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Schwinger model on S^2

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Introduction

The Schwinger model [1] is quantum electrodynamics with a single massless fermion in two-dimensional space-time. Since it was shown to be exactly solvable by Schwinger in 1962, there has been a lot of interest investigating various aspects of the model. The model has been shown to illustrate several phenomena which are vital to an understanding of particle physics. Spontaneous breakdown of local gauge symmetry, breakdown of global chiral symmetry through the $U(1)$-anomaly, charge shielding and 'quark' trapping are among these. For a discussion of these aspects see [1, 2, 15, 16].

Various generalizations of the model have been considered. Thus Coleman et al. [15] have investigated a massive theory by giving the fermion a small mass and then doing mass perturbation theory. Order parameter estimates for a massive model are given in [18]. The case when one has non-Abelian gauge fields has also been studied [17, 10]. In neither of these two cases is the model exactly solvable.

On the other hand, the model has also become a tool for testing various techniques of quantum field theory. The earlier investigations were based on Green's function methods [1, 16]. Operator methods were then successfully employed to study various aspects of the model. See [2, 4] for an account of the results. The vacuum structure of the Schwinger model has been investigated in [3, 6] using operator techniques. In the recent years, functional integral methods have been used to solve the model [5, 7, 8, 9, 11, 13]. Using Fujikawa's ideas about fermionic path integrals [14], the Schwinger model has been re-considered by a few authors [9, 12, 19]. In [19], an extension to the curved space has been used to suggest that the main features of the model persist even in the presence of a background gravitational field.

Although the best insight to the solution is probably provided by using the functional integral methods, no complete, satisfactory account seems to exist in the literature. It is known that naive calculations using the path integral in the flat space produce incorrect results [7, 8]. The reason becomes clear when one uses a compact manifold: The Dirac operator which has a discrete spectrum on a compact manifold, is found to possess zero modes for certain gauge field configurations. The result of the Grassmann integration in this case is not the same as when there are no zero modes. As we show here, when one takes the necessary modifications into account, one does indeed get the correct results. And one should also keep in mind that the fermion path integrals have no meaning unless defined using a discrete basis [14, 7].

In this work we present a rigorous calculation to reproduce the known results of the Schwinger model using the functional integral method, and illustrate the role of zero modes and gauge field configurations with non-trivial topology. More precisely, we consider quantum electrodynamics on S^2, i.e., the surface of a sphere, which in the limit of the radius R going to infinity becomes QED in Euclidean two dimensions. One may say that the Euclidean two dimensional plane is compactified to S^2. This kind of compactification is desirable for studying the above mentioned aspects. Technically speaking, this is equivalent to introducing an infrared cutoff. In addition, the choice of the sphere as the compact manifold has the great advantage that it does not destroy the solvability of the model. The symmetry of the sphere permits the application of a wealth of familiar mathematical tools enabling one to obtain exact results.

Because of the non-trivial topology of S^2, the gauge fields fall into classes characterized by the winding number k, defined by

$$k = \frac{q}{2\pi} \int_{S^2} d^2x F_{01}$$

which is an integer. Here q is the electron charge and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength of the gauge field A_μ. The number of zero modes of the Dirac operator then turns out to be equal to $|k|$. Thus neglecting the zero modes is equivalent to neglecting all the non-trivial topological sectors and so leads to incorrect results even in the limit $R \to \infty$. In particular, this leads to $\langle \bar{\psi}\psi \rangle = 0$.

The presentation that follows is divided into three main parts. The first part contains the definition of the model in the general formalism and in an $SU(2)$-invariant formalism. The two formalisms are equivalent though the latter is more suitable for explicit calculation since it incorporates the symmetry of the sphere. The importance of the fermionic determinant for calculating various expectation values is emphasized in Section 4. After regularizing the theory using Pauli–Villars regulator fields, a closed expression for the fermionic determinant is obtained in Section 6. This is the main result of Part I.

Objects with physical significance, i.e., the expectation values of various operators, is the subject of Part II. The contributions to the expectation values from each topological sector are calculated separately. All the results are obtained for finite R. At the end, the flat-space limit is taken. The properties of two- and four-point functions establish the cluster property. Considering the $R \to \infty$ limit of the two-point function of the field strength $F_{\mu\nu}$, one verifies that the theory is equivalent to that of a free scalar particle. This is in agreement with the interpretation that the theory describes a scalar meson consisting of a quark-antiquark pair.

Finally, in Part III, details of essential mathematical results used in the main calculation are presented. They are organized in four appendices. After considering the general properties of the Dirac operator, its complete spectrum for a special case is constructed in Appendix A. This is essential for obtaining a closed expression for the fermionic determinant. The method used there also gives the explicit expressions for the zero modes. How to evaluate fermionic path integrals in the presence of zero modes, is shown in Appendix B using first principles. As examples, several special cases, which are relevant to the main calculation are also illustrated. Appendix C shows the equivalence of the two formalisms mentioned above. Thereafter, the Green's function of the Dirac operator for an arbitrary gauge field is obtained. That this can be found explicitly lies at the heart of the solvability of the Schwinger model on the sphere. An infinite series which repeatedly occurs in the calculations of Part II is summed in Appendix D.

Part I

1. General formalism

Consider an orientable d-dimensional manifold $\mathcal{M}$ with positive definite metric $g_{\mu\nu}$. In this section we describe how to define quantum electrodynamics on such a manifold.

1.1. *Euclidean γ-matrices*

The d-dimensional Euclidean Dirac-algebra is

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab} \tag{1.1}$$

$$\gamma_a^\dagger = \gamma_a \tag{1.2}$$

where the indices $a, b, \ldots$ run from 1 to d. Without proof we state the following theorem:

Theorem 1. *Every representation of the Dirac-algebra is an orthogonal sum of irreducible representations. Up to unitary equivalence, there is exactly* 1 *irreducible representation if d is even, and exactly* 2 *if d is odd. The dimension of the representation is $2^{[d/2]}$, where $[d/2]$ is the largest integer $\leq d/2$. If d is odd, the two irreducible representations are distinguished by the sign in the relation*

$$\gamma_1\gamma_2 \cdots \gamma_d = \pm \exp i\frac{\pi}{4} d(d-1)$$

1.2. *Fields on a patch*

In order to describe the manifold $\mathcal{M}$ we need in general more than one coordinate patch. A Dirac field on the patch $U \subset \mathcal{M}$ covered by the coordinates

x^μ, $\mu = 1, \ldots, d$ is a (complex) spinor field $\psi_\alpha(x)$, $\alpha = 1, \ldots, 2^{[d/2]}$. Under a change $x \to x'(x)$ of coordinates in U, ψ_α transforms as a scalar field, i.e.,

$$\psi'_\alpha(x') = \psi_\alpha(x) \tag{1.3}$$

if x and x' are the old and new coordinates of the same point $p \in \mathcal{M}$.

A gauge field on $\mathcal{M}$ is a vector field A_μ taking values in the Lie algebra of the gauge group G. Under the above coordinate transformation it transforms like[1])

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x) \tag{1.4}$$

1.3. *Dirac operator*

The Dirac operator on $\mathcal{M}$ can be given by

$$D = \gamma^\mu(\partial_\mu + \tfrac{1}{4}\gamma^\nu \gamma_{\mu;\nu}) \tag{1.5}$$

Here $\gamma^\mu = g^{\mu\nu}\gamma_\nu$ satisfies

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{1.6}$$

and $\gamma_{\nu;\mu}$ is the (Riemann) covariant derivative of the vector field γ_ν:

$$\gamma_{\nu;\mu} = \partial_\mu \gamma_\nu - \Gamma^\lambda_{\mu\nu}\gamma_\lambda \tag{1.7}$$

$$\Gamma^\lambda_{\mu\nu} = \tfrac{1}{2}g^{\lambda\rho}\{\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\} \tag{1.8}$$

In terms of a fixed set $\mathring{\gamma}_a$ of (constant) γ-matrices we have

$$\gamma^\mu = e_{a\mu}\mathring{\gamma}_a, \tag{1.9}$$

where $e_{a\mu}$ depends on x and satisfies

$$e_{a\mu}e_{a\nu} = g_{\mu\nu}, \tag{1.10}$$

$$e_{a\mu}e_{b\nu}g^{\mu\nu} = \delta_{ab}. \tag{1.11}$$

For any fixed a, the coefficients $e_{a\mu}$ may be considered components of a vector e_a. From (1.11) it follows that these vectors are orthonormal. Thus we have a moving frame which can be chosen to have the same orientation at each point since $\mathcal{M}$ is an orientable manifold.

The Dirac operator described above has the following properties:

- It is a first order differential operator acting on Dirac fields.
- It is invariant under general coordinates transformations.
- It reduces to the ordinary Dirac operator in local normal coordinates.
- Under frame rotations $\tilde{e}_{a\mu} = \Lambda_{ab}e_{b\mu}$ where $\Lambda \in SO(d)$ and x-dependent, it transforms as $\tilde{D} = uDu^{-1}$ where u is a unitary matrix of determinant 1 and is related to Λ by $u\mathring{\gamma}_a u^{-1} = \Lambda_{ab}\mathring{\gamma}_b$.

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[1]) The Greek indices $\mu, \nu, \ldots$ run from $1, \ldots, d$. We also adopt the summation convention that all repeated indices are summed over.

If we also transform Dirac fields ψ under frame rotations according to

$$\psi' = u\psi \tag{1.12}$$

the action of the Dirac operator becomes frame independent.

In the presence of a gauge field, the Dirac fields may carry a further index corresponding to some representation $R(g)$ of the gauge group G. In this case, the Dirac operator may be written as

$$D = \gamma^\mu(\partial_\mu + R(A_\mu) + \tfrac{1}{4}\gamma^\nu\gamma_{\mu;\nu}) \tag{1.13}$$

1.4. *Global aspects*

Suppose that $\mathcal{M}$ is covered by a set of coordinate patches $U^{(i)}$. A point $p \in U^{(i)}$ then has coordinates $x^{(i)\mu}$ in $U^{(i)}$. Suppose now that in each patch $U^{(i)}$ we have chosen a moving frame $e^{(i)}_{a\mu}$. A collection of Dirac fields $\psi^{(i)}_\alpha(x^{(i)})$ is then said to be a Dirac field on $\mathcal{M}$, if on the overlap of any two patches $U^{(i)}$ and $U^{(j)}$ the fields $\psi^{(i)}$, $e^{(i)}_a$ and $\psi^{(j)}$, $e^{(j)}_a$ are related by a 'gauge transformation'. Let us first consider the case when there are no gauge fields. Thus for $x^{(i)} = h_{ij}(x^{(j)})$ we have,

$$\psi^{(i)}(x^{(i)}) = u_{ij}\psi^{(j)}(x^{(j)}) \tag{1.14}$$

$$e^{(i)}_{a\mu}(x^{(i)}) = \Lambda_{ab}(u_{ij})e^{(j)}_{b\nu}(x^{(i)})\frac{\partial x^{(j)\nu}}{\partial x^{(i)\mu}} \tag{1.15}$$

where u_{ij} depends on $x^{(j)}$, and is related to Λ by $u\mathring{\gamma}_a u^{-1} = \Lambda_{ab}\mathring{\gamma}_b$.

Consistency requires that

$$\left.\begin{aligned} x^{(i)} &= h_{ij}(h_{ji}(x^{(i)})) \\ 1 &= u_{ij}(x^{(j)})u_{ji}(x^{(i)}) \end{aligned}\right\} \quad \text{if} \quad p \in U^{(i)} \cap U^{(j)}$$

$$\left.\begin{aligned} h_{ik}(x^{(k)}) &= h_{ij}(h_{jk}(x^{(k)})) \\ u_{ik}(x^{(k)}) &= u_{ij}(x^{(j)})u_{jk}(x^{(k)}) \end{aligned}\right\} \quad \text{if} \quad p \in U^{(i)} \cap U^{(j)} \cap U^{(k)}$$

A manifold $\mathcal{M}$ for which a set of transition matrices u_{ij} with the above properties exists is called a *spin manifold*. Spin manifolds are thus the manifolds which admit globally defined non-zero Dirac fields. All d-dimensional spheres and tori are spin manifolds.

A similar transition rule holds also for a gauge field A_μ: For a point on the overlap $U^{(i)} \cap U^{(j)}$ we have

$$A^{(i)}_\mu(x^{(i)}) = \left\{g_{ij}A^{(j)}_\nu(x^{(j)})g_{ij}^{-1} + g_{ij}\frac{\partial}{\partial x^{(j)\nu}}g_{ij}^{-1}\right\}\frac{\partial x^{(j)\nu}}{\partial x^{(i)\mu}} \tag{1.16}$$

where $g_{ij} \in G$ depends on $x^{(j)}$ and satisfies the conditions analogous to the ones above.

In the presence of a gauge field, the transition rule for the Dirac field becomes

$$\psi^{(i)}(x^{(i)}) = R(g_{ij})u_{ij}\psi^{(j)}(x^{(j)}) \tag{1.17}$$

where $R(g)$ is the corresponding representation of G.

On a spin manifold the Dirac operator D defined above maps a globally defined Dirac field onto a new one. The eigenvalue equation

$$iD\psi = E\psi \tag{1.18}$$

therefore has a well defined global meaning with all eigenvalues E being invariant under gauge transformations and local frame rotations.

1.5. *The action for QED on the manifold $\mathcal{M}$*

The Euclidean action for a massless Dirac particle, represented by the Dirac field ψ interacting with an Abelian gauge field A_μ on the manifold $\mathcal{M}$ can be given by

$$S = \int_{\mathcal{M}} d^d x \sqrt{g}\, \bar{\psi} D \psi + \frac{1}{4} \int_{\mathcal{M}} d^d x \sqrt{g}\, F^{\mu\nu} F_{\mu\nu} \tag{1.19}$$

where

$$g = \det (g_{\mu\nu}), \tag{1.20}$$

$$D = \gamma^\mu (\partial_\mu + iqA_\mu + \tfrac{1}{4} \gamma^\nu \gamma_{\mu;\nu}), \tag{1.21}$$

and the field strength tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1.22}$$

q is the charge of the particle, which is described by the field ψ. We assume that $q > 0$.

Then S is invariant under

- general coordinate transformations $x \to x'$
- frame rotations $e_{a\mu} \to \Lambda_{ab} e_{b\mu}$ and
- local gauge transformations
  1. $\psi \to h(x)\psi$
  2. $A_\mu \to h A_\mu h^{-1} + \frac{1}{iq} h\, \partial_\mu h^{-1}$ where $h(x) \in U(1)$

On conformally flat spaces, like the d-sphere, coordinates may be chosen such that

$$g_{\mu\nu}(x) = \Omega(x) \delta_{\mu\nu} \qquad \Omega > 0 \tag{1.23}$$

A natural choice for the moving frame is then,

$$e_{a\mu} = \Omega^{-1/2} \delta_{a\mu} \tag{1.24}$$

and the Dirac operator reduces to

$$D = \Omega^{-(d-1)/4} \{ \Omega^{-1/2} \mathring{\gamma}_a (\partial_a + iqA_a) \} \Omega^{(d-1)/4} \tag{1.25}$$

1.6. *Schwinger model on S^2*

By setting $d=2$ in (1.19) we obtain the action for the Schwinger model on a curved 2-dimensional space. We choose the 2-sphere S^2 to be this space. On S^2, the stereographic coordinates provide a system of conformal coordinates: we have for $\mathbf{r} \in S^2$

$$r_1 = 2R^2x^1(R^2+\mathbf{x}^2)^{-1} \tag{1.26}$$

$$r_2 = 2R^2x^2(R^2+\mathbf{x}^2)^{-1} \tag{1.27}$$

$$r_3 = R(R^2-\mathbf{x}^2)(R^2+\mathbf{x}^2)^{-1} \tag{1.28}$$

$$g_{\mu\nu} = \partial_\mu \mathbf{r} \cdot \partial_\nu \mathbf{r} \tag{1.29}$$

$$\Omega = \frac{4R^4}{(R^2+\mathbf{x}^2)^2} \tag{1.30}$$

where R is the radius of the sphere.

For the $\mathring{\gamma}_a$-matrices we may choose two of the Pauli matrices:

$$\mathring{\gamma}_1 = \sigma_1 \qquad \mathring{\gamma}_2 = \sigma_2 \tag{1.31}$$

so the Dirac operator becomes

$$D = \Omega^{-3/4}\{\sigma_a(\partial_a + iqA_a)\}\Omega^{1/4} \tag{1.32}$$

There is an alternative way to formulate the whole theory, which incorporates the symmetry of the manifold where the fields live. In the next section this $SU(2)$-invariant formalism is described. It proves to be more convenient for the explicit calculation.

2. $SU(2)$-invariant formalism

In the $SU(2)$-invariant formalism, we require the rotations of S^2 to be symmetries of the model. In the limit $R \to \infty$ it should also reduce to the ordinary Schwinger model in flat space. To fulfill the first of these properties, we need rotation invariant Dirac matrices.

2.1. *The Γ-matrices*

Suppose $\mathbf{r} = \mathbf{r}(x)$ is any (local) coordinatization of S^2 by coordinates x^μ; $\mu = 0, 1$. (The orientation is always taken to be such that $\mathbf{r} \cdot (\partial_0 \mathbf{r} \times \partial_1 \mathbf{r})$ is positive). Define

$$g_{\mu\nu} = \partial_\mu \mathbf{r} \cdot \partial_\nu \mathbf{r} \qquad g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho \tag{2.1}$$

$$\Gamma_\mu = \frac{1}{R} \boldsymbol{\sigma} \cdot (\mathbf{r} \times \partial_\mu \mathbf{r}) \tag{2.2}$$

Here, σ_i are the Pauli matrices. Then

$$\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu} \tag{2.3}$$

$$\Gamma_\mu^\dagger = \Gamma_\mu \qquad \mathrm{Tr}\,\Gamma_\mu = 0 \tag{2.4}$$

With this definition for Γ-matrices it can easily be shown that

$$\Gamma_\mu|_{r\to Sr} = u\Gamma_\mu u^{-1} \tag{2.5}$$

where $u \in SU(2)$ and $S \in SO(3)$ are related by $u\sigma_i u^{-1} = \sigma_l S_{li}$. Thus, a rotation of the sphere is equivalent to a unitary transformation in the spinor space.

Furthermore, under a change of coordinates, Γ_μ transforms like a vector field, i.e., $\Gamma_\mu dx^\mu$ is invariant. In normal coordinates around $r = (0, 0, R)$, we have

$$\Gamma_\mu = \gamma_\mu + O\left(\frac{x}{R}\right) \tag{2.6}$$

$$\gamma_0 = \sigma_2 \qquad \gamma_1 = -\sigma_1 \tag{2.7}$$

so that

$$\lim_{R\to\infty} \Gamma_\mu = \gamma_\mu \tag{2.8}$$

at fixed x.

2.2. *Fields on S^2*

In local coordinates, the electron field ψ is a two-component spinor:

$$\psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \tag{2.9}$$

The gauge field is just a vector field A_μ. Under a gauge transformation,

$$\psi' = h\psi \tag{2.10}$$

$$A'_\mu = A_\mu - \frac{i}{q} h\,\partial_\mu h^{-1} \tag{2.11}$$

where $h(x) = e^{i\Lambda(x)}$ (Λ is defined mod 2π), belongs to $U(1)$.

Under a coordinate transformation, ψ and A_μ transform as scalar and vector fields. Since one needs at least two coordinate patches to cover the sphere, one has the following transformation rules when one goes from one coordinate patch to another (cf. 1.16, 1.17):

$$\psi'(y) = h(x)u(x)\psi(x) \tag{2.12}$$

$$A'_\mu(y) = \left\{A_\nu(x) - \frac{i}{q} h\,\partial_\nu h^{-1}\right\} \frac{\partial x^\nu}{\partial y^\mu} \tag{2.13}$$

where unprimed and primed quantities are the fields on coordinate patches characterized by the coordinates x and y respectively. Furthermore, since the gamma matrices are specified through equation (2.2), there is no need to consider frame rotations as before.

2.3. *The action*

We take the action to be

$$S=\frac{1}{4}\int_{S^2} d^2x\sqrt{g}\,g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}+\int_{S^2} d^2x\sqrt{g}\,\bar{\psi}\left(\mathscr{D}+\frac{i}{R}\right)\psi \tag{2.14}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu \tag{2.15}$$

$$\mathscr{D}=g^{\mu\nu}\Gamma_\nu\mathscr{D}_\mu \tag{2.16}$$

$$\mathscr{D}_\mu=\partial_\mu+iqA_\mu \tag{2.17}$$

Note that $\bar{\psi}$ is a spinor field independent of ψ as we are in Euclidean space.

S has been so constructed that it is invariant under general coordinate transformations, gauge transformations and rotations of S^2.

The extra term $(i/R)\bar{\psi}\psi$ is included to make S also chiral invariant (see below). We also note that the operator $\tilde{D}=(\mathscr{D}+i/R)$ here is, in fact, unitarily equivalent to the operator D introduced in the general formalism (see equation (1.21)). That is, there is a unitary matrix u such that

$$u\tilde{D}u^{-1}=D$$

See Appendix C for the proof. Thus we are indeed dealing with the same theory.

3. Expectation value of an operator $O(\bar{\psi}, \psi, A_\mu)$

In the path integral formulation of the Schwinger model, the Euclidean expectation value of an operator O given by

$$\langle O\rangle=Z^{-1}\int[D\bar{\psi}][D\psi][DA_\mu]e^{-S[\bar{\psi},\psi,A_\mu]}O(\bar{\psi},\psi,A_\mu) \tag{3.1}$$

where

$$Z=\int[D\bar{\psi}][D\psi][DA_\mu]e^{-S[\bar{\psi},\psi,A_\mu]} \tag{3.2}$$

In the following we show that the sum over all gauge fields can be written as a sum over classes of gauge fields characterized by the topological charge k. Indeed, define

$$k=\frac{q}{2\pi}\int_{S^2} d^2xF_{01} \tag{3.3}$$

Then, k is well defined, gauge invariant and independent of the coordinate system used. For all gauge fields, k is in fact an integer. To see this choose the following coordinates for upper and lower half-spheres:

$$r_1 = x^0 \qquad r_2 = x^1 \qquad r_3 = (R^2 - x^2)^{1/2} \qquad (|x| \le R) \tag{3.4}$$

$$r_1 = y^0 \qquad r_2 = -y^1 \qquad r_3 = -(R^2 - y^2)^{1/2} \qquad (|y| \le R) \tag{3.5}$$

These coordinate systems cover the sphere. They meet at the great circle $r_3 = 0$. Suppose $A_\mu(x)$ and $A'_\mu(y)$ are the gauge potentials in these coordinates. Then using Stoke's theorem,

$$k = \frac{q}{2\pi}\left(\oint_{r_3=0} A_\mu(x)\, dx^\mu - \oint_{r_3=0} A'_\mu(y)\, dy^\mu\right) \tag{3.6}$$

where the two line integrals are evaluated along the *same* direction on the great circle $r_3 = 0$.

By definition, the components of a gauge field defined on the overlap of two coordinate patches are related to each other by a gauge transformation (cf. 1.16), i.e., there exists $h \in U(1)$ such that,

$$A'_\mu(y) = \left(A_\nu(x) - \frac{i}{q} h\, \partial_\nu h^{-1}\right)\frac{\partial x^\nu}{\partial y^\mu} \tag{3.7}$$

or

$$dy^\mu A'_\mu(y) = \left(A_\nu(x) - \frac{i}{q} h(x) \frac{\partial}{\partial x^\nu} h^{-1}(x)\right) dx^\nu \tag{3.8}$$

Thus

$$k = \frac{i}{2\pi} \oint_{r_3=0} dx^\mu h(x)\, \partial_\mu h^{-1}(x) \tag{3.9}$$

It follows that k is an integer, the winding number of the gauge function $h(x)$ around $U(1)$ as x runs around the great circle $r_3 = 0$.

The gauge fields A_μ therefore fall into topological classes labelled by k. In the functional integral one must sum over all fields. A restriction to $k = 0$ would be a non-local 'boundary condition'.

Denote the set of gauge fields with topological charge k by $\mathscr{A}_k$. Thus we can write the expectation value of an operator O as

$$\langle O \rangle = Z^{-1} \sum_{k=-\infty}^{\infty} \int_{\mathscr{A}_k} [D\bar{\psi}][D\psi][DA_\mu] e^{-S[\bar{\psi}, \psi, A_\mu]} O(\bar{\psi}, \psi, A_\mu) \tag{3.10}$$

with

$$Z = \sum_{k=-\infty}^{\infty} \int_{\mathscr{A}_k} [D\bar{\psi}][D\psi][DA_\mu] e^{-S[\bar{\psi}, \psi, A_\mu]} \tag{3.11}$$

where $\int_{\mathscr{A}_k}$ denotes the integral over only those gauge fields which have topological charge k.

In fact, A_μ may be expressed in a nice form where this dependency on k is explicitly displayed. To this end, we first construct a rotational symmetric gauge potential.

Suppose x^μ is any coordinates. Choose a spinor $z(x)$ such that

$$r_i\sigma_i z = Rz, \qquad |z|^2 = 1 \tag{3.12}$$

Then z is uniquely defined up to a gauge transformation

$$z(x) \to h(x)z(x), \qquad h \in U(1) \tag{3.13}$$

The vector field

$$C_\mu(x) = \frac{1}{iq}\bar{z}(x)\,\partial_\mu z(x) \tag{3.14}$$

is hence a gauge potential as described above. In the coordinates x^μ and y^μ introduced in (3.4) and (3.5) we may choose

$$z(x) = [r_1^2 + r_2^2 + (R + r_3)^2]^{-1/2}\begin{pmatrix} R + r_3 \\ r_1 + ir_2 \end{pmatrix} \tag{3.15}$$

$$z(y) = [r_1^2 + r_2^2 + (R - r_3)^2]^{-1/2}\begin{pmatrix} r_1 - ir_2 \\ R - r_3 \end{pmatrix} \tag{3.16}$$

so that for $r_3 = 0$

$$z(y) = \frac{1}{R}(r_1 - ir_2)z(x) \tag{3.17}$$

It follows that C_μ has charge $k = 1$.

When $\mathbf{r}$ is rotated, z rotates with the corresponding $SU(2)$ matrix; i.e., if

$$r_i \to S_{ij}r_j$$

then

$$z \to u^{-1}z$$

where $u\sigma_j u^{-1} = \sigma_i S_{ij}$. Hence C_μ is rotation invariant. In particular, the associated field strength $F_{01}/\sqrt{g}$ must be proportional to $\sqrt{g}$. Because $k = 1$, we deduce

$$F_{01} = \frac{\sqrt{g}}{2qR^2} \tag{3.18}$$

Symmetric gauge potentials for any k are obtained just by multiplying C_μ by k.

Now suppose that A_μ is any gauge potential with topological charge k. Then $F_{01}/\sqrt{g}$ is a scalar field on S^2 and we may define the scalar potential ϕ by

$$-\Delta\phi = \frac{F_{01}}{\sqrt{g}} - \frac{k}{2qR^2} \tag{3.19}$$

$$\int_{S^2} d^2x\sqrt{g}\,\phi = 0 \tag{3.20}$$

where Δ is the Laplacian on S^2:

$$\Delta = \frac{1}{\sqrt{g}}\,\partial_\mu\sqrt{g}\,g^{\mu\nu}\,\partial_\nu \tag{3.21}$$

The definition (3.19) needs some explanation: If we exand the function $F_{01}/\sqrt{g}$ in a complete orthonormal set of eigenfunctions of the Laplacian Δ, the piece $k/2qR^2$ is just the term corresponding to the zero mode. Hence the function $F_{01}/\sqrt{g} - k/2qR^2$ no more contains a term proportional to the zero mode. This guarantees that Δ is invertible on the space of functions orthogonal to the zero mode so that equation (3.19) may be solved to obtain ϕ.

The field ϕ is gauge invariant and scalar under coordinate transformations. Define

$$A'_\mu = A_\mu - kC_\mu - \sqrt{g}\, \epsilon_{\mu\nu} g^{\nu\rho}\, \partial_\rho \phi \tag{3.22}$$

By construction, we have

$$\partial_\mu A'_\nu - \partial_\nu A'_\mu = 0 \tag{3.23}$$

so that A'_μ is a pure gauge. It follows that the representation

$$A_\mu = kC_\mu + \sqrt{g}\, \epsilon_{\mu\nu} g^{\nu\rho}\, \partial_\rho \phi + \frac{1}{iq} h\, \partial_\mu h^{-1} \tag{3.24}$$

holds. With the constraint (3.20) for ϕ, and upto a constant phase in h, the mapping

$$A \rightarrow (k, \phi, h)$$

is 1:1.

4. Fermionic determinant

In the following, all the calculations are done in the rotational invariant formalism introduced in Section 3. Later on, in Appendix C, we will use Dirac operator of the general formalism to obtain Green's functions of various operators.

Since the action (2.14) is quadratic in fermion fields, the fermionic integration can be easily done for many important operators. Before illustrating this let us note that it is more convenient to use a dimensionless operator in the action. This can be achieved by setting

$$\eta = \frac{\psi}{\sqrt{R}} \qquad \bar{\eta} = \frac{\bar{\psi}}{\sqrt{R}} \qquad \mathbb{D} = R\left(\mathscr{D} + \frac{i}{R}\right) \tag{4.1}$$

Then we have,

$$\langle O \rangle = Z^{-1} \sum_{k=-\infty}^{\infty} \int_{\mathscr{A}_k} [D\bar{\eta}][D\eta][DA_\mu]$$
$$\times \exp\left(-S[A_\mu] - \int_{S^2} d^2x \sqrt{g}\, \bar{\eta} \mathbb{D} \eta\right) O(\sqrt{R}\, \bar{\eta}, \sqrt{R}\, \eta, A_\mu)\bigg) \tag{4.3}$$

where

$$Z = \sum_{k=-\infty}^{\infty} \int_{\mathscr{A}_k} [D\bar{\eta}][D\eta][DA_\mu] \exp\left(-S[A_\mu] - \int_{S^2} d^2x\sqrt{g}\,\bar{\eta}\mathbb{D}\eta\right) \tag{4.4}$$

Now take, for instance, the two operators $O = O(A)$ and $O = \bar{\psi}(x)\psi(y)$. The result of the fermionic integration depends crucially on the number n of zero modes of the operator $\mathscr{D}$ (see Appendix B). As shown in Appendix A, this number is closely related to the topological charge of the gauge field A_μ present in $\mathbb{D} = R(\Gamma^\mu(\partial_\mu + iqA_\mu) + i/R)$. The following theorem summarizes the results of Appendix B.

Theorem 2. *Corresponding to every eigenfunction η_ν of the operator $i\mathbb{D}$ with a non zero eigenvalue E_ν, there is another eigenfunction $\eta_{-\nu} = \Gamma_5\eta_\nu$ with the eigenvalue $E_{-\nu} = -E_\nu$.*

Furthermore, all the zero modes χ_i have definite chirality, i.e., $\Gamma_5\chi_i = \pm\chi_i$ and the number of the zero modes $n = n_+ + n_-$, where n_+ and n_- denote the positive and negative chirality zero modes respectively, is given by

$$n_+ = 0 \qquad n_- = |k| \quad \text{if} \quad k \geq 0$$
$$n_+ = |k| \qquad n_- = 0 \quad \text{if} \quad k \leq 0$$

The chirality operator Γ_5 mentioned above is defined by $\Gamma_5 = (1/R)\sigma_i r_i$ and has the following properties.

$$\Gamma_5^\dagger = \Gamma_5, \qquad \Gamma_5^2 = 1, \qquad \{\Gamma_5, \Gamma_\mu\} = 0, \tag{4.5}$$

$$\Gamma_5\Gamma_\mu = i\sqrt{g}\,\epsilon_{\mu\nu}g^{\nu\rho}\Gamma_\rho, \qquad \{\mathbb{D}, \Gamma_5\} = 0. \tag{4.6}$$

Now we can perform the Grassmann integration over the fermionic fields (see Appendix B) and we get, formally,

$$\langle O(A)\rangle = Z^{-1}\int_{\mathscr{A}_k} [DA_\mu] \det \mathbb{D} e^{-S[A_\mu]} O(A_\mu)\Big|_{k=0} \tag{4.7}$$

$$\langle \bar{\psi}_\alpha(x)\psi_\beta(y)\rangle = RZ^{-1}\Big\{\int_{\mathscr{A}_k} [DA_\mu] e^{-S[A_\mu]}(-\det \mathbb{D})\mathscr{G}_{\beta\alpha}(x, y \mid \mathbb{D})\Big|_{k=0}$$
$$+ \sum_{k=\pm 1} \int_{\mathscr{A}_k} [DA_\mu] e^{-S[A_\mu]}(-\det{}' \mathbb{D})(\det N)^{-1}\hat{\bar{\chi}}_\alpha^{(k)}(x)\hat{\chi}_\beta^{(k)}(y)\Big\} \tag{4.8}$$

Here

$$Z = \int_{\mathscr{A}_k} [DA_\mu] \det \mathbb{D} e^{-S[A_\mu]}\Big|_{k=0} \tag{4.9}$$

In this expression $\det \mathbb{D}$ denotes the product of all the eigenvalues of $\mathbb{D}$. When there are zero modes (i.e., $k \neq 0$), the product of all *non-zero* eigenvalues is denoted by $\det' \mathbb{D}$. $\mathscr{G}_{\alpha\beta}$ is the Green's function of $\mathbb{D}$, $\hat{\chi}^{(k)}$ the only zero mode of $\mathbb{D}$

present for $|k| = 1$, and $\det N = \langle \hat{\chi}, \hat{\chi} \rangle$. The scalar product $\langle \; , \; \rangle$ is defined as

$$\langle \psi, \varphi \rangle = \int_{S^2} d^2x \sqrt{g}\, \bar{\psi}(x) \varphi(x) \tag{4.10}$$

where $(\bar{\psi})^\alpha = (\psi^\alpha)^*$.

The above expressions for expectation values are only formal because we have not yet defined what these infinite products mean. To make them well defined we have to regularize the theory. At any rate, we recognize $\det' \mathbb{D}$ and $\det \mathbb{D}$ as quantities of central importance in our calculation. In the following we give a proper meaning to them by using Pauli-Villars regulator fields with masses $M_i R$.

5. Regularized theory

The theory is regularized by introducing regulator fields ψ_i, whose contribution to the action is

$$\sum_i \int_{S^2} d^2x \sqrt{g}\, \bar{\psi}_i (\mathbb{D} - M_i R) \psi_i \qquad M_i > 0 \tag{5.1}$$

where ψ_i is fermionic or bosonic depending on e_i being $+1$ or -1.

$$\sum_{i=1}^{r} e_i = -1 \qquad \sum_{i=1}^{r} e_i (M_i R)^{2p} = 0 \quad \text{for} \quad p = 1, \ldots, r-1. \tag{5.2}$$

As a result, $\det' \mathbb{D}$ is now replaced by

$$\det' \mathbb{D} \prod_{i=1}^{r} \det (\mathbb{D} - M_i R)^{e_i} \doteq \exp \tfrac{1}{2} \Gamma^{(k)}_{reg} \tag{5.3}$$

It is more convenient to work with the hermitean operator Ω with positive (or zero) eigenvalues, defined by $\Omega = -\mathbb{D}^2$. It is easy to show that

$$\exp \tfrac{1}{2} \Gamma^{(k)}_{reg} = (-1)^{|k|} (\det' \Omega)^{1/2} \prod_i \{\det (\Omega + M_i^2 R^2)\}^{e_i/2} \tag{5.4}$$

where k is the topological charge corresponding to the gauge field A_μ present in $\mathbb{D}$. $\Gamma^{(k)}_{reg}$ is finite for finite M_i. Taking the logarithm we obtain

$$\Gamma^{(k)}_{reg} = 2i\pi\, |k| + \mathrm{Tr}' \ln \Omega + \mathrm{Tr} \sum_i e_i \ln (\Omega + M_i^2 R^2) \tag{5.5}$$

The 'prime' on the trace means that the zero modes are excluded.

6. Calculation of $\Gamma^{(k)}_{reg}$

This section is devoted to calculating $\Gamma^{(k)}_{reg}$ explicitly for any given k and for any A_μ. The strategy will be as follows. We first recall that (equation (3.24)) a gauge field with any topological charge may be represented as

$$A_\mu = kC_\mu + \sqrt{g}\,\epsilon_{\mu\nu}g^{\nu\rho}\,\partial_\rho\phi + \frac{1}{iq}h\,\partial_\mu h^{-1}$$

The variation $\delta\Gamma^{(k)}_{reg}$ of $\Gamma^{(k)}_{reg}$ for a change $\delta\phi$ of the scalar field ϕ is then calculated. (Notice that $\Gamma^{(k)}_{reg}$ is independent of h). This turns out to be integrable and thus gives $\Gamma^{(k)}_{reg}$ up to a term α independent of ϕ. Finally α is obtained by explicitly calculating $\Gamma^{(k)}_{reg}$ for the special gauge field kC_μ.

6.1. *Variation of* $\Gamma^{(k)}_{reg}$

Before taking the variation of $\Gamma^{(k)}_{reg}$, let us write it in a way which is more convenient for further manipulation:

$$\Gamma^{(k)}_{reg} = 2\pi i\,|k| + |k|\sum_i e_i \ln(M_i^2R^2) + \mathrm{Tr}'\left\{\sum_i e_i \ln(\Omega + M_i^2R^2) + \ln\Omega\right\}$$

$$= 2\pi i\,|k| + |k|\sum_i e_i \ln(M_i^2R^2) + \sum_{\lambda_n\neq 0}\sum_i e_i[\ln(\lambda_n + M_i^2R^2) - \ln\lambda_n]$$

Here, λ_n are the eigenvalues of the operator Ω. The quantity in the square bracket may be represented as an integral. Thus we get

$$\Gamma^{(k)}_{reg} = 2\pi i\,|k| + |k|\sum_i e_i \ln(M_i^2R^2) + \sum_{\lambda_n\neq 0}\sum_i e_i\int_0^\infty \frac{dt}{t}(e^{-t\lambda_n} - e^{-t(\lambda_n+M_i^2R^2)})$$

$$= 2\pi i\,|k| + |k|\sum_i e_i \ln(M_i^2R^2) - \int_0^\infty \frac{dt}{t}(\mathrm{Tr}\,e^{-t\Omega} - |k|)\left(1 + \sum_i e_i e^{-tM_i^2R^2}\right)$$

Under a variation $\delta\phi$ of ϕ, the change in $\Gamma^{(k)}_{reg}$ is thus given by

$$\delta\Gamma^{(k)}_{reg} = -\int_0^\infty \frac{dt}{t}\,\delta(\mathrm{Tr}\,e^{-t\Omega})\left(1 + \sum_i e_i e^{-tM_i^2R^2}\right) \tag{6.1}$$

To evaluate $\delta(\mathrm{Tr}\,e^{-t\Omega})$ we use the following results, the proofs of which are straight foward.

$$\delta\mathbb{D} = -q\Gamma_5[\mathbb{D}, \delta\phi] \qquad \delta\Omega = -\delta\mathbb{D}^2 = q\Gamma_5\{2\mathbb{D}\delta\phi\mathbb{D} + \delta\phi\Omega + \Omega\delta\phi\} \tag{6.2}$$

Now

$$\delta(\mathrm{Tr}\,e^{-t\Omega}) = \mathrm{Tr}\int_0^1 ds e^{-st\Omega}(-t\delta\Omega)e^{-(1-s)t\Omega}$$

$$= -t\,\mathrm{Tr}\,(\delta\Omega e^{-t\Omega})$$

$$= -4qt\,\mathrm{Tr}\,(\Gamma_5\delta\phi\Omega e^{-t\Omega})$$

$$= 4qt\frac{d}{dt}\mathrm{Tr}\,(\Gamma_5\delta\phi e^{-t\Omega}) \tag{6.3}$$

It is crucial that the variation of $\mathrm{Tr}\,e^{-t\Omega}$ can be written in this way, i.e., t times a total derivative with respect to t. This enables integration by parts leading to a rather simplified result. Inserting (6.3) in the expression for $\delta\Gamma_{reg}^{(k)}$ and then integrating by parts we obtain,

$$\delta\Gamma_{reg}^{(k)} = -4q \operatorname{Tr}(\Gamma_5 \delta\phi e^{-t\Omega})\left(1+\sum_i e_i e^{-tM_i^2R^2}\right)\Big|_0^\infty$$

$$-4q\sum_i M_i^2R^2\int_0^\infty dt \operatorname{Tr}\{\Gamma_5\delta\phi e^{-t\Omega}\}e^{-tM_i^2R^2} \tag{6.4}$$

To simplify this further to a form which will be more suitable for explicit calculation we now wish to use the heat kernel techniques. So let us recall some of the basic properties of the heat kernel $K_t(x, y \mid \hat{\Delta})$ of a differential operator $\hat{\Delta}$, which is of the form

$$\hat{\Delta} = -\frac{1}{\sqrt{g}} D_\mu \sqrt{g}\, g^{\mu\nu} D_\nu + C; \qquad D_\mu = \partial_\mu + A_\mu$$

where $C(x)$ is a field of hermitian $n \times n$ matrices and A_μ is a vector field of *anti-hermitian* $n \times n$ matrices. The heat kernel is defined by,

$$\left(\frac{\partial}{\partial t} + \hat{\Delta}\right) K_t(x, y \mid \hat{\Delta}) = 0 \quad \text{for} \quad t > 0, \tag{6.5}$$

$$\lim_{t\to 0} K_t(x, y \mid \hat{\Delta}) = |g|^{-1/2}\,\delta(x-y) \tag{6.6}$$

If $P_0(x, y \mid \hat{\Delta})$ is the projector on the zero mode space of the operator $\hat{\Delta}$, we have

$$\lim_{t\to\infty} K_t(x, y \mid \hat{\Delta}) = P_0(x, y \mid \hat{\Delta}) \tag{6.7}$$

Note also the relation

$$\operatorname{Tr} e^{-t\hat{\Delta}} = \int d^dx \sqrt{g} \operatorname{tr} K_t(x, x) \tag{6.8}$$

One of the most important properties of the heat kernel is that [25] it has an asymptotic expansion

$$K_t(x, y \mid \hat{\Delta}) \overset{t\to 0}{\sim} (4\pi t)^{-d/2} e^{-r^2/4t} \sum_{k=0}^{\infty} t^k c_k(x, y \mid \hat{\Delta}) \tag{6.9}$$

where all the coefficients c_k are calculable using the defining equations (6.5), (6.6) for K_t. Here d is the dimension of the (compact) manifold on which $\hat{\Delta}$ is defined, and r is the geodesic distance between the two points x and y. For the case at hand we notice that

$$\hat{\Delta} = R^{-2}\Omega = = \frac{1}{\sqrt{g}} \hat{D}_\mu \sqrt{g}\, g^{\mu\nu} \hat{D}_\nu + q\Gamma_5 \frac{F_{01}}{\sqrt{g}} + \frac{1}{R^2} \tag{6.10}$$

where $\hat{D}_\mu = D_\mu + (i/2R)\Gamma_\mu$, so it is in the general form mentioned above.

Furthermore, $K_t(x, y \mid \Omega) = K_{(R^2 t)}(x, y \mid \hat{\Delta})$. Here we conclude the remarks on the properties of the heat kernel and refer the reader to the detailed account given in [20].

Now we observe that the first term in (6.4) is zero at the lower limit $t = 0$ since the singular part of $\mathrm{Tr}\,(\Gamma_5 \delta\phi e^{-t\Omega})$ is proportional to $\mathrm{tr}\,\Gamma_5$ which is zero, and the regular part is multiplied by $(1 + \sum e_i)$ which is also zero due to equation (5.2). Here 'tr' stands for the usual matrix trace. At the upper limit we have

$$\mathrm{Tr}\,\{\Gamma_5 \delta\phi e^{-t\Omega}\} \overset{t\to\infty}{=} \mathrm{Tr}\,\{\Gamma_5 \delta\phi P_0(\Omega)\} \tag{6.11}$$

where $P_0(\Omega)$ is the projection on to the zero-mode subspace of Ω.

The second term in $\delta\Gamma^{(k)}_{reg}$ can be calculated by noting that in fact we are interested in the case where $M_i \to \infty$, so, only small t close to zero will contribute to the integral. Hence we can replace the range of integration by $\int_0^\epsilon$ where ϵ is an arbitrarily small finite value. Thus,

$$\begin{aligned}\int_0^\infty dt\, \mathrm{Tr}\,[\delta\phi\Gamma_5 e^{-t\Omega}] e^{-M_i^2 R^2 t} &= \int_0^\epsilon dt\, \mathrm{tr}\,[\delta\phi\Gamma_5 e^{-t\Omega}] e^{-M_i^2 R^2 t} \quad (M_i \to \infty)\\ &= \int_0^\epsilon \int_{S^2} d^2x \sqrt{g}\, \delta\phi(x)\, \mathrm{tr}\,[\Gamma_5(x) K_t(x, x \mid \Omega)] e^{-M_i^2 R^2 t}\end{aligned}$$

Since the integration over t is now done in the vicinity of $t = 0$, we can replace K_t by its asymptotic expansion (6.9). Hence,

$$\begin{aligned}&\int_0^\infty dt\, \mathrm{Tr}\,[\delta\phi\Gamma_5 e^{-t\Omega}] e^{-M_i^2 R^2 t}\\ &= \int_{S^2} d^2x \sqrt{g}\, \delta\phi(x) \int_0^\epsilon dt\, \mathrm{tr}\left[\Gamma_5(x) \frac{1}{4\pi R^2 t} \sum_{k=0}^\infty (R^2 t)^k c_k(x, x \mid \hat{\Delta})\right] e^{-M_i^2 R^2 t}\\ &= \int_{S^2} d^2x \sqrt{g}\, \delta\phi(x) \int_0^\epsilon dt \frac{1}{4\pi}\, \mathrm{tr}\left[\Gamma_5(x)\left(\frac{c_0}{R^2 t} + c_1\right)\right] e^{-M_i^2 R^2 t} \quad (M_i \to \infty)\\ &= \int_{S^2} d^2x \sqrt{g}\, \delta\phi(x) \int_0^\epsilon dt \frac{1}{4\pi}\, \mathrm{tr}\,[\Gamma_5(x) c_1(x, x \mid \hat{\Delta})] e^{-M_i^2 R^2 t} \quad (c_0 = 1)\\ &= \frac{1}{4\pi M_i^2 R^2} \int_{S^2} d^2x \sqrt{g}\, \delta\phi(x)\, \mathrm{tr}\,[\Gamma_5(x) c_1(x, x \mid \Omega)] \quad (M_i \to \infty)\end{aligned} \tag{6.12}$$

As mentioned above, the coefficients of the heat kernel expansion are calculable explicitly. We obtain for c_1, in particular,

$$c_1(x, x \mid \hat{\Delta}) = -q\Gamma_5 \frac{F_{01}}{\sqrt{g}} - \frac{1}{6R^2} \tag{6.13}$$

Putting everything together and also taking the relation $\sum_i e_i = -1$ into account we finally get

$$\delta\Gamma^{(k)}_{reg} = -4q\,\mathrm{Tr}\,[\Gamma_5\delta\phi P_0(\Omega)] - \frac{q}{\pi}\int_{S^2} d^2x\sqrt{g}\,\delta\phi(x)\,\mathrm{tr}\left[\Gamma_5(x)\left(q\Gamma_5\frac{F_{01}}{\sqrt{g}} + \frac{1}{6R^2}\right)\right]$$

$$= -4q\,\mathrm{Tr}\,[\Gamma_5\delta\phi P_0(\Omega)] - \frac{2q^2}{\pi}\int_{S^2} d^2x\sqrt{g}\,\delta\phi(x)\frac{F_{01}}{\sqrt{g}}$$

$$= 2\delta(\ln\det N) + \delta\left(\frac{q^2}{\pi}\int_{S^2} d^2x\sqrt{g}\,\phi\Delta\phi\right) \tag{6.14}$$

where $\Delta = \frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}\,g^{\mu\nu}\,\partial_\nu$ is the Laplacian on S^2, and N_{ij} is the $k \times k$ 'zero-mode matrix' of the operator $\mathbb{D}$ defined as

$$N_{ij} = \int_{S^2} d^2x\sqrt{g}\,\hat{\chi}_i(x)\hat{\chi}_j(x) \tag{6.15}$$

The spinors $\hat{\chi}_i$ form a complete set of independent zero modes of $\mathbb{D}$ which have the form (see Appendix A)

$$\hat{\chi}_i(x) = e^{-q\sigma\phi(x)}h(x)\chi_i(x) \tag{6.16}$$

Here $\{\chi_i(x)\}$ is a complete set of orthonormal zero modes of $\mathbb{D}_0$ which is obtained by putting $\phi = 0$, $h = 1$ in $\mathbb{D}$ and σ is the chirality of $\chi_i(x)$. Integrating back, we find $\Gamma^{(k)}_{reg}$ up to a ϕ-independent constant α;

$$\Gamma^{(k)}_{reg} = 2\ln\det N + \frac{q^2}{\pi}\int_{S^2} d^2x\sqrt{g}\,\phi\Delta\phi + \alpha(k, M_i, R). \tag{6.17}$$

(terms vanishing for $M_i \to \infty$ have been neglected). The constant α, for a given k, can be found by calculating $\Gamma^{(k)}_{reg}$ explicitly for a special value of ϕ. To this end we consider the case of a spherically symmetric gauge field with topological charge k. This corresponds to $\phi = 0$.

6.2. *Evauation of $\alpha(k, M_i, R)$*

Consider the gauge field corresponding to $\phi = 0$, $h = 1$. We have $A_\mu = kC_\mu$, i.e., a rotation symmetric gauge field with topological charge k. Let us denote by $\mathbb{D}_0$, Ω_0 and $\Gamma^{0(k)}_{reg}$ the operators $\mathbb{D}$, Ω and $\Gamma^{(k)}_{reg}$ respectively, corresponding to this special gauge field. As shown in Appendix A, the non-zero eigenvalues of $\mathbb{D}_0$ are of the form $\epsilon = \pm\sqrt{\nu(\nu + |k|)}$ with multiplicity $2\nu + |k|$ where $\nu = 1, 2, \ldots$. In addition, there are $|k|$ zero modes. Hence Ω_0 has the non-zero eigenvalues $\nu(\nu + |k|)$ with multiplicity $2(2\nu + |k|)$, apart from the same $|k|$ zero modes.

Thus,

$$\Gamma_{reg}^{0(k)} = 2\pi i\,|k| + \mathrm{Tr}'\ln\Omega_0 + \mathrm{Tr}\sum_{i=1}^{r} e_i \ln(\Omega_0 + M_i^2R^2)$$

$$= 2\pi i\,|k| + |k| \sum_{i=1}^{r} e_i \ln(M_iR)^2$$

$$+ \sum_{i=0}^{r} e_i \sum_{\nu=1}^{\infty} 2(2\nu + |k|)\ln[\nu(\nu + |k|) + (M_iR)^2] \tag{6.18}$$

where we have defined $e_0 = 1$, $M_0 = 0$.

By introducing a new variable $\mu = \nu - 1$, we can bring this to the following standard form:

$$\Gamma_{reg}^{0(k)} = 2\pi i\,|k| + |k| \sum_{i=1}^{r} e_i \ln(M_iR)^2 + \sum_{i=0}^{r} e_i \sum_{\mu=0}^{\infty} 2(2\mu + 2 + |k|)$$
$$\ln[(\nu+1)(\nu+1+|k|) + (M_iR)^2] \tag{6.19}$$

The sum over μ can now be performed by using the formula [21],

$$\sum_{\mu=0}^{\infty}\sum_{i=0}^{r} e_i(a\mu + b)\ln[(\mu+\alpha_1)(\mu+\alpha_2) + (M_iR)^2]$$
$$= \sum_{i=1}^{r} e_i\Big[-a(M_iR)^2\ln(M_iR) + \frac{\pi}{2}[2b - a(\alpha_1+\alpha_2)](M_iR)$$
$$+ \left[\frac{a}{2}(\alpha_1^2+\alpha_2^2) - \frac{a}{6} - b(\alpha_1+\alpha_2-1)\right]\ln(M_iR)\Big]$$
$$+ \frac{a}{4}(\alpha_1-\alpha_2)^2 + (a\alpha_1 - b)\xi'(0,\alpha_1)$$
$$+ (a\alpha_2 - b)\xi'(0,\alpha_2) - a[\xi'(-1,\alpha_1) + \xi'(-1,\alpha_2)] \tag{6.20}$$

Here $\xi(z, q)$ is the Riemann's zeta function (see [23, Sec. 9.5]). Thus we get for the case at hand, for $|k| \neq 0$,

$$\Gamma_{reg}^{0(k)} = 2\pi i\,|k| + |k|^2 + 2\,|k|\ln\Gamma(1+|k|) - 4\sum_{n=1}^{|k|} n\ln n + \beta(M_iR) \tag{6.21}$$

where

$$\beta = -2\sum_{i=1}^{r} e_i(M_iR)^2\ln(M_iR)^2 - \tfrac{1}{3}\sum_{i=1}^{r} e_i\ln(M_iR)^2 - 8\xi'(-1) \tag{6.22}$$

does not contain k, and is irrelevant because it cancels from the numerator and denominator in the expectation values. For $k = 0$, we simply have

$$\Gamma_{reg}^{0(k=0)} = \beta(M_i, R). \tag{6.23}$$

Now, the general expression (6.17) for $\Gamma_{reg}^{(k)}$ gives, by setting $\phi=0$ (and $h=1$),

$$\Gamma_{reg}^{0(k)} = 2 \ln \det N_0 + \alpha(k, M_i, R) \tag{6.24}$$

where $\det N_0 = \det N|_{\phi=0}$. Substituting back the expression for α found in this way, we finally obtain

$$\Gamma_{reg}^{(k)} = 2 \ln \left(\frac{\det N}{\det N_0}\right) + \frac{q^2}{\pi} \int_{S^2} d^2x \sqrt{g}\, \phi \Delta \phi$$
$$+ 2\pi i\, |k| + |k|^2 + 2\,|k| \ln \Gamma(1+|k|) - 4 \sum_{n=1}^{|k|} n \ln n + \beta(M_i, R) \tag{6.25}$$

Thus we have an explicit expression for the effective action, or the fermionic determinant of the theory for an arbitrary gauge field configuration characterized by the topological charge k. Note also that $\Gamma_{reg}^{(k)}$ is independent of h.

Part II

7. Expectation values

Having obtained an explicit expression for $\Gamma_{reg}^{(k)}$ (or, equivalently for the fermionic determinant) we are now in a position to calculate a number of physically interesting expectation values.

With the introduction of the new variables h and ϕ in place of A_μ ($\mu = 0, 1$) we can, first of all, write expressions (4.7), (4.8) for the expectation values of the regularized theory as

$$\langle O(h, \phi)\rangle = Z^{-1} \int [Dh][D\phi] \exp\left(\tfrac{1}{2}\Gamma_{reg}^{(k)}[\phi] - S^{(k)}[\phi]\right) O(h, \phi)|_{k=0} \tag{7.1}$$

$$\langle \bar{\psi}_\alpha(x) \psi_\beta(y)\rangle$$
$$= RZ^{-1}\Big\{-\int [Dh][D\phi] \exp\left(\tfrac{1}{2}\Gamma_{reg}^{(k)}[\phi] - S^{(k)}[\phi]\right) \mathscr{G}_{\beta\alpha}(y, x \mid \mathbb{D})|_{k=0}$$
$$- \sum_{k=\pm 1} \int [Dh][D\phi] \exp\left(\tfrac{1}{2}\Gamma_{reg}^{(k)}[\phi] - S^{(k)}[\phi]\right)(\det N)^{-1} \bar{\chi}_\beta^{(k)}(y)\Big\} \tag{7.2}$$

with

$$Z = \int [Dh][D\phi] \exp\left(\tfrac{1}{2}\Gamma_{reg}^{(k)}[\phi] - S^{(k)}[\phi]\right)|_{k=0} \tag{7.3}$$

Here $S^{(k)}[\phi]$ denotes the pure gauge part of the action for a gauge field in the topological sector k:

$$S^{(k)}[\phi] = \frac{\pi k^2}{2q^2R^2} + \frac{1}{2} \int_{S^2} d^2x \sqrt{g}\, \phi \Delta^2 \phi \tag{7.4}$$

and $\mathcal{G}(x, y \mid \mathbb{D})$ is the Green's function of the operator $\mathbb{D}$. Similar explicit expressions can be given for many other operators. We shall calculate some of them in later sections. Expectation values of fermionic operators contain explicitly the zero modes of the operator $\mathbb{D}$, which are $|k|$ in number in the topological sector k. The zero modes $\hat{\chi}_i$ of the operator $\mathbb{D}$ can be expressed in terms of the zero modes χ_i of $\mathbb{D}_0$ in the following way (see Appendix A):

$$\begin{aligned}\hat{\chi}_i(x) &= e^{-q\Gamma_5(x)\phi(x)}h(x)\chi_i(x)\\ &= e^{-q\sigma\phi(x)}h(x)\chi_i(x)\end{aligned} \tag{7.5}$$

In fact, $\sigma = \pm 1$ is the chirality of χ_i. The last step follows because all χ_i's have definite chirality.

Let us denote by χ and φ the normalized zero modes of $\mathbb{D}_0$ for $k = +1$ and $k = -1$, respectively. According to Theorem 1, they have negative and positive chirality, respectively.

$$k = +1 \qquad \mathbb{D}_0\chi = 0 \qquad (\chi, \chi) = 1 \qquad \Gamma_5\chi = -\chi \tag{7.6}$$

$$k = -1 \qquad \mathbb{D}_0\varphi = 0 \qquad (\varphi, \varphi) = 1 \qquad \Gamma_5\varphi = +\varphi \tag{7.7}$$

In fact, χ and φ may be found explicitly to be

$$\chi(x) = \frac{1}{2R\sqrt{\pi}}\begin{pmatrix} -z_2^*(x) \\ z_1(x) \end{pmatrix} \tag{7.8}$$

$$\varphi(x) = \frac{1}{2R\sqrt{\pi}}\begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} \tag{7.9}$$

where

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{7.10}$$

is the spinor (see equation (3.12)) used to construct the spherically symmetric gauge field C_μ whose components satisfy $z_1^*z_1 + z_2^*z_2 = 1$.

8. $\langle \bar{\psi}\psi \rangle$ of the Schwinger model

As the first application of the machinery we have developed so far, let us calculate $\langle \bar{\psi}\psi \rangle$ of the Schwinger model. Operator methods [2] give a non–zero value for $\langle \bar{\psi}\psi \rangle$. It is however known [7, 8] that naive attempts to obtain $\langle \bar{\psi}\psi \rangle$ using functional integral methods give the wrong result $\langle \bar{\psi}\psi \rangle = 0$. Assuming the cluster property, the correct magnitude of $\langle \bar{\psi}\psi \rangle$ has been obtained indirectly, within the path integral formulation [7]. Taking the presence of zero modes of the Dirac operator into account, here we show how to calculate $\langle \bar{\psi}\psi \rangle$ directly. In Section 10, we then also verify that the cluster property is indeed satisfied.

We first do a formal calculation to obtain $\langle \bar{\psi}\psi \rangle$ of the model. The calculation is formal because we are just setting $x = y$ in the non-gauge-invariant operator $\bar{\psi}(x)\psi(y)$, which then becomes a composite operator that has to be handled with care. In a later part of this section we calculate $\langle\bar{\psi}\psi\rangle$ in a more careful way paying proper attention to subtleties.

Define

$$\psi_L = \tfrac{1}{2}(1-\Gamma_5)\psi \qquad \bar{\psi}_L = \tfrac{1}{2}\bar{\psi}(1+\Gamma_5) \tag{8.1}$$

$$\psi_R = \tfrac{1}{2}(1+\Gamma_5)\psi \qquad \bar{\psi}_R = \tfrac{1}{2}\bar{\psi}(1-\Gamma_5) \tag{8.2}$$

Then it follows that

$$\bar{\psi}_L\psi_R = \tfrac{1}{2}\bar{\psi}(1-\Gamma_5)\psi \tag{8.3}$$

$$\bar{\psi}_R\psi_L = \tfrac{1}{2}\bar{\psi}(1+\Gamma_5)\psi \tag{8.4}$$

$$\bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L \tag{8.5}$$

$$\bar{\psi}\left(D+\frac{i}{R}\right)\psi = \bar{\psi}_L\left(D+\frac{i}{R}\right)\psi_L + \bar{\psi}_R\left(D+\frac{i}{R}\right)\psi_R \tag{8.6}$$

We can find $\langle\bar{\psi}_L\psi_R\rangle$ and $\langle\bar{\psi}_R\psi_L\rangle$ separately. We shall see that the contributions to them come exclusively from the topological sectors with $k=-1$ and $k=+1$ respectively.

To this end, first of all notice that

$$\left\langle\bar{\psi}\frac{1\mp\Gamma_5}{2}\psi\right\rangle = RZ^{-1}$$

$$\times\left\{-\int[Dh][D\phi]\exp\left(\tfrac{1}{2}\Gamma^{(k)}_{reg}[\phi]-S^{(k)}[\phi]\right)\sum_\nu\frac{1}{iE_\nu}\bar{\eta}_\nu\frac{(1\mp\Gamma_5)}{2}\eta_\nu\bigg|_{k=0}\right.$$

$$\left.-\sum_{k=\pm1}\int[Dh][D\phi](\det N)^{-1}\exp\left(\tfrac{1}{2}\Gamma^{(k)}_{reg}[\phi]-S^{(k)}[\phi]\right)\hat{\bar{\chi}}^{(k)}\frac{(1\mp\Gamma_5)}{2}\hat{\chi}^{(k)}\right\} \tag{8.7}$$

(see Appendix B – Evaluation of Grassmann integrals) where η_ν are the eigenfunctions of $\mathbb{D}$ for $k=0$ and iE_ν are the corresponding eigenvalues.

It is easy to see that the sum over ν gives zero. This follows directly from the properties of the eigenfunctions and eigenvalues of $\mathbb{D}$ (cf. Theorem 2). Furthermore, since all the zero modes have definite chirality, one of the projectors $(1\mp\Gamma_5)/2$ acting on a zero mode $\hat{\chi}^{(k)}$ always gives zero. Hence we find

$$\langle\bar{\psi}_R\psi_L\rangle = \left\langle\bar{\psi}\frac{1-\Gamma_5}{2}\psi\right\rangle$$

$$= -RZ^{-1}\int[Dh][D\phi](\det N)^{-1}\exp\left(\tfrac{1}{2}\Gamma^{(k)}_{reg}[\phi]-S^{(k)}[\phi]\right)\hat{\bar{\chi}}^{(k)}\hat{\chi}^{(k)}\Big|_{k=-1} \tag{8.8}$$

and

$$\langle \bar{\psi}_L \psi_R \rangle = \left\langle \bar{\psi} \frac{1+\Gamma_5}{2} \psi \right\rangle$$

$$= -RZ^{-1} \int [Dh][D\phi](\det N)^{-1} \exp\left(\tfrac{1}{2}\Gamma^{(k)}_{reg}[\phi] - S^{(k)}[\phi]\right) \hat{\bar{\chi}}^{(k)} \hat{\chi}^{(k)} \Big|_{k=+1} \tag{8.9}$$

i.e., only $k=-1$ and $k=+1$ contribute to $\langle \bar{\psi}_R \psi_L \rangle$ and $\langle \bar{\psi}_L \psi_R \rangle$ respectively.

Recall that $\hat{\chi}^{(k)}$ here are the zero modes of the operator $\mathbb{D}$. They have the form

$$\hat{\chi}^{(-1)} = e^{-q\phi(x)} h(x) \varphi(x) \tag{8.10}$$

$$\hat{\chi}^{(+1)} = e^{+q\phi(x)} h(x) \chi(x) \tag{8.11}$$

where φ and χ are the corresponding zero modes of $\mathbb{D}_0$.

Now substituting the explicit expressions for $\Gamma^{(k)}_{reg}$, $S^{(k)}[\phi]$ and $\chi^{(k)}$ we obtain,

$$\langle \bar{\psi}_R(x) \psi_L(x) \rangle = RZ^{-1} \int [Dh][D\phi](\det N)^{-1} \exp\left(\frac{1}{2} + \frac{\beta}{2} - \frac{\pi}{2q^2R^2}\right)$$
$$\times \exp\left(-\frac{1}{2} \int_{S^2} d^2x \sqrt{g}\, \phi \left(\Delta^2 - \frac{q^2}{\pi} \Delta\right) \phi\right)$$
$$\times \left(\frac{\det N}{\det N_0}\right)^{(k=-1)} e^{-2q\phi(x)} \frac{1}{2R^2\pi} \tag{8.12}$$

The factors $(\det N)$ cancel eath other. Furthermore,

$$\det N_0 = \det N|_{\phi=0} = (\varphi, \varphi) = 1 \tag{8.13}$$

Taking into account also the fact that

$$Z = \int [Dh][D\phi] \exp\left(\tfrac{1}{2}\Gamma^{(k)}_{reg}[\phi] - S^{(k)}[\phi]\right)\Big|_{k=0}$$

$$= \int [Dh][D\phi] e^{\beta/2} e^{-(\phi, \mathcal{O}\phi)} \tag{8.14}$$

where $\mathcal{O} = \frac{1}{2}(\Delta^2 - (q^2/\pi)\Delta)$, we thus have

$$\langle \bar{\psi}_R(x) \psi_L(x) \rangle = \frac{\exp\left(\dfrac{1}{2} - \dfrac{\pi}{2q^2R^2}\right) \displaystyle\int [D\phi] e^{-(\phi, \mathcal{O}\phi) - 2q\phi(x)}}{\displaystyle\int [D\phi] e^{-(\phi, \mathcal{O}\phi)}} \tag{8.15}$$

The h-integral drops out, for the integrands both in the numerator and in the denominator are now h-independent. Note also that the regulator masses M_i have disappeared in the last expression. The integrals both in the denominator and in the numerator are of Gaussian type and can be performed to give

$$\langle \bar{\psi}_R(x)\psi_L(x)\rangle = \frac{\exp\left(\frac{1}{2}-\frac{\pi}{2q^2R^2}\right)}{4R\pi} e^{q^2G(x,x\mid\mathcal{O})} \tag{8.16}$$

where $G(x, y \mid \mathcal{O})$ is the Green's function of the operator $\mathcal{O}$.

The eigenfunctions of the Laplace operator on S^2 are the spherical harmonics $Y_{lm}(\theta, \varphi)$ with eigenvalues $l(l+1)/R^2$; $l = 0, 1, 2, \ldots$ and $m = -l, \ldots, l$. Hence Y_{lm} also form a complete set of eigenfunctions of $\mathcal{O}$. We remark, however, that becuase of the restriction $\int_{S^2} d^2x\sqrt{g}\,\phi = 0$, ϕ can always be expanded in the set of *non-zero*, modes. Hence the path integral is performed only over these modes. As a result, in the Green's function the mode corresponding to $l = 0$ is excluded. Thus, we have

$$\begin{aligned} G(x, y \mid \mathcal{O}) &= \sum_{\substack{l\neq 0\\ m=-l,\ldots,l}} \frac{Y_{lm}(\theta, \varphi)Y^*_{lm}(\theta', \varphi')}{\frac{l(l+1)}{2R^2}\left\{\frac{l(l+1)}{2R^2}+\frac{q^2}{\pi}\right\}} \\ &= \sum_{l\neq 0} \frac{(2l+1)R^2}{2\pi l(l+1)\left\{l(l+1)+\frac{q^2R^2}{\pi}\right\}} P_l(\cos\omega) \\ &= \frac{1}{2q^2} F(\omega) \end{aligned} \tag{8.17}$$

where

$$F(\omega) = \frac{q^2R^2}{\pi} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)\left\{l(l+1)+\frac{q^2R^2}{\pi}\right\}} P_l(\cos\omega) \tag{8.18}$$

Here (θ, φ) and (θ', φ') represent the polar coordinates corresponding to the points x^μ, y^μ $(\mu = 0, 1)$ on the sphere. ω is the angle between $\mathbf{r}(x)$ and $\mathbf{r}(y)$ and $P_l(x)$ is the Legendre polynomial of order l.

Thus we have

$$\langle \bar{\psi}_R(x)\psi_L(x)\rangle = \frac{\exp\left(\frac{1}{2}-\frac{\pi}{2q^2R^2}\right)}{4R\pi} e^{F(0)/2} \tag{8.19}$$

Obviously for $\langle \bar{\psi}_L\psi_R\rangle$ we get exactly the same result:

$$\langle \bar{\psi}_L\psi_R\rangle = \frac{\exp\left(\frac{1}{2}-\frac{\pi}{2q^2R^2}\right)}{4R\pi} e^{F(0)/2} \tag{8.20}$$

The properties of sums of the type (8.18) are discussed in the Appendix D. In particular, when $R \to \infty$, i.e., when we come to the limit of the usual flat-space Schwinger model we have the asymptotic expansion

$$F(0) \overset{R\to\infty}{\sim} 2\gamma - 1 + 2\ln\left(\frac{q}{\sqrt{\pi}}R\right) + O\left(\frac{1}{R^2}\right) \tag{8.21}$$

where $\gamma = 0.577\cdots$ is the Euler constant. Hence in that limit we finally get

$$\langle \bar{\psi}_R \psi_L \rangle = \langle \bar{\psi}_L \psi_R \rangle = \frac{e^{\gamma}}{4\pi}\frac{q}{\sqrt{\pi}} \tag{8.22}$$

These results are identical to the results obtained by other methods [2].

8.1. $\langle \bar{\psi}\psi \rangle$ *as the limit of a gauge invariant operator*

Now we want to do the same calculation more carefully. Instead of the gauge-*non-invariant* operator $\langle \bar{\psi}(x)\psi(y)\rangle$, we now start with the gauge-*invariant* operator $\langle \bar{\psi}(x)U(x, y)\psi(y)\rangle$ and set

$$\langle \bar{\psi}\psi \rangle = \lim_{x\to y} \langle \bar{\psi}(x)U(x, y)\psi(y)\rangle \tag{8.23}$$

Here $U(x, y)$ is the phase factor,

$$U(x, y) = \exp\left(-iq\int_y^x dz^{\mu}A_{\mu}(z)\right) \tag{8.24}$$

The presence of U in the expectation value introduces an extra term to the exponent in the path integral. However, since A_μ depends *linearly* on ϕ, the integral remains Gaussian so that the path integral can be evaluated. In place of equation (4.8) we now have

$$\begin{aligned}
&\langle \bar{\psi}_\alpha(x)U(x, y)\psi_\beta(y)\rangle \\
&\quad = RZ^{-1}\Big\{-\int [Dh][D\phi]\exp\left(\tfrac{1}{2}\Gamma^{(k)}_{reg}[\phi] - S^{(k)}[\phi]\right) \\
&\qquad \times \exp\left(-iq\int_y^x dz^{\mu}A_{\mu}(z)\right)\mathcal{G}_{\beta\alpha}(y, x \mid \mathbb{D})\big|_{k=0} \\
&\qquad - \sum_{k=\pm 1}\int [Dh][D\phi](\det N)^{-1} \\
&\qquad \times \exp\left(\tfrac{1}{2}\Gamma^{(k)}_{reg}[\phi] - S^{(k)}[\phi]\right)\exp\left(-iq\int_y^x dz^{\mu}A_{\mu}(z)\right)\hat{\bar{\chi}}^{(k)}_{\alpha}(x)\hat{\chi}^{(k)}_{\beta}(y)\Big\}
\end{aligned} \tag{8.25}$$

Denote by $I(x, y)$ the line integral:

$$
\begin{aligned}
I(x, y) &= \int_y^x dz^\mu A_\mu(z) \\
&= \int_y^x dz^\mu\Big(kC_\mu + \sqrt{g}\,\epsilon_{\mu\nu}g^{\nu\rho}\,\partial_\rho\phi + \frac{1}{iq}h\,\partial_\mu h^{-1}\Big) \\
&= kI_C(x, y) + I_\phi(x, y) + \frac{1}{iq}\ln\frac{h(y)}{h(x)}
\end{aligned}
\tag{8.26}
$$

where we have also defined $I_C(x, y) = \int_y^x dz^\mu C_\mu(z)$, which is independent of both h and ϕ, and $I_\phi = \int_y^x dz^\mu\sqrt{g}\,\epsilon_{\mu\nu}g^{\nu\rho}\,\partial_\rho\phi$ which has the sole ϕ-dependence. Recalling that

$$\hat{\chi}^{(-1)} = e^{-q\phi}h\varphi \tag{8.27}$$

$$\hat{\chi}^{(+1)} = e^{+q\phi}h\chi \tag{8.28}$$

we now rewrite the above expectation value in terms of ϕ and h:

$$
\begin{aligned}
&\langle\bar{\psi}_\alpha(x)U(x, y)\psi_\beta(y)\rangle \\
&\quad= RZ^{-1}\Big\{-\int[Dh][D\phi]\exp\big(\tfrac{1}{2}\Gamma^{(k)}_{reg}[\phi] - S^{(k)}[\phi] - iqI_\phi[\phi]\big) \\
&\qquad\times\frac{h(x)}{h(y)}\,\mathcal{G}_{\beta\alpha}(y, x \mid \mathbb{D})\big|_{k=0} - e^{-iqI_C}\int[Dh][D\phi](\det N)^{-1} \\
&\qquad\times\exp\big(\tfrac{1}{2}\Gamma^{(k)}_{reg}[\phi] - S^{(k)}[\phi] - iqI_\phi[\phi]\big)\exp\big(+q(\phi(x) + \phi(y))\big)\bar{\chi}_\alpha(x)\chi_\beta(y)\big|_{k=+1} \\
&\qquad- e^{+iqI_C}\int[Dh][D\phi](\det N)^{-1}\exp\big(\tfrac{1}{2}\Gamma^{(k)}_{reg}[\phi] - S^{(k)}[\phi] - iqI_\phi[\phi]\big) \\
&\qquad\times\exp\big(-q(\phi(x) + \phi(y))\big)\bar{\varphi}_\alpha(x)\varphi_\beta(y)\big|_{k=-1}\Big\}
\end{aligned}
\tag{8.29}
$$

Let us denote the three terms in this sum by α_0, α_+ and α_-, respectively. For instance,

$$
\begin{aligned}
\alpha_+ = -RZ^{-1}e^{-iqI_C}\int[Dh][D\phi](\det N)^{-1}\exp\big(\tfrac{1}{2}\Gamma^{(k)}_{reg}[\phi] - S^{(k)}[\phi] - iqI_\phi[x, y]\big) \\
\times\exp\big(+q(\phi(x) + \phi(y))\big)\bar{\chi}_\alpha(x)\chi_\beta(y)\big|_{k=+1}
\end{aligned}
\tag{8.30}
$$

8.1.1. *Evaulation of α_+ and α_-.* Expressing $\Gamma^{(k)}_{reg}$, S and Z in terms of h and ϕ, we get

$$\alpha_+ = R\exp\Big(\frac{1}{2} - \frac{\pi}{2q^2R^2}\Big)e^{-iqI_C}\bar{\chi}_\alpha(x)\chi_\beta(y)\,\frac{\int[D\phi]e^{-(\phi,\mathcal{O}\phi) - iqI_\phi + q(\phi(x)+\phi(y))}}{\int[D\phi]e^{-(\phi,\mathcal{O}\phi)}} \tag{8.31}$$

To evaluate the path integrals we first notice that for products of Gaussian integrals, the following is true:

$$\frac{\int \prod_n da_n \exp\left(-\sum_n (\epsilon_n a_n^2 + b_n a_n)\right)}{\int \prod_n da_n \exp\left(-\sum_n \epsilon_n a_n^2\right)} = \exp\left(\sum_n \frac{b_n^2}{4\epsilon_n}\right) \tag{8.32}$$

where ϵ_n and b_n are independent of a_n.

In our case we have a similar expression when we expand the real field $\phi(x)$ in a set of real eigenfunctions of the Laplace operator on S^2. As mentioned earlier, the zero mode is excluded.

Suppose $\phi(x) = \sum_n a_n \phi_n$ where $\{\phi_n\}$ is a complete, orthonormal set of real eigenfunctions of L with non zero eigenvalues and let $\mathcal{O}\phi_n = \epsilon_n \phi_n$. Then we have

$$(\phi, \mathcal{O}\phi) = \sum_n \epsilon_n a_n^2 \tag{8.33}$$

$$\phi(x) + \phi(y) = \sum_n a_n(\phi_n(x) + \phi_n(y)) \tag{8.34}$$

$$I_\phi[\phi] = \sum_n a_n\left(\int_y^x dz^\mu \sqrt{g}\, \epsilon_{\mu\nu} g^{\nu\rho}\, \partial_\rho^z \phi_n(z)\right) \tag{8.35}$$

so that we have precisely the same Gaussian integral as above when we identify

$$b_n = iq \int_y^x dz^\mu \sqrt{g}\, \epsilon_{\mu\nu} g^{\nu\rho}\, \partial_\rho^z \phi_n(z) - q(\phi_n(x) + \phi_n(y)) \tag{8.36}$$

to find the result of our path integral we now only have to calculate $\sum_n (b_n^2/4\epsilon_n)$. Recalling that the Green's function of the operator $\mathcal{O}$ was defined as

$$G(x, y \mid \mathcal{O}) = \sum_n \frac{\phi_n(x)\phi_n(y)}{\epsilon_n} \tag{8.37}$$

where the zero mode is excluded from the sum, we get, after some algebra,

$$\begin{aligned}\sum_n (b_n^2/4\epsilon_n) = \frac{q^2}{4}\Big[&2G(0, 0) + 2G(x, y) \\ &- 2i \int_y^x dz^\mu \sqrt{g}\, \epsilon_{\mu\nu} g^{\nu\rho}\, \partial_\rho^z (G(x, z) + G(y, z)) \\ &- \int_y^x dz^\mu \sqrt{g}\, \epsilon_{\mu\nu} g^{\nu\rho}\, \partial_\rho^z \int_y^x dz'^{\mu'} \sqrt{g'}\, \epsilon_{\mu'\nu'} g'^{\nu'\rho'}\, \partial_{\rho'}^{z'} G(z, z')\Big]\end{aligned} \tag{8.38}$$

We have used (and shall use hereafter) the shorthand notation $G(x, y)$ for $G(x, y \mid \mathcal{O})$, and g' denotes the metric tensor in z' coordinates.

Consider a typical integral in the above expression;

$$I = \int_y^x dz^\mu \sqrt{g}\, \epsilon_{\mu\nu} g^{\nu\rho}\, \partial_\rho^z G(x, z) \tag{8.39}$$

To evaluate this integral, let us choose the path $x \to y$ to be a geodesic on the sphere. Then z lies always on the same geodesic, as we are evaluating the *line* integral along the path $x \to y$ on the geodesic. Furthermore, we can fix one of the points x and y on the sphere. Let us fix x to be the north pole. It is convenient to use spherical polar coordinates for the explicit calculation.

$$x \leftrightarrow (0, \varphi) \tag{8.40}$$

$$z \leftrightarrow (\theta, \varphi) \tag{8.41}$$

Then $G(x, z)$ depends only on θ, so we write

$$G(\theta) = G(x, z) = \sum_{l=1}^{\infty} \frac{(2l+1)R^2}{2\pi l(l+1)\left\{l(l+1) + \frac{q^2R^2}{\pi}\right\}} P_l(\cos\theta) \tag{8.42}$$

For I, in polar coordinates we now have,

$$I = \int_y^x d\theta \sqrt{g}\, \epsilon_{\theta\varphi} g^{\varphi\varphi} \frac{d}{d\varphi} G(\theta) = 0 \tag{8.43}$$

Similarly, the integral containing $G(y, z)$ is also zero. Recalling that both points z and z' have to lie on the $x \to y$ geodesic, we immediately see that the last double integral also gives zero.

Now we can give the final result for α_+:

$$\alpha_+ = R \exp\left(\frac{1}{2} - \frac{\pi}{2q^2R^2} - iqI_C\right) \bar{\chi}_\alpha(x) \chi_\beta(y) \exp\left(\frac{q^2}{2}(G(0, 0) + G(x, y))\right) \tag{8.44}$$

It is easy to see that the path integral in α_- gives the same result as the path integral in α_+ which we have just calculated. Since I_C is also zero, we can directly write

$$\alpha_- = R \exp\left(\frac{1}{2} - \frac{\pi}{2q^2R2} - iqI_C\right) \bar{\varphi}_\alpha(x) \varphi_\beta(y) \exp\left(\frac{q^2}{2}(G(0, 0) + G(x, y))\right) \tag{8.45}$$

Thus the contribution from the sectors $k = \pm 1$ is given by

$$\begin{aligned} &\langle \bar{\psi}_\alpha(x) U(x, y) \psi_\beta(y) \rangle \big|_{|k|=1} \\ &\quad = \alpha_+ + \alpha_- \\ &\quad = R \exp\left(\frac{1}{2} - \frac{\pi}{2q^2R^2}\right) (\bar{\chi}_\alpha(x) \chi_\beta(y) e^{-iqI_c} \\ &\qquad - \bar{\varphi}_\alpha(x) \varphi_\beta(y) e^{+iqI_C}) \exp\left(\frac{q^2}{2}(G(0, 0) + G(x, y))\right) \end{aligned} \tag{8.46}$$

From the definition (8.23) it follows that, for the sectors $k = \pm 1$,

$$\langle \bar{\psi}\psi \rangle = \lim_{x \to y} \langle \bar{\psi}(x) U(x, y) \psi(y) \rangle$$

$$= R \exp\left(\frac{1}{2} - \frac{\pi}{2q^2R^2}\right)(\bar{\chi}_\alpha(x)\chi_\beta(y) + \bar{\varphi}_\alpha(x)\varphi_\beta(y))e^{q^2G(0,0)} \tag{8.47}$$

where we have used the fact that $\lim_{x \to y} I_C = 0$. The explicit expressions for $\chi(x)$ and $\varphi(x)$ (see equations (7.8), (7.9)) give

$$\bar{\chi}(x)\chi(x) = \bar{\varphi}(x)\varphi(x) = \frac{1}{4R^2\pi} \tag{8.48}$$

Hence we get

$$\langle \bar{\psi}\psi \rangle \big|_{|k|=1} = \frac{\exp\left(\frac{1}{2} - \frac{\pi}{2q^2R^2}\right)}{2R\pi} e^{F(0)/2} \tag{8.49}$$

where $G(0, 0)$ has been replaced by $F(0)$ according to the definition $G(x, y) = F(\omega)/2q^2$ (see equation (8.17)). Note that this is just twice the value we had for $\langle \bar{\psi}_R \psi_L \rangle$. In the limit as $R \to \infty$ one thus gets

$$\langle \bar{\psi}\psi \rangle \big|_{|k|=1} = \frac{e^\gamma}{2\pi} \frac{q}{\sqrt{\pi}} \tag{8.50}$$

What now remains to be calculated is α_0.

8.1.2. *Evaluation of* α_0. In order to complete the calculation of $\langle \bar{\psi} U \psi \rangle$ we still have to find α_0. To this end, we first obtain an expression for $\mathscr{G}_{\beta\alpha}(y, x \mid \mathbb{D})$ and then substituting this we evaluate the path integral.

The Green's function $\mathscr{G}$ *of* $\mathbb{D}$

Recall that

$$\mathbb{D} = R\tilde{D} \tag{8.51}$$

where

$$\tilde{D} = \Gamma^\mu(\partial_\mu + iqA_\mu) + \frac{i}{R} \tag{8.52}$$

We also have

$$A_\mu = kC_\mu + \sqrt{g}\, \epsilon_{\mu\nu} g^{\nu\rho}\, \partial_\rho \phi + \frac{1}{iq} h\, \partial_\mu h^{-1} \tag{8.53}$$

It is possible to find $\mathscr{G}(x, y \mid \mathbb{D})$ explicitly for any given k, ϕ and h. This is done in Appendix C using the stereographic coordinates on the sphere. Here we merely state the result for $k = 0$:

$$\mathscr{G}(x, y \mid \mathbb{D}) = \frac{1}{R} \frac{h(x)}{h(y)} u^{-1}(x) e^{-q\sigma_3\phi(x)} S(x, y) e^{-q\sigma_3\phi(y)} u(y) \tag{8.54}$$

Here $S(x, y)$ is the Green's function of the operator D of the general formalism, in the absence of any gauge field:

$$S(x, y) = \frac{1}{2\pi} (\Omega_x \Omega_y)^{-1/4} \frac{\sigma_a(x^a - y^a)}{|\mathbf{x} - \mathbf{y}|^2} \tag{8.55}$$

where Ω is the conformal factor defined by $g_{\mu\nu}(x) = \Omega(x)\delta_{\mu\nu}$ in stereographic coordinates and $u(x)$ the unitary matrix which relates the operator $\mathbb{D}$ in $SU(2)$-invariant formalism to the Dirac operator D in the general formalism. It is also calculated in Appendix C:

$$u(x) = a_0(x)(1 + i\alpha_i(x)\sigma^i) \tag{8.56}$$

with

$$a_0 = \frac{R}{\sqrt{2(R^2 + x^2)}} \qquad \alpha_1 = \frac{x_1 - x_2}{R} \qquad \alpha_2 = \frac{x_1 + x_2}{R} \qquad \alpha_3 = 1 \tag{8.57}$$

Calculation of α_0

In fact, what we are interested in is the *trace* of the matrix $\langle \bar\psi_\alpha(x) U(x, y) \psi_\beta(y) \rangle$. Therefore, we only need $\operatorname{tr} \mathcal{G}(y, x \mid \mathbb{D}) = \mathcal{G}_{\alpha\alpha}(y, x \mid \mathbb{D})$. Since the traces of the Pauli matrices are zero, the only contribution comes from that term of $\mathcal{G}$ which is proportional to the unit matrix. A straight forward calculation yields

$$\mathcal{G}_{\alpha\alpha}(y, x \mid \mathbb{D}) = -\frac{1}{R} \frac{h(y)}{h(x)} (\Omega(y)\Omega(x))^{-1/4} a_0(y) a_0(x) \frac{2i}{\pi R} \cosh \theta_{yx} \tag{8.58}$$

where $\theta_{xy} = q(\phi(x) - \phi(y))$. The explicit use of Ω and a_0 also gives the relation $\Omega^{-1/4}(x) a_0(x) = 1/2$ so that we have

$$\mathcal{G}_{\alpha\alpha}(y, x \mid \mathbb{D}) = -\frac{i}{2\pi R^2} \frac{h(y)}{h(x)} \cosh \theta_{yz} \tag{8.59}$$

For $\operatorname{tr} \alpha_0$, we thus obtain

$$\begin{aligned} \operatorname{tr} \alpha_0 &= \frac{i}{2\pi R} e^{\beta/2} Z^{-1} \int [Dh][D\phi] e^{-(\phi, \mathcal{O}\phi) - iqI_\phi[x,y]} \cosh \theta_{yx} \\ &= \frac{i}{2\pi R} e^{\beta/2} Z^{-1} \int [Dh][D\phi] e^{-(\phi, \mathcal{O}\phi) - iqI_\phi[x,y] + q(\phi(y) - \phi(x))} \end{aligned} \tag{8.60}$$

Proceeding exactly the same was as was done for α_+ we can evaluate the path integral to get

$$\operatorname{tr} \alpha_0 = \frac{i}{2\pi R} \exp\left(\frac{q^2}{2} [G(0, 0) - G(x, y)]\right) \tag{8.61}$$

where $G(x, y)$ is again the Green's function of the operator $\mathcal{O}$. Denoting the geodesic distance between the two points x and y by s we can write for large R,

$$G(0,0)-G(x,y)=G(0)-G(\theta)$$
$$=-\frac{1}{q^2}\left[\ln 2-\gamma-\ln\frac{qs}{\sqrt{\pi}}\right]-K_0\left(\frac{qs}{\sqrt{\pi}}\right) \qquad (8.62)$$

Thus

$$\alpha_0=\frac{i}{2\pi R}\frac{1}{\sqrt{2}}e^{\gamma/2}\left(\frac{qs}{\sqrt{\pi}}\right)\exp\left(\tfrac{1}{2}K_0\left(\frac{qs}{\sqrt{\pi}}\right)\right) \qquad (8.63)$$

and as $s\to 0$,

$$\alpha_0\to\frac{i}{2\pi R}\frac{1}{\sqrt{2}}e^{\gamma/2}\sqrt{2}\,e^{-\gamma/2}=\frac{i}{2\pi R}. \qquad (8.64)$$

For finite R we obviously have a non-zero value for α_0. However, it is proportional to $1/R$ so that for $\langle\bar{\psi}\psi\rangle$ we do not get any contribution in the flat-space limit.

Now that we have calculated all three terms α_0, α_+ and a_- of 8.29 we can write down the final result for $\langle\bar{\psi}\psi\rangle$ in the flat space limit:

$$\langle\bar{\psi}\psi\rangle=\lim_{R\to\infty}\lim_{x\to y}\langle\bar{\psi}_\alpha(x)U(x,y)\psi_\beta(y)\rangle$$
$$=\lim_{R\to\infty}\lim_{x\to y}(\alpha_0+\alpha_++\alpha_-)$$
$$=\frac{e^\gamma}{2\pi}\frac{q}{\sqrt{\pi}} \qquad (8.65)$$

This completes the rigorous calculation of $\langle\bar{\psi}\psi\rangle$, giving no different result than in the previous formal one in the $R\to\infty$ limit. The non-zero value of $\langle\bar{\psi}\psi\rangle$ indicates the breakdown of chiral symmetry. It is known that this is due to a $U(1)$ anomaly present in the theory.

9. $\langle F_{01}(x)F_{01}(y)\rangle$: Interpretation as a meson theory

Another interesting quantity with direct physical significance is the two point function of the field strength operator F_{01}. The path integral remains Gaussian in the presence of $F_{01}(x)F_{01}(y)$ so we can again evaluate the expectation value explicitly. Recall that (see equation (3.19))

$$F_{01}=k\frac{\sqrt{g}}{2qR^2}-\sqrt{g}\,\Delta\phi \qquad (9.1)$$

Instead of F_{01}, we will use the more convenient quantity

$$\chi(x)=\frac{F_{01}(x)}{\sqrt{g(x)}}=\frac{k}{2qR^2}-\Delta\phi(x) \qquad (9.2)$$

The expectation value of $\chi(x)\chi(y)$ is given by

$$\langle\chi(x)\chi(y)\rangle = \frac{\sum_k \int [Dh][D\phi] \exp\left(\frac{1}{2}\Gamma^{(k)}_{reg}[\phi] - S^{(k)}[\phi]\right)\chi(x)\chi(y)}{\int [Dh][D\phi] \exp\left(\frac{1}{2}\Gamma^{(k)}_{reg}[\phi] - S^{(k)}[\phi]\right)\Big|_{k=0}} \tag{9.3}$$

In fact, only $k=0$ sector contributes, as our operator does not contain any Fermi fields (see equation (4.7)). Thus

$$\langle\chi(x)\chi(y)\rangle = \frac{\int [Dh][D\phi] \exp\left(\frac{1}{2}\Gamma^{(k)}_{reg}[\phi] - S^{(k)}[\phi]\right)\Delta_x\phi(x)\Delta_y\phi(y)}{\int [Dh][D\phi] \exp\left(\frac{1}{2}\Gamma^{(k)}_{reg}[\phi] - S^{(k)}[\phi]\right)}\Bigg|_{k=0}$$

$$= \Delta_x\Delta_y \frac{\int [D\phi] e^{-(\phi,\mathcal{O}\phi)}\phi(x)\phi(y)}{\int [D\phi] e^{-(\phi,\mathcal{O}\phi)}}$$

Here $\mathcal{O} = \frac{1}{2}(\Delta^2 - (q^2/\pi)\Delta)$ as before. We can perform the path integral by expanding ϕ in a complete set of real eigenfunctions of the Laplace operator Δ.

$$\phi(x) = \sum_n a_n \phi_n \tag{9.4}$$

The path integral is now defined over the expansion coefficients a_n:

$$\int [D\phi] e^{-(\phi,\mathcal{O}\phi)}\phi(x)\phi(y) \to \int \prod_n da_n e^{-\epsilon_n a_n^2} \sum_{i,j} \phi_i(x)\phi_j(y) a_i a_j$$
$$= \sum_{i,j} \phi_i(x)\phi_j(y) \int \prod_n da_n e^{-\epsilon_n a_n^2} a_i a_j \tag{9.5}$$

Here ϵ_n is given by $\mathcal{O}\phi_n = \epsilon_n \phi_n$. If $i \neq j$ we have in the product two odd-integrals of the form $\int_{-\infty}^{\infty} dx e^{-\alpha x^2} x$ and the whole product becomes zero. If $i=j$, all the integrals are simple Gaussian integrals, except the one

$$\int_{-\infty}^{\infty} da_i e^{-\epsilon_n a_n^2} a_i^2$$

which gives $(1/2\epsilon_i)\sqrt{\pi/\epsilon_i}$. Thus

$$\int \prod_n da_n e^{\epsilon_n a_n^2} a_i a_j = \delta_{ij} \frac{1}{2\epsilon_i} \frac{\pi^{N/2}}{(\det \mathcal{O})^{1/2}} \tag{9.6}$$

where N denotes the number of eigen modes, and $\det \mathcal{O}$ is the product of all eigenvalues. However, the factor $\pi^{N/2}/(\det \mathcal{O})^{1/2}$ is exactly cancelled by a similar factor coming from the path integral in the denominator. We thus have

$$\langle \chi(x)\chi(y)\rangle = \Delta_x \Delta_y \sum_i \frac{\phi_i(x)\phi_i(y)}{2\epsilon_i}. \tag{9.7}$$

The real eigenfunctions of Δ, written here rather symbolically as ϕ_n have in fact two indices, like in $\mathcal{Y}_{lm}(\theta, \varphi)$. Let us denote them by $f_{lr}(x)$. They are obtained from $\mathcal{Y}_{lm}$ by making linear combinations. More precisely, for a given l, the functions f_{lr} ($r = 1, \ldots, 2l + 1$) are obtained by linearly combining $\mathcal{Y}_{l,-l}, \ldots, \mathcal{Y}_{l,l}$ which span a subspace $\mathcal{R}_l$. Thus we have

$$\Delta_x f_{lr}(x) = -\frac{l(l+1)}{R^2} f_{lr}(x) \qquad r = 1, \ldots, 2l+1 \tag{9.8}$$

Expressed in terms of f_{lr}, the above two point function is given by

$$\langle \chi(x)\chi(y)\rangle = \Delta_x \Delta_y \sum_{r=1}^{2l+1} \frac{f_{lr}(x) f_{lr}(y)}{2\epsilon_l} \tag{9.9}$$

where

$$\epsilon_l = \frac{1}{2}\frac{l(l+1)}{R^2}\left\{\frac{l(l+1)}{R^2} + \frac{q^2}{\pi}\right\} \tag{9.10}$$

Notice also that, since Δ is a hermitian operator the change of basis between $\{f_{lr}\}$ and $\{\mathcal{Y}_{lm}\}$, is achieved by a constant *unitary* matrix in each subspace $\mathcal{R}_l$. Hence,

$$\begin{aligned}\sum_{r=1}^{2l+1} f_{lr}(x) f_{lr}(y) &= \sum_{m=-l}^{l} \mathcal{Y}^*_{lm}(x)\mathcal{Y}_{lm}(y) \\ &= \frac{1}{R^2}\sum_{m=-l}^{l} Y^*_{lm}(x) Y_{lm}(y) \\ &= \frac{2l+1}{4\pi R^2} P_l(\cos\omega)\end{aligned} \tag{9.11}$$

where ω is the angle between $\mathbf{r}(x)$ and $\mathbf{r}(y)$. Here we have identified the points x and y with (θ, φ) and (θ', γ') respectively.

Letting Δ_y in (9.9) act on the argument, we get

$$\begin{aligned}\langle \chi(x)\chi(y)\rangle &= \Delta_x \sum_{l=1}^{\infty} -\frac{l(l+1)}{2R^2\epsilon_l}\sum_{r=1}^{2l+1} f_{lr}(x) f_{lr}(y) \\ &= -\Delta_x \sum_{l=1}^{\infty} \frac{(l+1)}{2R^2\epsilon_l}\frac{2l+1}{4\pi R^2} P_l(\cos\omega)\end{aligned} \tag{9.12}$$

Let us also fix y to be the north pole so that $\cos\omega = \cos\theta$. It follows that

$$\langle \chi(x)\chi(0)\rangle = -\Delta_x S_b(\theta) \tag{9.13}$$

where

$$S_b(\theta) = \frac{1}{4\pi}\sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)/bR^2} P_l(\cos\theta) \tag{9.14}$$

with $b = q^2/\pi$. Properties of $S_b(\theta)$ are studied in detail in Appendix D. Here we only use the results obtained there. Since in polar coordinates

$$\Delta = \frac{1}{R^2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\} \tag{9.15}$$

we can write

$$\langle\chi(x)\chi(0)\rangle = -\frac{1}{R^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}S_b(\theta)\right). \tag{9.16}$$

The flat-space limit is obtained by letting $R \to \infty$ as $s = R\theta$ is held fixed, s being the geodesic distance between the point x and the north pole. This yields

$$\langle\chi(x)\chi(0)\rangle \overset{R\to\infty}{=} -\frac{b}{2\pi}K_0(\sqrt{b}\,s) \tag{9.17}$$

The function K_0 is actually proportional to the free Green's function, i.e.,

$$\langle\chi(x)\chi(0)\rangle = -m\int_{-\infty}^{\infty}\frac{d^2p}{(2\pi)^2}\frac{e^{ipx}}{p^2+m^2} \tag{9.18}$$

where $m = b = q/\sqrt{\pi}$, and x^μ are flat space coordinates such that $x^2 = s$. When continued back to Minkowski space, one gets

$$\langle 0 \mid F_{01}(x)F_{01}(0) \mid 0\rangle = +m\int_{-\infty}^{\infty}\frac{dp}{2\pi 2p_0}e^{-ip_0x^0+ipx^1} \tag{9.19}$$

where $p_0 = \sqrt{m^2+p^2}$. Thus, F_{01} is a free field describing a massive pseudoscalar particle. In particular, this particle has no interactions with itself or with any other particle which could be in the theory.

10. The cluster property

The cluster property states that the vacuum matrix element of a product of local operators factorizes when their space-like separations become large. This property has been well established for massive theories. In this section we show that also for the Schwinger model, where we have a massless fermion interacting with a gauge field, this property holds.

We start by calculating the four-point function $\langle\bar\psi(x)\psi(x)\bar\psi(y)\psi(y)\rangle$. This poses no big complications; the techniques we have developed so far can be applied in a straightforward way. Nevertheless, the results turn out to be quite illuminating. The necessity of considering *all* topological sectors, not just the one with topological charge zero, becomes clear. In fact, it will be proven that the sector $k = 0$ gives only one half of the value expected. The sector $|k| = 2$ provide the other half whereas $|k| = 1$ sectors give no contribution in the $R \to \infty$ limit.

Notice that the expectation value of $\bar\psi(x)\psi(x)\bar\psi(y)\psi(y)$ is given by [cf.

equation (4.3)]

$$\langle \bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y)\rangle = R^2 Z^{-1} \sum_{k=-\infty}^{\infty} \int_{\mathcal{A}_k} [D\bar{\eta}][D\eta][DA_\mu]$$

$$\times \exp\left(-S[A_\mu] - \int_{S^2} d^2x\sqrt{g}\,\bar{\eta}\mathbb{D}\eta\right)\bar{\eta}(x)\eta(x)\bar{\eta}(y)\eta(y) \tag{10.1}$$

In fact, we must be considering an operator like $\bar{\psi}(x)U(x, z)\psi(z)\bar{\psi}(y)U(y, t)\psi(t)$ which is the product of two gauge invariant operators and take the limit $x \to z$, $y \to t$. However, the experience with the calculation of the two point function tells us that the result we get will be the same. Thus we directly calculate $\langle \bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y)\rangle$. As explained in detail in the Appendix B, we have to consider each topological sector separately since, depending on the number of zero modes of $\mathbb{D}$ (which is equal to the absolute value of the topological charge), the fermionic integral takes a different shape.

10.1 *Contribution from the $k = 0$ sector*

The fermionic part of the path integral in this case is given by (see equation (B-38))

$$I_4^0 = \int [D\bar{\eta}][D\eta]e^{-(\eta,\mathbb{D}\eta)}\bar{\eta}(x)\eta(x)\bar{\eta}(y)\eta(y)$$

$$= \det \mathbb{D}\{\mathcal{G}^{\alpha\alpha}(x, x)\mathcal{G}^{\beta\beta}(y, y) - \mathcal{G}^{\alpha\beta}(x, y)\mathcal{G}^{\beta\alpha}(y, x)\} \tag{10.2}$$

For the sake of simplicity, let us consider the two terms separately. Define

$$P = R^2 Z^{-1}\int [Dh][D\phi] \det \mathbb{D}e^{-S[\phi]} \operatorname{tr} \mathcal{G}(x, x) \operatorname{tr} \mathcal{G}(y, y) \tag{10.3}$$

$$Q = R^2 Z^{-1}\int [Dh][D\phi] \det \mathbb{D}\mathcal{G}^{\alpha\beta}(x, y)\mathcal{G}^{\beta\alpha}(y, x) \tag{10.4}$$

Then we have the contribution for the 4-point function from the sector $k = 0$ as

$$\langle \bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y)\rangle|_{k=0} = P - Q \tag{10.5}$$

Recall that

$$\det \mathbb{D} = \exp\left(\tfrac{1}{2}\Gamma_{reg}^{(k=0)}\right) \tag{10.6}$$

$$\Gamma_{reg}^{k=0} = \frac{q^2}{\pi}\int_{S^2} d^2x\sqrt{g}\,\phi\Delta\phi + \beta(M_i, R) \tag{10.7}$$

$$S[\phi]|_{k=0} = \frac{1}{2}\int_{S^2} d^2x\sqrt{g}\,\phi\Delta^2\phi \tag{10.8}$$

$$Z = \int [Dh][D\phi] \exp\left(\tfrac{1}{2}\Gamma_{reg}^{(k=0)} - S[\phi]\right)$$

$$= e^{\beta/2}\int [Dh][D\phi]e^{-(\phi,\mathcal{O}\phi)} \tag{10.9}$$

where

$$\mathcal{O}=\frac{1}{2}\left(\Delta^2-\frac{q^2}{\pi}\Delta\right) \tag{10.10}$$

Also recall that (equation (C-40))

$$\mathcal{G}(x, y)=\frac{1}{R}u^{-1}(x)h(x)e^{-q\sigma_3\phi(x)}S(x, y)e^{-q\sigma_3\phi(y)}h^{-1}(y)u(y) \tag{10.11}$$

where

$$S(x, y)=\frac{1}{2\pi}(\Omega(x)\Omega(y))^{-1/4}\frac{\sigma_a(x^a-y^a)}{|\mathbf{x}-\mathbf{y}|^2} \tag{10.12}$$

Using the explcit expression for u we find

$$\mathcal{G}(x, y)=\frac{i}{4\pi R^3}\frac{h(x)}{h(y)}\frac{1}{|\mathbf{x}-\mathbf{y}|^2}\begin{pmatrix} Ry^+(x^--y^-)e^{-\Phi} & -R^2(x^--y^-)e^{-\Phi} \\ -Rx^-(x^+-y^+)e^{+\Phi} & -x^-y^-(x^+-y^+)e^{+\Phi} \\ x^+y^+(x^--y^-)e^{-\Phi} & -Rx^+(x^--y^-)e^{-\Phi} \\ +R^2(x^+-y^+)e^{+\Phi} & +Ry^-(x^+-y^+)e^{+\Phi} \end{pmatrix} \tag{10.13}$$

where Φ stands for $q(\phi_x-\phi_y)$. Thus

$$\operatorname{tr}\mathcal{G}(x, y)= -\frac{i}{4\pi R^2}\frac{h(x)}{h(y)}(e^{-\Phi}+e^{+\Phi}) \tag{10.14}$$

Hence we have

$$\operatorname{tr}\mathcal{G}(x, x)=\operatorname{tr}\mathcal{G}(y, y)= -\frac{i}{2\pi R^2} \tag{10.15}$$

Furthermore,

$$\mathcal{G}^{\alpha\beta}(x, y)\mathcal{G}^{\beta\alpha}(y, x)=\left(\frac{i}{4\pi R^3}\right)^2\frac{1}{|\mathbf{x}-\mathbf{y}|^2}(R^2+\mathbf{x}^2)(R^2+\mathbf{y}^2)(e^{+2\Phi}+e^{-2\Phi}) \tag{10.16}$$

It follows that

$$P=R^2Z^{-1}\left(\frac{-i}{2\pi R^2}\right)^2\int[Dh][D\phi]\exp\left(\tfrac{1}{2}\Gamma_{reg}^{(k=0)}-S^{(k=0)}[\phi]\right)$$

$$= -\frac{1}{4\pi^2R^2} \tag{10.17}$$

and

$$Q = 2R^2 Z^{-1}\left(\frac{i}{4\pi R^3}\right)^2 \frac{(R^2+\mathbf{x}^2)(R^2+\mathbf{y}^2)}{|\mathbf{x}-\mathbf{y}|^2}$$

$$\times \int [Dh][D\phi] \exp\left(\tfrac{1}{2}\Gamma_{reg}^{(k=0)} - S^{(k=0)}[\phi]\right) \exp\left(2q(\phi(x)-\phi(y))\right)$$

$$= -\frac{1}{8\pi^2 R^4}\frac{(R^2+\mathbf{x}^2)(R^2+\mathbf{y}^2)}{|\mathbf{x}-\mathbf{y}|^2} \exp\left(q^2\{G(x,x)+G(y,y)-2G(x,y)\}\right) \tag{10.18}$$

Here G stands for the Green's function of the operator $\mathcal{O}$. Let us fix the point y to be the north pole; i.e., $y^a = 0$, and let s be the geodesic distance between x and y. If $s = R\theta$ we can use the notation in the Appendix D to write

$$Q = -\frac{1}{8\pi^2 R^2}\frac{R^2+\mathbf{x}^2}{|\mathbf{x}|^2} e^{\{F(0)-F(\theta)\}} \tag{10.19}$$

Now let us fix s and let $R \to \infty$. (θ will also go to zero, however, $s = R\theta$ remaining fixed). In this limit, we also have

$$|\mathbf{x}| = \frac{s}{2} + O\left(\frac{1}{R}\right) \tag{10.20}$$

The contribution from P is clearly zero whereas, using the results of Appendix D, for Q we get

$$Q \overset{R\to\infty}{=} -\frac{1}{8\pi^2} b e^{2\gamma + 2K_0(\sqrt{b}\, s)} \tag{10.21}$$

where $b = q^2/\pi$. Thus the sole contribution to the 4-point function from the $k = 0$ sector in the flat-space limit is given by

$$\langle \bar\psi(x)\psi(x)\bar\psi(y)\psi(y)\rangle|_{k=0} \overset{R\to\infty}{=} \frac{1}{8\pi^2} b e^{2\gamma + 2K_0(\sqrt{b}\, s)} \tag{10.22}$$

As the distance between the two points x and y is now made large, i.e., $s \to \infty$, $K_0(\sqrt{b}\, s)$ goes to zero and we have

$$\lim_{s\to\infty}\left\{\lim_{R\to\infty} Q\right\} = -\frac{e^{2\gamma}}{8\pi^2}\left(\frac{q^2}{\pi}\right) \tag{10.23}$$

Thus finally,

$$\lim_{s\to\infty}\left\{\lim_{R\to\infty} \langle \bar\psi(x)\psi(x)\bar\psi(y)\psi(y)\rangle|_{k=0}\right\} = \frac{e^{2\gamma}}{8\pi^2}\left(\frac{q^2}{\pi}\right) \tag{10.24}$$

10.2. *Contribution from the $k = \pm 1$ sectors*

The contribution from the sectors $k = \pm 1$ is given by

$$\langle \bar{\psi}(x)\psi(y)\bar{\psi}(z)\psi(t)\rangle|_{|k|=1} = R^2 Z^{-1} \sum_{|k|=1} \int_{\mathscr{A}_k} [D\bar{\eta}][D[\eta][DA_\mu]$$
$$\times \exp\left(-S[A_\mu] - \int_{S^2} d^2x\sqrt{g}\,\bar{\eta}\mathbb{D}\eta\right)\bar{\eta}(x)\eta(y)\bar{\eta}(z)\eta(t) \tag{10.25}$$

In each sector, the operator $\mathbb{D}$ has one zero mode. The fermionic integral gives (see equation (B-39)) in each sector,

$$\int [D\bar{\eta}][D\eta] e^{-(\eta,\mathbb{D}\eta)}\bar{\eta}^{\alpha}(x)\eta^{\beta}(y)\bar{\eta}^{\gamma}(z)\eta^{\delta}(t)$$
$$= \det{}' \mathbb{D}(\det N)^{-1}\{\hat{\bar{\chi}}^{\alpha}(x)\hat{\chi}^{\beta}(y)\mathscr{G}^{\delta\gamma}(t,z) + \mathscr{G}^{\beta\alpha}(y,x)\hat{\bar{\chi}}^{\gamma}(z)\hat{\chi}^{\delta}(t)$$
$$- \hat{\bar{\chi}}^{\gamma}(z)\hat{\chi}^{\beta}(y)\mathscr{G}^{\delta\alpha}(t,x) - \mathscr{G}^{\beta\gamma}(y,z)\hat{\bar{\chi}}^{\alpha}(x)\hat{\chi}^{\delta}(t)\} \tag{10.26}$$

where $\hat{\chi}$ and $\mathscr{G}$ are the corresponding zero mode and the Green's function of the operator $\mathbb{D}$, respectively.

For $k = +1$, these two quantities are given by (see Appendix C)

$$\hat{\chi}^{(+1)}(x) = e^{+q\phi(x)}h(x)\chi(x) \tag{10.27}$$

$$\chi(x) = \frac{1}{2R\sqrt{\pi}}\begin{pmatrix} -z_2^* \\ z_1 \end{pmatrix} = \frac{1}{2R\sqrt{\pi}\sqrt{R^2+\mathbf{x}^2}}\begin{pmatrix} -x^- \\ R \end{pmatrix} \tag{10.28}$$

$$\mathscr{G}(x,y) = \frac{1}{R}u^{-1}(x)h(x)e^{-q\sigma_3\phi(x)}$$
$$\times \frac{1}{4\pi R^2}\frac{R^2 + x^- y^+}{|\mathbf{x}-\mathbf{y}|^2}\sigma_a(x^a - y^a)e^{-q\sigma_3\phi(y)}h^{-1}(y)u(y) \tag{10.29}$$

whereas for $k = -1$ we have

$$\hat{\chi}^{(-1)} = e^{-q\phi(x)}h(x)\varphi(x) \tag{10.30}$$

$$\varphi(x) = \frac{1}{2R\sqrt{\pi}}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{2R\sqrt{\pi}\sqrt{R^2+\mathbf{x}^2}}\begin{pmatrix} R \\ x^+ \end{pmatrix} \tag{10.31}$$

$$\mathscr{G}(x,y) = \frac{1}{R}u^{-1}(x)h(x)e^{-q\sigma_3\phi(x)}$$
$$\times \frac{1}{4\pi R^2}\frac{R^2 + x^+ y^-}{|\mathbf{x}-\mathbf{y}|^2}\sigma_a(x^a - y^a)e^{-q\sigma_3\phi(y)}h^{-1}(y)u(y) \tag{10.32}$$

Consider a typical term in (10.25) after the fermionic integration:

$$I^{\alpha_1\alpha_2\alpha_3\alpha_4}(x_1,x_2,x_3,x_4) = R^2 Z^{-1}\int [Dh][D\phi]\det{}'\mathbb{D}(\det N)^{-1}$$
$$\times \hat{\bar{\chi}}^{\alpha_1}(x_1)\mathscr{G}^{\alpha_2\alpha_3}(x_2,x_3)\hat{\chi}^{\alpha_4}(x_4)e^{-S[\phi]} \tag{10.33}$$

In terms of I, we can write

$$\langle \bar{\psi}^{\alpha}(x)\psi^{\beta}(y)\bar{\psi}^{\gamma}(z)\psi^{\delta}(t)\rangle\big|_{|k|=1} = \sum_{k=\pm 1} \{I^{\alpha\delta\gamma\beta}(x, t, z, y) + I^{\gamma\beta\alpha\delta}(z, y, x, t) - I^{\gamma\delta\alpha\beta}(z, t, x, y) - I^{\alpha\beta\gamma\delta}(x, y, z, t)\} \tag{10.34}$$

Recall that,

$$\det{}' \mathbb{D} = \exp\left(\tfrac{1}{2}\Gamma^{(k)}_{reg}\right) \tag{10.35}$$

$$\Gamma^{(k=\pm 1)}_{reg} = 2 \ln\left(\frac{\det N}{\det N_0}\right)^{(k=\pm 1)} + \frac{q^2}{\pi}\int_{S^2} d^2x\sqrt{g}\,\phi\Delta\phi + 2\pi i \tag{10.36}$$

so that

$$I^{\alpha_1\alpha_2\alpha_3\alpha_4}(x_1, x_2, x_3, x_4) = -R^2 Z^{-1} \exp\left(\frac{1}{2} + \frac{\beta}{2} - \frac{\pi}{2q^2R^2}\right)\bar{\Psi}^{\alpha_1}(x_1)\Psi^{\alpha_4}(x_4) \times \int [Dh][D\phi]\frac{h(x_4)}{h(x_1)}\frac{h(x_2)}{h(x_3)} \exp\left(\pm q(\phi(x_1) + \phi(x_4)) - (\phi, \mathcal{O}\phi)\right)\mathcal{G}^{\alpha_2\alpha_3}(x_2, x_3) \tag{10.37}$$

where $\pm$ sign in front of $q\phi$ corresponds to the cases $\Psi = \chi$ and $\Psi = \varphi$, respectively.

For $k = +1$ we thus have

$$\begin{aligned} I^{\alpha_1\alpha_2\alpha_3\alpha_4}&(x_1, x_2, x_3, x_4) \\ &= -R^2 Z^{-1} \exp\left(\frac{1}{2} + \frac{\beta}{2} - \frac{\pi}{2q^2R^2}\right)\bar{\chi}^{\alpha_1}(x_1)\chi^{\alpha_4}(x_4) \\ &\times \int [Dh][D\phi]\frac{h(x_4)}{h(x_1)}\frac{h(x_2)}{h(x_3)}\, e^{+q(\phi_{x_1}+\phi_{x_4}) - (\phi,\mathcal{O}\phi)} \\ &\times \left\{\frac{1}{R}u^{-1}(x_2)e^{-q\sigma_3\phi_{x_2}}\frac{1}{4\pi R^2}\frac{R^2 + x_2^- x_3^+}{|\mathbf{x}_2 - \mathbf{x}_3|^2}\,\sigma_a(x_2^a - x_3^a)e^{-q\sigma_3\phi_{x_3}}u(x_3)\right\} \end{aligned} \tag{10.38}$$

and for $k = -1$,

$$\begin{aligned} I^{\alpha_1\alpha_2\alpha_3\alpha_4}&(x_1, x_2, x_3, x_4) \\ &= -R^2 Z^{-1} \exp\left(\frac{1}{2} + \frac{\beta}{2} - \frac{\pi}{2q^2R^2}\right)\bar{\varphi}^{\alpha_1}(x_1)\varphi^{\alpha_4}(x_4) \\ &\times \int [Dh][D\phi]\frac{h(x_4)}{h(x_1)}\frac{h(x_2)}{h(x_3)} \exp\left(-q(\phi_{x_1} + \phi_{x_4}) - (\phi, \mathcal{O}\phi)\right) \\ &\times \left\{\frac{1}{R}u^{-1}(x_2)e^{-q\sigma_3\phi_{x_2}}\frac{1}{4\pi R^2}\frac{R^2 + x_2^+ x_3^-}{|\mathbf{x}_2 - \mathbf{x}_3|^2}\,\sigma_a(x_2^a - x_3^a)e^{-q\sigma_3\phi_{x_3}}u(x_3)\right\} \end{aligned} \tag{10.39}$$

Using the explicit expression for u (see Appendix C) we obtain

$$u^{-1}(x_2)e^{-q\sigma_3\phi_{x_2}}\sigma_a(x_2^a - x_3^a)e^{-q\sigma_3\phi_{x_3}}u(x_3)$$

$$= \frac{i}{2R^2}(\Omega_{x_2}\Omega_{x_3})^{1/4}\begin{pmatrix} Rx_3^+(x_2^- - x_3^-)e^{-\Phi} & -R^2(x_2^- - x_3^-)e^{-\Phi} \\ -Rx_2^-(x_2^+ - x_3^+)e^{+\Phi} & -x_2^-x_3^-(x_2^+ - x_3^+)e^{+\Phi} \\ x_2^+x_3^+(x_2^- - x_3^-)e^{-\Phi} & -Rx_2^+(x_2^- - x_3^-)e^{-\Phi} \\ +R^2(x_2^+ - x_3^+)e^{+\Phi} & -Rx_3^-(x_2^+ - x_3^+)e^{+\Phi} \end{pmatrix}$$

$$= \frac{i}{2R^2}(\Omega_{x_2}\Omega_{x_3})^{1/4}A \tag{10.40}$$

where Φ stands for $q(\phi_{x_2} - \phi_{x_3})$ and the matrix A is defined via the last step.

In fact, we are interested in the quantity $\langle\bar{\psi}(x)\psi(y)\bar{\psi}(z)\psi(t)\rangle$; i.e., we can set $\alpha = \beta$, $\gamma = \delta$. From the equation (10.34) we thus have

$$\langle\bar{\psi}(x)\psi(y)\bar{\psi}(z)\psi(t)\rangle|_{|k|=1} = \sum_{k=\pm 1}\{I^{\alpha\gamma\gamma\alpha}(x, t, z, y) + I^{\gamma\alpha\alpha\gamma}(z, y, x, t) - I^{\gamma\gamma\alpha\alpha}(z, t, x, y) - I^{\alpha\alpha\gamma\gamma}(x, y, z, t)\} \tag{10.41}$$

For $k = +1$,

$$I^{\alpha\gamma\gamma\alpha}(x_1, x_2, x_3, x_4) = R^2Z^{-1}\exp\left(\frac{1}{2} + \frac{\beta}{2} - \frac{\pi}{2q^2R^2}\right)\int[Dh[D\phi]\frac{h(x_4)}{h(x_1)}\frac{h(x_2)}{h(x_3)}$$
$$\times \exp(+q(\phi_{x_1} + \phi_{x_4}) - (\phi, \mathcal{O}\phi))\frac{1}{4\pi R^3}\frac{R^2 + x_2^-x_3^+}{|\mathbf{x}_2 - \mathbf{x}_3|^2}\frac{i}{2R^2}(\Omega_{x_2}\Omega_{x_3})^{1/4}$$
$$\times (\bar{\chi}^1(x_1)\chi^1(x_4) + \bar{\chi}^2(x_2)\chi^2(x_4))(A_{11} + A_{22}) \tag{10.42}$$

Noteice that

$$A_{11} + A_{22} = -R\,|\mathbf{x}_2 - \mathbf{x}_3|^2\,(e^{-\Phi} + e^{+\Phi}) \tag{10.43}$$

$$\bar{\chi}^1(x_1)\chi^1(x_4) = \frac{1}{4\pi R^2}z_2(x_1)z_2^*(x_4) \tag{10.44}$$

$$\bar{x}^2(x_1)\chi^2(x_4) = \frac{1}{4\pi R^2}z_1^*(x_1)z_1(x_4) \tag{10.45}$$

Thus

$$I^{\alpha\gamma\gamma\alpha}(x_1, x_2, x_3, x_4) = Z^{-1}\exp\left(\frac{1}{2} + \frac{\beta}{2} - \frac{\pi}{2q^2R^2}\right)\int[Dh][D\phi]\frac{h(x_4)}{h(x_1)}\frac{h(x_2)}{h(x_3)}$$
$$\times \frac{i}{8\pi R^2}(R^2 + x_2^- + x_3^+)(\Omega_{x_2}\Omega_{x_3})^{1/4}\frac{1}{4\pi R^2}(z_2(x_1)z_2^*(x_4) + z_1(z_1)z_1^*(x_4))$$
$$\times \exp(-(\phi, \mathcal{O}\phi) + q(\phi_{x_1} + \phi_{x_4}))(e^{-\Phi} + e^{+\Phi}) \tag{10.46}$$

Similarly,

$$I^{\alpha\alpha\gamma\gamma}(x_1, x_2, x_3, x_4)$$
$$= -R^2Z^{-1}\exp\left(\frac{1}{2}+\frac{\beta}{2}-\frac{\pi}{2q^2R^2}\right)\int[Dh][D\phi]\frac{h(x_4)}{h(x_1)}\frac{h(x_2)}{h(x_3)}$$
$$\times\exp\left(+q(\phi_{x_2}+\phi_{x_4})-(\phi, \mathcal{O}\phi)\right)\frac{1}{4\pi R^2}\frac{R^2+x_2^-x_3^+}{|\mathbf{x}_2-\mathbf{x}_3|^2}\cdot\frac{i}{2R^2}(\Omega_{x_2}\Omega_{x_3})^{1/4}$$
$$\times\{\bar{\chi}^1(x_1)\chi^4(x_1)A_{11}+\bar{\chi}^1(x_1)\chi^2(x_4)A_{12}+\bar{\chi}^2(x_1)\chi^1(x_4)A_{21}+\bar{\chi}^2(x_1)\chi^2(x_4)A_{22}\} \tag{10.47}$$

Analogous results are obtained for $k = -1$ also. Knowing that there is no singularity of $I^{\alpha\gamma\gamma\alpha}(x_1, x_2, x_3, x_4)$ for $x_2 = x_3$, as seen from (10.46), we can furthermore set $x = y$, $z = t$ in (10.41):

$$\langle\bar{\psi}(x)\psi(x)\bar{\psi}(t)\psi(t)\rangle|_{|k|=1} = \sum_{k=\pm1}\{I^{\alpha\gamma\gamma\alpha}(x, t, t, x) + I^{\gamma\alpha\alpha\gamma}(t, x, x, t)$$
$$+ I^{\gamma\gamma\alpha\alpha}(t, t, x, x) + I^{\alpha\alpha\gamma\gamma}(x, x, t, t)\} \tag{10.48}$$

Let us again concentrate on the case $k = +1$. Let us also fix the point t to be the north pole; i.e., $t^a = 0$. Using the explicit form for χ (equation (10.28)), and performing the Gaussian integral over ϕ, we now obtain

$$I^{\alpha\gamma\gamma\alpha}(x, 0, 0, x) = \frac{i}{8\pi^2R^2}\exp\left(\frac{1}{2}-\frac{\pi}{2q^2R^2}\right)\exp\left(\tfrac{1}{2}F(0)\right) = I^{\gamma\alpha\alpha\gamma}(0, x, x, 0) \tag{10.49}$$

Similarly, using (10.47) one obtains,

$$I^{\alpha\alpha\gamma\gamma}(x, x, 0, 0) = 0 = I^{\gamma\gamma\alpha\alpha}(0, 0, x, x) \tag{10.50}$$

Analogous calculations for the case $k = -1$ yield

$$I^{\alpha\gamma\gamma\alpha}(x, 0, 0, x) = \frac{i}{8\pi^2R^2}\exp\left(\frac{1}{2}-\frac{\pi}{2q^2R^2}\right)\exp\left(\tfrac{1}{2}F(0)\right) = I^{\gamma\alpha\alpha\gamma}(0, x, x, 0) \tag{10.51}$$

$$I^{\alpha\alpha\gamma\gamma}(x, x, 0, 0) = 0 = I^{\gamma\gamma\alpha\alpha}(0, 0, x, x) \tag{10.52}$$

Putting all these results together, from the equation (10.48) we thus get

$$\langle\bar{\psi}(x)\psi(x)\bar{\psi}(t)\psi(t)\rangle|_{|k|=1} = \frac{1}{2\pi^2R^2}\exp\left(\frac{1}{2}-\frac{\pi}{2q^2R^2}\right)\exp\left(\tfrac{1}{2}F(0)\right) \tag{10.53}$$

Since for large R, $F(0) = 2\ln((q/\sqrt{\pi})R) + 2\gamma - 1 + O(1/R^2)$, in the flat-space limit we have

$$\langle\bar{\psi}(x)\psi(x)\bar{\psi}(t)\psi(t)\rangle|_{|k|=1} \sim \frac{1}{R} \tag{10.54}$$

i.e., the conttibution from the sectors $k = \pm1$ is zero.

10.3. *Contribution from the $k = \pm 2$ sectors*

In this case, we have

$$\langle \bar{\psi}(x)\psi(y)\bar{\psi}(z)\psi(t)\rangle = R^2 Z^{-1} \sum_{k=\pm 2} \int_{\mathcal{A}_k} [D\bar{\eta}][D\eta][DA_\mu] \times \exp\left(-S[A_\mu^k] - \int_{S^2} d^2x\sqrt{g}\,\bar{\eta}\mathbb{D}\eta\right)\bar{\eta}(x)\eta(y)\bar{\eta}(z)\eta(t) \quad (10.55)$$

For $|k| = 2$, the operator $\mathbb{D}$ has two zero modes in each of the sectors $k = +2$ and $k = -2$. Performing the Gaussian integral over fields $\bar{\eta}$ and η (see equation (B-42)) we get,

$$\int [D\bar{\eta}][D\eta]e^{-(\eta,\mathbb{D}\eta)}\bar{\eta}(x)\eta(y)\bar{\eta}(z)\eta(t)|_{k=-2}$$
$$= \det \mathbb{D}^{k=-2}(\det N)^{-1}\{\hat{\bar{\chi}}^0(x)\hat{\chi}^0(y)\hat{\bar{\chi}}^1(z)\hat{\chi}^1(t) + \hat{\bar{\chi}}^1(x)\hat{\chi}^1(y)\hat{\bar{\chi}}^0(z)\hat{\chi}^0(t) - \hat{\bar{\chi}}^0(x)\hat{\chi}^1(y)\hat{\bar{\chi}}^1(z)\hat{\chi}^0(t) - \hat{\bar{\chi}}^1(x)\hat{\chi}^0(y)\hat{\bar{\chi}}^0(z)\hat{\chi}^1(t)\} \quad (10.56)$$

where $\hat{\chi}^0$ and $\hat{\chi}^1$ are two independent zero modes of $\mathbb{D}$ for $k = -2$. As mentioned in Appendix A they have the form

$$\hat{\chi}^i(x) = e^{-q\phi(x)}h(x)\chi^i(x); \qquad i = 0, 1 \quad (10.57)$$

where χ^i $(i = 0, 1)$ denote two independent zero modes of $\mathbb{D}_0$. Two orthonormal zero modes of $\mathbb{D}_0$ are explicitly found to be

$$\chi^0(x) = \frac{1}{R\sqrt{2\pi}} z_1(x)\begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} \quad (10.58)$$

$$\chi^1(x) = \frac{1}{R\sqrt{2\pi}} z_2(x)\begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} \quad (10.59)$$

Similarly for $k = +2$ we have

$$\int [D\bar{\eta}][D\eta]e^{-(\eta,\mathbb{D}\eta)}\bar{\eta}(x)\eta(y)\bar{\eta}(z)\eta(t)|_{k=+2}$$
$$= \det \mathbb{D}^{k=+2}(\det N)^{-1}\{\hat{\bar{\varphi}}^0(x)\hat{\varphi}^0(y)\hat{\bar{\varphi}}^1(z)\hat{\varphi}^1(t) + \hat{\bar{\varphi}}^1(x)\hat{\varphi}^1(y)\hat{\bar{\varphi}}^0(z)\hat{\varphi}^0(t) - \hat{\bar{\varphi}}^0(x)\hat{\varphi}^1(y)\hat{\bar{\varphi}}^1(z)\hat{\varphi}^0(t) - \hat{\bar{\varphi}}^1(x)\hat{\varphi}^0(y)\hat{\bar{\varphi}}^0(z)\hat{\varphi}^1(t)\} \quad (10.60)$$

where $\hat{\varphi}^0$ and $\hat{\varphi}^1$ are two independent zero modes of $\mathbb{D}$ for $k = +2$. Now we have

$$\hat{\varphi}^i(x) = e^{+q\phi(x)}h(x)\varphi^i(x); \qquad i = 0, 1 \quad (10.61)$$

where two orthonormal zero modes of $\mathbb{D}_0$ are given by

$$\varphi^0(x) = \frac{1}{R\sqrt{2\pi}} z_2^*(x) \begin{pmatrix} -z_2^*(x) \\ z_1(x) \end{pmatrix} \tag{10.62}$$

$$\varphi^1(x) = \frac{1}{R\sqrt{2\pi}} z_1(x) \begin{pmatrix} -z_2^*(x) \\ z_1(x) \end{pmatrix} \tag{10.63}$$

Recall that

$$\det \mathbb{D}^{(k)} = \exp\left(\tfrac{1}{2}\Gamma_{reg}^{(k)}\right) \tag{10.64}$$

$$\Gamma_{deg}^{(k)} = 2\ln\left(\frac{\det N}{\det N_0}\right)^{(k)} + \frac{q^2}{\pi}\int_{S^2} d^2x\sqrt{g}\,\phi\Delta\phi$$
$$+ 2\pi i\,|k| + |k|^2 + 2\,|k| \ln\Gamma(1+|k|) - \sum_{n=1}^{|k|} n\ln n + \beta(M_i, R) \tag{10.65}$$

Here, N is the zero mode matrix; $N_0 = N|_{\phi=0}$. Thus

$$\det \mathbb{D}^{(k=\pm 2)} = \left(\frac{\det N}{\det N_0}\right)^{(k=\pm 2)} \exp\left(\frac{q^2}{2\pi}\int_{S^2} d^2x\sqrt{g}\,\phi L\phi + 2 - 2\ln 2 + \frac{\beta}{2}\right) \tag{10.66}$$

When these values are substituted in the above expressions for the path integrals, the factor $\det N$ drops out. Notice also that in each case $\det N_0 = 1$, since the entries of N_0 are scalar products of *orthonormal* functions. Replacing $[DA_\mu]$ by $[Dh][D\phi]$, and inserting the explicit expressions for the zero modes, we thus get

$$\langle \bar\psi(x)\psi(x)\bar\psi(y)\psi(y)\rangle|_{|k|=2} = \frac{Z^{-1}}{4\pi R^2}\int [Dh][D\phi] e^{-S^{(|k|=2)}[\phi]}$$
$$\times \exp\left(\frac{q^2}{2\pi}\int_{S^2} d^2x\sqrt{g}\,\phi\Delta\phi + 2 - 2\ln 2 + \frac{\beta}{2}\right)[e^{-2q(\phi(x)+\phi(y))} + e^{+2q(\delta(x)+\phi(y))}]$$
$$\times \{(z_1^*z_1)(x)(z_2^*z_2)(y) + (z_2^*z_2)(x)(z_1^*z_1)(y)$$
$$- (z_1^*z_2)(x)(z_2^*z_1)(y) - (z_2^*z_1)(x)(z_1^*z_2)(y)\} \tag{10.67}$$

Substituting $\phi \to -\phi$, we also see that the contribution from the two terms are equal. Thus we obtain, after dropping the h-integral

$$\langle \bar\psi(x)\psi(x)\bar\psi(y)\psi(y)\rangle = \frac{1}{2\pi^2 R^2}\exp\left(-\frac{2\pi}{q^2R^2} + 2 - 2\ln 2\right)$$
$$\times \{(z_1^*z_1)(x)(z_2^*z_2)(y) + (z_2^*z_2)(x)(z_1^*z_1)(y)$$
$$- (z_1^*z_2)(x)(z_2^*z_1)(y) - (z_2^*u_1)(x)(z_1^*z_2)(y)\}$$
$$+ \frac{\int [D\phi] e^{-(\phi,\mathcal{O}\phi) - 2q(\phi(x)+\phi(y))}}{\int [D\phi] e^{-(\phi,\mathcal{O}\phi)}} \tag{10.68}$$

The result of the path integral is given by

$$I = \frac{\int [D\phi] e^{-(\phi, \mathcal{O}\phi) - 2q(\phi(x)+\phi(y))}}{\int [D\phi] e^{-(\phi, \mathcal{O}\phi)}}$$
$$= \exp\left(q^2\{G(x, x) + G(y, y) + 2G(x, y)\}\right)$$
$$= \exp\left([F(\theta) + F(0)]\right) \tag{10.69}$$

where we have identified $x \leftrightarrow (\theta, \varphi)$ and y with the north pole. We are also using the notation of Appendix D.

If we now fix the geodesic distance between the two points to be $s = R\theta$ and let $R \to \infty$ (and $\theta \to 0$), using the results of Appendix D we obtain

$$I = \frac{(\sqrt{b}\,R)^4}{(\sqrt{b}\,s)^2} \exp\left(2 \ln 2 + 2\gamma - 2 - 2K_0(\sqrt{b}\,s)\right) \tag{10.70}$$

where $b = q^2/2\pi$. Using the explicit form of z_1 and z_2 we can also easily find that

$$\{(z_1^* z_1)(x)(z_2^* z_2)(y) + (z_2^* z_2(x)(z_1^* z_1)(y) - (z_1^* z_2)(x)(z_2^* z_1)(y) - (z_2^* z_1)(x)(z_1^* z_2)(y)\} = \sin^2 \frac{\theta}{2} \tag{10.71}$$

which is equal to $s^2/4R^2$ in the above limit.

Putting these results together, and neglecting terms which go to zero as $R \to \infty$, we get

$$\langle \bar{\psi}(x)\psi(x)\bar{\psi}(0)\psi(0)\rangle|_{|k|=2} \overset{R\to\infty}{=} \frac{b}{8\pi^2} \exp\left(2\gamma - 2K_0(\sqrt{b}\,s)\right) \tag{10.72}$$

As the distance s between the two points is now made large, we get an analogous equation to equation (10.24)

$$\lim_{s\to\infty}\left\{\lim_{R\to\infty} \langle \bar{\psi}(x)\psi(x)\bar{\psi}(0)\psi(0)\rangle|_{|k|=2}\right\} = \frac{e^{2\gamma}}{8\pi^2}\left(\frac{q^2}{\pi}\right) \tag{10.73}$$

This is the contribution to the above 4-point function from the topological sectors $k = +2$ and $k = -2$, in the limit $s \to \infty$.

Adding contributions from all sectors (in fact only $k = 0$ and $k = \pm 2$ sectors contribute), we finally get

$$\lim_{s\to\infty}\left\{\lim_{R\to\infty} \langle \bar{\psi}(x)\psi(x)\bar{\psi}(0)\psi(0)\rangle\right\} = \frac{e^{2\gamma}}{4\pi^2}\left(\frac{q^2}{\pi}\right) \tag{10.74}$$

Comparing with the value obtained (8.65) for the two-point function $\langle \bar{\psi}\psi \rangle$, we now see that in the flat space limit the cluster property is indeed satisfied, i.e.,

$$\langle \bar{\psi}(x)\psi(x)\bar{\psi}(0)\psi(0)\rangle \overset{s\to\infty}{=} \langle \bar{\psi}\psi\rangle^2 \tag{10.75}$$

11. Summary and conclusions

In this work we have shown how to use the functional integral method to solve the Euclidean Schwinger model on S^2. Apart from being a generalization to the ordinary model, the compact manifold S^2 also enables one to proceed in a mathematically more satisfactory way. Since the relevant differential operators have discrete spectra, one can use their eigenfunctions to define the path integrals properly. The radius R of the sphere plays the role of an infrared cutoff; we use Pauli–Villars regulators to remove the ultraviolet divergences occurring in the fermionic path integral. On S^2, the model is still exactly solvable. It is, of course, essential that the mass of the fermion is zero and the Pauli–Villars regulator masses are sent to infinity. We have obtained all the results for finite R, before taking the $R\to\infty$ limit.

The model nicely illustrates the relevance of the notion of topology in field theory. Abelian gauge fields on S^2 can be classified according to their topological charge. The path integral decomposes into a sum of path integrals involving gauge fields of fixed topological charge $k=0, \pm1, \pm2, \ldots$. On the other hand, the non-trivial gauge field topology implies the occurrence of fermionic zero modes. For gauge fields of topological charge k, the massless Dirac operator possesses exactly $|k|$ zero modes which require special treatment. We have obtained explicit expressions for the propagator and the fermionic effective action in the presence of an external gauge field of topological charge k.

Since the effective fermionic action turns out to be quadratic in the gauge field, the functional integral over this field does not present any problem. We have explicitly calculated several expectation values of physical interest. As it is seen in these examples, only a limited number of topological sectors contribute to a given expectation value. Thus, for the two-point function of the field strength $\langle F_{01}(x)F_{01}(0)\rangle$, the only contribution comes from the $k=0$ sector. For the two- and four-point functions of fermion fields, however, all the sectors with $|k|\le 1$ and $|k|\le 2$, respectively, contribute. In general, one can say that if the operator under consideration contains a product of n pairs of $(\bar{\psi}, \psi)$, then the contributions come from the sectors with $|k|\le n$. Some of these contributions vanish in the flat-space limit.

In contrast to the earlier calculations using the path integral method, where the presence of the zero modes were not properly accounted for, we get the non-zero value $(e^{\gamma}/2\pi)(q/\sqrt{\pi})$ for $\langle\bar{\psi}\psi\rangle$. This agrees with the value obtained by operator methods. After calculating the two-point function of the field strength, we have finally calculated, the four-point function $\langle \bar{\psi}\psi(x)\bar{\psi}\psi(0)\rangle$ in Section 10. The cluster property, which has been always assumed in earlier works to obtain $\langle\bar{\psi}\psi\rangle$, is directly verified here.

Part III

A. Properties of the operators $\mathbb{D}$ and $\mathbb{D}_0$

In this Appendix we study the general features of the operators $\mathbb{D}$ and $\mathbb{D}_0$. Recall that

$$\mathbb{D} = R\left(\mathscr{D} + \frac{i}{R}\right) \tag{A-1}$$

where $\mathscr{D} = \Gamma^\mu \mathscr{D}_\mu = \Gamma^\mu(\partial_\mu + iqA_\mu)$. We will also use the representation

$$A_\mu = kC_\mu + \sqrt{g}\,\epsilon_{\mu\nu} g^{\nu\rho}\,\partial_\rho \phi + \frac{1}{iq} h\,\partial_\mu h^{-1} \tag{A-2}$$

for the gauge field A_μ. The operator $\mathbb{D}_0$ is obtained from $\mathbb{D}$ by setting $\phi = 0$, $h = 1$. For the operator $\mathbb{D}_0$, one can obtain the spectrum, zero-modes, etc. Although the spectrum of $\mathbb{D}$ cannot be given explicitly, the zero modes of $\mathbb{D}$ may be given in terms of the zero modes of $\mathbb{D}_0$.

A.1. *The spectrum of the operator* $\mathbb{D}$

Notice that $i\mathbb{D}$ is a hermitean operator with respect to the scalar product

$$(\psi, \chi) = \int_{S^2} d^2x \sqrt{g}\,\bar{\psi}(x)\chi(x) \tag{A-3}$$

where ψ and χ are 2-component spinors. If

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \tag{A-4}$$

$\bar{\psi}$ denotes the row vector (ψ^{1*}, ψ^{2*}). Furthermore, since the elliptic operator $i\mathbb{D}$ is defined on the *compact* manifold S^2, its eigenvalues are discrete.

Let $\{\eta_\nu\}$ $(\nu = 1, 2, \ldots)$ be a set of independent eigenfunctions of $i\mathbb{D}$ with *positive* eigenvalues E_ν. Since $\mathbb{D}$ anticommutes (see equation (4.9)) with Γ_5 it follows that $\Gamma_5\eta_\nu$ are also independent eigenfunctions with eigenvalues $-E_\nu$. Let us denote $\eta_{-\nu} = \Gamma_5\eta_\nu$, $E_{-\nu} = -E_\nu$. Together with the zero modes, the set $\{\eta_\nu\}$, $\nu = \pm 1, \pm 2, \ldots$ forms a complete set of eigenfunctions of $i\mathbb{D}$.

In other words, the *anti*-hermitean operator $\mathbb{D}$ has the independent eigenfunctions η_ν $(\nu = \pm 1, \pm 2, \ldots)$ with corresponding non-zero eigenvalues iE_ν, where $\eta_{-\nu} = \Gamma_5\eta_\nu$ and $E_{-\nu} = -E_\nu$, and the same zero modes as $i\mathbb{D}$.

Zero-modes of $\mathbb{D}$

Since $\mathbb{D}$ anti-commutes with Γ_5 we can choose the zero modes to have definite chirality. Suppose we have chosen such a basis in the zero-mode subspace. Define

$$T = \mathbb{D}|_{\mathscr{H}_+} \qquad T^\dagger = -\mathbb{D}|_{\mathscr{H}_-} \tag{A-5}$$

where $\mathcal{H}_+$ and $\mathcal{H}_-$ represent the positive and negative chirality subspaces, respectively. Thus the zero mode equations become

$$T\chi = 0 \quad \text{or} \quad T^\dagger \chi = 0 \tag{A-6}$$

depending on whether χ belongs to $\mathcal{H}_+$ or $\mathcal{H}_-$. Now, if ϕ is varied, and χ is also varied according to

$$\delta\chi = -q\delta\phi\chi \quad \text{or} \quad \delta\chi = +q\delta\phi\chi \tag{A-7}$$

respectively, the zero mode equations (A-6) remain invariant. It follows that the numbers of zero modes n_+ and n_- in $\mathcal{H}_+$ and $\mathcal{H}_-$ are independent of ϕ, and may hence be determined at $\phi = 0$. This is done explicitly in the next section, and the result is,

$$\begin{aligned} &n_+ = 0 \qquad n_- = |k| \quad \text{if} \quad k \geq 0 \\ &n_+ = |k| \qquad n_- = 0 \quad \text{if} \quad k \leq 0 \end{aligned} \tag{A-8}$$

The form of the zero modes may also be easily found. If χ_i are the zero modes of $\mathbb{D}_0 = \mathbb{D}(\phi = 0, h = 1)$, then the zero modes of $\mathbb{D}$ are given by

$$\begin{aligned} \hat{\chi}_i(x) &= e^{-q\Gamma_5(x)\phi(x)} h(x)\chi_i(x) \\ &= e^{-q\sigma\phi(x)} h(x)\chi_i(x) \end{aligned} \tag{A-9}$$

where $\sigma = \pm 1$ is the chirality of χ_i.

A.2. *The spectrum and the zero-modes of* $\mathbb{D}_0$

The eigenvalue equation for $\mathbb{D}_0$ can be solved for any k, because of the rotation invariance, which allows to reduce $\mathbb{D}_0$ to angular momentum operators. To achieve this we make use of a manifestly covariant formalism [21].

Let $\mathcal{H}_k$, $k \in \mathbb{Z}$, be the space of square integrable two-component spinor fields $\hat{\psi}(g)$, $g \in SU(2)$, which are homogeneous:

$$\hat{\psi}(ge^{i\omega\sigma_3}) = e^{-ik\omega}\hat{\psi}(g) \tag{A-10}$$

A complete set of such spinors is given by

$$\hat{\psi}_{slm}(g) = e^{im\pi}\chi_s \langle l, -m\, |R^{(l)}(g)|\, k, -\frac{k}{2}\rangle \tag{A-11}$$

where

$$\begin{aligned} &s = \pm\tfrac{1}{2} \\ &l = \tfrac{1}{2}\,|k|, \tfrac{1}{2}\,|k| + 1, \ldots \\ &m = -l, \ldots, l \end{aligned} \tag{A-12}$$

and

$$\chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{A-13}$$

Here, $R^{(l)}$ denotes the representation with angular momentum l and $|l, m\rangle$ are the usual basis vectors in that representation.

'Angular momentum' operators acting on the functions in $\mathscr{H}_k$ may be defined through

$$I_a\hat{\psi}(g) = \frac{1}{i}\frac{d}{dt}\,\hat{\psi}(e^{-i(t/2)\sigma_a}g)\big|_{t=0} \qquad a = 1, 2, 3 \tag{A-14}$$

These operators I_a have algebraic properties that are completely analogous to the properties of the angular momentum operators $I_a^{(l)}$ which are the generators of the Lie algebra of $SU(2)$, or of rotation group defined by

$$R^{(l)}(e^{i\omega\sigma_a}) = e^{2i\omega I_a^{(l)}} \tag{A-15}$$

Namely, we have

(i) $[I_a, I_b] = i\epsilon_{abc}I_c$
(ii) $I^2\hat{\psi}_{slm} = \sum_a I_aI_a\hat{\psi}_{slm} = l(l+1)\hat{\psi}_{slm}$
(iii) $I_3\hat{\psi}_{slm} = m\hat{\psi}_{slm}$
(iv) $I_\pm\hat{\psi}_{slm} = (I_1 \pm iI_2)\hat{\psi}_{slm} = \sqrt{l(l+1) - m(m+1)}\;\hat{\psi}_{sl(m\pm1)}$

Our next step is to show that the operator $i\mathbb{D}_0$ may be mapped to an operator acting on the functions in $\mathscr{H}_k$.

Choose coordinates x^μ on S^2 and define $z(x)$ as in Section 3. Set

$$u(x) = \begin{pmatrix} z_1(x) & -z_2^*(x) \\ z_2(x) & z_1^*(x) \end{pmatrix} \in SU(2) \tag{A-16}$$

and, for any $\hat{\psi} \in \mathscr{H}_k$,

$$\psi(x) = \hat{\psi}(u(x)) \tag{A-17}$$

Because of homogeneity, ψ and $\hat{\psi}$ contain the same information. Noting that any $g \in SU(2)$ can be written uniquely as,

$$g = u(x)e^{i(\tau/2)\sigma_3} \qquad x \in \mathbb{R}^2 \qquad -2\pi \le \tau \le 2\pi \tag{A18}$$

and also making use of the homogeneity property it is straightforward, though tedious, to verify that

$$i\mathbb{D}_0\psi(x) = -\left\{\sigma_aI_a - \frac{k}{2R}\,\sigma_ar_a + 1\right\}\hat{\psi}(u(x)) \tag{A-19}$$

where $\mathbf{r} \in S^2$ corresponds to the point x. Thus the eigenvalue equation $i\mathbb{D}_0\psi = E\psi$ translates to

$$M\hat{\psi} = -E\hat{\psi} \tag{A-20}$$

where $M = \sigma_aI_a - (k/2R)\sigma_ar_a + 1$. Because Γ_5 anticommutes with M, we have

$$M\Gamma_5\hat{\psi} = E\Gamma_5\hat{\psi} \tag{A-21}$$

Suppose $\{\hat{\psi}_\nu, \hat{\chi}_i\}$, $\nu = \pm1, \pm2, \ldots$; $i = 1, \ldots, n$ is a complete set of eigenfunctions of M in $\mathscr{H}_k$, where $\hat{\psi}_\nu$ are eigenfunctions with non-zero eigenvalues and $\hat{\chi}_i$ are the zero modes. Furthermore, we have identified $\hat{\psi}_{-\nu} = \Gamma_5 \hat{\psi}_\nu$. Now define the positive and negative chirality wave functions,

$$\hat{\chi}_\nu^\pm = \hat{\psi}_\nu \pm \hat{\psi}_{-\nu} \qquad \nu = 1, 2, \ldots \tag{A-22}$$

Although $\hat{\chi}_\nu^\pm$ are not eigenfunctions of M, the set $\{\hat{\chi}_\nu^\pm, \hat{\chi}_i\}$, $\nu = 1, 2, \ldots$ still forms a basis in $\mathscr{H}_k$. In fact,

$$M\hat{\chi}_\nu^+ = -E_\nu \hat{\chi}_\nu^- \tag{A-23}$$

$$M\hat{\chi}_\nu^- = -E_\nu 7\chi_\nu^+ \tag{A-24}$$

However, we see that

$$M^2 \hat{\chi}_\nu^\pm = E_\nu^2 \hat{\chi}_\nu^\pm \tag{A-25}$$

so $\hat{\chi}_\nu^\pm$ are eigenfunctions of M^2 with eigenvalues E_ν^2. In face of the fact that the zero modes $\hat{\chi}_i$ can always be chosen to have a definite chirality (because $\{M, \Gamma_5\} = 0$), what we have done here is to construct a complete set of eigenfunctions of M^2 with definite chirality. This was, of course, possible because M^2 *commutes* with Γ_5.

Recalling that $\Gamma_5 = (1/R)\sigma_a r_a$, we see that all these eigenfunctions satisfy,

$$\{(\sigma_a l_a + 1)^2 - \tfrac{1}{4}k^2\}\hat{\chi} = E^2 \hat{\chi} \tag{A-26}$$

This eigenvalue problem can be solved using the method of adding angular momenta. Define the 'total angular momentum'

$$J_a = l_a + \tfrac{1}{2}\sigma_a \tag{A-27}$$

Then we have

$$\sigma_a l_a = J_a J_a - l_a l_a - \tfrac{3}{4} \tag{A-28}$$

The eigenfunctions of J_a are readily obtained by taking linear combinations of $\hat{\psi}_{slm}$'s (which are the eigenfunctions of l_a and $\tfrac{1}{2}\sigma_a$) in such a way as to obtain eigenfunctions of J^2, J_3 and l^2. According to the rules of adding angular momenta, the simultaneous eigenvalues of these operators are $j(j+1)$, m_j and $l(l+1)$, respectively, where

$$l = \tfrac{1}{2}|k| + n; \qquad n = 0, 1, 2, \ldots \tag{A-29}$$

$$j = l \pm \tfrac{1}{2} \quad \text{if} \quad l \neq 0 \tag{A-30}$$

$$j = \tfrac{1}{2} \quad \text{if} \quad l = 0 \tag{A-31}$$

$$m_j = -j, -j+1, \ldots, j \tag{A-32}$$

The corresponding eigenfunctions can be labelled by $\hat{\chi}_{ljm_j}$. It follows that

$$\begin{aligned} E^2 &= \{j(j+1) - l(l+1) + \tfrac{1}{4}\}^2 - \tfrac{1}{4}k^2 \\ &= \tfrac{1}{4}\{(2j+1)^2 - k^2\} \end{aligned} \tag{A-33}$$

The following table shows the values taken by l and j, and the corresponding multiplicity of the eigenvalue.

l	j	multiplicity $(2j+1)$
$\frac{1}{2}\lvert k\rvert$	$\frac{1}{2}\lvert k\rvert-\frac{1}{2}$ $\frac{1}{2}\lvert k\rvert+\frac{1}{2}$	$\lvert k\rvert$ $\lvert k\rvert+2$
$\frac{1}{2}\lvert k\rvert+1$	$\frac{1}{2}\lvert k\rvert+\frac{1}{2}$ $\frac{1}{2}\lvert k\rvert+\frac{3}{2}$	$\lvert k\rvert+2$ $\lvert k\rvert+4$
$\frac{1}{2}\lvert k\rvert+2$	$\frac{1}{2}\lvert k\rvert+\frac{3}{2}$ $\frac{1}{2}\lvert k\rvert+\frac{5}{2}$	$\lvert k\rvert+4$ $\lvert k\rvert+6$
$\vdots$	$\vdots$	$\vdots$

For $l=0$, which is only possible if $k=0$, j only takes the value $\frac{1}{2}$ corresponding to $\frac{1}{2}|k|+\frac{1}{2}$. Thus we come to the following conclusions:

1. There are $|k|$ zero modes which correspond to $l=|k|/2$, $j=|k|/2-1/2$ and $m_j=-j,\ldots,j$.
2. The non-zero eigenvalues may be labelled by the values taken by $(2j+1)$. Indeed, introduce

$$(2j+1)=|k|+2\nu; \qquad \nu=1,2,\ldots$$

then

$$E_\nu^2=\nu(\nu+|k|); \qquad \nu=1,2,\ldots$$

with multiplicity $2(|k|+2\nu)$.

The above is a statement on the eigenvalues of the operator M^2 which, translated back to operators acting on spinors $\psi(x)$, holds for the operator $\Omega_0=-(\mathbb{D}_0)^2$.

Let us denote the subspace of $\mathcal{H}_k$ spanned by the eigenfunctions of M^2 corresponding to the eigenvalues E_ν^2 by $\mathcal{R}_\nu$. The functions $\hat{\chi}_{lj}m_j$ ($j=(|k|/2)+\nu-\frac{1}{2}$, $l=j\pm\frac{1}{2}$, $m_j=-j,\ldots,j$) may not have definite chirality. However, it is possible to make a basis with definite chirality by taking linear combinations of them. It is also easy to show that any such basis contains equal numbers of positive and negative chirality elements. This enables us to construct the eigenfunctions of M. Suppose, for instance, we have found a set of positive chirality basis elements $\hat{\chi}_\nu^+$. To each of them we can assign a negative chirality vector $\hat{\chi}_v^- = -(1/E_v)M\hat{\chi}_v^+$, where $E_v = \sqrt{v(v+|k|)}$. Then the functions $\hat{\psi}_{\pm v} = \frac{1}{2}(\hat{\chi}_v^+ \pm \hat{\chi}_v^-)$, $v = 1, 2, \ldots$ are the eigenfunctions of M.

$$M\hat{\psi}_v = -E_v\hat{\psi}_n \tag{A-34}$$

$$M\hat{\psi}_{-v} = +E_v\hat{\psi}_{-v} \tag{A-35}$$

Furthermore, all the zero modes of M^2 are also the zero modes of M itself. Thus we have shown that, given the functions $\hat{\psi}_{slm}$, we can completely find out the spectrum of M, and hence of $i\mathbb{D}_0$.

There is an interesting thing about the zero modes; for a given k they all have the *same* chirality. To prove this, consider the zero modes $\hat{\chi}_{ljm_j}$; $l = \frac{1}{2}|k|$, $j = \frac{1}{2}|k| - \frac{1}{2}$, $m_j = -j, \ldots, j$.

$$M\hat{\psi} = 0 \tag{A-36}$$

or

$$\{\sigma_a I_a + 1 - \tfrac{1}{2}k\Gamma_5\}\hat{\chi} = 0 \tag{A-37}$$

or

$$\{j(j+1) - l(l+1) - \tfrac{3}{4} + 1 - \tfrac{1}{2}k\Gamma_5\}\hat{\chi} = 0 \tag{A-38}$$

After substituting $j = \frac{1}{2}|k| - \frac{1}{2}$, $l = \frac{1}{2}|k|$, this gives $\Gamma_5\hat{\psi} = -(|k|/k)\hat{\chi}$, or in other words, depending on whether k is positive or negative, all the zero modes $\hat{\psi}_{ljm_j}$ have either negative or positive chirality. Denote the number of positive and negative chirality zero modes by n_+ and n_-, respectively. Then, for any k, positive or negative, we have

$$n_+ - n_- = -k \tag{A-39}$$

This is in accordance with the Atiyah–Singer index theorem [25].

B. Evaluation of Grassmann integrals

Evaluation of Grassmann integrals with a Gaussian integrand $\exp(a, Aa)$ is quite familiar in the case when A has no zero modes. When A has zero modes, a little more care must be taken although the basic principles are precisely the same. In this Appendix we derive some related results, and illustrate their application to our main calculation. (For a nice account on the basic concepts of Grassmann integrals see the book by Berèzin [22]). The contents will be the following:

- Evaluation of the Grassmann integrals
  1. $I_0 = \int \prod_{i=1}^N da_i^*\, da_i a_{i_1} a_{j_1}^* \cdots a_{i_N} a_{j_N}^*$
  2. $I = \int [D\bar{\psi}][D\psi] e^{-(\bar{\psi}, A\psi)} \psi(x_1)\bar{\psi}(y_1) \cdots \psi(x_n)\bar{\psi}(y_n)$ where A is a self-adjoint operator with respect to the scalar product (φ, χ), for the cases when
     (a) A has precisely n zero modes.
     (b) A has $k(<n)$ zero modes.
- Illustration of the results in special cases.

B.1. *Evaluation of I_0*

Consider the Grassmann algebra generated by the Grassmann variables $(a_1, \ldots, a_N, a_1^*, \ldots, a_N^*)$. We wish to evaluate the following integral:

$$I_0 = \int \prod_{i=1}^{N} da_i^* \, da_i a_{i_1} a_{j_1}^* \cdots a_{i_N} a_{j_N}^* \tag{B-1}$$

According to the rules of Grassmann integration it is clear that the value of I_0 is either $+1$ or -1 for given sets of *distinct* indices $(i_1, \ldots, i_N)$ and $(j_1, \ldots, j_N)$. If they are *not* distinct, I_0 is of course identically zero. Furthermore, notice that I_0 is totally antisymmetric with respect to the indices $(i_1, \ldots, i_N)$ and $(j_1, \ldots, j_N)$ separately, i.e., if we exchange i_k and i_l, the value of I_0 changes the sign. The same is true with the indices $(j_1, \ldots, j_N)$.

Thus, if $\epsilon_{\mu_1 \cdots \mu_N}$ denotes the totally anti-symmetric Levi–Civita symbol of nth rank, I_0 has to be proportional to $\epsilon_{i_1 \cdots i_N}$ as well as to $\epsilon_{j_1 \cdots j_N}$. This implies that

$$I_0 = c \epsilon_{i_1 \cdots i_N} \epsilon_{j_1 \cdots j_N} \tag{B-2}$$

By setting $(i_1, \ldots, i_N) = (j_1, \ldots, j_N) = (1, \ldots, N)$ we find that $c = 1$. Hence,

$$I_0 = \int \prod_{i=1}^{N} da_i^* \, da_i a_{i_1} a_{j_1}^* \cdots a_{i_N} a_{j_N}^* = \epsilon_{i_1 \cdots i_N} \epsilon_{j_1 \cdots j_N} \tag{B-3}$$

B.2. *Evaluation of I*

$$I = \int [D\bar{\psi}][D\psi] e^{-(\bar{\psi}, A\psi)} \psi(x_1) \bar{\psi}(y_1) \cdots \psi(x_n) \bar{\psi}(y_n) \tag{B-4}$$

Let $\{\psi_i\}$ be a complete set of orthonormal eigenfunctions of the operator A:

$$A\psi_i = \epsilon_i \psi_i \tag{B-5}$$

and for any ψ and $\bar{\psi}$,

$$\psi(x) = \sum_i a_i \psi_i(x) \tag{B-6}$$

$$\bar{\psi}(x) = \sum_i a_i^* \bar{\psi}_i(x) \tag{B-7}$$

where $\bar{\psi}_i = (\psi_i^*)^T$. The set $\{a_i, a_i^*\}$ may be considered as an orthonormal basis for the Grassmann algebra generated by them. The scalar product (,) is defined by

$$(\bar{\psi}, \psi) = \sum_i a_i^* a_i \tag{B-8}$$

We define the measure $[D\bar{\psi}][D\psi]$ so that,

$$I = \int \prod da^* \, da e^{-\Sigma' \epsilon_i a_i^* a_i} \sum_{\substack{i_1 \cdots i_n \\ j_1 \cdots j_n}} a_{i_1} a_{j_1}^* \cdots a_{i_n} a_{j_n}^* \psi_{i_1}(x_1) \bar{\psi}_{j_1}(y_1) \cdots \psi_{i_n}(x_n) \bar{\psi}_{j_n}(y_n) \tag{B-9}$$

where $\sum'$ denotes the sum over only those i's which corresponds to non-zero modes. Suppose there are k zero modes. If $k > n$, I is identically zero since the product $a_{i_1} \cdots a^*_{j_n}$ cannot produce all k variables corresponding to the missing zero modes in the exponent. Thus we only have to consider the two cases $k = n$ and $k < n$.

Case (i): $k = n$
In this case, in order to get a non-zero value for I, all the indices $(i_1, \ldots, i_n)$, $(j_1, \ldots, j_n)$ must correspond to zero modes. This is so, because $\exp(\sum' \epsilon_i a_i a_i^*)$ could not provide for any of the n pairs (a_i, a_i^*) which correspond to zero modes. Thus the integral decomposes into two parts.

$$I = \int \prod_{i=1}^{n} da_i^* \, da_i \sum a_{i_1} a^*_{j_1} \cdots a_{i_n} a^*_{j_n} \psi_{i_1}(x_1) \bar{\psi}_{j_1}(y_1) \cdots \psi_{i_n}(x_n) \bar{\psi}_{j_n}(y_n)$$

$$\times \int \prod{}' da^* \, da e^{+\sum' \epsilon_i a_i a_i^*} \quad \text{(B-10)}$$

where the sum in the first factor is over the zero modes only. The second factor, on the other hand, contains only non-zero modes, and gives the result $\det' A = \prod' \epsilon_i$ where $\prod'$ denotes the product over non-zero modes. Thus,

$$I = \det' A \sum_{\substack{i_1 \cdots i_n \\ j_1 \cdots j_n}} I_0 \psi_{i_1}(x_1) \bar{\psi}_{j_1}(y_1) \cdots \psi_{i_n}(x_n) \bar{\psi}_{j_n}(y_n) \quad \text{(B-11)}$$

where I_0 is the integral (B-1) considered in the previous section.

$$I_0 = \epsilon_{i_1 \cdots i_n} \epsilon_{j_1 \cdots j_n} \quad \text{(B-12)}$$

Thus

$$I = \det' A \epsilon_{i_1 \cdots i_n} \epsilon_{j_1 \cdots j_n} \psi_{i_1}(x_1) \bar{\psi}_{j_1}(y_1) \cdots \psi_{i_n}(x_n) \bar{\psi}_{j_n}(y_n) \quad \text{(B-13)}$$

where the repeated indices are summed over zero modes. Recall that the ψ's appearing here are all *orthonormal* zero modes.

Suppose instead of orthonormal zero modes, we are provided with a set of only linearly independent zero modes, i.e., instead of an orthonormal basis in the zero mode subspace, we only have an independent basis $\{\chi_i\}$.

To construct orthonormal zero modes out of them we could, for instance, use the Gram–Schmidt procedure. However, this method is not very practical when the number of basis elements is large. Therefore we rather resort to general arguments. It will be proven that for the special combination of basis elements we have above, there exists a nice expression in terms of independent zero modes.

Expand χ_i in terms of orthonormal basis elements:

$$\chi_i = S_{ij} \psi_j \quad \text{(B-14)}$$

It follows that

$$\begin{aligned} \langle \chi_i, \chi_j \rangle &= S^*_{ik} S_{jl} \langle \psi_k, \psi_l \rangle \\ &= S^*_{ik} S_{jl} \delta_{kl} \end{aligned} \quad \text{(B-15)}$$

Define the 'zero-mode matrix' by $N_{ij} = \langle \chi_i, \chi_j \rangle$. Thus

$$N_{ij} = S_{jk} S^{\dagger}_{ki} \quad \text{or} \quad N^T = SS^{\dagger} \tag{B-16}$$

It follows that

$$\det N = \det S \det S^{\dagger} \tag{B-17}$$

Expressing χ's in the product

$$\epsilon_{i_1 \cdots i_n} \epsilon_{j_1 \cdots j_n} \chi_{i_1}(x_1) \bar{\chi}_{j_1}(y_1) \cdots \chi_{i_n}(x_n) \bar{\chi}_{j_n}(y_n) \tag{B-18}$$

in terms of ψ's and also making use of the identity

$$\epsilon_{i_1 \cdots i_n} \det B = B_{i_1 i'_1} \cdots B_{i_n i'_n} \epsilon_{i'_1 \cdots i'_n} \tag{B-19}$$

which is valid for any matrix B, it is easy to prove that

$$I = \det{}' A (\det N)^{-1} \epsilon_{i_1 \cdots i_n} \epsilon_{j_1 \cdots j_n} \chi_{i_1}(x_1) \bar{\chi}_{j_1}(y_1) \cdots \chi_{i_n}(x_n) \bar{\chi}_{j_n}(y_n) \tag{B-20}$$

Case (ii): $k < n$

Let us denote the integral I by I^k_{2n} in order to remind us that n and k are different in this case; we are calculating the $2n$-point function in the presence of k zero modes of the operator A.

Like in the Case (i), using an orthonormal set of eigenfunctions of A we can write the above integral as

$$I^k_{2n} = \int \prod da^* \, da e^{-\sum \epsilon_{i'} a^*_{i'} a_{i'}} \sum_{\substack{i_1 \cdots i_n \\ j_1 \cdots j_n}} a_{i_j} a^*_{j_1} \cdots a_{i_n} a^*_{j_n} \times \psi_{i_1}(x_1) \bar{\psi}_{j_1}(y_1) \cdots \psi_{i_n}(x_n) \bar{\psi}_{j_n}(y_n) \tag{B-21}$$

The primed indices here correspond to the non-zero eigenmodes. Consider the following facts:

- The expansion of the exponential always gives terms with pairs $a_i a^*_i$.
- The above expansion does not give any one of the k pairs $a_i a^*_i$ that correspond to the zero modes.
- Therefore, the rest of the integrand should provide for the product $a_1 a^*_1 \cdots a_k a^*_k$ corresponding to the zero modes.
- Since there remain $(n-k)$ pairs that correspond to non-zero modes, we have to consider only those terms in the expansion of the exponent, where just these modes are lacking.
- The terms in the expansion of $\exp(\sum_{i=1}^N \epsilon_i a_i a^*_i)$ which contain $N-1$ products of distinct $a_i a^*_i$ pairs and lack the remaining l pairs are given by

$$\sum_{\{\sigma_l\}} \prod_{i \in \bar{\sigma}_l} \epsilon_i a_i a^*_i$$

where, σ_l denote sets of l distinct indices (which are missing in the product) and $\bar{\sigma}_l$ is the complement of σ_l: it contains all the other $N-l$ indices.

Thus we have, setting $l = n - k$,

$$I_{2n}^k = \int \prod da^* \, da \sum_{\{\sigma_{n-k}\}} \prod_{i' \in \bar{\sigma}_{n-k}} \epsilon_{i'} a_{i'} a_{i'}^* \sum_{\substack{i_1 \cdots i_n \\ j_1 \cdots j_n}} a_{i_1} a_{j_1}^* \cdots a_{i_n} a_{j_n}^*$$
$$\times \psi_{i_1}(x_1) \bar{\psi}_{j_1}(y_1) \cdots \psi_{i_n}(x_n) \bar{\psi}_{j_n}(y_n) \tag{B-22}$$

The indices $i_1, \ldots, i_n, j_1, \ldots, j_n$ take values in the set

$$S = \{1, \ldots, k\} \cup \sigma_{n-k}$$

where the indices $1, \ldots, k$ label zero modes. Indices in σ and $\bar{\sigma}$ correspond to non-zero modes. Since the sum over σ means the sum over all *dinstant* indices, we can rewrite it as a sum over all the indices $i'_1, \ldots, i'_{n-k} \in \sigma \cup \bar{\sigma}$, i.e., over all non-zero modes which satisfy the condition $i'_1 < \cdots < i'_{n-k}$. Thus

$$I_{2n}^k = \int \prod da^* \, da \sum_{\substack{i'_1 \cdots i'_{n-k} \\ i'_1 < \cdots < i'_{n-k}}} \prod_{i' \in \bar{\sigma}_{n-k}} \epsilon_{i'} a_{i'} a_{i'}^*$$
$$\times \sum_{\substack{i_1, \ldots, i_n \in S \\ j_1, \ldots, j_n \in S}} a_{i_1} a_{j_1}^* \cdots a_{i_n} a_{j_n}^* \psi_{i_1}(x_1) \bar{\psi}_{j_1}(y_1) \cdots \psi_{i_n}(x_n) \bar{\psi}_{j_n}(y_n) \tag{B-23}$$

The set S is now given by $S = \{1, \ldots, k, i'_1, \ldots, i'_{n-k}\}$. Notice also that the product $a_{i_1} a_{j_1}^* \cdots a_{in} a_{j_n}^*$ is anti-symmetric with respect to exchange of two indices in $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_n)$ separately. Hence we can write this product as

$$\epsilon_{i_1 \cdots i_n} \epsilon_{j_1 \cdots j_n} a_1 a_1^* \cdots a_k a_k^* a_{i'_1} a_{i'_1}^* \cdots a_{i'_{(n-k)}} a_{i'_{n-k}}^* \tag{B-24}$$

where $\epsilon_{i_1 \cdots i_n}$ is again the totally antisymmetric Levi–Civita symbol with

$$\epsilon_{1 \cdots k i'_1 \cdots i'_{(n-k)}} = 1 \tag{B-25}$$

Thus we finally have

$$I_{2n}^k = \sum_{\substack{i'_1 \cdots i'_{n-k} \\ i'_1 < \cdots < i'_{n-k}}} \int \prod da^* \, da \prod_{i' \in \bar{\sigma}} \epsilon_{i'} a_{i'} a_{i'}^* \sum_{\substack{i_1, \ldots, i_n \in S \\ j_1, \ldots, j_n \in S}} \epsilon_{i_1 \cdots i_n} \epsilon_{j_1 \cdots j_n}$$
$$\times a_1 a_1^* \cdots a_k a_k^* a_{i'_1} a_{i'_1}^* \cdots a_{i'_{(n-k)}} a_{i'_{(n-k)}}^* \psi_{i_1}(x_1) \bar{\psi}_{j_1}(y_1) \cdots \psi_{i_n}(x_n) \bar{\psi}_{j_n}(y_n)$$
$$= \sum_{\substack{i'_1 \cdots i'_{n-k} \\ i'_1 < \cdots < i'_{n-k}}} \left[\sum_{\substack{i_1, \ldots, i_n \in S \\ j_1, \ldots, j_n \in S}} \det{}' A \, \frac{\epsilon_{i_1 \cdots i_n} \epsilon_{j_1 \cdots j_n}}{\epsilon_{i'_1} \cdots \epsilon_{i'_{(n-k)}}} \psi_{i_1}(x_1) \bar{\psi}_{j_1}(y_1) \cdots \psi_{i_n}(x_n) \bar{\psi}_{j_n}(y_n) \right] \tag{B-26}$$

where $\det' A$ is the product of all non-zero eigenvalues, and $i'_1, \ldots, i'_{(n-k)}$ take values corresponding to the non-zero modes.

Notice that the expression in the square brackets is symmetric with respect to the indices $(i'_1, \ldots, i'_{(n-k)})$. Also notice that we can allow the possibility that any two indices be equal, since the ϵ-symbols take care that such a contribution gives zero. Both of these considerations result in the identities:

$$\sum_{i'_1 \cdots i'_{(n-k)}} [\ ] \equiv \sum_{\substack{i'_1 \cdots i'_{(n-k)} \\ \text{no two} \\ \text{indices equal}}} [\ ] \equiv (n-k)! \sum_{i'_1 < \cdots < i'_{(n-k)}} [\ ] \tag{B-27}$$

where [ ] denotes the square bracket in the above equation. Thus

$$I_{2n}^k = \frac{1}{(n-k)!} \times \sum_{i'_1,\ldots,i'_{(n-k)}} \left[\sum_{\substack{i_1\cdots i_n \\ j_1\cdots j_n}} \det{}' A \frac{\epsilon_{i_1\cdots i_n}\epsilon_{j_1\cdots j_n}}{\epsilon_{i'_1}\cdots\epsilon_{i'_{(n-k)}}} \psi_{i_1}(x_1)\bar{\psi}_{j_1}(y_1)\cdots\psi_{i_n}(x_n)\bar{\psi}_{j_n}(y_n) \right] \tag{B-28}$$

In this expression, the unprimed indices take values corresponding to all the eigenmodes while the primed indices take only values that correspond to non-zero modes.

B.3. *Illustration*

Here we shall illustrate the results of the previous subsection for the cases of 2- and 4-point functions.

B.3.1. *The 2-point function.*

$$I_2 = \int [D\bar{\psi}][D\psi] e^{-(\psi, A\psi)} \psi(x_1)\bar{\psi}(y_1) \tag{B-29}$$

When the operator A has no zero modes we get from equation (B-28)

$$I_2^0 = \sum_{i'_1} \sum_{i_1, j_1} \det A \frac{\epsilon_{i_1}\epsilon_{j_1}}{\epsilon_{i'_1}} \psi_{i_1}(x_1)\bar{\psi}_{j_1}(y_1) \tag{B-30}$$

where $i_1, j_1 \in \{i'_1\}$, which in this case means, that $i_1 = j_1 = i'_1 \in \{1, 2, \ldots\}$. It follows that

$$I_2^0 = \sum_{i'} \det A \frac{\psi_{i'}(x_1)\bar{\psi}_{i'}(y_1)}{\epsilon_{i'}} = \det A\, \mathcal{G}(x_1, y_1) \tag{B-31}$$

where

$$\mathcal{G}(x, y) = \sum_{i=1,2,\ldots} \frac{\psi_i(x)\bar{\psi}_i(y)}{\epsilon_i} \tag{B-32}$$

When A has one zero mode, equation (B-28) gives

$$I_2^1 = \sum_{i_1, j_1} \det{}' A \epsilon_{i_1}\epsilon_{j_1}\psi_{i_1}(x_1)\bar{\psi}_{j_1}(y_1) \tag{B-33}$$

where $i_1, j_1 \in \{1\}$ or $i_1 = j_1 = 1$. Thus we have

$$I_2^1 = \det{}' A \psi_1(x_1)\bar{\psi}_1(y_1) \tag{B-34}$$

where ψ_1 is the normalized zero mode of A. Notice that the same result may be obtained via equation (B-20) which gives

$$I_2^1 = \det{}' A \frac{\chi(x_1)\bar{\chi}(y_1)}{\langle \chi, \chi \rangle} \tag{B-35}$$

where χ is *any* (i.e., not necessarily a normalized) zero mode.

B.3.2. The 4-point function

$$I_4 = \int [D\bar{\psi}][D\psi] e^{-(\psi, A\psi)} \psi^\alpha(x_1)\bar{\psi}^\beta(y_1)\psi^\gamma(x_2)\bar{\psi}^\delta(y_2) \tag{B-36}$$

In this case equation (B-28) gives

$$I_4^0 = \frac{1}{2!} \sum_{i_1', i_2'} \sum_{\substack{i_1, i_2 \\ j_1, j_2}} \det A \frac{\epsilon_{i_1 i_2} \epsilon_{j_1 j_2}}{\epsilon_{i_1'} \epsilon_{i_2'}} \psi_{i_1}^\alpha(x_1)\bar{\psi}_{j_1}^\beta(y_1)\psi_{i_2}^\gamma(x_2)\bar{\psi}_{j_2}^\delta(y_2) \tag{B-37}$$

where $i_1, i_2, j_1, j_2 \in \{i_1', i_2'\}$ while i_1', i_2' take all values $1, 2, \ldots$. Simplifying this one gets

$$I_4^0 = \det A \{\mathcal{G}^{\alpha\beta}(x_1, y_1)\mathcal{G}^{\gamma\delta}(x_2, y_2) - \mathcal{G}^{\alpha\delta}(x_1, y_2)\mathcal{G}^{\gamma\beta}(x_2, y_1)\} \tag{B-38}$$

Similarly, when A has one zero mode the result is

$$\begin{aligned} I_4^1 = \det{}' A \{ &\psi_1^\alpha(x_1)\bar{\psi}_1^\beta(y_1)\mathcal{G}^{\gamma\delta}(x_2, y_2) + \mathcal{G}^{\alpha\beta}(x_1, y_1)\psi_1^\gamma(x_2)\bar{\psi}_1^\delta(y_2) \\ &- \psi_1^\alpha(x_1)\bar{\psi}_1^\delta(y_2)\mathcal{G}^{\gamma\beta}(x_2, y_1) - \mathcal{G}^{\alpha\delta}(x_1, y_2)\psi_1^\gamma(x_2)\bar{\psi}_1^\beta(y_1)\} \end{aligned} \tag{B-39}$$

where

$$\mathcal{G}^{\alpha\beta}(x, y) = \sum_{i'=2,3,\ldots} \frac{\psi_{i'}^\alpha(x)\bar{\psi}_{i'}^\beta(y)}{\epsilon_{i'}} \tag{B-40}$$

and ψ_1 is the normalized zero mode of A. In the case when there are 2 zero modes, we obtain

$$\begin{aligned} I_4^2 = \det{}' A \{ &\psi_1(x_1)\bar{\psi}_1(y_1)\psi_2(x_2)\bar{\psi}_2(y_2) + \psi_2(x_1)\bar{\psi}_2(y_1)\psi_1(x_2)\bar{\psi}_1(y_2) \\ &- \psi_1(x_1)\bar{\psi}_2(y_1)\psi_2(x_2)\bar{\psi}_1(y_2) - \psi_2(x_1)\bar{\psi}_1(y_1)\psi_1(x_2)\bar{\psi}_2(y_2)\} \end{aligned} \tag{B-41}$$

where ψ_1 and ψ_2 are two orthonormal zero modes. Using equation (B-20) we can give the answer in terms of any two independent zero modes χ_1 and χ_2:

$$\begin{aligned} I_4^2 = \det{}' A (\det N)^{-1} \{ &\chi_1(x_1)\bar{\chi}_1(y_1)\chi_2(x_2)\bar{\chi}_2(y_2) \\ &+ \chi_2(x_1)\bar{\chi}_2(y_1)\chi_1(x_2)\bar{\chi}_1(y_2) - \chi_1(x_1)\bar{\chi}_2(y_1)\chi_2(x_2)\bar{\chi}_1(y_2) \\ &- \chi_2(x_1)\bar{\chi}_1(y_1)\chi_1(x_2)\bar{\chi}_2(y_2)\} \end{aligned} \tag{B-42}$$

where $N_{ij} = \langle \chi_i, \chi_j \rangle$ is the zero mode matrix.

C. Certain operators and their Green's functions

Here we investigate how certain operators mentioned in the main calculation are related to each other and how the corresponding Green's functions reflect these relations. This will make clear why different formalisms are equivalent. Expressing the Green's functions of complicated operators in terms of those of simple operators will also facilitate the explicit calculation.

C.1. *Equivalence of the operators D and* $\mathbb{D}$

As mentioned in the Section 1, the Dirac operator in the general formalism is given by

$$D = \gamma^a e_a^\mu(x) D_\mu = \gamma^\mu D_\mu \tag{C-1}$$

where $D_\mu = \partial_\mu + A_\mu + \frac{1}{4}\gamma^\nu \gamma_{\nu;\mu}$. γ^a are globally defined γ-matrices. Multiplying γ^a with the d-bein $e_a^\mu(x)$ (assuming we have a d-dimensional manifold) we obtain the position-dependent γ-matrices: $\gamma^\mu(x) = \gamma^a e_a^\mu(x)$. If $g_{\mu\nu}$ is the metric tensor we define $\gamma_\mu(x) = g_{\mu\nu}\gamma^\nu$ and ";" denotes the covariant derivative with respect to the index that follows it. The quantity $\frac{1}{4}\gamma^\nu\gamma_{\nu;\mu}$ is the so-called spin-connection. Finally A_μ is the gauge field, which takes values of the Lie algebra of the gauge group G.

On conformally flat spaces like the d-sphere, the metric tensor may be written as

$$g_{\mu\nu}(x) = \Omega(x)\delta_{\mu\nu} \qquad \Omega > 0 \tag{C-2}$$

by choosing suitable coordinates. In this case, the natural choice for the d-bein is

$$e_{a\mu}(x) = g_{\mu\nu} e_a^\mu(x) = \Omega^{1/2}\delta_{a\mu} \tag{C-3}$$

or

$$e_a^\mu = \Omega^{-1/2}\delta_{a\mu} \tag{C-4}$$

Hence, in these specific coordinates, we get

$$D = \Omega^{-(d-1)/4}\{\Omega^{-1/2}\gamma_a(\partial_a + iqA_a)\}\Omega^{(d-1)/4} \tag{C-5}$$

where we have writeen $A_a = A_\mu$ for these specific coordinates.

On the 2-sphere S^2, of radius R, the sterographic coordinates (x^1, x^2) defined by

$$r_1 = 2R^2x^1(R^2 + \mathbf{x}^2)^{-1} \tag{C-6}$$

$$r_2 = 2R^2x^2(R^2 + \mathbf{x}^2)^{-1} \tag{C-7}$$

$$r_3 = R(R^2 - \mathbf{x}^2)(R^2 + \mathbf{x}^2)^{-1} \tag{C-8}$$

provide such a coordinate system. Here (r_1, r_2, r_3) denotes the point (x^1, x^2) on the sphere in terms of the coordinates of the flat 3-dimensional space in which S^2 is embedded. Indeed, for the metric tensor we have

$$g_{\mu\nu}(x) = \partial_\mu \mathbf{r} \cdot \partial_\nu \mathbf{r} = \Omega(x)\delta_{\mu\nu} \tag{C-9}$$

where

$$\Omega(x) = \frac{4R^4}{(R^2 + \mathbf{x}^2)^2} \tag{C-10}$$

Hence

$$D = \Omega^{-3/4}\gamma_a(\partial_a + iqA_a)\Omega^{1/4} \tag{C-11}$$

Now let us take the operator defined in the $SU(2)$-invariant formalism:

$$\tilde{D} = \left(\mathscr{D} + \frac{i}{R}\right) \tag{C-12}$$

with $\mathscr{D} = \Gamma^\mu \mathscr{D}_\mu$ where Γ_μ and $\mathscr{D}_\mu$ are defined by

$$\Gamma_\mu = \frac{1}{R}\boldsymbol{\sigma} \cdot (\mathbf{r} \times \partial_\mu \mathbf{r}) \quad \text{and} \quad \mathscr{D}_\mu = \partial_\mu + iqA_\mu \tag{C-13}$$

respectively. It is easy to see that in stereographic coordinates defined above

$$\Gamma_1 = \boldsymbol{\sigma} \cdot \partial_2 \mathbf{r} \qquad \Gamma_2 = -\boldsymbol{\sigma} \cdot \partial_1 \mathbf{r} \tag{C-14}$$

and

$$\tilde{D} = \Omega^{-1/2}\tilde{\gamma}_a(\partial_a + iqA_a) + \frac{i}{R} \tag{C-15}$$

where

$$\tilde{\gamma}_1(x) = \Omega^{-1/2}\boldsymbol{\sigma} \cdot \partial_2 \mathbf{r} \qquad \tilde{\gamma}_2(x) = -\Omega^{-1/2}\boldsymbol{\sigma} \cdot \partial_1 \mathbf{r} \tag{C-16}$$

These 2×2 γ-matrices satisfy the Clifford algebra

$$\{\tilde{\gamma}_a, \tilde{\gamma}_b\} = 2\delta_{ab} \tag{C-17}$$

Since any two irreducible representations of this Clifford algebra are unitarily equivalent to one another, there must be a matrix $u(x) \in SU(2)$ which relates the matrices $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ to $\gamma_1 = \sigma_1$, $\gamma_2 = \sigma_2$ which also satisfy the same Clifford algebra: i.e., there exists $u \in SU(2)$ such that

$$\gamma_a = u\tilde{\gamma}_a u^{-1} \tag{C-18}$$

Now we can prove that the operators D, in the general formalism and $\tilde{D}$, in the $SU(2)$-invariant formalism are unitarily equivalent.

The proof of the above equivalence goes as follows: The matrices 1, $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ and $\tilde{\gamma}_1\tilde{\gamma}_2$ for any given x are linearly independent and hence form a basis for 2×2 matrices. This can be used to expand $\partial_1\tilde{\gamma}_a$, $\partial_2\tilde{\gamma}_a$ in this basis. For example, we get

$$\partial_1\tilde{\gamma}_1 = -\tfrac{1}{2}\Omega^{-1}\,\partial_2\Omega\tilde{\gamma}_2 \tag{C-19}$$

$$\partial_1\tilde{\gamma}_2 = \tfrac{1}{2}\Omega^{-1}\,\partial_2\Omega\tilde{\gamma}_1 + \frac{1}{iR}\Omega^{1/2}\tilde{\gamma}_1\tilde{\gamma}_2 \tag{C-20}$$

for $\partial_1\tilde{\gamma}_a$. Inserting $\tilde{\gamma}_a = u^{-1}\gamma_a u$ we obtain

$$[u\,\partial_1 u^{-1}, \gamma_1] = -\tfrac{1}{2}\Omega^{-1}\,\partial_2\Omega\gamma_2 \tag{C-21}$$

$$[u\,\partial_1 u^{-1}, \gamma_2] = \tfrac{1}{2}\Omega^{-1}\,\partial_2\Omega\gamma_1 + \frac{1}{R}\Omega^{1/2}\sigma_3 \tag{C-22}$$

where we have used the fact that $[\gamma_1, \gamma_2] = [\sigma_1, \sigma_2] = 2i\sigma_3$. Since $u \in SU(2)$ it follows that $u\,\partial_a u^{-1}$ is an element of the *Lie algebra* of $SU(2)$. The above equations may be used to find the coefficients of the expansion $u\,\partial_1 u^{-1} = \alpha_i\sigma_i$. This gives

$$u\,\partial_1 u^{-1} = \frac{1}{2i}\left\{-\tfrac{1}{2}\Omega^{-1}\,\partial_2\Omega\sigma_3 + \frac{1}{R}\Omega^{1/2}\sigma_1\right\} \tag{C-23}$$

or

$$\gamma_1(u\,\partial_1 u^{-1}) = \frac{1}{2i}\left\{\frac{i}{2}\Omega^{-1}\,\partial_2\Omega\sigma_2 + \frac{1}{R}\Omega^{1/2}\mathbb{1}\right\} \tag{C-24}$$

Proceeding in the same way one can prove that

$$\gamma_2(u\,\partial_2 u^{-1}) = \frac{1}{2i}\left\{\frac{i}{2}\Omega^{-1}\,\partial_1\Omega\sigma_1 + \frac{1}{R}\Omega^{1/2}\mathbb{1}\right\} \tag{C-25}$$

Thus we finally have

$$\begin{aligned} u\tilde{D}u^{-1}\psi &= u\left\{\Omega^{-1/2}\tilde{\gamma}_a(\partial_a + iqA_a) + \frac{i}{R}\right\}u^{-1}\psi \\ &= \Omega^{-3/4}\sigma_a(\partial_a + iqA_a)\Omega^{1/4}\psi \end{aligned} \tag{C-26}$$

i.e., $u\tilde{D}u^{-1} = D$.

Explicit form of the matrix $u(x)$

We can find $u(x)$ explicitly using stereographic coordinates. Writing

$$u = a_0\mathbb{1} + ia_i\sigma_i \tag{C-27}$$

with a_0, a_i all real and $a_0^2 + \mathbf{a}^2 = 1$ and using the relations

$$u\tilde{\gamma}_a u^{-1} = \gamma_a \quad \text{or} \quad u\tilde{\gamma}_a = \gamma_a u, \tag{C-28}$$

where $\gamma_1 = \sigma_1$, $\gamma_2 = \sigma_2$, $\tilde{\gamma}_1 = \Omega^{-1/2}\boldsymbol{\sigma}\cdot\partial_2\mathbf{r}$, $\tilde{\gamma}_2 = -\Omega^{-1/2}\boldsymbol{\sigma}\cdot\partial_1\mathbf{r}$, one can solve a set of linear equations to obtain

$$\frac{a_1}{a_0} = \frac{x_1 - x_2}{R} \qquad \frac{a_2}{a_0} = \frac{x_1 + x_2}{R} \qquad \frac{a_3}{a_0} = 1 \tag{C-29}$$

where $a_0 = R/\sqrt{2(R^2 + \mathbf{x}^2)}$. In obtaining these results it is helpful to notice that $\mathbf{r}$, $\partial_1\mathbf{r}$, $\partial_2\mathbf{r}$ form an orthogonal set of axes at every point $\mathbf{r}$ on the sphere, and from $g_{\mu\nu}(x) = \partial_\mu\mathbf{r}\cdot\partial_\nu\mathbf{r} = \Omega(x)\delta_{\mu\nu}$ it follows that $|\partial_1\mathbf{r}|^2 = |\partial_2\mathbf{r}|^2 = \Omega(x)$.

Thus $u(x)$ in stereographic coordinates becomes

$$u(x) = \frac{1}{\sqrt{2(R^2 + \mathbf{x}^2)}}\begin{pmatrix} R(1+i) & (x_1 - ix_2)(1+i) \\ -(x_1 + ix_2)(1-i) & R(1-i) \end{pmatrix} \tag{C-30}$$

whereas

$$u^{-1}(x) = u^\dagger(x) = \frac{1}{\sqrt{2(R^2 + \mathbf{x}^2)}}\begin{pmatrix} R(1-i) & -(x_1 - ix_2)(1+i) \\ (x_1 + ix_2)(1-i) & R(1+i) \end{pmatrix} \tag{C-31}$$

C.2. *Green's functions*

The Green's function of a differential operator $\mathscr{A}$ on a Riemann manifold is defined by

$$\mathscr{A}G(x, y \mid \mathscr{A}) = g^{-1/2}\delta(x-y) \tag{C-32}$$

when $\mathscr{A}$ has no zero modes. In this case, the following representation for the Green's function may be given:

$$G(x, y \mid \mathscr{A}) = \sum_i \frac{\psi_i(x)\bar{\psi}_i(y)}{E_i} \tag{C-33}$$

Here $\{\psi_i\}$ form a complete set of orthonormal eigenfunctions of the operator $\mathscr{A}$ with the corresponding eigenvalues E_i. However, when $\mathscr{A}$ has zero modes this definition loses its meaning. In this case one may still define the Green's function as

$$G(x, y \mid \mathscr{A}) = \sum_{\substack{i \\ (E_i \neq 0)}} \frac{\psi_i(x)\bar{\psi}_i(y)}{E_i} \tag{C-34}$$

The Green's function so defined satisfies the differential equation

$$\mathscr{A}G(x, y \mid \mathscr{A}) = g^{-1/2}\delta(x-y) - P(x, y \mid \mathscr{A}) \tag{C-35}$$

where $P(x, y)$ is the projector onto the zero-mode subspace:

$$P((x, y \mid \mathscr{A}) = \sum_{\substack{i \\ (E_i = 0)}} \chi_i(x)\bar{\chi}_i(y) \tag{C-36}$$

The functions $\{\chi_i\}$ here form a complete set of orthonormal zero modes of $\mathscr{A}$.

As shown in Appendix A, the operator $\mathbb{D}$, defined as

$$\mathbb{D} = R\left(\mathscr{D} + \frac{i}{R}\right) \tag{C-37}$$

where $\mathscr{D} = \Gamma^\mu(\partial_\mu + iqA_\mu)$, $A_\mu = kC_\mu + \sqrt{g}\,\epsilon_{\mu\nu}g^{\nu\rho}\,\partial_\rho\phi + (1/iq)h\,\partial_\mu h^{-1}$ has precisely $|k|$ zero modes for a given k. Recall the foregoing discussion where the equivalence of the operators $\tilde{D} = \mathbb{D}/R$ and D was proven:

$$D = u\tilde{D}u^{-1} \quad \text{or} \quad \tilde{D} = u^{-1}Du \tag{C-38}$$

If $\{\psi_i\}$ is a complete set of orthonormal eigenfunctions of the operator D with corresponding eigenvalues iE_i, $\{u^{-1}\psi_i\}$ form a complete orthonormal set of eigenfunctions of the operator $\tilde{D}$ with the same eigenvalues. It follows that the Green's functions of the two operators are related by

$$G(x, y \mid \tilde{D}) = u^{-1}(x)G(x, y \mid D)u(y) \tag{C-39}$$

Since $\mathbb{D} = R\tilde{D}$, we also have

$$\mathscr{G}(x, y \mid \mathbb{D}) = \frac{1}{R}G(x, y \mid \tilde{D}) = \frac{1}{R}u^{-1}G(x, y \mid D)u(y) \tag{C-40}$$

In the following we construct the Green's function $G(x, y \mid D)$ of the operator D explicitly for any given k, using stereographic coordinates. Recall that in these coordinates,

$$\mathbf{r} = \frac{R}{R^2 + \mathbf{x}^2}(2Rx^1, 2Rx^2, R^2 - \mathbf{x}^2), \tag{C-41}$$

the operator D is given by (equation (1.32))

$$D = \Omega^{-3/4}\sigma_a(\partial_a + iqA_a)\Omega^{1/4} \tag{C-42}$$

where

$$A_a = kC_a + \epsilon_{ab}\,\partial_b\phi + \frac{1}{iq}h\,\partial_a h^{-1} \tag{C-43}$$

and

$$\Omega(x) = \frac{4R^4}{(R^2 + \mathbf{x}^2)^2} \tag{C-44}$$

Denote the Green's function of D for $\phi = 0$, $h = 1$ by $S(x, y)$. Then it is easy to show that the Green's function in the general case is given by

$$G(x, y, D) = h(x)e^{-q\sigma_3\phi(x)}S(x, y)e^{-q\sigma_3\phi(y)}h^{-1}(y) \tag{C-45}$$

Thus the problem of finding the Green's function of $\mathbb{D}$ is further reduced to finding it for a special gauge field kC_a which has constant field strength. Denoting the operator D for $\phi = 0$, $h = 1$ by D_0, we have

$$D_0 = \Omega^{-3/4}\sigma_a(\partial_a + iqkC_a)\Omega^{1/4} \tag{C-46}$$

$$C_a = \frac{1}{iq}\bar{z}\,\partial_a z = \frac{1}{4q}\epsilon_{ab}\,\partial_b \ln\Omega = -\frac{\Omega^{1/2}}{2qR^2}\epsilon_{ab}x^b \tag{C-47}$$

After some algebra one can write D_0 in the form

$$D_0 = \Omega^{-1/2}\sigma_a\left(\partial_a - \frac{\Omega^{1/2}}{2R^2}(1 + k\sigma_3)x^a\right) \tag{C-48}$$

which, in turn, may be put in to the form

$$D_0 = \Omega^{-3/4}\exp\left(\frac{k}{4}\sigma_3 \ln\Omega\right)\sigma_a\,\partial_a\Omega^{1/4}\exp\left(\frac{k}{4}\sigma_3 \ln\Omega\right) \tag{C-49}$$

Because of the equivalence of the operators in the general and $SU(2)$-invariant formalisms, the eigenfunctions and the eigenvalues of the two operators are in one-to-one correspondence. Thus, if $\tilde{\psi}$ is an eigenfunction of $\tilde{D}$ with eigenvalue iE, it follows that

$$D(u\tilde{\psi}) = iE(u\tilde{\psi}) \tag{C-50}$$

i.e., $u\tilde{\psi}$ is an eigenfunction of D with the same eigenvalue iE. Furthermore, the chirality operator Γ_5 of the $SU(2)$-invariant formalism turns out to be equivalent to the Pauli matrix σ_3:

$$u\Gamma_5 u^{-1} = \sigma_3 \tag{C-51}$$

In other words, if $\tilde{\psi}$ is a spinor with chirality τ with respect to Γ_5, then $u\tilde{\psi}$ is a spinor with the same chirality τ with respect to σ_3. From the discussion in Appendix A it then follows that

1. the non-zero eigenfunctions of the anti-hermitean operator D may be labelled as η_ν; $\nu = \pm 1, \pm 2, \ldots$ where $\eta_{-\nu} = \sigma_3 \eta_\nu$
2. if η_ν is an eigenfunction of iD with eigenvalue iE_ν, the eigenvalue corresponding to $\eta_{-\nu}$ is $-iE_\nu$
3. there are precisely $|k|$ zero modes of D for a given k; they all have either positive or negative chirality depending on whether $k<0$ or $k>0$, respectively.

Now it is easy to see that the Green's function of D, defined as in (C-34) has the form

$$S(x, y) = \begin{pmatrix} 0 & S_+(x, y) \\ S_-(x, y) & 0 \end{pmatrix} \tag{C-52}$$

and satisfies

$$S^\dagger(x, y) = -S(y, x) \tag{C-53}$$

Furthermore, if χ_n is any zero mode of D, we have the relations

$$\int_{S^2} d^2x \sqrt{g}\, \bar{\chi}_n(x) S(x, y) = 0 \tag{C-54}$$

$$\int_{S^2} S(x, y) \chi_n(y) \sqrt{g}\, d^2y = 0 \tag{C-55}$$

All these statements about D are, in particular, true for D_0 which is obtained by setting $\phi = 0$, $h = 1$ in D. The zero mode equation for D_0 thus becomes

$$\sigma_a\, \partial_a(\Omega^{1/4} e^{(k/4)\sigma_3 \ln \Omega} \chi_n) = 0 \tag{C-56}$$

$$\chi_n = \begin{pmatrix} \chi_n^+ \\ \chi_n^- \end{pmatrix} \tag{C-57}$$

whereby either χ_n^+ or χ_n^- is identically zero depending on whether $k>0$ or $k<0$, respectively. Hence, the zero-mode equation for the two cases may be written as

$$\partial_+(\Omega^{(1-k)/4} \chi_n^-) = 0 \tag{C-58}$$

$$\partial_-(\Omega^{(1+k)/4} \chi_n^+) = 0 \tag{C-59}$$

respectively, where we have defined

$$\partial_\pm = \tfrac{1}{2}(\partial_1 \mp i\, \partial_2) \tag{C-60}$$

$$x^\pm = x^1 \pm i x^2 \tag{C-61}$$

Case (i): $k>0$

Let us concentrate on the case $k>0$. An independent set of solutions to the equation (C-58) is given by

$$\chi_n^- = a_n (x^-)^n \Omega^{(k-1)/4}(x) \tag{C-62}$$

The condition that these zero modes must be normalizable

$$\int_{S^2} d^2x \sqrt{g}\, \bar{\chi}_n^- \chi_n^- = 1 \tag{C-63}$$

restricts the values n can take. It is easy to show that

$$n = 0, 1, \ldots, k-1 \tag{C-64}$$

$$|a_n|^2 = \frac{k}{4\pi}\binom{k-1}{n} 2^{1-k} R^{-(2n+1)} \tag{C-65}$$

This is in agreement with the fact that D has precisely $|k|$ zero modes for a given k. Let us set $R=1$ for simplicity; at the end we can restore the R factors.

The projection operator onto the zero-mode subspace is thus given by

$$P(x, y) = \sum_n \begin{pmatrix} 0 \\ \chi_n^-(x) \end{pmatrix} (0 \quad \chi_n^-(x)^*) \tag{C-66}$$

$$= \tfrac{1}{2}(1-\sigma_3) \frac{k}{4\pi} \{(1+\mathbf{x}^2)(1+\mathbf{y}^2)\}^{(1-k)/2} (1+x^- y^+)^{k-1} \tag{C-67}$$

Thus the differential equation (C-35) satisfied by the Green's function reduces to the two equations

$$\Omega^{(k-3)/4}\, \partial_+ (\Omega^{(1-k)/4} S_-) = \tfrac{1}{2}\Omega_x^{-1}\, \delta(x-y) \tag{C-68}$$

$$\Omega^{-(k+3)/4}\, \partial_- (\Omega^{(k+1)/4} S_+) = \tfrac{1}{2}\Omega_x^{-1}\, \delta(x-y) \tag{C-69}$$

$$- \frac{k}{4\pi} 2^{1-k} (\Omega_x \Omega_y)^{(k-1)/4} (1+x^- y^+)^{k-1} \tag{C-70}$$

where, for simplicity, we have used the notation $\Omega_x = \Omega(x)$, $\Omega_y = \Omega(y)$. To simplify them further, define

$$S_-(x, y) = \Omega_x^{(k-1)/4} T_-(x, y) \Omega_y^{-(k+1)/4} \tag{C-71}$$

$$S_+(x, y) = \Omega_x^{-(k+1)/4} T_+(x, y) \Omega_y^{(k-1)/4} \tag{C-72}$$

Thus we obtain the differential equations for T_- and T_+:

$$\partial_+^x T_-(x, y) = \tfrac{1}{2}\, \delta(x-y) \tag{C-73}$$

$$\partial_-^x T_+(x, y) = \tfrac{1}{2}\, \delta(x-y) - \frac{k}{\pi} 2^{-(k+1)} \Omega_x^{(k+1)/2} (1+x^- y^+)^{k-1} \tag{C-74}$$

Furthermore, the properties (C-54), (C-55) lead to the conditions

$$\int d^2x\bar{\chi}_n^-(x)\Omega_x^{(k-1)/4}T_-(x, y) = 0 \tag{C-75}$$

$$\int d^2y\chi_n^-(y)\Omega_y^{(k-1)/4}T_+(x, y) = 0 \tag{C-76}$$

for $n = 0, 1, \ldots, k-1$. Finally, from

$$S(x, y)^\dagger = -S(y, x) \tag{C-77}$$

follows the relation

$$T_+(x, y) = -T_-(y, x)^* \tag{C-78}$$

Thus we can write two further differential equations for T_- and T_+:

$$\partial_+^y T_-(x, y) = -\tfrac{1}{2}\,\delta(x-y) + \frac{k}{8\pi}2^{1-k}\Omega_y^{(k+1)/2}(1 + x^-y^+)^{k-1} \tag{C-79}$$

$$\partial_-^y T_+(x, y) = -\tfrac{1}{2}\,\delta(x-y) \tag{C-80}$$

The differential equations (C-73) and (C-79) determine $T_-(x, y)$ up to an additive function of x^- and y^-. The solution is

$$T_-(x, y) = \frac{1}{2\pi}\frac{1}{x^- - y^-}(1 + y^+y^-)^{-k}(1 + x^-y^+)^k + t_-(x^-, y^-) \tag{C-81}$$

The condition (C-75) leads to

$$t_-(x^-, y^-) = 0 \tag{C-82}$$

We prove that $t_-(x^-, y^-) = 0$ by showing that the function

$$\tilde{T}(x, y) = \frac{1}{2\pi}\frac{1}{x^- - y^-}(1 + y^+y^-)^{-k}(1 + x^-y^+)^k \tag{C-83}$$

alone satisfies the condition (C-75) for $0 \le n \le k-1$. Uniqueness of the Green's function then implies that $T(x, y) = \tilde{T}(x, y)$ or $t_-(x^-, y^-) = 0$. The proof goes as follows: Define

$$J_{n,k}(y) = \int d^2x\frac{(x^+)^n(1 + x^-y^+)^k}{(1 + x^+x^-)^{k+1}(x^- - y^-)} \tag{C-84}$$

This contains all the relevant factors in the integrand (C-75). The substitution

$$x^1 = r\cos\varphi \qquad x^2 = r\sin\varphi$$

yields

$$J_{n,k} = \int_0^\infty dr\frac{r^{n+1}}{(1 + r^2)^{k+1}}\sum_{q=0}^{k}\binom{k}{q}(y_+r)^q I_{n-q} \tag{C-85}$$

where

$$I_p = \int_0^{2\pi} d\varphi e^{ip\varphi} \frac{1}{re^{-i\varphi} - y_-}$$

$$= 2\pi \frac{r^p}{(y_-)^{p+1}} \begin{cases} -\theta(|y_-| - r) & \text{for} \quad p \geq 0 \\ \theta(r - |y_-|) & \text{for} \quad p < 0 \end{cases} \tag{C-86}$$

Setting $r^2 = \rho$, $|y_-|^2 = z$ one obtains

$$J_{n,k} = \frac{\pi}{(y_-)^{n+1}} \left\{ -\sum_{q=0}^{\pi} \binom{k}{q} z^q \int_0^z d\rho \frac{\rho^n}{(1+\rho)^{k+1}} + \sum_{q=n+1}^{k} \binom{k}{q} z^q \int_z^\infty d\rho \frac{\rho^n}{(1+\rho)^{k+1}} \right\} \tag{C-87}$$

Using [23, Eq. 3.194.3]

$$\int_0^\infty d\rho \frac{\rho^n}{(1+\rho)^{k+1}} = B(n+1, k-n) = \frac{n!\,(k-n-1)!}{k!} \tag{C-88}$$

one can write this as

$$J_{n,k} = \frac{\pi}{(y_-)^{n+1}} \left\{ -\sum_{q=0}^{n} \frac{n!\,(k-n-1)!}{q!\,(k-q)!} z^q + (1+z)^k \int_z^\infty d\rho \frac{\rho^n}{(1+\rho)^{k+1}} \right\} \tag{C-89}$$

To evaluate the remaining integral, make the substitution

$$(1+\rho) = (1+z)\omega \tag{C-90}$$

Then

$$(1+z)^k \int_z^\infty d\rho \frac{\rho^n}{(1+\rho)^{k+1}} = \sum_{q=0}^{n} \binom{n}{q} z^q \int_1^\infty d\omega (\omega-1)^{n-q} \omega^{q-k-1} \tag{C-91}$$

Using [23, Eq. 3.191.2]

$$\int_1^\infty d\omega \omega^{q-k-1} (\omega-1)^{n-q} = B(k-n, n-q-1) = \frac{(k-n-1)!\,(n-q)!}{(k-q)!} \tag{C-92}$$

one sees that the two terms in (C-89) precisely cancel each other thus giving

$$J_{n,k} = 0 \tag{C-93}$$

for all $0 \leq n \leq k-1$. As mentioned above, this implies that $t_-(x^-, y^-) = 0$. Then $T_+(x, y)$ is given by (C-78):

$$T_+(x, y) = \frac{1}{2\pi} \frac{1}{x^+ - y^+} (1 + x^+ x^-)^{-k} (1 + x^- y^+)^k \tag{C-94}$$

Thus the Green's function of D_0 is given by

$$S = \frac{2^{1-k}}{4\pi} \frac{(1 + x^- y^+)^k}{|\mathbf{x} - \mathbf{y}|^2} (\Omega_x \Omega_y)^{(k-1)/4} \begin{pmatrix} 0 & x^- - y^- \\ x^+ - y^- & 0 \end{pmatrix}$$

or, restoring $R \neq 1$,

$$S(x, y) = \frac{2^{1-k} R^{-2k}}{4\pi} \frac{(R^2 + x^- y^+)^k}{|\mathbf{x} - \mathbf{y}|^2} (\Omega_x \Omega_y)^{(k-1)/4} \sigma_a (x^a - y^a) \tag{C-95}$$

Case (ii): $k < 0$
Zero modes are now given by (see equation (C-59))

$$\partial_-(\Omega^{(1-|k|)/4} \chi_n^+) = 0 \tag{C-96}$$

An independent set of normalizable zero modes may thus be given as

$$\chi_n^+(x) = b_n (x^+)^n \Omega^{(|k|-1)/4}(x) \tag{C-97}$$

where $n = 0, 1, \ldots, |k| - 1$. The normalization condition leads to

$$|b_n|^2 = \frac{|k|}{4\pi} \binom{|k| - 1}{n} 2^{1-|k|} \tag{C-98}$$

The projection operator on to the zero-mode subspace is

$$P = \sum_n \begin{pmatrix} \chi_n^+(x) \\ 0 \end{pmatrix} (\chi_n^+(y)^* \quad 0)$$
$$= \tfrac{1}{2}(1 + \sigma_3) \frac{|k|}{4\pi} \{(1 + \mathbf{x}^2)(1 + \mathbf{y}^2)\}^{(1-|k|)/2} (1 + x^+ y^-)^{|k|-1} \tag{C-99}$$

The two non-zero components S_+, S_- of the Green's function are now given by

$$\Omega_x^{-(|k|+3)/4} \partial_+^x (\Omega_x^{(|k|+1)/4} S_-) = \tfrac{1}{2} \Omega_x^{-1} \delta(x - y) - \frac{|k|}{8\pi} \{(1 + \mathbf{x}^2)(1 + \mathbf{y}^2)\}^{(1-|k|)/4} (1 + x^+ y^-)^{|k|-1} \tag{C-100}$$

$$\Omega_x^{-(3-|k|)/4} \partial_-^x (\Omega_x^{(1-|k|)/4} S_+) = \tfrac{1}{2} \Omega_x^{-1} \delta(x - y) \tag{C-101}$$

To simplify these equations further we use the same substitutions as before (see C-71, C-72):

$$S_-(x, y) = \Omega_x^{-(|k|+1)/4} T_-(x, y) \Omega_y^{(|k|-1)/4} \tag{C-102}$$
$$S_+(x, y) = \Omega_x^{(|k|-1)/4} T_+(x, y) \Omega_y^{-(|k|+1)/4} \tag{C-103}$$

Thus we obtain

$$\partial_+^x T_-(x, y) = \tfrac{1}{2} \delta(x - y) - \frac{|k|}{8\pi} 2^{1-|k|} \Omega_x^{(|k|+1)/2} (1 + x^+ y^-)^{|k|-1} \tag{C-104}$$

$$\partial_-^x T_+(x, y) = \tfrac{1}{2} \delta(x - y) \tag{C-105}$$

Taking the complex conjugation of the two equations we get

$$\partial_-^x T_-^*(x, y) = \tfrac{1}{2}\,\delta(x-y) - \frac{|k|}{8\pi} 2^{1-|k|}\Omega_x^{(|k|+1)/2}(1 + x^- y^+)^{|k|-1} \tag{C-106}$$

$$\partial_+^x T_+^*(x, y) = \tfrac{1}{2}\,\delta(x-y) \tag{C-107}$$

Comparison of these equations with equations (C-73) and (C-74) in $k>0$ case shows that the solutions $T_-^*(x, y)$ and $T_+^*(x, y)$ for $k<0$ are the same as $T_+(x, y)$ and $T_-(x, y)$ in the previous case. Hence we find

$$T_-^*(x, y) = \frac{1}{2\pi}\frac{1}{x^+ - y^+}(1 + x^+x^-)^{-|k|}(1 + x^-y^+)^{|k|} \tag{C-108}$$

$$T_+^*(x, y) = \frac{1}{2\pi}\frac{1}{x^- - y^-}(1 + y^+y^-)^{-|k|}(1 + x^-y^+)^{|k|} \tag{C-109}$$

or

$$T_-(x, y) = \frac{1}{2\pi}\frac{1}{x^- - y^-}(1 + x^+x^-)^{-|k|}(1 + x^+y^-)^{|k|} \tag{C-110}$$

$$T_+(x, y) = \frac{1}{2\pi}\frac{1}{x^+ - y^+}(1 + y^+y^-)^{-|k|}(1 + x^+y^-)^{|k|} \tag{C-111}$$

The Green's function for $k<0$ is thus given by

$$S(x, y) = \frac{2^{1-|k|}R^{-2|k|}}{4\pi}\frac{(R^2 + x^+y^-)^{|k|}}{|\mathbf{x}-\mathbf{y}|^2}(\Omega_x\Omega_y)^{(|k|-1)/4}\sigma_a(x^a - y^a) \tag{C-112}$$

Case (III): $k=0$

In this case, there are no zero modes; the projector P is zero. The Green's function may be obtained by simply setting $k=0$ in either of (C-95) or (C-112). Thus we get

$$S(x, y) = \frac{1}{2\pi}(\Omega_x\Omega_y)^{-1/4}\frac{\sigma_a(x^a - y^a)}{|\mathbf{x}-\mathbf{y}|^2} \tag{C-113}$$

Thus we have solved the problem of finding the Green's function of the operator $\mathbb{D}$ for any given k. Namely, from equations (C-40) and (C-45) we have

$$\mathcal{G}(x, y \mid \mathbb{D}) = \frac{1}{R}u^{-1}(x)h(x)e^{-q\sigma_3\phi(x)}S(x, y)e^{-q\sigma_3\phi(y)}h^{-1}(y)u(y) \tag{C-114}$$

where $S(x, y)$ is given by the equation (C-95), (C-112) or (C-113) depending on whether $k>0$, $k<0$, or $k=0$.

D. Various sums and their limits appearing in the main calculation

Most of the sums resulting in the ϕ-integral of the path integrals can be expressed through simple relations between sums of the form

$$S_b(\theta) = \frac{1}{4\pi} \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1) + bR^2} P_l(\cos\theta) \tag{D-1}$$

Therefore we study the properties of $S_b(\theta)$ in detail so that all the other sums and their limits may be obtained easily. All the relations among various special functions used here may be found in [23] and [24].

D.1. $S_b(\theta)$ *and the Green's function of the Laplacian on* S^2

The Laplacian operator on a curved manifold is given by

$$\Delta = \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g}\, g^{\mu\nu} \partial_\nu \tag{D-2}$$

where $g_{\mu\nu}$ is the metric tensor of the manifold and $g = \det(g_{\mu\nu})$. If we choose polar coordinates on S^2,

$$x = (\theta, \varphi)$$

we can write

$$\Delta = -\frac{1}{R^2} \mathbf{L}^2 \tag{D-3}$$

where $\mathbf{L}^2$ is the angular momentum operator and R is the radius of the sphere. Now consider the Green's function defined by

$$(-\Delta_x + b) G_b(x, y \mid -\Delta) = g^{-1/2} \delta(x-y) \qquad (b > 0) \tag{D-4}$$

In polar coordinates $x = (\theta, \varphi)$, $y = (\theta', \varphi')$ we also have $g(x) = R^4 \sin^2\theta$. If $Y_{lm}(\theta, \varphi)$ are the spherical harmonics, the functions $\mathcal{Y}_{lm} = Y_{lm}/R$ form a complete, orthogonal set of eigenfunctions of $-\Delta$ which are normalized to 1 on S^2. Hence we can write

$$G_b(x, y \mid -\Delta) = \sum_{\substack{l=0,1,\ldots \\ m=-l,\ldots,l}} \frac{\mathcal{Y}_{lm}(\theta, \varphi) \mathcal{Y}^*_{lm}(\theta', \varphi')}{\dfrac{l(l+1)}{R^2} + b} \tag{D-5}$$

and using the 'addition theorem' for spherical harmonics

$$\sum_m Y_{lm}(\theta, \varphi) Y^*_{lm}(\theta', \varphi') = \frac{2l+1}{4\pi} P_l(\cos\omega) \tag{D-6}$$

we get

$$G_b(x, y \mid -\Delta) = \frac{1}{4\pi} \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1) + bR^2} P_l(\cos\omega) \tag{D-7}$$

where ω is the angle between $\mathbf{r}(x)$ and $\mathbf{r}(y)$. Let us choose the point $y = (\theta', \varphi')$ to be the north pole. This implies that $\cos\omega = \cos\theta$ so that G depends only on θ. Let us denote it by $G(\theta \mid -\Delta)$. Comparison with the sum $S_b(\theta)$ gives the following relation:

$$G_b(\theta \mid -\Delta) = S_b(\theta) + \frac{1}{4\pi b R^2} \tag{D-8}$$

We wish to find a closed expression for $S_b(\theta)$, or for $G_b(\theta \mid -\Delta)$. Since $G_b(\theta \mid -\Delta)$ satisfies equation (D-4) written in polar coordinates

$$\left\{ -\frac{1}{R^2 \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + b \right\} G_b(\theta \mid -\Delta) = 0 \tag{D-9}$$

for $\theta \neq 0$. Thus $G_b(\theta \mid -\Delta)$ satisfies the differential equation for *associated Legendre functions*. Therefore, let us examine the associated Legendre functions with the property that they are singular at $x = \cos\theta = 1$. These are usually denoted by $Q_\nu(x)$. Expand $Q_\nu(x)$ in Legendre polynomials:

$$Q_\nu(x) = \sum_{l=0}^{\infty} a_l P_l(x). \tag{D-10}$$

Using $\int_{-1}^{1} dz P_l(z) P_m(z) = 2\delta_{lm}/(2l+1)$ we find that [23]

$$a_l = (l + \tfrac{1}{2}) \frac{\{1 - \cos\pi(\nu - l)\}}{(l - \nu)(l + 1 + \nu)} \tag{D-11}$$

If we require that the denominator here to be equal to the denominator in the sum for $G(\theta \mid -\Delta)$

$$(l - \nu)(l + 1 + \nu) = l(l+1) + bR^2 \tag{D-12}$$

we get the two solutions

$$\nu_1 = -\tfrac{1}{2} + \sqrt{\tfrac{1}{4} - bR^2} \qquad \nu_2 = -\tfrac{1}{2} - \sqrt{\tfrac{1}{4} - bR^2} \tag{D-13}$$

for ν, with $\nu_1 + \nu_2 = -1$. It is simple to see that

$$Q_{\nu_1}(x) + Q_{\nu_2}(x) = \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1) + bR^2} P_l(x) \tag{D-14}$$

which is just $4\pi G_b(\theta \mid -\Delta)$. Thus we have $S_b(\theta)$ in terms of known functions:

$$S_b(\theta) = \frac{1}{4\pi} \{Q_{\nu_1}(x) + Q_{\nu_2}(x)\} - \frac{1}{4\pi b R^2}. \tag{D-15}$$

D.2. *Different limits of the Green's function*

We can now check if our Green's function reproduces the known flat-space limit. We can also see what happens as $b \to 0$.

Flat-space Green's function of $(-\Delta + b)$

To check the flat-space limit let us fix the geodesic distance $s = R\theta$ and let $R \to \infty$. So we have to let $\theta \to 0$ as well. Define $\lambda = \sqrt{bR^2 - \frac{1}{4}}$ which is real since R is large. Now

$$G_b(\theta \mid -\Delta) = \frac{1}{4\pi} \{Q_{-1/2+i\lambda}(\cos\theta) + Q_{-1/2-i\lambda}(\cos\theta)\}$$

$$= \frac{1}{4\cosh\lambda\pi} P_{-1/2+i\lambda}(-\cos\theta) \tag{D-16}$$

We can use the integral representation

$$P_\nu(\cos\theta) = \frac{2}{\pi}\int_0^\infty \frac{\cos(\nu+\frac{1}{2})\varphi}{\sqrt{2(\cos\varphi - \cos\theta)}}\, d\varphi \tag{D-17}$$

to find the behaviour of $P_{-1/2-i\lambda}(-\cos\theta)$ for large R and small θ. This leads to the asymptotic expansion

$$P_{-1/2+i\lambda}(-\cos\theta) = \frac{e^{\lambda\pi}}{\pi} K_0(\lambda\theta) + \frac{1}{R} O\left(\frac{s}{R}\right)^2. \tag{D-18}$$

where K_0 is a Bessel function. Furthermore,

$$\lambda\theta = \sqrt{b}\, s - \frac{s}{8\sqrt{b}}\frac{1}{R^4} \tag{D-19}$$

and hence

$$K_0(\lambda\theta) \simeq K_0\left(\sqrt{b}\, s - \frac{s}{8\sqrt{b}\, R^2}\right)$$

$$= K_0(\sqrt{b}\, s) + K_1(\sqrt{b}\, s)\frac{s}{8\sqrt{b}\, R^2} + O\left(\frac{1}{R^4}\right) \tag{D-20}$$

The last step follows from a Taylor expansion and using the fact that $(d/dz)K_0(x) = -K_1(z)$. Thus, for the Green's function we have

$$G_b(\theta \mid -\Delta) = \frac{1}{2\pi} K_0(\sqrt{b}\, s) + K_1(\sqrt{b}\, s)\frac{s}{16\sqrt{b}\, R^2\pi} + O\left(\frac{1}{R^4}\right). \tag{D-21}$$

In the limit $R \to \infty$, this agrees with the flat-space Green's function of $(-\Delta + b)$ in two dimensions, which is

$$G(\mathbf{r} - \mathbf{r}') = \frac{1}{2\pi} K_0(\sqrt{b}\, |\mathbf{r} - \mathbf{r}'|) \tag{D-22}$$

For S, we have

$$S_b(\theta) \overset{\substack{R\to\infty \\ s\text{ fixed}}}{=} \frac{1}{2\pi} K_0(\sqrt{b}\, s) + O\left(\frac{1}{R^2}\right) \tag{D-23}$$

Green's function of $-\Delta$ and its flat-space limit

Since $l=0$ is a zero mode of $-\Delta$ we cannot represent the Green's function of $-\Delta$ as a sum over *all* eigen modes. In this case, a meaningful way to define the Green's function is through (cf. equation (C-35))

$$-\Delta G(x, y \mid -\Delta) = g^{-1/2}\,\delta(x-y) - P_0(x, y \mid -\Delta) \tag{D-24}$$

where P_0 is the projector into the zero mode space. Then the following relation holds.

$$G_0(\theta \mid -\Delta) = \frac{1}{4\pi}\sum_{l=1}^{\infty}\frac{2l+1}{l(l+1)}P_l(\cos\theta) \tag{D-25}$$

This is indentical with $S_0(\theta)$. Notice that $l=0$ is excluded. This series is convergent and the sum is given by ([23], Eqs. 8.926)

$$S_0(\theta) = G_0(\theta \mid -\Delta) = -\frac{1}{2\pi}\ln\sin\frac{\theta}{2} - \frac{1}{4\pi} \tag{D-26}$$

We can also deduce this by noticing that

$$G_0(\theta \mid -\Delta) = \lim_{b\to 0}\left\{G_b(\theta \mid -\Delta) - \frac{1}{4\pi bR^2}\right\} \tag{D-27}$$

Define $\nu = \nu_1 = -\frac{1}{2} + \sqrt{\frac{1}{4} - bR^2}$, then $\nu_2 = -1-\nu$. Expanding ν for small b we get

$$\nu = -bR^2 - b^2R^4 + O(b^3) \tag{D-28}$$

which gives zero as $b \to 0$. Therefore

$$G_0(\theta \mid -\Delta) = \lim_{b\to 0}\frac{1}{4\pi}\left\{Q_\nu(\cos\theta) + Q_{-1-\nu}(\cos\theta) - \frac{1}{bR^2}\right\} \tag{D-29}$$

may be expanded for small ν and we get

$$G_0(\theta \mid -\Delta) = \frac{1}{4\pi}\left[2Q_0 - \left.\frac{\partial P_\nu}{\partial \nu}\right|_{\nu=0}\right] + \frac{1}{4\pi}\lim_{b\to 0}\left[-\frac{1}{\nu} - \frac{1}{bR^2}\right] \tag{D-30}$$

In the limit $b \to 0$, $[-1/\nu - 1/bR^2] \to -1$. Furthermore,

$$Q_0(x) = \tfrac{1}{2}\ln\frac{1+x}{1-x} \tag{D-31}$$

$$\left.\frac{\partial P_\nu}{\partial \nu}\right|_{\nu=0} = \ln\frac{1+x}{2} \tag{D-32}$$

where $x = \cos\theta$, so we get

$$G_0(\theta \mid -\Delta) = S_0(\theta)$$

$$= -\frac{1}{4\pi}\ln\frac{1-x}{2} - \frac{1}{4\pi} \tag{D-33}$$

which is, in fact, the same result as before. This is the Green's function of $-\Delta$ on the sphere S^2.

Now we can check for the flat-space limit. Again we fix $s = R\theta$ and let $R \to \infty$, $\theta \to 0$.

$$\frac{1-x}{2} = \sin\frac{\theta}{2} = \sin\frac{s}{2R} = \frac{s}{2R} + O\left(\frac{1}{R^3}\right) \tag{D-34}$$

Hence

$$G_0(\theta \mid -\Delta) = -\frac{1}{2\pi}\ln\frac{s}{R} + const + O\left(\frac{1}{R}\right) \tag{D-35}$$

This is the familier result for the Laplacian in flat 2-dimensional space.

D.3. *Various sums in the main calculation*

i) While calculating $\langle \bar{\psi}\psi \rangle$ we encountered the sum

$$F(\theta) = \sum_{l=1}^{\infty} \left\{\frac{(2l+1)bR^2}{l(l+1)\{l(l+1)+bR^2\}}\right\} P_l(\cos\theta) \tag{D-36}$$

where we had $b = q^2/\pi$. Notice that

$$\begin{aligned} F(\theta) &= \sum_{l=1}^{\infty} (2l+1)\left\{\frac{1}{l(l+1)} - \frac{1}{l(l+1)+bR^2}\right\} P_l(\cos\theta) \\ &= 4\pi\{S_0(\theta) - S_b(\theta)\} \end{aligned} \tag{D-37}$$

Using the results in the previous section now it is trivial to give the large R behaviour of $F(\theta)$:

$$F(\theta) \overset{R\to\infty}{=} -2\ln\frac{s}{2R} - 1 - 2K_0(\sqrt{b}\,s) + O\left(\frac{1}{R^2}\right) \tag{D-38}$$

where $s = R\theta$ is held fixed.

ii) We also had the sum

$$F(0) = \sum_{l=1}^{\infty} \frac{(2l+1)bR^2}{l(l+1)\{l(l+1)+bR^2} \tag{D-39}$$

in the formal calculation of $\langle \bar{\psi}\psi \rangle$. We give two ways to find this sum.

a) If we want to take $F(\theta)$ and set $\theta = 0$, we are alarmed by the fact that $S_b(\theta)$ as well as $S_0(\theta)$ is singular for $\theta = 0$. Indeed,

$$S_0(\theta) = -\frac{1}{4\pi}\ln\frac{1-x}{2} - \frac{1}{4\pi} \tag{D-40}$$

$$S_b(\theta) = \frac{1}{4\pi}\{Q_{-1/2+i\lambda}(x) + Q_{-1/2-i\lambda}(x)\} - \frac{1}{4bR^2} \tag{D-41}$$

where $x = \cos\theta$, and $Q_\nu(x)$ is singular for $x = 1$. However, we expect the *difference* $S_0(\theta) - S_b(\theta)$ to be regular there. Using the properties of $Q_\nu(x)$ we indeed have (see [23, 24])

$$Q_{-1/2+i\lambda}(x) + Q_{-1/2-i\lambda}(x) = -\ln\frac{1-x}{2} - 2\gamma - \psi(\tfrac{1}{2}+i\lambda) - \psi(\tfrac{1}{2}-i\lambda) + O(1-x) \tag{D-42}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ denotes the Riemann's ψ-function. The singular piece in this expression exactly cancels the singular term in $S_0(\theta)$. Hence

$$F(0) = \psi(\tfrac{1}{2}+i\lambda) + \psi(\tfrac{1}{2}-i\lambda) + 2\gamma - 1 + \frac{1}{bR^2} \tag{D-43}$$

b) There is also a direct way to calculate $F(0)$. Notice that

$$\begin{aligned} F(0) &= \sum_{l=1}^{\infty}\left\{\frac{1}{l(l+1)} - \frac{1}{l(l+1)+bR^2}\right\} \\ &= \sum_{l=1}^{\infty}\left\{\frac{1}{l} + \frac{1}{l+1} - \frac{1}{l+\frac{1}{2}+i\lambda} - \frac{1}{l+\frac{1}{2}-i\lambda}\right\} \end{aligned} \tag{D-44}$$

where $\lambda = \sqrt{b^2R^2 - \frac{1}{4}}$ as before.

In the limit $R \to \infty$, $|\frac{1}{2}+i\lambda| = |\frac{1}{2}-i\lambda| = b^2R^2$ also tends to infinity. Hence we can use the asymptotic expansion

$$\psi(z) = \ln z - \frac{1}{2z} + O\left(\frac{1}{R^2}\right) \tag{D-45}$$

We immediately get

$$F(0) \overset{R\to\infty}{=} 2\ln\sqrt{b}\,R + 2\gamma - 1 + O\left(\frac{1}{R^2}\right) \tag{D-46}$$

iii) Another occasion where we encounter the sum $S_b(\theta)$ is in the calculation of $\langle\chi(x)\chi(0)\rangle$ in Section 9. There we have

$$\langle\chi(x)\chi(0)\rangle = -\Delta_x S_b(\theta) \tag{D-47}$$

where Δ_x is the Laplacian. From the relation (D-8) we have

$$S_b = G_b - \frac{1}{4\pi bR^2} \tag{D-48}$$

where, for $\theta \neq 0$, G_b satisfies the equation

$$(-\Delta + b)G_0 = 0 \tag{D-49}$$

Hence

$$(-\Delta + b)S_b = -\frac{1}{4\pi R^2} \quad \text{for} \quad \theta \neq 0 \tag{D-50}$$

This implies that

$$\langle \chi(x)\chi(0)\rangle = -bS_b - \frac{1}{4\pi R^2}$$

$$\overset{\substack{R\to\infty \\ s \text{ fixed}}}{=} \quad -\frac{b}{2\pi} K_0(\sqrt{b}\, s) + O\left(\frac{1}{R^2}\right) \tag{D-51}$$

When s is then also made large, we can use an asymptotic expansion for K_0:

$$K_0(\sqrt{b}\, s) \overset{s\to\infty}{=} \sqrt{\frac{\pi}{2\sqrt{b}\, s}} \exp -\sqrt{b}\, s\left(1 + O\left(\frac{1}{s}\right)\right) \tag{D-52}$$

It follows that

$$\lim_{s\to\infty}\left\{\lim_{R\to\infty} \{\langle \chi(x)\chi(0)\rangle\}\right\} = -\frac{1}{2^{3/2}\pi^{5/4}s^{1/2}} e^{-qs/\sqrt{\pi}}\left(1 + O\left(\frac{1}{s}\right)\right) \tag{D-53}$$

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REFERENCES

[1] J. Schwinger, Phys. Rev., *128*, 2425 (1962).
[2] J. H. Lowenstein and J. A. Swieca, Ann. Phys. (N.Y.), *68*, 172 (1971).
[3] A. K. Raina and G. Wanders, Ann. Phys. (N.Y.), *132*, 404 (1981).
[4] P. Becher, Ann. Phys. (N.Y.), *146*, 223 (1983).
[5] J. Balog and P. Hrasko, KFKI preprint, KFKI-1983-39.
[6] A. Z. Capri and R. Ferrari, Nuovo Cimento, *A62*, 273 (1981).
[7] N. K. Nielsen and B. Schroer, *Nucl. Phys.*, *B120*, 62 (1977).
[8] K. D. Rothe and J. A. Swieca, Ann. Phys. (N.Y.), *117*, 382 (1979).
[9] R. Roskies and F. Schaposnik, Phys. Rev., *D23*, 558 (1981).
[10] R. E. Gamboa Saravi, F. Schaposnik and J. E. Solomin, Nucl. Phys., *B185*, 239 (1981).
[11] M. Hortacsu, K. D. Rothe and B. Schroer, Phys. Rev., *D20*, 3203 (1979).
[12] G. Maiella, Lett. Nuovo Cimento, *28*, 261 (1981).
[13] N. V. Krasnikov, V. A. Matveev, V. A. Rubakov, A. N. Tavkhelidze and V. F. Tokarev, Phys. Lett., *B97*, 103 (1980).
[14] K. Fujikawa, Phys. Rev., *D21*, 2848 (1980).
[15] S. Coleman, R. Jakiw and L. Susskind, Ann. Phys., *93*, 267 (1975).
[16] L. S. Brown, Nuovo Cimento, *29*, 617 (1963).
[17] Y. Frishman, Nucl. Phys., *B148*, 74 (1979).
[18] H. M. Fried and T. Grandou, Phys. Rev., *D33*, 1151 (1986).
[19] J. Barcelos and Ashok Das, Phys. Rev., *D33*, 2262 (1986).
[20] M. Lüscher, Ann. Phys. (N.Y.), *142*, 359 (1982).
[21] B. Berg and M. Lüscher, Commun. Math. Phys., *69*, 57–80 (1979).
[22] Berezin, *The Method of Second Quantization*. Academic Press Inc, New York, 1966.

[23] I. S. GRADSTHEYN and I. M. RYZHIK, *Table of Integrals Series and Products.* Academic Press, New York, 1980.
[24] A. ERDÈLYI, *Editor, Tables of Integral Transforms* Vol. II. McGraw-Hill Book Company Inc, 1954.
[25] P. B. GILKEY, *The Index Theorem and the Heat Equation.* Publish or Perish, Boston, 1974.