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The continuum limit of dissipative dynamics in H. Fröhlich's pumped phonon system

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Abstract. The kinetics of H. Fröhlich's pumped phonon system is analysed within the framework of Quantum Dynamical Semigroups. The linear evolution of the hierarchy of reduced density matrices is shown to support a decorrelated solution in the limit of continuously many phonon modes. Thus, a measure theoretic analogue of Fröhlich's non-linear kinetic equation for the one-particle distribution function is obtained.

1. Introduction

In 1968 H. Fröhlich proposed a model of coherent excitations in biological systems.[1] The model comprises a finite number of polarisation waves immersed in a heat bath, but maintained away from equilibrium by external pumping. Fröhlich described this by means of a non-linear kinetic equation for the occupation numbers (equation (1.1) below) and argued that for sufficiently strong pumping the stationary state undergoes Bose condensation into the mode of lowest frequency.

In this paper we examine Fröhlich's kinetic equation in the rigorous setting of Quantum Dynamical Semigroups (QDSG's). We show how the non-linearity arises naturally in the limit of continuously many modes, through the decorrelation of the hierarchy of kinetic equations governing the evolution of reduced density matrices. The kinetic equation thereby obtained for the one-particle distribution function is the measure-theoretic generalisation of Fröhlich's equation to the continuum model.

Fröhlich's Model. We briefly review the original formulation of the model. Let there be V modes with frequencies x_k: $0 < x_1 \leq x_2 \leq \cdots \leq x_V$. Denote by n_k the occupation number of the kth mode. The system heat-bath interaction is assumed to lead to spontaneous emission and absorption of phonons (with transition probabilities ξ_k) and two phonon exchanges (with probability χ_{jk}/V). Detailed balance at heat-bath temperature T is assumed for these processes. Energy is pumped into the kth mode at a rate s_k. In units for which $\hbar = k_B T$ these

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assumptions give rise to the following kinetic equation for the n_k:

$$\frac{d}{dt}n_k = s_k - \xi_k[n_k e^{x_k} - (1+n_k)] - \sum_{j=1}^{V} \chi_{jk}[n_k(1+n_j)e^{x_k} - n_j(1+n_k)e^{x_j}] \tag{1.1}$$

The analysis of the stationary solution of (1.1) is by now well known. Assuming the transition probabilities to be uniform (i.e. $s_k = s$, $\xi_k = \xi$ and $\chi_{jk} = \chi$), a self-consistent expression can be found for the stationary distribution of (1.1) which we denote by m_k:

$$m_k = \left[1 + \frac{\xi}{\chi}(1 - e^{-\mu})\right]\frac{1}{e^{x_k - \mu} - 1} \tag{1.2}$$

where μ is the "effective chemical potential", determined from (1.1) by the requirement that $\sum_k \dot{m}_k = 0$:

$$\xi + s = \frac{\xi}{V}\sum_{k=1}^{V} m_k(e^{x_k} - 1) \tag{1.3}$$

The stationary density $\rho = V^{-1}\sum_k m_k$ can be shown to increase indefinitely with s, which is only possible if μ approaches x_1 from below. Thus in the limit of continuously many modes we expect to find non-equilibrium boson condensation into the mode of lowest energy.

How, exactly, does Fröhlich's non-linear kinetic equation arise? This question has been addressed by Wu and Austin [2, 3]. From a Hamiltonian model of the phonon system coupled to a reservoir, the system dynamics are obtained by elimination of the reservoir variables in terms of their two-point correlation functions in the thermal state. The non-linearity of the kinetic equation arises through the assumption that observables over different phonon modes are uncorrelated. Whereas this condition can be specified for an initial state, it cannot be expected to hold for all time.

In this paper we shall proceed rather in the spirit of Fröhlich. We shall analyse the dissipative dynamics of the phonon system alone, but in the rigorous setting for irreversible quantum processes: that of the theory of Quantum Dynamical Semigroups [4, 5]. Decorrelation (and hence the non-linear dynamics for the occupation numbers) will arise naturally when we proceed to the limit of continuously many phonon modes. States which are initially decorrelated remain so. This property is sometimes called the "propagation of molecular chaos". Such behaviour is, of course, familiar from the decorrelation of the Boltzmann hierarchy to yield the Vlasov equation [6].

The scheme is as follows. Let us start with a finite number V of phonon modes. We take a V-dimensional Hilbert space $\mathscr{H}_V$ with an orthonormal basis $\{f_i: i = 1, 2, \ldots, V\}$ whose elements are in 1-1 correspondence with the phonon modes. Let $\mathscr{F}_V$ be the boson Fock space over $\mathscr{H}_V$, and $\mathscr{B}_V$ the trace-class operators on $\mathscr{F}_V$. Let N_i be the number operator over the mode f_i and ρ a state in $\mathscr{B}_V$. Identifying the expectation values of the N_i in the state ρ, $\langle N_i; \rho\rangle$, with the n_i of (1.1), we seek a QDSG (i. a strongly continuous semigroup of completely positive trace-preserving linear maps on $\mathscr{B}_V$) $\{T_t : t \in \mathscr{R}^+\}$ whose dynamics resembles (1.1) in the (so far formal) sense that

$$\frac{d}{dt}\langle N_k; T_t\rho\rangle = s_k - \xi_k(e^{x_k}\langle N_k; T_t\rho\rangle - \langle (1+N_k); T_t\rho\rangle)$$
$$-\frac{1}{V}\sum_{j\neq k}\chi_{jk}(e^{x_k}\langle N_k(1+N_j); T_t\rho\rangle - e^{x_j}\langle N_j(1+N_k); T_t\rho\rangle) \quad (1.4)$$

The factor $1/V$ in the last term is chosen to make the derivative an intensive quantity. (1.4) can be formally reproduced by a putative generator K for the semigroup T_t whose form is

$$K\rho = \sum_{j=1}^{V}(s_j+\xi_j)(a_j^*\rho a_j - \tfrac{1}{2}(a_j a_j^*\rho + \rho a_j a_j^*))$$
$$+\sum_{j=1}^{V}(s_j+\xi_j e^{x_j})(a_j\rho a_j^* - \tfrac{1}{2}(a_j^* a_j\rho + \rho a_j^* a_j))$$
$$+\frac{1}{V}\sum_{\substack{i,j=1\\ i\neq j}}^{V}\chi_{ij}e^{x_j}(a_i^* a_j\rho a_j^* a_i - \tfrac{1}{2}(a_i a_j^* a_j a_i^*\rho + \rho a_i a_j^* a_j a_i^*)) \quad (1.5)$$

where $a_i = a(f_i)$ and $a_i^* = a^*(f_i)$ are the annihilation and creation operators for the phonon modes. Formally, $\langle N_k; KT_t\rho\rangle$ yields (1.4). Of course, if a state ρ is decorrelated in the sense that

$$\langle N_k N_j; \rho\rangle = \langle N_k; \rho\rangle\langle N_j; \rho\rangle \qquad j\neq k \quad (1.6)$$

then the identification of n_k with $\langle N_k; \rho\rangle$ yields (1.1) exactly. Thus the problem of deriving (1.1) reduces to that of substantiating (1.4) and (1.6) in an appropriate sense.

The paper will be organised as follows. In Section 2 we will show that the densely defined operator K in (1.5) gives rise to a QDSG $\{T_t : t \in \mathscr{R}^+\}$ which preserves finiteness of expectation values of powers of the particle density. In Section 3 this property will allow us to write down the kinetic equation for the evolution of the hierarchy of reduced density matrices under T_t. In Section 4 we will show that the continuum limit of this hierarchy supports a unique decorrelated solution for decorrelated initial conditions: the n-particle distribution function is simply a product of copies of the 1-particle distribution function. The kinetic equation for the 1-particle distribution is the continuum analogue of (1.1) which we seek.

2. Irreversible quantum dynamics

We now construct the irreversible quantum evolution specified by the putatative generator (1.5). The main result of this section, Theorem 2.4, is that if expectation values of powers of the particle density are initially finite, they remain so.

First we specify our model exactly. Let $X = [\underline{x}, \bar{x}]: 0 < \underline{x} < \bar{x} < \infty$. For each $V \in \mathbf{N}$, let $X_V \subset X$, $X_V = \{x_i : i = 1, 2, \ldots, V\}$ be the set of phonon mode energies. We connect with the rate constants s_i, ξ_i and χ_{ij} of (1.1) by defining the functions

$$\gamma_V : X_V \times X_V \to \mathscr{R}^+ \qquad \gamma_V(x_i, x_j) = e^{x_i}\chi_{ij} \tag{2.1a}$$

$$\alpha_V^+ : X_V \to \mathscr{R}^+ \qquad \alpha_V^+(x_i) = s_i + \xi_i \tag{2.1b}$$

$$\alpha_V^- : X_V \to \mathscr{R}^+ \qquad \alpha_V^-(x_i) = s_i + \xi_i e^{x_i} \tag{2.1c}$$

We assume that for each V, γ_V (resp. $\alpha_V^\pm$) is the restriction to $X_V \times X_V$ (resp. X_V) of a strictly positive continuous function γ on $X \times X$ (resp. $\alpha^\pm$ on X). In what follows we shall sometimes write γ_{ij} for $\gamma(x_i, x_j)$ and $\alpha_i^\pm$ for $\alpha^\pm(x_i)$. Then (1.4) yields the family $\{K_V : V \in \mathbf{N}\}$ of densely defined operators on $\{\mathscr{B}_V : V \in \mathbf{N}\}$.

$$\begin{aligned} K_V\rho = {} & \tfrac{1}{2}\sum_{i=1}^{V} \alpha_i^+([a_i^*\rho, a_i] + [a_i^*, \rho a_i]) \\ & + \tfrac{1}{2}\sum_{i=1}^{V} \alpha_i^-([a_i\rho, a_i^*] + [a_i, \rho a_i^*]) \\ & + \tfrac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{V} \gamma_{ij}([a_i^* a_j\rho, a_i a_j^*] + [a_i^* a_j, \rho a_i a_j^*]) \end{aligned} \tag{2.2}$$

For each V we have the following proposition:

Proposition 2.1. *There exists a strongly continuous semigroup of positive trace-preserving maps $\{T_{V,t} : t \in \mathscr{R}^+\}$ whose generator is an extension of K_V.*

Proof. [5, Theorems 3.1 and 4.1].

In order to prove that the expectation values of powers of the density remain finite under the evolution we introduce some notation. Since we are interested only in observables which are polynomials in the N_i we will restrict our attention to $\mathscr{E}_V$, the subset of states in $\mathscr{B}_V$ which are functions of the N_i alone. To each V-tuple of non-negative integers $\mathbf{n} = (n_1, n_2, \ldots, n_V)$ there corresponds a state in $\mathscr{E}_V$ with n_i particles in mode i. Signify the projection onto the corresponding vector in $\mathscr{F}_V$ by $P_\mathbf{n}$. The $P_\mathbf{n}$ form a basis for $\mathscr{E}_V$, and the norm on $\mathscr{E}_V$ inherited from $\mathscr{B}_V$ is

$$\|\rho\| = \sum_n |\text{trace } P_n\rho| \tag{2.3}$$

Now any state in $\mathscr{E}_V$ can be viewed as a probability measure on $\mathbf{N}^V$. In what follows we shall use N_i to denote both a number operator on $\mathscr{F}_V$ and the operator on $\mathscr{E}_V$ corresponding to multiplication by that operator. Dom (N_i) will denote the dense domain of this latter operator in $\mathscr{E}_V$.

We now write down the action of K_V on $\mathscr{E}_V$. Denote by I_i^+ the shift on $\mathscr{E}_V$ for which

$$I_i^+(P_{\mathbf{n}}) = P_{\mathbf{n}+1_i} \qquad \text{where } \mathbf{n} + 1_i = (n_1, \ldots, n_i + 1, \ldots, n_V) \tag{2.4}$$

and let I_i^- denote the inverse shift (with $I_i^-(P_{\mathbf{n}}) = 0$ when $n_i = 0$). Define $I_{ij} = I_i^+ I_j^- (i \neq j)$ and set $I_{ii} = I_i^- I_i^+ = 1$. Then the action of K_V on $\mathscr{E}_V$ is

$$K_V \rho = J_V \rho + Z_V \rho \tag{2.5}$$

where

$$J_V \rho = \sum_{i=1}^{V} (\alpha_i^+ I_i^+ (N_i + 1) + \alpha_i^- I_i^- N_i) \rho + \frac{1}{V} \sum_{i,j=1}^{V} \gamma_{ji} I_{ij} (N_i + 1) N_j \rho \tag{2.6a}$$

$$Z_V \rho = -\sum_{i=1}^{V} (\alpha_i^+ (N_i + 1) + \alpha_i^- N_i) \rho - \frac{1}{V} \sum_{i,j=1}^{V} \gamma_{ji} (N_i + 1) N_j \rho \tag{2.6b}$$

For clarity, the volume subscript V will be suppressed for the remainder of this section. From [5] and [7] we can extract the following facts:

(i) For $r \in [0, 1)$, $K_r = Z + rJ$ is closable, and its closure is the generator of a strongly continuous positive contraction semigroup $\{T_{r,t} : t \in \mathscr{R}^+\}$ on $\mathscr{E}_V$.

(ii) For $\lambda > 0$, $(\lambda - K)^{-1} = \lim_{r \to 1} (\lambda - K_r)^{-1}$ densely defines a closed operator K on $\mathscr{E}_V$.

(iii) For $\rho \in \mathscr{E}_V^+$ if $0 \le r \le s < 1$ then $0 \le T_{r,t}\rho \le T_{s,t}\rho$.

(iv) For $\rho \in \mathscr{E}_V$, $T_t \rho = \lim_{r \to 1} T_{r,t} \rho$ defines a strongly continuous semigroup of probability preserving positive contractions whose generator is K.

Now we define the mean-number (or density) operator $\underset{\sim}{N} = 1/V \sum_{i=1}^{V} N_i$. In Appendix A we prove the following lemma:

Lemma 2.2. *For $r \in [0, 1)$, $p \in \mathbf{N}$,* Dom $(\underset{\sim}{N}^{2p}) =$ Dom $((K_r)^p)$.

Corollary 2.3. *For $r \in [0, 1)$, $p \in \mathbf{N}$,* Dom $(\underset{\sim}{N}^{2p})$ *is invariant under $T_{r,t}$.*

Proof. [8, Lemma 1.1] can be trivially modified to show that for any contraction semigroup S_t with generator Q, Dom (Q^p) is invariant under S_t for all $p \in \mathbf{N}$.

Theorem 2.4. *Let $\rho \in C^\infty(\underset{\sim}{N}) = \bigcap_{n \in \mathbf{N}}$ Dom $(\underset{\sim}{N}^n)$. Then for all $p \in \mathbf{N}$, $t \in [0, \infty)$, $\|\underset{\sim}{N}^p T_t \rho\| < \infty$.*

Proof. By Corollary 2.3, $C^\infty(\underset{\sim}{N})$ is invariant under $T_{r,t}$, and $\langle \underset{\sim}{N}^p ; T_{r,t}\rho \rangle$ is continuously differentiable. In Appendix A we show that for $\rho \in \mathscr{E}_V^+$, $p \in \mathbf{N}$ and $r \in (\frac{1}{2}, 1)$

$$\frac{d}{dt} \langle \underset{\sim}{N}^p ; T_{r,t}\rho \rangle < \langle (-p\underset{\sim}{\alpha}\underset{\sim}{N}^p + (\bar{\alpha} 2^{p+2} \underset{\sim}{N}^{p-1}) ; T_{r,t}\rho \rangle \tag{2.6}$$

for some positive constants $\underset{\sim}{\alpha}$ and $\bar{\alpha}$. By (i) above, $\|T_{r,t}\rho\| \le \|\rho\|$, so for $p = 1$ (2.6) can be integrated and we conclude that $\langle \underset{\sim}{N} ; T_{r,t}\rho \rangle$ is bounded on any compact interval $[0, T]$, uniformly in r. Now proceed by induction. Suppose that $\langle \underline{N}^{p-1}; T_{r,t}\rho \rangle$ is bounded on $[0, T]$. Then we can integrate (2.6) on $[0, T]$ to obtain the same property for $\langle \underline{N}^p; T_{r,t}\rho \rangle$. Furthermore, the bound is independent of r. Thus, by (iii) above, $\|\underline{N}^p T_{r,t}\rho\| = \langle \underline{N}^p; T_{r,t}\rho \rangle$ is an increasing function of r, bounded above, and $\underline{N}^p T_{r,t}\rho$ is a family in $\mathscr{E}_V$, monotone in r, converging to some limit. Since $\underline{N}^p$ is a closed operator, then by (iv) above

$$\lim_{r\to 1} \|\underline{N}^p T_{r,t}\rho\| = \|\underline{N}^p T_t\rho\| < \infty \tag{2.7}$$

Finally, since $\underline{N}^p$ is positive, its domain is positively generated, so the result can be extended to the whole of $\mathscr{E}_V$.

Remark 2.5. From (2.6) one sees immediately that the bounds on $\langle \underline{N}^p; T_{r,t}\rho \rangle$ are independent of V.

3. The continuum limit

For each $n \in \mathbf{N}$, let X^n denote the Cartesian product of n copies of X, $C(X^n)$ the Banach space of continuous functions on X^n with supremum norm, and $\mathscr{M}(X^n)$ the Banach dual of $C(X^n)$, i.e. the Baire measures on X^n. Let $\{\rho_V : V \in \mathbf{N}, \rho_V \in C^\infty(\underline{N})\}$ be a sequence of initial states. We define the hierarchy of reduced density measures (or multiparticle distribution functions) $\{\phi_{V,t} : V \in \mathbf{N}, t \in \mathscr{R}^+\}$ where $\phi_{V,t} = \{\phi^n_{V,t} : n \in \mathbf{N}\}$ as a sequence of positive linear functionals on $\{C(X^n) : n \in \mathbf{N}\}$: for $f \in C(X^n)$ define

$$\phi^n_{V,t}(f) = \frac{1}{V^n} \sum_{x_{i_1},\ldots,x_{i_n} \in X_V} f(x_{i_1}, \ldots, x_{i_n}) \langle N_{i_1} N_{i_2} \ldots N_{i_n}; T_{V,t}\rho_V \rangle \tag{3.1}$$

The existence of the measures for all finite time is guaranteed by the fact, established in the previous section, that $C^\infty(\underline{N})$ is invariant under $T_{V,t}$. We also define the family $\{\nu_V : V \in \mathbf{N}\}$ of means on $C(X)$:

$$\nu_V(f) = \frac{1}{V} \sum_{x \in X_V} f(x) \tag{3.2}$$

Furthermore, we assume that the ν_V have a weak-* limit ν. Note that $\nu_V(\mathbf{1}) = \nu(\mathbf{1}) = 1$, where $\mathbf{1}(x) = 1$ for all $x \in X$.

In Appendix B we show

$$\begin{aligned} \frac{d}{dt}\phi^n_{V,t}(f) &= \phi^{n+1}_{V,t}(B^n f - \tilde{B}^n f) + (\phi^n_{V,t} \otimes \nu_V)(B^n f - \tilde{B}^n f) \\ &\quad + \phi^n_{V,t}(A^{+n} f - A^{-n} f) + (\phi^{n-1}_{V,t} \otimes \nu_V)(A^{+n} f) \\ &\quad + \langle R^n_V(f); T_{V,t}\rho_V \rangle \end{aligned} \tag{3.3}$$

for $n \in \mathbf{N}$ (with the convention that $\phi^0_{V,t} \otimes \nu_V = \nu_V$) where B^n, $\tilde{B}^n : C(X^n) \to$

$C(X^{n+1})$, and $A^{\pm n}: C(X^n) \to C(X^n)$ are defined by

$$(B^n f)(x_1, \ldots, x_n) = \sum_{p=1}^{n} f(x_1, \ldots, x_{p-1}, x_{n+1}, x_{p+1}, \ldots, x_n)\gamma(x_p, x_{n+1})$$

$$(\tilde{B}^n f)(x_1, \ldots, x_n) = \sum_{p=1}^{n} f(x_1, \ldots, x_p, \ldots, x_n)\gamma(x_p, x_{n+1}) \tag{3.4}$$

$$(A^{\pm n} f)(x_1, \ldots, x_n) = \sum_{p=1}^{n} f(x_1, \ldots, x_{p-1}, x_n, x_p, \ldots, x_{n-1})\alpha^{\pm}(x_p)$$

and

$$\|R_V^n(f)\rho\| \leq \frac{n(n-1)k}{V} \|f\| \cdot \|(\underset{\sim}{\mathrm{N}} + \tilde{k})^n \rho\| \tag{3.5a}$$

for some positive constants k and $\tilde{k}$. The second derivative $\ddot{\phi}^n_{V,t}(f)$ can be calculated in a similar way. It contains measures $\phi^m_{V,t}$ for $m \in \{n-2, n-1, n, n+1, n+2\}$ and a remainder term $\langle S^n_V(f); T_{V,t}\rho_V \rangle$ for which there is the bound

$$\|S_V^n(f)\rho\| \leq \frac{(n+1)n^2(n-1)c}{V} \|f\| \cdot \|(\underset{\sim}{\mathrm{N}} + \tilde{c})^{n+1} \rho\| \tag{3.5a}$$

for some positive constants c and $\tilde{c}$.

We now state the main theorem of this section.

Theorem 3.1. *Let the family $\{\rho_V : V \in \mathbf{N}\}$ be such that for all $n \in \mathbf{N}$ there exists a constant $k_n \in \mathscr{R}^+$ such that $\|\phi^n_{V,0}\| < k_n$ for all V. Then for all $n \in \mathbf{N}$ and $t \in \mathscr{R}^+$, $\phi^n_{V,t}$ converges pointwise on $C(X^n)$ to a measure ϕ^n_t, and correspondingly $\dot{\phi}^n_{V,t}$ converges pointwise on $C(X^n)$ to $\dot{\phi}^n_t$, as $V \to \infty$ over some subsequence of integers which is independent of n and t. Furthermore, the limit measures satisfy the following hierarchy of evolution equations*:

$$\begin{aligned} \frac{d}{dt}\phi^n_t(f) = (L\phi_t)^n(f) &:= \phi^{n+1}_t(B^n f - \tilde{B}^n f) + (\phi^n_t \otimes v)(B^n f - \tilde{B}^n f) \\ &\quad + \phi^n_t(A^{+n} f - A^{-n} f) + (\phi^{n-1}_t \otimes v)(A^{+n} f) \end{aligned} \tag{3.6}$$

The idea of proving convergence over a subsequence has been used to treat the Boltzmann hierarchy [7]. In the present case the proof of the theorem also relies on properties of certain compact subsets of $C(X^n)$.

Let

$$D^n = \left\{ f \in C(X^n) : f > 0,\ \|f\| \leq 1,\ \sup_{x, y \in X^n} |f(x) - f(y)| \leq \|x - y\|_{\mathscr{R}^n} \right\} \tag{3.7}$$

We collect together the required properties of the D^n in the following proposition:

Proposition 3.2. [9, 10]

(a) *D^n is closed and equicontinuous, and thus a compact subset of $C(X^n)$.*

(b) *Let $\{\mu_i : i \in \mathbf{N}\}$ be a sequence in $\mathcal{M}(X^n)$ for which $\|\mu_i\|$ is bounded. Then μ_i converges weak-* (i.e. pointwise on $C(X^n)$) iff it converges uniformly on D^n. (The existence of a limit measure is guaranteed by the fact that $\mathcal{M}(X)$ is weak-* sequentially complete.)*

Proof of Theorem 3.1. By Remark 2.4, for each n the V-indexed families of functions

$$(f, t) \mapsto \phi^n_{V,t}(f); \qquad (f, t) \mapsto \dot{\phi}^n_{V,t}(f); \qquad (f, t) \mapsto \ddot{\phi}^n_{V,t}(f) \tag{3.8}$$

are uniformly bounded in V for $(f, t) \in D^n \times [0, T]$ for any positive finite T. Furthermore, all the terms in the derivative (3.3) are linear and bounded on D^n for $t \in [0, T]$, so that the functions in (3.8) are uniformly Lipschitz continuous on D^n, uniformly on $[0, T]$. Hence the functions $(f, t) \mapsto \phi^n_{V,t}(f)$ and $(f, t) \mapsto \dot{\phi}^n_{V,t}(f)$ are uniformly equicontinuous on the compact metric space $D^n \times [0, T]$ (where D^n is equipped with the metric topology inherited from the norm topology of $C(X^n)$). Thus by the Arzela–Ascoli theorem, there is a subsequence of integers such that for each $f \in D^n$ there is a function $\phi^n(f) : [0, T] \to \mathcal{R}^+$ for which

$$\phi^n_t(f) = \lim_{V \to \infty} \phi^n_{V,t}(f) \tag{3.9a}$$

uniformly on $D^n \times [0, T]$, and for which

$$\frac{d}{dt} \phi^n_t(f) = \lim_{V \to \infty} \dot{\phi}^n_{V,t}(f) \tag{3.9b}$$

since the derivatives also converge uniformly. Since $\phi^n_{V,t}$ are positive, $\|\phi^n_{V,t}\|$ and hence $\|\dot{\phi}^n_{V,t}\|$ are bounded in V, uniformly for $t \in [0, T]$. Thus by (b) of Proposition 3.2, ϕ^n_t can be extended to define a measure on the whole of $C(X^n)$, convergence and differentiability in (3.9) being understood pointwise on $C(X^n) \times [0, T]$. The same argument can be used over the increasing family of sets $\{C(X^n) \times [0, mT] : m \in \mathbf{N}\}$. Then by the diagonalisation argument we can find a subsequence of volumes V for which convergence is pointwise on $C(X^n) \times \mathcal{R}^+$. Having made the construction for $n = 1$ we then repeat for $n = 2, 3$, etc., using the subsequence obtained by the above procedure at the nth stage as the initial sequence for the $(n + 1)$th stage. A final diagonalisation yields the desired subsequence.

Finally, the limiting form of the derivative follows straightforwardly from (3.3) using the fact that the remainder term is of order V^{-1}. □

4. Decorrelation and non-linearity

In the previous section we constructed a hierarchy $\{\phi_t : t \in \mathcal{R}^+\}$, $\phi_t = \{\phi^n_t : n \in \mathbf{N}\}$ of positive reduced density measures satisfying the linear differential equations (3.3). In this section we prove the following result:

Theorem 4.1. *Solutions of the hierarchy* (3.6) *which initially satisfy the decorrelation property*

$$\phi_0^n = \phi_0^1 \otimes \ldots \otimes \phi_0^1 \ (n \text{ copies}) \tag{4.1}$$

are unique, and remain decorrelated for all time. Furthermore the one particle measure ϕ_t^1 *satisfies the following measure-theoretic analogue of Fröhlich's non-linear kinetic equation on* $\mathcal{M}(X)$:

$$\frac{d}{dt}\phi_t^1(f) = (\phi_t^1 \otimes \phi_t^1)(B^1 f - \tilde{B}^1 f) + (\phi_t^1 \otimes v)(B^1 f - \tilde{B}^1 f)$$
$$+ \phi_t^1(A^{+1} f - A^{-1} f) + v(A^{+1} f) \tag{4.2}$$

We proceed via a number of lemmas and propositions. For any n we use $\mathbf{1}$ to denote the function which takes the value 1 throughout X^n.

Lemma 4.2. *For any positive solution of* (3.6)

$$\phi_t^n(\mathbf{1}) \le (p/q)^n + e^{-nqt} \sum_{r=1}^{n} \binom{n}{r} (\phi_0^r(\mathbf{1}) - (p/q)^r)(e^{qt} - 1)^{n-r}(p/q)^{n-r} \tag{4.3}$$

where

$$p = \sup_{x \in X} \alpha^+(x), \qquad q = \inf_{x \in X} \{\alpha^-(x) - \alpha^+(x)\} > 0 \tag{4.4}$$

Proof. Insert $f = \mathbf{1}$ into (3.6). The terms in B^1 and $\tilde{B}^1$ cancel. Using the fact that $v(\mathbf{1}) = 1$:

$$\dot{\phi}_t^n(\mathbf{1}) \le -np\phi_t^n(\mathbf{1}) + nq\phi_t^{n-1}(\mathbf{1}) \tag{4.5}$$

for $n \ge 1$. For $n = 1$ this integrates to

$$\phi_t^1(\mathbf{1}) \le (p/q) + (\phi_0^1(\mathbf{1}) - (p/q))e^{-qt} \tag{4.6}$$

which is just (4.3) for $n = 1$. Now proceed by induction. Assume that (4.3) is true for n. Then by (4.5)

$$\frac{d}{dt}\{\phi_t^{n+1}(\mathbf{1})e^{q(n+1)t}\} \le \phi_t^n(\mathbf{1}) \cdot p(1+n) \cdot e^{q(1+n)t} \le (p/q)^n \cdot p(1+n) \cdot e^{q(1+n)t}$$
$$+ \sum_{r=1}^{n} (n+1)\binom{n}{r} e^{qt}(e^{qt}-1)^{n-r}(p/q)^{n-r} p(\phi_0^r(\mathbf{1}) - (p/q)^r) \tag{4.7}$$

Equation (4.7), integrated between 0 and t yields (4.3) for $n + 1$.

Remark 4.3. For initially decorrelated states we have that

$$\phi_0^n(\mathbf{1}) = (\phi_0^1(\mathbf{1}))^n \tag{4.8}$$

which, substituted into (4.3) yields

$$\phi_1^n(\mathbf{1}) \le (\phi_0^1(\mathbf{1}) + (p/q))^n \tag{4.9}$$

Proposition 4.4. *Positive solutions of* (3.6) *for which the initial hierarchy is decorrelated are unique on some interval* $[0, \tau]$ *where* τ *depends only on* $\phi_0^1(\mathbf{1})$.

Proof. We show that any positive solution with decorrelated initial condition can be expressed as a convergent power series in ϕ_0^1, and is hence unique. We iterate (3.3):

$$\phi_t^n(f) = \phi_0^n(f) + \sum_{k=1}^{l-1} \frac{t^k}{k!} (L^k\phi_0)^n(f) + (\Delta_l\phi_t)^n(f) \tag{4.10}$$

where

$$(\Delta_l\phi_t)^n(f) = \int_0^t dt_1 \cdots \int_0^{t_{l-1}} dt_l (L^l\phi_{t_l})^n(f) \tag{4.11}$$

(a) *The remainder term*: First estimate $|(L^l\phi_t)^n(f)|$. Let C^n stand for B^n, $\tilde{B}^n$ or $A^{\pm n}$. From (3.6) $(L\phi_t)^n(f)$ contains 7 terms each involving a C^{n+1}, a C^n, or a C^{n-1}. Similarly, $(L^l\phi_t)^n(f)$ contains 7^l terms each of which is of the form

$$Y_t(q; q_1, q_2, \ldots, q_l) = \phi_t^q(C^{q_l}C^{q_{l-1}} \cdots C^{q_1}f) \tag{4.12}$$

where $0 \leq q_p \leq n + p - 1$ for $p = 1, 2, \ldots, l$ and $0 \leq q \leq n + l$. Let $\|\cdot\|$ denote the supremum norm on the $C(X^n)$. Then from the definition (3.4) of the C^n

$$\|C^n f\| \leq n \|f\| c \qquad \text{with} \qquad c = \max\{\|\gamma\|, \|\alpha^+\|, \|\alpha^-\|\} \tag{4.13}$$

So by the assumed positivity of the ϕ_t^q,

$$\begin{aligned} |Y_t(q; q_1, \ldots, q_l)| &\leq \phi_t^q(\mathbf{1})c^l \|f\| q_1 q_2 \cdots q_l \\ &\leq \phi_t^q(\mathbf{1})c^l \|f\| \frac{(n+l-1)!}{(n-1)!} \end{aligned} \tag{4.14}$$

Adding all possible terms together

$$|(L^l\phi_t)^n(f)| \leq \frac{(n+l-1)!}{(n-1)!} (7c)^l \|f\| \sup_{m=0,1,\ldots,n+l} \{\phi_t^m(\mathbf{1})\} \tag{4.15}$$

and so in (4.11)

$$\begin{aligned} |(\Delta_l\phi_t)^n(f)| &\leq \frac{t^l}{l!} \sup_{0 \leq s \leq t} \{|(L^l\phi_t)^n(f)|\} \\ &\leq \binom{n+l-1}{n-1} \|f\| (7ct)^l \sup_{0 \leq s \leq t} \sup_{m=0,1,\ldots,n+l} \{\phi_s^m(\mathbf{1})\} \end{aligned} \tag{4.16}$$

Defining $E(\phi_0) = \max\{1, (\phi_0^1(\mathbf{1}) + (p/q))\}$, then by Remark 4.3.:

$$|(\Delta_l\phi_t)^n(f)| \leq \|f\| 2^{n-1}(14ct)^l E(\phi_0)^{n+l} \tag{4.17}$$

so that for $t < \tau = (14cE(\phi_0))^{-1}$, $(\Delta_l\phi_t)^n(f) \to 0$, *for all* n, as $l \to \infty$.

(b) *The power series*: By (4.15)

$$|(L^k\phi_0)^n(f)| \leq \frac{(n+l-1)!}{(n-1)!} \|f\| E(\phi_0)^{n+l} \tag{4.18}$$

so that for $t \leq \tau$,

$$\sum_{k=1}^{l-1} \frac{t^k}{k!} |(L^k \phi_0)^n(f) \leq 2^{n-1} E(\phi_0)^n \tag{4.19}$$

which is, of course, bounded as $l \to \infty$

Proposition 4.5. *The non-linear kinetic equation* (4.2) *has a unique local solution in the norm topology of* $\mathcal{M}(X)$.

Proof. The derivative in (4.2) is bounded:

$$\left|\left(\frac{d}{dt}\phi_t^l\right)(f)\right| \leq \|f\| \,(p\, \|\phi_t^1\|^2 + q\, \|\phi_t^1\| + r); \qquad \text{some } p, q, r > 0 \tag{4.20}$$

and uniformly Lipschitz continuous in any norm ball of $\mathcal{M}(X)$, so by standard existence theory [11] a local solution exists in some interval $[0, \tau]$.

Proposition 4.6. *The local solution of* (4.2) *preserves positivity.*

Proof. We model our proof on one in [12, 13]. Let the local solution exist on an interval $[0, \tau]$. Since the local solution is norm-bounded, there is a positive constant $\tilde{c}$ such that $\phi_t^1(\mathbf{1}) < \tilde{c}$ on $[0, \tau]$. We set

$$c = \max\,\{\tilde{c}, 2\phi_0^1(\mathbf{1})\} \tag{4.21}$$

Since $\alpha^{\pm}$ and γ are bounded, we can find positive numbers a and b for which $a + \alpha^+(x) - \alpha^-(x)$ and $b + \gamma(x, y) - \gamma(y, x)$ are positive for all x and y. Now write (4.2) as

$$\dot{\phi}_t^1(f) = (J\phi_t^1)(f) + (K\phi_t^1)(f) + (N\phi_t^1)(f) \tag{4.22a}$$

where

$$(J\phi_t^1)(f) = -(a + bc)\phi_t^1(f) \tag{4.22b}$$

$$(K\phi_t^1)(f) = (\phi_t^1 \otimes \nu)(Bf - \tilde{B}f) \tag{4.22c}$$

$$\begin{aligned}(N\phi_t^1)(f) = {} & (\phi_t^1 \otimes \phi_t^1)(b\mathbf{1} \otimes f + B^1 f - B^1) + b(c - \phi_t^1(\mathbf{1}))\phi_t^1(f) \\ & + \phi_t^1(af + A^{+1}f - A^{-1}f) + \nu(A^{+1}f)\end{aligned} \tag{4.22d}$$

K generates a norm-continuous positive contraction semigroup on $\mathcal{M}(X)$. This is most straightforwardly seen by considering the action of the predual semigroup with generator $\tilde{K}$ defined by $\phi^1(\tilde{K}f) = (K\phi^1)(f)$:

$$(\tilde{K}f)(x) = \int_X d\nu(y)\gamma(y, x)(f(y) - f(x)) \tag{4.23}$$

As is well known [14], operators of this form generate a norm-continuous positive contraction semigroup on $C(X)$. By duality the same is true for K on $\mathcal{M}(X)$. Clearly J also generates such a semigroup on $\mathcal{M}(X)$, and commutes with K. Thus

$J+K$ generates a positive contraction semigroup $\{S_t : t \in \mathscr{R}^+\}$ on $\mathscr{M}(X)$, and clearly $\|S_t\| \leq e^{-(a+bc)t}$.

The point of writing (4.2) in the form (4.22) is the following. Define

$$(\bar{N}_t\phi^1)(f) = (\phi^1 \otimes \phi^1)(b\mathbf{1}\otimes f + B^1 f - \tilde{B}^1 f) + b(c - \bar{\phi}_t^1(\mathbf{1}))\phi^1(f) \\ + \phi^1(af + A^{+1}f - A^{-1}f) + v(A^{+1}f) \tag{4.24}$$

where $\bar{\phi}_t^1$ is the local solution on $[0, \tau]$. If we define a map Θ on $C([0, \tau]; \mathscr{M}(X))$ by

$$(\Theta\psi)_t = S_t\psi_0 + \int_0^t ds S_{t-s}\bar{N}_s\psi_s; \qquad 0 \leq t \leq \tau \tag{4.25}$$

then the local solution is a fixed point of θ. S_t and N_t both preserve positive order in $\mathscr{M}(X)$. Now consider the sequence of measure valued functions on $[0, \tau]$

$$\psi_t^{(0)} = 0, \qquad \psi_t^{(n)} = (\Theta\psi^{(n-1)})_t; \qquad n = 1, 2, \ldots \tag{4.26}$$

Since $\psi_t^{(1)} > \psi_t^{(0)}$, $\psi_t^{(n)}$ is an increasing sequence of positive measures. In fact, the sequence is bounded above: let

$$P_r^{(n)} = \sup_{0 \leq t \leq \tau} \psi_t^{(n)}(\mathbf{1}) \tag{4.27}$$

Then a simple estimate on (4.25) shows that

$$P_\tau^{(n+1)} \leq \phi_t^1(\mathbf{1}) + (u + vP_\tau^{(n)} + w(P_\tau^{(n)})^2)\frac{(1 - e^{-(a+bc)\tau})}{a + bc} \tag{4.28}$$

for some positive u, v and w. Thus we can always choose τ small enough so that

$$P_\tau^{(n)} \leq 2\psi_0^{(1)} \Rightarrow P_\tau^{(n+1)} \leq 2\psi_0^{(1)} \tag{4.29}$$

Since $P_\tau^{(0)} = 0$, the monotone sequence $\{\psi_t^{(n)}(f) : n \in \mathbf{N}\}$ is bounded above for all n and so has a limit $\psi_t(f)$ which defines a positive linear functional on $C(X)$ which is equal to the local solution.

Proof of Theorem 4.1. Construct the hierarchy $\phi_t^n = \phi_t^l \otimes \cdots \otimes \phi_t^l$ (n copies) of decorrelated reduced density measures from the local solution of (4.2) in $[0, \tau]$. By differentiation the ϕ_t^n satisfy the differential equations (3.3) of the hierarchy, and are thus the unique local solution of the hierarchy. Since the hierarchy is decorrelated at $t = \tau$ we can repeat the construction on successive intervals $[\tau, \tau + \tau_1]$, $[\tau + \tau_1, \tau + \tau_1 + \tau_2]$ etc. We are done if we can show that the elements of the sequence $\{\tau_1, \tau_2, \ldots\}$ are bounded below. But by remark (4.3) $\phi_t^1(\mathbf{1})$ and hence $E\{\phi_t\}$ are bounded for all $t > 0$; likewise since ϕ_t^l are bounded, the interval over which the local solutions to (4.2) are constructed are bounded below for all time.

5. Conclusions

We have seen how the non-linear kinetics in Fröhlich's arises naturally in the continuum limit in the form of the propagation of molecular chaos, rather than through any decorrelation condition imposed at finite volume. In another paper [15], we find the stationary measure for kinetic equation (4.2) (for a certain family of parametrisations $\alpha^{\pm}$ and γ), demonstrate the existence of a critical pumping rate above which the stationary measure displays condensation, and show this measure is **globally** stable with respect to perturbations.

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Appendix A

The basis of the estimates are the following commutation relations

$$N_j I_i^{\pm} = I_i^{\pm}(N_j \pm \delta_{ij}) \qquad \text{on Dom}\,(N_j) \tag{A1}$$

from which it follows that

$$\underset{\sim}{\mathrm{N}} I_j^{\pm} = I_i^{\pm}(\underset{\sim}{\mathrm{N}} \pm V^{-1}) \qquad \text{on Dom}\,(\underset{\sim}{\mathrm{N}}) \tag{A2}$$

Proof of Lemma 2.2. (a) Dom $(\underset{\sim}{\mathrm{N}}^{2p}) \subset$ Dom $((K_r)^p)$: from the commutation relations (A1) and (A2) it is not too difficult to see that

$$(K_r)^p = \sum_{q=0}^{2p} R_q M_q \tag{A3}$$

where R_q is a polynomial function of the $I_i^{\pm}$, and M_q is a monomial of order q in the N_i. Thus there exist positive constants $\{r_q : q = 1, 2, \ldots, p\}$ such that

$$\|(K_r)^p \rho\| \leq \sum_{q=0}^{2p} r_q \, \|\underset{\sim}{\mathrm{N}}^q \rho\| \tag{A4}$$

(b) Dom $(\underset{\sim}{\mathrm{N}}^{2p}) \supset$ Dom $((K_r)^p)$: we write

$$K_r = \sum_{i=1}^{V} (A_i^{+}(N_i + 1) + A_i^{-} N_i) + \sum_{i,j=1}^{V} B_{ij}(N_i + 1)N_j \tag{A5}$$

where

$$A_i^{\pm} = \alpha_i^{\pm}(rI_i^{\pm} - 1) \qquad B_{ij} = \frac{1}{V}\gamma_{ji}(rI_{ij} - 1) \tag{A6}$$

Since $\|I_i^{\pm}\| = 1$ and $r < 1$, $A_i^{\pm}$ and B_{ij} have bounded inverses. Let $C = (\prod_{i=1}^{V} A_i^{+})^{-1}(\prod_{i=1}^{V} A_i^{-})^{-1}(\prod_{i,j=1}^{V} B_{ij})^{-1}$. Then

$$CK_r\rho = \left(\sum_{i=1}^{V} C_i^{+}(N_i + 1) + \sum_{i=1}^{V} C_i^{-} N_i + \sum_{i,j=1}^{V} C_{ij}(N_i + 1)N_j\right)\rho \tag{A7}$$

on Dom K_r, where $C_i^{\pm} = CA_i^{\pm}$ and $C_{ij} = CB_{ij}$. Hence

$$\begin{aligned}\|C\|\,\|K_\tau\rho\| &\geq \|CK_\tau\rho\| \\ &\geq \sum_{i=1}^{V} \frac{\|(N_i+1)\rho\|}{\|(C_i^+)^{-1}\|} + \sum_{i=1}^{V} \frac{\|N_i\rho\|}{\|(C_i^-)^{-1}\|} + \sum_{i,j=1}^{V} \frac{\|(N_i+1)N_j\rho\|}{\|(C_{ij})^{-1}\|}\end{aligned} \tag{A8}$$

so that

$$\|K_r\rho\| \geq a\,\|\underset{\sim}{N}^2\rho\| \tag{A9}$$

for some positive a. Hence $\|(K_r)^p\rho\| \geq a\,\|\underset{\sim}{N}^2(K_r)^{p-1}\rho\|$ on Dom $((K_r)^p)$. Again, from the commutation relations it follows that for $q \in \mathbf{N}$

$$\underset{\sim}{N}^{2q}K_r = K_r\underset{\sim}{N}^{2q} + \Delta_q \qquad \text{where} \qquad \|\Delta_q\rho\| \leq b\,\|\underset{\sim}{N}^{2q}\rho\| \tag{A10}$$

(the essential point here is that the B_{ij} commute with $\underset{\sim}{N}$) and hence

$$\|(K_r)^p\rho\| \geq a^2\,\|\underset{\sim}{N}^4(K_r)^{p-2}\rho\,\| - ab\,\|\underset{\sim}{N}^2(K_r)^{p-2}\rho\| \tag{A11}$$

so that by (A9)

$$\|\underset{\sim}{N}^4(K_r)^{p-2}\rho\| \leq c_p\,\|(K_r)^p\rho\| + c_{p-1}\,\|(K_r)^{p-1}\rho\|$$

with c_p, c_{p-1} positive. This procedure can be iterated to show that

$$\|\underset{\sim}{N}^{2p}\rho\| \leq \sum_{q=0}^{p} c_q\,\|(K_r)^q\rho\| \tag{A12}$$

Estimates for Theorem 2.4

Let $\rho \in C^\infty(\underset{\sim}{N})$, $\rho > 0$. Then for $p \in \mathbf{N}$, by (A1,2)

$$\begin{aligned}\frac{d}{dt}\langle \underset{\sim}{N}^p; T_{r,t}\rho\rangle &= \frac{(r-1)}{V}\sum_{i,j=1}^{V} \langle \gamma_{ji}\underset{\sim}{N}^p(N_i+1)N_j; T_{r,t}\rho\rangle \\ &\quad + \sum_{i=1}^{V} \langle \alpha_i^+(N_i+1)(r(\underset{\sim}{N}+V^{-1})^p - \underset{\sim}{N}^p); T_{r,t}\rho\rangle \\ &\quad + \sum_{i=1}^{V} \langle \alpha_i^- N_i(r(\underset{\sim}{N}-V^{-1})^p - \underset{\sim}{N}^p); T_{r,t}\rho\rangle\end{aligned} \tag{A13}$$

Discard all negative terms in (A13). Then

$$\frac{d}{dt}\langle \underset{\sim}{N}^p; T_{r,t}\rho\rangle < \left\langle r\sum_{q=0}^{p-1}\binom{p}{q}\left(\sum_{i=1}^{V}(\alpha_i^+ + (\alpha_i^+ - \alpha_i^-)N_i)\underset{\sim}{N}^q V^{q-p}\right); T_{r,t}\rho\right\rangle \tag{A14}$$

Now observe that if $0 < q_1 < q_2$, then $0 \leq \underset{\sim}{N}^{q_1}V^{q_2} \leq \underset{\sim}{N}^{q_2}V^{q_2}$ so that for $r \in (\frac{1}{2}, 1)$

and $p \geq 1$

$$\frac{d}{dt}\langle \underline{N}^p; T_{r,t}\rho\rangle < \Big\langle p\Big\{((\underline{\alpha}^+ + \underline{\alpha}^-)\underline{N}^{p-1} - \underline{\alpha}\underline{N}^p) + \sum_{q=0}^{p-2}\binom{p}{q}(\underline{\alpha}^+ + \underline{\alpha}^-)V^{-1}(\underline{N}^{p-2} + \underline{N}^{p-1}); T_{r,t}\rho\Big\rangle\Big\}$$
$$< \langle -p\underline{\alpha}\underline{N}^p + (\underline{\alpha}^+ + \underline{\alpha}^-)2^{p+2}\underline{N}^{p-1}; T_{r,t}\rho\rangle \quad \text{(A15)}$$

where $\underline{\alpha}^{\pm} = \sup_{x\in X}\alpha^{\pm}(x)$ and $\underline{\alpha} = \frac{1}{2}\inf_{x\in X}\{\alpha^-(x) - \alpha^+(x)\}$.

Appendix B

We calculate the derivative of the reduced density measure in (3.1).

$$\frac{d}{dt}\phi^n_{V,t}(f) = \frac{1}{V^n}\sum_{i_1,\ldots,i_n=1}^{V} f(x_{i_1},\ldots,x_{i_n})\langle N_{i_1}\cdots N_{i_n}; K_V T_{V,t}\rho_V\rangle$$
$$= \frac{1}{V^n}\sum_{i_1,\ldots,i_n=1}^{V} f(x_{i_1},\ldots,x_{i_n})\Big\langle N_{i_1}\cdots N_{i_n}; \Big[\sum_{i=1}^{V}(\alpha_i^+(I_i^+ - 1)(N_i+1) + \alpha_i^-(I_i^+ - 1)N_i) + \frac{1}{V}\sum_{i,j=1}^{V}\gamma_{ji}(I_{ij}-1)(N_i+1)N_j\Big]T_{V,t}\rho_V\Big\rangle \quad \text{(B1)}$$

Using the commutation relations (A1) and (A2) to pull the N_i through to the right of all the $I_i^{\pm}$, and the fact that I_i^+ is trace-preserving on $\mathscr{E}_V$, I_i^- trace-preserving on the range of N_i, (B1) is equal to

$$\frac{1}{V^n}\sum_{i_1,\ldots,i_n=1}^{V} f(x_{i_1},\ldots,x_{i_n})\Big\langle\Big[\sum_{i=1}^{V}\Big[\Big\{\prod_{\mu=1}^{n}(N_{i_\mu}+\delta_{i,i_\mu}) - \prod_{\mu=1}^{n}N_{i_\mu}\Big\}\alpha_i^+(N_i+1) + \Big\{\prod_{\mu=1}^{n}(N_{i_\mu}-\delta_{i,i_\mu}) - \prod_{\mu=1}^{n}N_{i_\mu}\Big\}\alpha_i^-N_i\Big]$$
$$+ \frac{1}{V}\sum_{i,j=1}^{V}\Big\{\prod_{\mu=1}^{n}(N_{i_\mu}+\delta_{i,i_\mu}-\delta_{j,i_\mu}) - \prod_{\mu=1}^{n}N_{i_\mu}\Big\}\gamma_{ji}(N_i+1)N_j\Big]; T_{V,t}\rho_V\Big\rangle \quad \text{(B2)}$$

The leading terms (in powers of V^{-1}) are those involving only one Kronecker delta, i.e.

$$\frac{1}{V^n}\sum_{i_1,\ldots,i_n=1}^{V} f(x_{i_1},\ldots,x_{i_n})\Big\langle\sum_{p=1}^{n}N_{i_1}\cdots\hat{N}_{i_p}\cdots N_{i_n} \times\Big[\sum_{i=1}^{V}\{\delta_{i,i_p}\alpha_i^+(N_i+1) - \delta_{i,i_p}\alpha_i^-N_i\} + \frac{1}{V}\sum_{i,j=1}^{V}\gamma_{ji}(N_i+1)N_j(\delta_{i,i_p}-\delta_{j,i_p})\Big]; T_{V,t}\rho_V\Big\rangle \quad \text{(B3)}$$

(a caret over an N_j denotes exclusion from the product.)

$$= (\phi_{V,t}^{n+1} + \phi_{V,t}^{n} \otimes \nu_V)(B^n f - \tilde{B}^n f)$$
$$+ (\phi_{V,t}^{n} + \phi_{V,t}^{n-1} \otimes \nu_V)(A^{+n} f) - \phi_{V,t}^{n}(A^{-n} f) \quad \text{(B4)}$$

as required. The remaining terms are bounded by

$$\frac{1}{V^{n-1}} \|f\| \Big\langle [(\underset{\sim}{\alpha}^{+} + \underset{\sim}{\alpha}^{-})(\underset{\sim}{N} + 1)$$
$$+ \sup_{x,y} \gamma(x, y) \underset{\sim}{N}(\underset{\sim}{N} + 1)] \sum_{q=0}^{n-2} \binom{n}{q} 2^q (V\underset{\sim}{N})^q ; T_{V,t}\rho_V \Big\rangle$$
$$\leq \text{const} \times \frac{1}{V} n(n-1) \langle (\underset{\sim}{N} + 1)^2 (\underset{\sim}{N} + 2V^{-1})^{n-2} ; T_{V,t}\rho_V \rangle \quad \text{(B5)}$$

Adding the two sets of terms together (i.e. leading terms and remainders) we achieve the form stated in (3.3), (3.4) and (3.5a). The calculation of the second derivatives is similar, but all that is required for Theorem 3.1 is that case is that $\|\ddot{\varphi}_{V,t}^{n}\|$ be bounded in V for all n and all time in any compact interval.

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