perp = p^\mu (e_\mu - \bar{e}_\mu)^\nu \frac{\partial}{\partial p^\nu}. \tag{8.65}$$

We write this result as follows:

$$\delta_\xi f^{(1)} = L_{(T\xi)^\parallel} f^{(0)} \tag{8.66}$$

with

$$(T\xi)^\parallel = T\xi - (T\xi)^\perp. \tag{8.67}$$

In order to work this out, we must calculate $\bar{e}_\mu - e_\mu$ for a gauge transformation given by ξ. From the transformation properties of the lapse function and the shift vector, equation (8.10), we find:

$$\begin{aligned}\bar{e}_0 = \bar{n} = \bar{\alpha}^{-1}(\partial_t - \bar{\boldsymbol{\beta}}) &= e_0 + (\alpha - \bar{\alpha})\partial_t + ((\beta^i - \bar{\beta}^i)\mathbf{e}_i^{(0)}\\ &= e_0 - [(\dot{\xi}^0 + (\dot{a}/a)\xi^0)e_0^{(0)} + (\dot{\xi}^i - \xi^{0|i})\mathbf{e}_i^{(0)}].\end{aligned} \tag{8.68}$$

For the horizontal basis vector fields we have the following transformation law:

$$\begin{aligned}\bar{\mathbf{e}}_i &= \mathbf{e}_i + (L_\xi \mathbf{e}_i)^{\text{hor}} = \mathbf{e}_i + [\xi, \mathbf{e}_i^{(0)}]^{\text{hor}}\\ &= \mathbf{e}_i + (\xi^0 (a^{-1})^{\cdot} a \mathbf{e}_i^{(0)} - \xi^\mu_{,i} e_\mu^{(0)})^{\text{hor}}\\ &= \mathbf{e}_i - [(\dot{a}/a)\xi^0 \mathbf{e}_i^{(0)} + \xi^j_{,i}\mathbf{e}_j^{(0)}].\end{aligned} \tag{8.69}$$

The second term on the right hand side of (8.69) is of first order. Therefore, it is sufficient that the square bracket is horizontal with respect to the background metric.

With the help of (8.68) and (8.69), equation (8.65) yields

$$(T\xi)^\perp = ((\dot{a}/a)\xi^0 + \dot{\xi}^0)p^0 \frac{\partial}{\partial p^0} + (\dot{\xi}^i - \xi^{0,i})p^0 \frac{\partial}{\partial p^i} + (\dot{a}/a)\xi^0 p^i \frac{\partial}{\partial p^i} + \xi^j_{,i} p^i \frac{\partial}{\partial p^j}.$$

Furthermore, in a coordinate frame $\{\partial_\mu\}$ we have

$$T\xi = \xi^\mu \partial_\mu + \xi^\mu_{,\nu} p^\nu \frac{\partial}{\partial p^\mu}.$$

This leads to

$$(T\xi)^{\parallel} = \xi^\mu \partial_\mu - ((\dot a/a)\xi^0 p^0 - \xi^0_{,i} p^i)\frac{\partial}{\partial p^0} - ((\dot a/a)\xi^0 p^i - \xi^{0,i} p^0)\frac{\partial}{\partial p^i}. \tag{8.70}$$

For a vector field of the form given in (8.8) we arrive at

$$(T\xi)^{\parallel} = TY\,\partial_t + LY^i\,\partial_i - T\left[((\dot a/a)Yp^0 + kY_i p^i)\frac{\partial}{\partial p^0} + ((\dot a/a)Yp^i + Y^i p^0)\frac{\partial}{\partial p^i}\right]. \tag{8.71}$$

Let us choose variables $(t, \mathbf{x}, v, \gamma)$ on $P_m^{(0)}$, where

$$v = (a/m_X)\mathbf{g}^{(0)}(\mathbf{p}, \mathbf{p})^{1/2},$$

and $\boldsymbol{\gamma}$ denotes the unit vector in the direction of $\mathbf{p}$. Then $f^{(0)}$ is a function of v alone: Since collisionless particles in a Friedman universe move on geodesics with respect to the Friedman metric, their distribution function changes only by redshifting the momenta and we have

$$f^{(0)}(t, \mathbf{p}) = f^{(0)}(t_0, (a/a_0)\mathbf{p})$$

for an arbitrary reference time t_0. Because of its isotropy, $f^{(0)}$ is thus a function of v alone and we obtain

$$(\partial_t f^{(0)})_{p^i} = 2(\dot a/a)v\frac{df^{(0)}}{dv}, \tag{8.72}$$

$$\left(\frac{\partial f^{(0)}}{\partial p^i}\right)_t = (v/p)\gamma^i\frac{df^{(0)}}{dv}, \tag{8.73}$$

where the variables which are kept constant are indicated by subscripts. Setting

$$q = (a^2/m_X)p^0 = (v^2 + a^2)^{1/2}, \tag{8.74}$$

equation (8.71) yields

$$L_{(T\xi)^{\parallel}} f^{(0)} = T\frac{df^{(0)}}{dv}[(\dot a/a)vY - kq\gamma^i Y_i]. \tag{8.75}$$

For the sake of simplicity, we specialize now to $K = 0$. The scalar harmonics are then given by $Y(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}$. Let us now perform the harmonic analysis also for $f^{(1)}$. For a specific mode Y we set

$$f^{(1)} = F(t, \mathbf{p})Y.$$

Defining

$$\mu = k^{-1} k_i \gamma^i, \tag{8.76}$$

equations (8.66) and (8.75) lead finally to the following transformation law for F:

$$F \to \bar{F} = f + \frac{df^{(0)}}{dv} [(\dot{a}/a)v + ik\mu q]T. \tag{8.77}$$

If one compares (8.77) with the transformation properties of the geometrical quantities given in (8.31), one finds that the first term in the bracket of (8.77) transforms like $v\mathcal{R}$ and the second term like $iq\mu\sigma_g$. Thus, the combination

$$\mathcal{F} = F - \frac{df^{(0)}}{dv} [v\mathcal{R} + iq\mu\sigma_g] \tag{8.78}$$

is gauge-invariant. Of course every linear combination of $\mathcal{F}$ with gauge-invariant quantities is again invariant, but (8.78) will turn out to be suitable for the calculation of the gauge-invariant fluid variables in the next subsection.

8.5.2. *Momentum integrals of* $\mathcal{F}$. In an orthonormal frame, the invariant volume element, $\pi(x)$ of the mass shell $P_m(x)$ is as in special relativity:

$$\pi(x) = \frac{p^2}{p^0} dp\, d\Omega.$$

Using the definitions of v and q, we therefore obtain

$$\pi(x) = T_X^2 \frac{v^2}{q} dv\, d\Omega, \tag{8.79}$$

where we have set

$$T_X = m_X / a. \tag{8.80}$$

Since we are treating collisionless particles, they are not in thermodynamical equilibrium, and thus T_X is not an equilibrium temperature. It is however related to the decoupling temperature (see, e.g., Ref. [7]).

Equation (8.79) leads to

$$\rho^{(0)} = T_X^4 4\pi \int f^{(0)} v^2 q\, dv, \tag{8.81}$$

$$p^{(0)} = \frac{T_X^4 4\pi}{3} \int f^{(0)} (v^4/q)\, dv. \tag{8.82}$$

The calculation of the energy momentum tensor from f yields the following equations for the gauge-invariant fluid variables defined in equations (8.40), (8.36), (8.34), and (8.35):

$$\Delta_g = \frac{T_X^4}{\rho^{(0)}} \int v^2 q \mathcal{F} \, dv \, d\Omega, \tag{8.83}$$

$$V = \frac{iT_X^4}{\rho^{(0)} + p^{(0)}} \int v^3 \mu \mathcal{F} \, dv \, d\Omega, \tag{8.84}$$

$$\Pi = \frac{T_X^4}{2p^{(0)}} \int (v^4/q)(1 - 3\mu^2) \mathcal{F} \, dv \, d\Omega, \tag{8.85}$$

$$\Gamma = \frac{T_X^4}{p^{(0)}} \int (v^4/3q - c_s^2 v^2 q) \mathcal{F} \, dv \, d\Omega, \tag{8.86}$$

where μ is defined by (8.76).

Let us derive the first two of these equations. The energy momentum tensor is given by the second moments of the distribution function:

$$T^\mu_\nu = T_X^2 \int p^\mu p_\nu f(t, \mathbf{x}, v, \gamma)(v^2/q) \, dv \, d\Omega.$$

Here the p^μ's and p_ν's are considered as functions of $(t, \boldsymbol{x}, v, \gamma)$. In first order we find

$$T^0_0 = -T_X^4 \int q v^2 (f^{(0)} + FY) \, dv \, d\Omega.$$

On the other hand we have by (8.15)

$$T^0_0 = -\rho^{(0)}(1 + \delta Y),$$

and thus

$$\rho^{(0)} \delta = T_X^4 \int q v^2 F \, dv \, d\Omega.$$

If we use this, the definition (8.78) and the isotropy of $f^{(0)}$, we obtain

$$T_X^4 \int q v^2 \mathcal{F} \, dv \, d\Omega = \rho^{(0)} \delta + \mathcal{R} T_X^4 \int \frac{d}{dv} (q v^3) f^{(0)} \, dv \, d\Omega. \tag{8.87}$$

The integral on the right hand side of (8.87) is

$$T_X^4 \int \frac{d}{dv} (q v^3) f^{(0)} \, dv \, d\Omega = 3(\rho^{(0)} + p^{(0)}). \tag{8.88}$$

Using finally the definition of Δ_g, (8.40), we obtain indeed (8.83).

Next we calculate

$$T^0_j = Y T_X^4 \int f \gamma_j v^3 \, dv \, d\Omega. \tag{8.89}$$

In flat space we have to first order for a general mode (see Section 8.1)

$$k^{-1}k^jT^0_j = (\rho^{(0)} + p^{(0)})(v - B)Y_jk^{-1}k^j = -i(\rho^{(0)} + p^{(0)})(v - B)Y, \tag{8.90}$$

so that by the definition of V, equation (8.36),

$$VY = i(\rho^{(0)} + p^{(0)})^{-1}k^{-1}k^jT^0_j - \sigma_g Y.$$

Inserting (8.89) here gives

$$V = -\sigma_g + i(\rho^{(0)} + p^{(0)})^{-1}T_X^4 \int v^3 \mu F\, dv\, d\Omega$$

and thus, with (8.78), we obtain (8.84). Equations (8.85) and (8.86) are derived similarly.

8.5.3. *Liouville's equation.* Liouville's equation provides the equation of motion for $\mathcal{F}$ which we derive now. Using equation (4.5) of Section 4 we can write the Liouville equation in the form

$$X_g f = 0, \tag{8.91}$$

with

$$X_g = \frac{p^0}{\alpha}\partial_t + \mathbf{p} - \frac{p^0}{\alpha}\boldsymbol{\beta} - \left[\boldsymbol{\omega}^i_j\left(\mathbf{p} - \frac{p^0}{\alpha}\boldsymbol{\beta}\right)p^j + (p^0)^2(\ln\alpha)^{|i} + \alpha^{-1}(\beta^{|i}_j - c^i_j)p^0p^j\right]\frac{\partial}{\partial p^i}. \tag{8.92}$$

Here the momentum components are those with respect to the tetrad $\{e_0 = n, \mathbf{e}_i\}$ adapted to the slicing $\{t = \text{const.}\}$. Let us consider, as before, f as a function of the variables $(t, \mathbf{x}, v, \gamma)$. Since

$$v = (a/m_X)p = T_X^{-1}p,$$

we have

$$\left(\frac{\partial f}{\partial t}\right)_p = \left(\frac{\partial f}{\partial t}\right)_v + (\dot a/a)v\frac{\partial f}{\partial v},$$

where the subscript p, v indicates which variable is kept constant while evaluating the t-derivative. Using in addition

$$\mathbf{p} = T_X v\boldsymbol{\gamma},\ p^0 = T_X q,$$

we can write (8.91) in the form

$$\frac{q}{\alpha}(\partial_t + (\dot a/a)v\,\partial_v)f + \left(v\boldsymbol{\gamma} - \frac{q}{\alpha}\boldsymbol{\beta}\right)f$$
$$- T_X\left[\boldsymbol{\omega}^i_j\left(v\boldsymbol{\gamma} - \frac{q}{\alpha}\boldsymbol{\beta}\right)\gamma^j v + (q)^2(\alpha^{|i}/\alpha) + \alpha^{-1}(\beta^{|i}_j - c^i_j)qv\gamma^j\right]\frac{\partial}{\partial p^i}f = 0. \tag{8.93}$$

To find the background and the first order contribution to (8.93) we use the decomposition

$$f = f^{(0)} + FY$$

and make use of the background quantities

$$\boldsymbol{\beta}^{(0)} = 0, \qquad \boldsymbol{\omega}_j^{(0)i} = 0, \tag{8.94}$$

$$\alpha^{(0)} = a, \qquad (c^{(0)})^i_j = (\dot{a}/a)\delta^i_j. \tag{8.95}$$

Taking into account the homogeneity and isotropy of $f^{(0)}$:

$$\partial_i f^{(0)} = 0, \ \frac{\partial f^{(0)}}{\partial p^i} = \gamma^i \frac{\partial f^{(0)}}{\partial p} = \gamma^i T_X^{-1} \frac{\partial f^{(0)}}{\partial v},$$

we obtain the background contribution to (8.93),

$$\partial_t f^{(0)} = 0. \tag{8.96}$$

Hence, $f^{(0)}$ is a function of v alone as we already reasoned in subsection 8.5.3.

To find the first order contribution we use (8.7) and

$$c^i_j = (\dot{a}/a)\delta^i_j + (\dot{H}_L Y\, \delta^i_j + \dot{H}_T Y^i_j)$$

as well as (8.94) and (8.95). Equation (8.93) then yields the following first order equation:

$$(q/a)(\partial_t F + (\dot{a}/a)v\, \partial_v F)Y + ivk\mu a^{-1} FY - (\dot{a}/a^2) qv T_X^{-1} \gamma^i \frac{\partial F}{\partial p^i} Y$$
$$-[AYqv(\dot{a}/a^2) + ia^{-1}AYq^2k\mu + a^{-1}BYk\mu^2 vq - AYqv(\dot{a}/a^2)$$
$$+ a^{-1}(\dot{H}_L Y\, \delta_{ij} + \dot{H}_T Y_{ij})\gamma^i\gamma^j vq] \frac{df^{(0)}}{dv} = 0.$$

After some rearrangements this leads to

$$\partial_t F + ik\mu \frac{v}{q} F = [iAk\mu q + Bk\mu^2 v + \dot{H}_L + (\tfrac{1}{3} - \mu^2)\dot{H}_T] \frac{df^{(0)}}{dv}. \tag{8.97}$$

Now we use

$$\dot{H}_L + \tfrac{1}{3}\dot{H}_T = \mathcal{R},$$
$$B - k^{-1}\dot{H}_T = -\sigma_g,$$

and add

$$-i\mu[\partial_t(q\sigma_g) + k(v^2/q)\mathcal{R}] \frac{df^{(0)}}{dv}$$

on both sides. With the help of (8.78) we then find

$$\partial_t \mathcal{F} + ik\mu \frac{v}{q} \mathcal{F} = ik\mu \frac{df^{(0)}}{dv} \left[qA - k^{-1} q\dot{\sigma}_g - \frac{a^2}{kq}(\dot{a}/a)\sigma_g - (v^2/q)\mathcal{R} \right].$$

Inserting $q^2 - v^2 = a^2$ in the third term of the square bracket, and making use of (8.38) and (8.39) gives finally

$$\partial_t \mathcal{F} + ik\mu \frac{v}{q} \mathcal{F} = ik\mu \frac{df^{(0)}}{dv} \left[q\Psi - \frac{v^2}{q} \Phi \right]. \tag{8.98}$$

This gauge-invariant perturbation equation, together with the field equations (8.47) and (8.49) and the momentum integrals (8.83), (8.84) and (8.85), form a closed system of ordinary differential equations which we have solved numerically for hot and cold dark matter, adding also photons and massless neutrinos to the matter content. A discussion of the numerical results is presented in [7].

8.5.4. *The nonrelativistic limit.* At this point we derive an integral equation for Δ_g in the nonrelativistic limit. Let us introduce the quantity

$$\Delta(t_1, t_2, v) = \int_{t_1}^{t_2} (v/q)\, dt, \tag{8.99}$$

which is the comoving distance travelled by a particle of momentum v in the time interval t_1 to t_2. With this notation (8.98) has the following integral representation:

$$\mathcal{F}(t, v, \mu) = \frac{df^{(0)}}{dv} \left[ik\mu \int_{t*}^{t} (q\Psi - (v^2/q)\Phi) \exp(-ik\mu\Delta(t', t, v))\, dt' \right.$$
$$\left. + \mathcal{F}(t_*, v, \mu) \exp(-ik\mu\Delta(t_*, t, v)) \right]. \tag{8.100}$$

$\mathcal{F}(t_*, v, \mu)$ is the initial value.

Let us proceed now to the nonrelativistic limit for which

$$q \approx a \gg v, \tag{8.101}$$

and therefore,

$$\Delta(t_*, t, v) \approx v \int_{t*}^{t} a^{-1} =: vs. \tag{8.102}$$

We then obtain

$$\mathcal{F}(s, v, \mu) = \mathcal{F}(0, v, \mu) e^{-is\mathbf{k}\cdot\mathbf{v}} + i\mathbf{k} \cdot \mathbf{v} v^{-1} \frac{df^{(0)}}{dv} \int_0^s a^2 \Psi e^{-i\mathbf{k}\cdot\mathbf{v}(s-s')}\, ds'. \tag{8.103}$$

In the nonrelativistic approximation ($p^{(0)}\Pi \ll \rho^{(0)}\Delta_g$, $kt \gg 1$), (8.47) and (8.49) give

$$a^2\Psi = -\frac{8\pi G a^4 \rho^{(0)}}{2k^2} \Delta_g.$$

Recalling that $\rho^{(0)}a^3$ is constant, leads to

$$\mathscr{F}(s, v, \mu) = \mathscr{F}(0, v, \mu)e^{-is\mathbf{k}\cdot\mathbf{v}} - i\mathbf{k}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}} 4\pi G\rho^{(0)}(0)a(0)^3\int_0^s a\Delta_g k^{-2}e^{-i\mathbf{k}\cdot\mathbf{v}(s-s')}\,ds'. \qquad (8.104)$$

After multiplication by q and integration over d^3v we obtain

$$\frac{\rho}{T_X^4}\Delta_g = a(s)\int \mathscr{F}(o, \mu, v)e^{-i\mathbf{k}\cdot\mathbf{v}s}\,d^3v + a(s)4\pi G\rho^{(0)}(0)a^3(0)\int_0^s a(s')\Delta_g(s')\phi(\mathbf{k}(s-s'))(s-s')\,ds', \qquad (8.105)$$

with

$$\phi(\mathbf{x}) = \int f^{(0)}(v)e^{-i\mathbf{v}\cdot\mathbf{x}}\,d^3v. \qquad (8.106)$$

For the left hand side of (8.105) we used (8.83). On the right hand side we applied (8.101), and in the second term we integrated by parts. With the help of (8.80), we finally bring (8.105) into the form

$$\Delta_g = \frac{m_X^4}{\rho^{(0)}(0)a^3(0)}\int \mathscr{F}(o, \mu, v)e^{-i\mathbf{k}\cdot\mathbf{v}s}\,d^3v + 4\pi(m_X^2/m_{pl})^2\int_0^s a(s')\Delta_g(s')\phi(\mathbf{k}(s-s'))(s-s')\,ds'. \qquad (8.107)$$

This integral equation coincides with Gilbert's equation (see [20], equations (32) and (33)).

Appendices

A. Connection and curvature forms

In this Appendix we calculate the connection and curvature forms in the orthonormal basis introduced in Section 2, equation (2.11).

A.1. *Connection forms.* From the first structure equation,

$$d\theta^\mu + \omega^\mu_\nu \wedge \theta^\nu = 0,$$

and the definition of the second fundamental form:

$$K_{ij} = -n_{i;j}, \qquad (A.1)$$

where n denotes the normal field of the slicing, one finds immediately the Gauss formulas (remember $n = e_0$):

$$\omega^i_k(\mathbf{e}_j) = \boldsymbol{\omega}^i_k(\mathbf{e}_j), \tag{A.2}$$

$$\omega^0_i(\mathbf{e}_j) = -K_{ij}. \tag{A.3}$$

We define the coefficients c^i_j by

$$\partial_t \boldsymbol{\vartheta}^i = c^i_j \boldsymbol{\vartheta}^j. \tag{A.4}$$

Then we can calculate the following quantities (details are given below):

$$\omega^0_i(e_0) = \alpha^{-1}\alpha_{|i}, \tag{A.5}$$

$$\omega^i_j(e_0) = -\alpha^{-1}\boldsymbol{\omega}^i_j(\boldsymbol{\beta}) + \frac{1}{2\alpha}(\beta^i_{\ |j} - \beta^j_{\ |i} - c^i_j + c^j_i), \tag{A.6}$$

$$K_{ij} = \frac{1}{2\alpha}(\beta_{i|j} + \beta_{j|i} - c^i_j - c^j_i), \tag{A.7}$$

where the vertical bar | denotes covariant derivation with respect to **g**. The last equation implies

$$\mathbf{K} = \frac{1}{2\alpha}[L_{\boldsymbol{\beta}}\mathbf{g} - \partial_t \mathbf{g}], \tag{A.8}$$

Using the general relation

$$\partial_t(\det \mathbf{g}) = \operatorname{tr}(\partial_t \mathbf{g}) \det \mathbf{g},$$

we find from (A.8)

$$\partial_t \operatorname{vol}(\mathbf{g}) = (\operatorname{div} \boldsymbol{\beta} - \alpha \operatorname{tr} \mathbf{K}) \operatorname{vol}(\mathbf{g}). \tag{A.9}$$

The derivation of (A.5), (A.6) and (A.7) proceeds as follows. First we note that

$$d\theta^0 = d(\alpha\, dt) = \mathbf{d}\alpha \wedge dt = \alpha_{|i}\boldsymbol{\vartheta}^i \wedge dt = \alpha^{-1}\alpha_{|i}\theta^i \wedge \theta^0.$$

From the first structure equation and (A.3) we conclude

$$\begin{aligned} i_{\mathbf{e}_l} i_{e_0}\, d\theta^i &= -i_{\mathbf{e}_l} i_{e_0}(\omega^i_0 \wedge \theta^0 + \omega^i_j \wedge \theta^j) \\ &= -(K_{il} + \omega^i_l(e_0)). \end{aligned}$$

The left hand side of this equation is

$$\begin{aligned} i_{\mathbf{e}_j} i_{e_0}\, d\theta^i &= i_{\mathbf{e}_j} i_{e_0} d(\boldsymbol{\vartheta}^i + \beta^i\, dt) \\ &= i_{\mathbf{e}_j} i_{e_0}(\mathbf{d}\boldsymbol{\vartheta}^i + dt \wedge \partial_t \boldsymbol{\theta}^i + \mathbf{d}\beta^i \wedge dt) \\ &= \alpha^{-1} i_{\mathbf{e}_j}(i_{\boldsymbol{\beta}}(\boldsymbol{\omega}^i_l \wedge \boldsymbol{\vartheta}^l) + \partial_t \boldsymbol{\vartheta}^i - \mathbf{d}\beta^i) \\ &= \alpha^{-1}[\boldsymbol{\omega}^i_j(\boldsymbol{\beta}) - \boldsymbol{\omega}^i_k(\mathbf{e}_j)\beta^k - \mathbf{d}\beta^i(\mathbf{e}_j) + \partial_t \boldsymbol{\vartheta}^i(\mathbf{e}_j)] \\ &= \alpha^{-1}[\boldsymbol{\omega}^i_j(\boldsymbol{\beta}) - \beta^i_{\ |j} + c^i_j]. \end{aligned}$$

The symmetric and antisymmetric contribution of the last identity yield the formulas (A.7) and (A.6) for K_{ij} and $\omega^i_j(e_0)$, respectively.

A.2. *The curvature forms.* From the second structure equation,

$$\Omega^{\mu}_{\nu} = d\omega^{\mu}_{\nu} + \omega^{\mu}_{\lambda} \wedge \omega^{\lambda}_{\nu},$$

and equations (A.2) to (A.7) one finds immediately

$$\Omega^{i}_{j}(\mathbf{e}_k, \mathbf{e}_l) = \mathbf{\Omega}^{i}_{j}(\mathbf{e}_k, \mathbf{e}_l) + K^i_k K_{jl} - K^i_l K_{jk} \qquad \text{(Gauss)}, \tag{A.10}$$

$$\Omega^{0}_{j}(\mathbf{e}_k, \mathbf{e}_l) = K_{jk|l} - K_{jl|k} \qquad \text{(Mainardi)}. \tag{A.11}$$

We need also the other components of Ω^0_j. The second structure equation gives with (A.3) and (A.5)

$$\Omega^i_0 = -d(K_{ij}\theta^j) + d(\alpha^{-1}\alpha_{|i}\theta^0) + \omega^i_l \wedge (-K_{lj}\theta^j + \alpha^{-1}\alpha_{|l}\theta^0).$$

A straightforward calculation leads to

$$\Omega^i_0 = \alpha^{-1}\alpha_{|ij}(\theta^j \wedge \theta^0) - dK^i_j \wedge \theta^j + K^i_j(\omega^j_l \wedge \theta^l - K^j_l\theta^l \wedge \theta^0) + \omega^i_j K^j_l \wedge \theta^l,$$

which yields (A.11) and the following components of Ω^i_0:

$$\Omega^i_0(\mathbf{e}_j, e_0) = \alpha^{-1}\alpha^{|i}_{|j} + dK^i_j(e_0) - K^i_s\omega^s_j(e_0) + (K^2)^i_j - \omega^i_s(e_0)K^s_j. \tag{A.12}$$

Now we can calculate the Ricci tensor, $R_{\beta\sigma} = \Omega^{\alpha}_{\beta}(e_\alpha, e_\sigma)$,

$$R_{00} = \frac{1}{\alpha}\Delta\alpha + \alpha^{-1}(\partial_t \operatorname{tr}(\mathbf{K}) - L_{\boldsymbol{\beta}} \operatorname{tr}(\mathbf{K})) + \operatorname{tr}\mathbf{K}^2 \tag{A.13}$$

and (A.11) gives

$$R_{0i} = (\operatorname{tr}\mathbf{K})_{|i} - K^j_{i|j}. \tag{A.14}$$

For the spatial components

$$R_{ij} = \Omega^0_i(e_0, \mathbf{e}_j) + \Omega^k_i(\mathbf{e}_k, \mathbf{e}_j).$$

(A.12) and (A.10) lead to

$$R_{ij} = \mathbf{R}_{ij} + \operatorname{tr}(\mathbf{K})K_{ij} - \alpha^{-1}\alpha_{|ij} - \alpha^{-1}(\partial_t K_{ij} - L_{\boldsymbol{\beta}}K_{ij}) + K_{is}\omega^s_j(e_0) + K_{js}\omega^s_i(e_0). \tag{A.15}$$

With help of (A.6), (A.7) and (A.8) one can bring (A.15) into the form

$$\operatorname{Ric}(g) = \mathbf{Ric}\,(\mathbf{g}) + \operatorname{tr}(\mathbf{K})\mathbf{K} - 2\mathbf{K}^2 - \alpha^{-1}(\partial_t\mathbf{K} - L_{\boldsymbol{\beta}}\mathbf{K}) - \alpha^{-1}\,\mathbf{Hess}\,(\alpha). \tag{A.16}$$

Using (A.13) and (A.15) we find

$$\begin{aligned} G_{00} &= \tfrac{1}{2}(R_{00} + \textstyle\sum_i R_{ii}) \\ &= \tfrac{1}{2}[\mathbf{R} + (\operatorname{tr}(\mathbf{K}))^2 - \operatorname{tr}(\mathbf{K}^2)]. \end{aligned} \tag{A.17}$$

B. *The Ricci tensor of the spatial slices of a perturbed Friedman universe*

To make use of (A.16) we need the Ricci tensor **Ric (g)** of the $\{t = \text{const.}\}$ slices for the perturbed Friedman universe. This is the Ricci tensor of the induced metric

$$\mathbf{g} = a^2[(1 + 2H_L Y)\,\delta_{ij} + \epsilon 2H_T Y_{ij}]\tau^i \otimes \tau^j, \tag{B.1}$$

where the $\{\tau^i\}$ denote orthonormal 1-forms of the metric γ_{ij}. We set

$$\mathbf{Ric}\,(\mathbf{g}) = (\mathbf{R}_{ij}^{(0)} + \delta\mathbf{R}_{ij})\tau^i \otimes \tau^j,$$

where (generalizing to n spatial dimensions)

$$\mathbf{R}_{ij}^{(0)} = K(n-1)\delta_{ij}.$$

By Palatini's identity (see for example [11] p. 217)

$$\delta R_{ij}(g^{(0)} + \delta g) = \tfrac{1}{2}[\delta g^{k}_{i|jk} - \delta g^{k}_{k|ij} + \delta g^{k}_{j|ik} - \Delta\,\delta g_{ij}]. \tag{B.2}$$

The indices are raised and lowered with respect to the background metric $g^{(0)}$. To calculate (B.2) for a perturbation of the form (B.1) we make use of equations (8.4), (8.5) and (8.6) of Section 8. Furthermore we have to apply the following identity which is derived like the equations (8.6)

$$Y^{m}_{i|jm} = \left(K(n-1) + \frac{n-1}{n}k^2\right)Y_{ij} - \frac{(n-1)(n+1)}{n^2}(nK - k^2)Y\,\delta_{ij}. \tag{B.3}$$

One then finds

$$\delta\,\mathbf{Ric} = \Bigg[H_L\left\{2\frac{n-1}{n}k^2 Y\,\delta_{ij} + (2-n)k^2 Y_{ij}\right\} + H_T\left\{2\frac{n-1}{n^2}(k^2 - nK)Y\,\delta_{ij} + \left(\frac{2-n}{n}k^2 + 2(n-1)K\right)Y_{ij}\right\}\Bigg]\tau^i \otimes \tau^j. \tag{B.4}$$

We want to write **Ric** (**g**) with respect to our adapted basis $\{\vartheta^i\}$ which is orthonormal with respect to the perturbed metric. The transformation matrix is given by

$$\tau^i = \frac{1}{a}[(1 - H_L)\vartheta^i - H_T Y^i_j \vartheta^j]. \tag{B.5}$$

A short calculation leads to

$$\mathbf{Ric}\,(\mathbf{g}) = a^{-2}\Bigg[K(n-1)\,\delta_{ij} + \left(H_L + \frac{1}{n}H_T\right) \times \left(2\frac{n-1}{n}(k^2 - nK)Y\,\delta_{ij} + (2-n)k^2 Y_{ij}\right)\Bigg]\vartheta^i \otimes \vartheta^j.$$

Setting

$$\mathcal{R} = H_L + H_T/n, \tag{B.6}$$

we find the following perturbation of the Ricci curvature with respect to our perturbed orthonormal frame:

$$\delta \, \mathbf{Ric}\,(\mathbf{g}) = a^{-2}\left[2\frac{n-1}{n}(k^2 - nK)Y\,\delta_{ij} + (2-n)k^2 Y_{ij}\right]\mathcal{R}\vartheta^i \otimes \vartheta^j. \tag{B.7}$$

From (B.8) we can extract the trace term and the traceless contribution:

$$\delta \, \mathbf{R} = a^{-2}2(n-1)(k^2 - nK)\mathcal{R}Y, \tag{B.8}$$

$$\delta \, \widetilde{\mathbf{Ric}} = a^{-2}(2-n)k^2\mathcal{R}Y_{ij}\vartheta^i \otimes \vartheta^j. \tag{B.9}$$

REFERENCES

[1] A. E. FISCHER and J. E. MARSDEN, *The Initial Value Problem and the Dynamical Formulation of General Relativity*, in *General Relativity, an Einstein Centenary Survey*, ed. Hawking and Israel, Cambridge University press (1979).
[2] Y. CHOQUET-BRUHAT and J. W. YORK, *The Cauchy Problem*, in *General Relativity and Gravitation*, Vol. 1, ed. A. Held, Plenum (1980).
[3] See, e.g., T. PIRAN, in *Gravitational Radiation*, ed. N. Deruelle and T. Piran, North Holland (1982).
[4] K. S. THORNE and D. A. MACDONALD, MNRAS *198*, 339 (1982).
[5] K. S. THORNE, R. H. PRICE and D. A. MACDONALD, *Black Holes: The Membrane Paradigm*, Yale University (1986).
[6] R. DURRER, Ph.D. Dissertation, University of Zürich, Zürich (1988).
[7] R. DURRER, *Gauge-Invariant Perturbation Theory of Friedman Universes Dominated by Collisionless Matter: Numerical Results*, submitted to *Astron. Astrophys.* (1988).
[8] J. M. BARDEEN, Phys. Rev. *D22*, 1882 (1980).
[9] H. KODAMA and M. SASAKI, Prog. Theor. Phys. Suppl. *78*, (1984).
[10] R. ABRAHAM and J. E. MARSDEN, *Foundations of Mechanics*, Second Edition, Benjamin (1978).
[11] N. STRAUMANN, *General Relativity and Relativistic Astrophysics*, Springer (1985).
[12] N. STRAUMANN, *Physics of Type II Supernova Explosions*, Lectures given at the XXVII. Internationale Universitätswochen für Kernphysik, Schladming (1988).
[13] R. MAYLE, Ph.D. dissertation, University of California, Berkeley (1985).
[14] A. LINDE, in *300 Years of Gravitation*, ed. S. W. Hawking and W. Israel, Cambridge University Press (1987).
[15] J. R. BOND and A. S. SZALAY, Ap.J. *274*, 443 (1983).
[16] P. J. E. PEEBLES, *The Large-Scale Structure of the Universe*, Princeton University Press (1980).
[17] P. J. E. PEEBLES, Ap. J. *259*, 442 (1982).
[18] D. T. WILKINSON, Science *232*, 1517 (1986).
[19] N. YA. VILENKIN and YA. A. SMORODINSKII, Sov. Phys.-JETP *19*, 1209 (1964).
[20] I. H. GILBERT, Ap. J. *144*, 223 (1965).