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From algebras of local observables to quantum fields: generalized H-bounds

Dedicated to the memory of Alexander Zabrodsky

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Abstract. Previous results on obtaining quantum fields as limits of sequences of bounded, local operators (local observables) are extended to generalized H-bounds and ultradistribution fields. A topology on the net of local observable algebras is specified such that each limit point of suitable sequences in the topology determines an ultradistribution (resp. tempered distribution) quantum field that is associated to the net in a certain strong sense and that satisfies an L^1-continuous generalized H-bound. And it is shown that an ultradistribution (or tempered distribution) quantum field that satisfies an L^1-continuous generalized H-bound is associated to a net of local algebras if and only if it is obtainable as such a limit.

I. Introduction

An outstanding problem in mathematical relativistic quantum theory is the connection between the two major axiom systems that were formulated to provide a physically motivated mathematical framework in terms of which the general structure of the theory could be investigated and within which physically interesting, concrete models could be accommodated. These two systems of axioms are those of the general theory of quantized fields [1, 2] and of the algebraic relativistic quantum theory [3, 4, 5]. In the former the primary object of study is a local, relativistic quantum field in some Hilbert space, and in the latter the primary object is a net of local observable algebras. In practice, the quantum fields have proven to be exceedingly useful to express and calculate dynamical quantities, while the algebraic approach has succeeded in bringing clarity to questions dealing with conceptual structure, e.g. superselection sectors [6], the admissible state space of gauge theories [7], Bell's inequalities [8–11], etc. Recently there has been significant progress in the mathematically rigorous determination of the relationship between the primary objects of the two approaches.

On the one hand, necessary and sufficient conditions have been found for fields satisfying a certain regularity condition (a generalized H-bound) in order that the fields have associated to them a net of observable algebras [12]. These conditions allow an explicit construction of a net of observable algebras that is unique in a definite sense. A very tight relationship between the field operators of the quantum field and the local algebras of the net was demonstrated, along with interesting consequences for both the net and the quantum field that follow from this relationship. Although some aspects of this work will be mentioned below, the reader is referred to [12–15] for the details.

At the same time, the problem of constructing relativistic quantum fields starting from a net of local algebras has been examined in a number of recent papers [16–21]. In [20–22] it has been shown that if a quantum field $\varphi(x)$ satisfying the axioms of [1, 2, 23] is associated in a certain sense with the net, then the field $\varphi(x)$ can be obtained as a limit of a sequence of elements of local observable algebras measurable in space-time regions shrinking to the point x. However, although the topologies on the net of local algebras considered in [18–21] are weak enough to obtain all associated quantum fields as limits of bounded, local observables, it is clear from [18–21] that the topologies studied there are, in fact, too weak, since not all of the limit points obtained are necessarily quantum fields in the sense of [1, 2, 23]. Among other problems, the "field operators" associated with such limits do not necessarily have invariant common domains, so that products of such operators and thus Wightman functions may not be defined. The optimal topology – that topology that yields all and only quantum fields in the sense of [1, 2, 23] that are associated with the net – is not known.

But it is in any case of interest to tighten the topologies considered, in order to study the relation between the manner of convergence of the sequence of local operators and properties of the associated field that go *beyond* those stated in the axioms [1, 2, 23], the latter being, after all, simply the *minimal* properties such fields should have. In [16, 17] Fredenhagen and Hertel identified the topology on the net of local algebras in which all (and *only*) associated tempered distribution fields satisfying polynomial H-bounds can be obtained. In this paper the primary purpose is to do the same for associated ultradistribution (and tempered distribution) fields satisfying (L^1-continuous) generalized H-bounds, which have been found to have very nice properties in [12, 13, 15] and which include the fields considered by Fredenhagen and Hertel as a special case. We mention that there are examples [24] of nets of local algebras that have associated to them both tempered distribution fields *and* nontempered ultradistribution fields satisfying generalized H-bounds. In fact, we show in the Appendix that out of almost every tempered distribution quantum field satisfying the axioms of [1, 2] and a given, arbitrary H-bound one can construct many *nontempered* ultradistribution fields satisfying the axioms of [23] and the same H-bound (with, however, possibly different continuity properties). We show, in addition, that if the original field is locally associated with a net of local algebras in the sense of [12], then the nontempered ultradistributions obtained from this field by the procedure discussed in the Appendix are also locally associated in the same sense to the net. Thus there are many examples of quantum fields that fall outside of the range of application of [16]. In this paper only fields satisfying L^1-continuous generalized

H bounds (see Section 3) will be considered. A subsequent paper will treat fields obeying generalized H-bounds that are not L^1-continuous.

This paper does not address the problem of the optimal topology suggested two paragraphs above nor does it consider the problem of the Lorentz covariance of the fields obtained as limit points of sequences of bounded observables. This latter problem has yet to be addressed in the literature; here, as in [16–21], we are only concerned with the translation covariance of such fields. For that reason we shall restrict our attention to hermitian, scalar quantum fields (although the individual components of higher-spin fields could be handled by combining the methods of this paper with those of [25]).

It is perhaps useful to mention that this paper and [16–21] are not motivated solely by the desire to understand the relation between the two axiom systems mentioned earlier. Since both basic objects of the two systems are known to be better behaved when they are locally associated to each other, particularly when the field satisfies a generalized H-bound, it is thus natural to try to determine when and how such a local association occurs. But also, as mentioned in [16], one would like to combine the given multiplicative structure of the local observable algebras with the methods of this paper and [16–21], along with a rigorous version of the operator product expansion used in heuristic quantum field theory, in order to determine from a given net of local observable algebras the underlying *dynamics* of the system.

After establishing notation and definitions in Section 2, we study in Section 3 the relationship between sesquilinear forms and ultradistribution (and tempered distribution) fields that both satisfy generalized H-bounds of a certain type. This relationship is one to one. Then in Section 4 a natural topology on the net of local algebras is proposed in which all limit points of suitable sequences of local operators are precisely such sesquilinear forms satisfying generalized H-bounds and the quantum fields they determine are locally associated to the net in a strong sense. Moreover, it is shown that all locally associated quantum fields satisfying said generalized H-bounds are determined by sesquilinear forms obtainable in precisely this manner. The technical core of the paper is Section 3, while those interested only in a precise statement of the main results are referred to Section 4. A difference between the behavior of the sesquilinear forms associated to H-bounded ultradistribution fields that have extensions to tempered fields and those associated to H-bounded, nontempered ultradistribution fields is identified (as a by-product we answer an open question in [21]). In the Appendix we produce examples of such nontempered ultradistribution fields that satisfy generalized H-bounds and are locally associated to nets of local algebras.

We mention that all results are valid for 2, 3 or 4 space-time dimensions.

II. Notation and definitions

We commence with the test function spaces that will be used in this paper. $\mathscr{S}(\mathbb{R}^4)$ will signify the space of tempered test functions on which the tempered distributions fields of [1, 2] are defined. Further, a function $\omega:\mathbb{R}\to\mathbb{R}$ will be called a Jaffe indicatrix function if it satisfies

(1) $e^{\omega(t)}$ is a real analytic function: $e^{\omega(t)} = \sum_{n=0}^{\infty} c_{2n}t^{2n}$, $c_0 \geq 1$, $c_{2n} \geqq 0$,
(2) $\int_0^\infty \frac{\omega(t)}{1+t^2}\,dt < \infty$ (Carleman's criterion),
(3) $\omega(t)$ is an increasing concave function on $[0, \infty)$,
(4) there is a real constant a and a positive constant b such that $\omega(t) \geq a + b \log(1+t^2)$ on $[0, \infty)$.

(Note: no attempt is made here to state the most general possible conditions on a Jaffe indicatrix function; but compare [21, 23, 26, 27] and references given there.) If $\omega(t)$ is a Jaffe indicatrix function, we denote by $\mathscr{C}_\omega(\mathbb{R}^4)$ the set of all functions $f \in L^1(\mathbb{R}^4)$ such that the Fourier transform $\tilde{f}$ is C^∞ and

$$\pi_{\alpha,\lambda}(f) \equiv \sup_{p\in\mathbb{R}^4} e^{\lambda\omega(|p|)}\,|D^\alpha \tilde{f}(p)| < \infty$$

for all multiindices α and all $\lambda \geq 0$. $\mathscr{C}_\omega(\mathbb{R}^4)$ is endowed with the locally convex topology generated by these seminorms $\pi_{\alpha,\lambda}$. The topological dual $\mathscr{C}'_\omega(\mathbb{R}^4)$ of this space is a space of ultradistributions. $\mathscr{C}_\omega(\mathbb{R}^4)$ has many of the properties of $\mathscr{S}(\mathbb{R}^4)$ [21, 23, 27]; indeed, if $\omega(t) = \ln(1+t^2)$, then $\mathscr{C}_\omega(\mathbb{R}^4) = \mathscr{S}(\mathbb{R}^4)$ [26]. Carleman's criterion (2) above assures the existence of sufficiently many functions with compact support in $\mathscr{C}_\omega(\mathbb{R}^4)$ [23]; if Carleman's criterion is violated by $\omega(t)$, then there are *no* nontrivial functions with compact support in the corresponding $\mathscr{C}_\omega(\mathbb{R}^4)$.

Next we describe what we shall mean by "quantum field" in this paper.

Definition 2.1. Let $\mathscr{H}$ be a Hilbert space on which a strongly continuous, unitary representation $U(\mathbb{R}^4)$ of the translation group acts. $U(\mathbb{R}^4)$ is assumed to satisfy the spectrum condition [1, 2, 23] (the uniqueness or even the existence of the vacuum is not assumed). Let $D \subset \mathscr{H}$ be linear, dense, $U(\mathbb{R}^4)$-invariant, and contain a core for e^{tH}, for any $t > 0$, where H is the positive, selfadjoint generator of the time translation subgroup of $U(\mathbb{R}^4)$. If $\mathscr{F}(\mathbb{R}^4)$ is one of the test function spaced defined above, assume that $\varphi(\cdot)$ is a linear map from $\mathscr{F}(\mathbb{R}^4)$ into the linear operators leaving D invariant such that

$$\langle \Phi, \varphi(\cdot)\Psi\rangle \in \mathscr{F}'(\mathbb{R}^4), \qquad \forall\, \Phi, \Psi \in D, \tag{2.1}$$

and $U(x)\varphi(f)U(x)^{-1} = \varphi(f_x)$ on D, where $f_x(y) = f(y-x)$. Then $\{\mathscr{H}, D, \varphi(\cdot), U\}$ (henceforth abbreviated to $\varphi(\cdot)$) will be called a quantum field.

It is well known that (2.1) and the invariance of D under $\varphi(f)$ imply that

$$\langle \Phi, \varphi(x_1)\cdots\varphi(x_n)\Psi\rangle \in \mathscr{F}'(\mathbb{R}^{4n}), \qquad \forall\, n \in \mathbb{N},\ \Phi, \Psi \in D, \tag{2.2}$$

since $\mathscr{C}_\omega(\mathbb{R}^k)$ is a nuclear space and $\mathscr{C}_\omega(\mathbb{R}^k) \otimes \mathscr{C}_\omega(\mathbb{R}^\ell) \cong \mathscr{C}_\omega(\mathbb{R}^{k+\ell})$ for any $k, \ell \in \mathbb{N}$ [27, 21]. Any quantum field satisfying the axioms of [1, 2] or [23] is a quantum field in the sense of Def. 2.1. If $\mathscr{F}(\mathbb{R}^4) = \mathscr{S}(\mathbb{R}^4)$, the field will be called tempered, and if $\mathcal{F}(\mathbb{R}^4) = \mathcal{C}_\omega(\mathbb{R}^4)$ for some Jaffe indicatrix ω, $\varphi(\cdot)$ will be said to be an ultradistribution field.

In Section 4 we shall presume the existence of a net of von Neumann algebras $\{\mathcal{A}(\mathcal{O})\}$ ($\mathcal{O} \subset \mathbb{R}^4$ open) on a Hilbert space $\mathcal{H}$, that satisfies the usual conditions of isotony, locality and translation covariance under a strongly continuous, unitary representation $U(\mathbb{R}^4)$ of the translation group satisfying the spectrum condition [3, 4, 5]. As a matter of shorthand, we shall denote such a net by $\{\mathcal{A}(\mathcal{O})\}$ and shall call it a net of local algebras. $\mathcal{A} \equiv \overline{\cup \mathcal{A}(\mathcal{O})}$ will denote the C^*-algebra generated by the algebras in $\{\mathcal{A}(\mathcal{O})\}$. $\mathcal{B}(\mathcal{H})$ will represent the algebra of all bounded operators on the Hilbert space $\mathcal{H}$.

We now can begin developing the argument of the paper.

III. *H*-bounded sesquilinear forms and quantum fields

In this section, starting from a sesquilinear form φ satisfying a generalized H-bound, we examine the properties of the "quantum field" defined by

$$\varphi(f) \equiv \int f(x) U(x) \varphi U(x)^{-1}\, d^4x, \tag{3.1}$$

where $\{U(x) \mid x \in \mathbb{R}^4\}$ is a strongly continuous, unitary representation of the translation group on $\mathcal{H}$ that satisfies the spectrum condition. We commence by proving a number of technical lemmas. Let $D_c(H)$ be the set of all vectors $\Psi \in \mathcal{H}$ such that for some compact set $K \subset \mathbb{R}$, $[\int_K E(dp)]\Psi = \Psi$, where $H = \int pE(dp)$ is the spectral decomposition of H from the spectral theorem; in other words, $D_c(H)$ is the set of all vectors in $\mathcal{H}$ with compact energy support. Moreover, let $\omega_\beta \equiv (I + H^2)^{\beta/2}$ and $D_H(\alpha) \equiv \text{span}\,\{e^{-\omega_\beta}\mathcal{H} \mid \beta > \alpha\}$. Note that if $\omega_\beta(t) = (1 + t^2)^{\beta/2}$ and $1 > \beta$, then $\omega_\beta(t)$ is a Jaffe indicatrix function. $C_{\omega_\beta}(\mathbb{R}^4)$ will denote the test function space corresponding to such an indicatrix function.

Lemma 3.1. (a) *For any $\alpha \geq 0$, $D_c(H)$ and $D_H(\alpha)$ are dense in $\mathcal{H}$, invariant under $U(\mathbb{R}^4)$, and are cores for e^{ω_α}.*

(b) *$e^{\omega_\alpha} D_H(\alpha) \subseteq D_H(\alpha)$, for any $\alpha \geq 0$.*

Proof. (a) is straightforward (see Lemma 5.2 in [12]). To see (b), note that for any $\Phi \in D_H(\alpha)$ there exist a $\beta > \alpha$ and a vector $\Psi_\beta(\Phi) \in \mathcal{H}$ such that $\Phi = e^{-\omega_\beta}\Psi_\beta$. But for any $\delta > 0$ with $\beta - \alpha > \delta$, $e^{\omega_{\alpha+\delta}} e^{\omega_\alpha} e^{-\omega_\beta}$ is bounded, by the spectral calculus. Thus, $e^{\omega_\alpha}\Phi \in D(e^{\omega_{\alpha+\delta}}) = e^{-\omega_{\alpha+\delta}}\mathcal{H} \subset D_H(\alpha)$.

Let $C^n(\mathbb{R}^d)$ denote the set of all n-times continuously differentiable, complex-valued functions on $\mathbb{R}^d$.

Lemma 3.2. *If $C \in \mathcal{B}(\mathcal{H})$ and $f \in L^1(\mathbb{R})$, let $C(f) \equiv \int f(t) e^{itH} C e^{-itH}\, dt$, the integral being understood in the weak operator topology.*

(1) *If $\tilde{f} \in C^2(\mathbb{R})$ and $|\tilde{f}(s)| < e^{-|s|^\beta}$ for some $\beta > \alpha \geq 0$ and all s with $|s|$ sufficiently large, then $C(f) D_H(\alpha) \subseteq D_H(\alpha)$ and $C(f)^* D_H(\alpha) \subseteq D_H(\alpha)$.*

(2) *If* $\tilde{f} \in C^2(\mathbb{R})$ *and* $|\tilde{f}(s)| < e^{-(1+\sigma)|s|^\alpha}$ *for some* $\sigma > 0$ *and all* s *with* $|s|$ *sufficiently large, then* $C(f)D_H(\alpha) \subseteq D(e^{\omega_\alpha})$.

Proof. Assume Case 1. It is first shown that the sesquilinear form $e^{\omega_\gamma}C(f)e^{-\omega_\delta}$ defined on $D(e^{\omega_\gamma}) \times \mathcal{H}$ is bounded for any $\gamma < \min\{\beta, \delta\}$. This is done by showing that each summand in

$$e^{\omega_\gamma}C(f)e^{-\omega_\delta} = P_\Omega e^{\omega_\gamma}C(f)e^{-\omega_\delta} + (I - P_\Omega)e^{\omega_\gamma}C(f)e^{-\omega_\delta}P_\Omega + (I - P_\Omega)e^{\omega_\gamma}C(f)e^{-\omega_\delta}(I - P_\Omega) \tag{3.2}$$

is bounded (P_Ω is the orthogonal projection onto the translation-invariant subspace of $\mathcal{H}$, which may be trivial). The first term on the right-hand side of (3.2) is trivially bounded by $\|C\| \cdot \|f\|_1$ ($\|\cdot\|_1$ is the $L^1(\mathbb{R})$-norm). Since for any $\Omega \in P_\Omega\mathcal{H}$, $e^{\omega_\gamma}C(f)e^{-\omega_\delta}\Omega = e^{-1}e^{\omega_\gamma}\tilde{f}(H)C\Omega$ and $e^{\omega_\gamma(s)}\tilde{f}(s) \in L^\infty(\mathbb{R})$, the second term in (3.2) is bounded by $\|e^{\omega_\gamma(s)}\tilde{f}(s)\|_\infty \cdot \|C\|$.

Next, note that for any $\Phi \in D(e^{\omega_\gamma}) \cap (I - P_\Omega)\mathcal{H}$ and $\Psi \in (I - P_\Omega)\mathcal{H}$,

$$\langle e^{\omega_\gamma}\Phi, C(f)e^{-\omega_\delta}\Psi\rangle = \int_0^\infty \int_0^\infty e^{\omega_\gamma(p)}e^{-\omega_\delta(k)}\tilde{f}(p-k)\, d_p\, d_k\langle\Phi, E(p)CE(k)\Psi\rangle, \tag{3.3}$$

where $\{E(p)\}_{p\in\mathbb{R}}$ is the spectral family associated with H and the indicated integral is interpreted as an iterated Stieltjes integral. Let $\chi \in \mathcal{D}(\mathbb{R})$ be a positive function such that $\chi(p) = 1$ for $|p| \leq 1$ and $\chi(p) = 0$ for $|p| \geq 2$. Then again using the functional calculus, if $\chi_m(p) \equiv \chi(p/m)$ the following quantity

$$\langle\chi_m(H)e^{\omega_\gamma}\Phi, C(f)\chi_m(H)e^{-\omega_\delta}\Psi\rangle = \int_0^\infty \int_0^\infty \chi_m(p)e^{\omega_\gamma(p)}\chi_m(k)e^{-\omega_\delta(k)}\tilde{f}(p-k)\, d_p\, d_k\langle\Phi, E(p)CE(k)\Psi\rangle \tag{3.4}$$

converges as $m \to \infty$ to (3.3). Since the integrand in (3.4) has compact support, one can use standard results on the integration by parts of Stieltjes integrals to obtain the following expression for (3.4):

$$\int_0^\infty \int_0^\infty \left[\frac{\partial^2}{\partial p\, \partial k}(\chi_m(p)e^{\omega_\gamma(p)}\chi_m(k)e^{-\omega_\delta(k)}\tilde{f}(p-k))\right]\langle\Phi, E(p)CE(k)\Psi\rangle\, dp\, dk.$$

Note that $E(0) = P_\Omega$ has been used to eliminate the boundary terms at $p = 0 = k$. Since $|\langle\Phi, E(p)CE(k)\Psi\rangle| \leq \|C\| \cdot \|\Phi\| \cdot \|\Psi\|$ uniformly in p and k, the absolute value of the integral above (and thus of (3.4)) is bounded by

$$\|C\| \cdot \|\Phi\| \cdot \|\Psi\| \int_0^\infty \int_0^\infty \left|\frac{\partial^2}{\partial p \partial k}(\chi_m(p)e^{\omega_\gamma(p)}\chi_m(k)e^{-\omega_\delta(k)}\tilde{f}(p-k))\right| dp\, dk,$$

which converges as $m \to \infty$ to

$$\|C\| \cdot \|\Phi\| \cdot \|\Psi\| \int_0^\infty \int_0^\infty \left|\frac{\partial^2}{\partial p \partial k}(e^{\omega_\gamma(p)}e^{-\omega_\delta(k)}\tilde{f}(p-k))\right| dp\, dk. \tag{3.5}$$

The validity of this assertion will become clear, if it is not already, in the following process of estimating (3.5).

Consider

$$\int_0^\infty \int_k^\infty e^{\omega_\gamma(p)} e^{-\omega_\delta(k)} |\tilde{f}(p-k)| \, dp \, dk.$$

Setting $s = p - k$ and performing a change of variables, one obtains

$$\int_0^\infty \int_0^\infty e^{\omega_\gamma(s+k)-\omega_\delta(k)} |\tilde{f}(s)| \, ds \, dk \le \int_0^\infty \int_0^\infty e^{\omega_\gamma(s)+\omega_\gamma(k)-\omega_\delta(k)} |\tilde{f}(s)| \, ds \, dk,$$

which is finite. (Note that $\omega_\gamma(s+k) \le \omega_\gamma(s) + \omega_\gamma(k)$ has been employed, and this subadditivity of $\omega_\gamma(s)$ follows because it is concave and monotone increasing.) The other contribution ($p < k$) to the integral is

$$\int_0^\infty \int_0^k e^{\omega_\gamma(p)} e^{-\omega_\delta(k)} |\tilde{f}(p-k)| \, dp \, dk,$$

which, since $p < k$ and $\omega_\gamma(p)$ is monotone increasing, is bounded by

$$\int_0^\infty \int_0^\infty e^{-\epsilon\omega_\gamma(p)+(1+\epsilon)\omega_\gamma(k)-\omega_\delta(k)} |\tilde{f}(p-k)| \, dp \, dk,$$

with $\epsilon > 0$. This integral is also finite. The derivatives in (3.5) can be handled with the same arguments after integrating by parts any derivatives on $\tilde{f}$.

Therefore, the sesquilinear form in (3.2) extends to a bounded operator, so that for each $\Psi \in D(e^{\omega_\delta})$,

$$\|e^{\omega_\gamma} C(f)\Psi\| \le \|e^{\omega_\gamma} C(f) e^{-\omega_\delta}\| \cdot \|e^{\omega_\delta}\Psi\| < \infty.$$

Since each $\Psi \in D_H(\alpha)$ is in the domain of e^{ω_δ} for some $\delta > \alpha$, the assertion of the lemma follows. The proof for $C(f)^*$ is similar.

The arguments for Case 2 are parallel, when one substitutes e^{ω_α} for e^{ω_γ} above. The crucial estimates are then $e^{\omega_\alpha(s)}\tilde{f}(s) \in L^\infty(\mathbb{R})$ and

$$\int_0^\infty \int_0^\infty e^{-\epsilon\omega_\alpha(p)+(1+\epsilon)\omega_\alpha(k)-\omega_\delta(k)} |\tilde{f}(p-k)| \, dp \, dk < \infty,$$

which are clearly true for all $\delta > \alpha$.

Remark. The above lemma is again true if $C(f) = \int f(x)U(x)CU(x)^{-1}d^4x$, where $f \in L^1(\mathbb{R}^4)$ is such that $\tilde{f} \in C^2(\mathbb{R}^4)$ and $|\tilde{f}(p)| < \text{const}.e^{-|p|^\beta}$ for some $\beta > \alpha$ (resp. $|\tilde{f}(p)| < \text{const}.e^{-(1+\epsilon)|p|^\alpha}$ for some $\epsilon > 0$) and all sufficiently large $|p|$ ($|p|$ is the Euclidean norm on $\mathbb{R}^4$). The details are left to the reader, since, due to the spectrum condition, the arguments are perfectly parallel. Note that if $f \in C_{\omega_\beta}(\mathbb{R}^4)$ with $\beta > \alpha$, f is in Case 1, and if $f \in C_{\omega_\alpha}(\mathbb{R}^4)$, it is certainly in Case 2.

In this section we are interested in sesquilinear forms φ satisfying the following condition:

(C1) φ is defined on $D_H(\alpha) \times D_H(\alpha)$ for some $\alpha \ge 0$, and $e^{-\omega_\alpha}\varphi e^{-\omega_\alpha}$ is bounded.

Definition. For a given form φ, if α_0 is the infimum of all α such that (C1) obtains, the sesquilinear form φ will be said to satisfy a generalized H-bound of order α_0. Polynomial H-bounds, i.e. bounds such as $\|(I+H)^{-n}\varphi(I+H)^{-n}\|<\infty$ for some $n \in \mathbb{N}$, which were studied in [16, 17], are of order 0 in this terminology.

Let φ satisfy (C1) and $f(x) \in L^1(\mathbb{R}^4)$. For any $\Phi, \Psi \in D_H(\alpha)$ there exist $\beta_1, \beta_2 > \alpha$ such that $\Phi = e^{-\omega_{\beta_1}}\Phi'$ and $\Psi = e^{-\omega_{\beta_2}}\Psi'$ for some $\Phi', \Psi' \in \mathcal{H}$. Then, understanding (3.1) as a weak integral, we have

$$\langle \Phi, \varphi(f)\Psi \rangle = \int f(x)\langle U(x)^{-1}\Phi', e^{-\omega_{\beta_1}}\varphi e^{-\omega_{\beta_2}}U(x)^{-1}\Psi' \rangle \, d^4x,$$

so that

$$|\langle \Phi, \varphi(f)\Psi \rangle| \le \|e^{-\omega_{\beta_1}}\varphi e^{-\omega_{\beta_2}}\| \cdot \|\Phi'\| \cdot \|\Psi'\| \cdot \|f\|_1 < \infty. \tag{3.6}$$

Therefore, $\varphi(f)$ is in this case a well-defined sesquilinear form on $D_H(\alpha) \times D_H(\alpha)$. Moreover, for any $\Phi, \Psi \in \mathcal{H}$ and $f \in L^1(\mathbb{R}^4)$,

$$|\langle \Phi, e^{-\omega_\alpha}\varphi(f)e^{-\omega_\alpha}\Psi \rangle| \le \|e^{-\omega_\alpha}\varphi e^{-\omega_\alpha}\| \cdot \|\Phi\| \cdot \|\Psi\| \cdot \|f\|_1. \tag{3.7}$$

This entails that $f \to e^{-\omega_\alpha}\varphi(f)e^{-\omega_\alpha}$ may be interpreted as determining a continuous map from $L^1(\mathbb{R}^4)$ into $\mathcal{B}(\mathcal{H})$, supplied with the uniform operator topology. In the next proposition it is shown that if f satisfies certain conditions, $\varphi(f)$ determines an operator with invariant domain $D_H(\alpha)$.

Proposition 3.3. *Let φ satisfy condition* (C1). *Then for every $f \in L^1(\mathbb{R}^4)$ such that $\tilde{f} \in C^2(\mathbb{R}^4)$ and $|\tilde{f}(p)| \le ce^{-(1+\epsilon)|p|^\alpha}$ for some constant c and $\epsilon > 0$ and for all sufficiently large $|p|$, the sesquilinear form $\varphi(f)$ on $D_H(\alpha) \times D_H(\alpha)$ determines an operator on $D_H(\alpha)$ (again denoted by $\varphi(f)$). If, in addition, $|\tilde{f}(p)| \le ce^{-|p|^\beta}$ for some $\beta > \alpha$ and all sufficiently large $|p|$, then $\varphi(f)D_H(\alpha) \subseteq D_H(\alpha)$.*

Proof. Let $\Phi \in D_H(\alpha)$ and, for some $\beta' > \alpha$, $\Psi = e^{-\omega_{\beta'}}\chi \in e^{-\omega_{\beta'}}\mathcal{H} \subset D_H(\alpha)$. Then $\langle \Phi, \varphi(f)\Psi \rangle$ is well defined. Furthermore,

$$\begin{aligned}\langle \Phi, \varphi(f)\Psi \rangle &= \langle e^{\omega_\alpha}\Phi, e^{-\omega_\alpha}\varphi(f)e^{-\omega_\alpha}e^{\omega_\alpha}e^{-\omega_{\beta'}}\chi \rangle \\ &= \int d^4x f(x)\langle e^{\omega_\alpha}\Phi, U(x)e^{-\omega_\alpha}\varphi e^{-\omega_\alpha}U(x)^{-1}e^{\omega_\alpha - \omega_{\beta'}}\chi \rangle.\end{aligned}$$

Since $C \equiv e^{-\omega_\alpha}\varphi e^{-\omega_\alpha} \in \mathcal{B}(\mathcal{H})$, the above is equal to

$$\langle e^{\omega_\alpha}\Phi, C(f)e^{\omega_\alpha - \omega_{\beta'}}\chi \rangle.$$

But $e^{\omega_\alpha - \omega_{\beta'}}\chi \in D_H(\alpha)$ (use arguments of proof of Lemma 3.1), so Lemma 3.2 entails that even under the weaker assumption on $|\tilde{f}|$ one must have $C(f)e^{\omega_\alpha - \omega_{\beta'}}\chi \in D(e^{\omega_\alpha})$. Thus,

$$|\langle \Phi, \varphi(f)\Psi \rangle| \le \|\Phi\| \cdot \|e^{\omega_\alpha}C(f)e^{\omega_\alpha - \omega_{\beta'}}\chi\| < \infty,$$

for any $\Phi \in D_H(\alpha)$, $\chi \in \mathcal{H}$, $\beta' > \alpha$. Therefore, $\varphi(f)$ defines an operator on $D_H(\alpha)$. Under the stronger assumption on $|\tilde{f}|$, Lemma 3.2 yields $C(f)e^{\omega_\alpha - \omega_{\beta'}}\chi \in D_H(\alpha)$, thus $\varphi(f)D_H(\alpha) \subseteq D_H(\alpha)$, since $e^{\omega_\alpha}D_H(\alpha) \subseteq D_H(\alpha)$.

Remark. Henceforth, if φ and f satisfy the hypothesis of Proposition 3.3, $\varphi(f)$ will denote the operator on $D_H(\alpha)$ determined by the corresponding sesquilinear form as above. Moreover, $\overline{\varphi(f)}$ will signify the closure of this operator. Note that, by definition, the operators $\varphi(f)$ are covariant under the translation group $U(\mathbb{R}^4)$.

Since such "quantum field" operators have an invariant, common domain, products of such operators are well-defined on $D_H(\alpha)$. The next proposition shows that such operators satisfy generalized H-bounds themselves.

Proposition 3.4. *Let* φ *satisfy* (C1). *Then for any* $f \in L^1(\mathbb{R}^4)$ *such that* $\tilde{f} \in C^2(\mathbb{R}^4)$ *and* $|\tilde{f}(p)| \leq ce^{-(1+\epsilon)|p|^\alpha}$ *for some constant c and some* $\epsilon > 0$ *for all sufficiently large* $|p|$, *the following holds*:

$$\|\varphi(f)e^{-\omega_\gamma}\| \leq c_\gamma(f) < \infty, \qquad \textit{for any } \gamma > \alpha.$$

Proof. It will be shown that given any $\gamma > \alpha$, $\varphi(f)e^{-\omega_\gamma}$ is defined on all of $\mathcal{H}$ and has a closed graph. The closed graph theorem then yields the claim. To begin, note that $e^{-\omega_\gamma}\mathcal{H} \subset D_H(\alpha)$, so that by Proposition 3.3, $\varphi(f)e^{-\omega_\gamma}$ is well-defined on all of $\mathcal{H}$. Let $\Phi \in \mathcal{H}$ and $\{\Phi_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ converging strongly to Φ such that $\{\varphi(f)e^{-\omega_\gamma}\Phi_n\}_{n\in\mathbb{N}}$ is strongly Cauchy and converges to $\Psi \in \mathcal{H}$. It is sufficient to show that $\varphi(f)e^{-\omega_\gamma}\Phi = \Psi$.

Let $u \in D_H(\alpha)$. Consider

$$\begin{aligned}\langle u, \varphi(f)e^{-\omega_\gamma}(\Phi_n - \Phi)\rangle &= \langle e^{\omega_\alpha}u, e^{-\omega_\alpha}\varphi(f)e^{-\omega_\alpha}e^{-\omega_\gamma+\omega_\alpha}(\Phi_n - \Phi)\rangle \\ &= \langle e^{\omega_\alpha}u, C(f)e^{-\omega_\gamma+\omega_\alpha}(\Phi_n - \Phi)\rangle\end{aligned}$$

(where, as before, $C \equiv e^{-\omega_\alpha}\varphi e^{-\omega_\alpha}$)

$$= \langle e^{-\omega_\gamma+\omega_\alpha}C(f)^*e^{\omega_\alpha}u, \Phi_n - \Phi\rangle.$$

Thus,

$$|\langle\{u, \varphi(f)e^{-\omega_\gamma}(\Phi_n - \Phi)\rangle| \leq \|\Phi_n - \Phi\| \cdot \|e^{-\omega_\gamma+\omega_\alpha}C(f)^*e^{\omega_\alpha}u\|.$$

If the second factor on the right-hand side above is bounded by $d_\gamma(f)\ \|u\|$, with $d_\gamma(f) < \infty$ dependent only on f and $\gamma > \alpha$, the desired conclusion follows, since $D_H(\alpha)$ is dense in $\mathcal{H}$. This bound, however, is a straightforward consequence of the argument used in the proof of Lemma 3.2, because the integral

$$\int_0^\infty \int_0^\infty |e^{-\omega_\gamma(|p|)+\omega_\alpha(|p|)}\widetilde{f^*}(p-k)e^{\omega_\alpha(|k|)}|\, d^4p\, d^4k$$

is finite. The assertion is therefore proven.

Definition. A quantum field $\varphi(\cdot)$ satisfying

$$\|\overline{\varphi(f)}e^{-\omega_\gamma}\| \leq c_\gamma(f) < \infty, \tag{3.8}$$

for all f in its test function space and some fixed $\gamma \geq 0$, will be said to satisfy a generalized H-bound. ($\overline{\varphi(f)}$ denotes the closure of $\varphi(f)$ on D.) If α_0 is the infimum of all γ such that (3.8) holds, the field will be said to satisfy a generalized H-bound of order α_0.

We remark then that Proposition 3.4 implies that if φ satisfies (C1), then the associated "quantum field" $\varphi(x)$ satisfies a generalized H-bound (GHB) of order $\alpha_0 \leq \alpha$. And if φ satisfies a GHB of order $\alpha_0 \geq 0$, then $\varphi(x)$ satisfies a GHB of order α_0.

We next note that if φ satisfies condition (C1) and we pick Φ, $\Psi \in D(e^{\omega_\alpha})$ (in particular, both $\Phi = \Psi$ could be a vacuum vector, so that $H\Phi = 0$ and $\Phi \in D(e^{\omega_\beta})$ for all β), then by Proposition 3.3

$$W_{\Phi,\Psi}(f_1, \ldots, f_n) \equiv \left\langle \Phi, \left(\prod_{i=1}^{n} \varphi(f_i)\right)\Psi \right\rangle \tag{3.9}$$

is well-defined for any $n \in \mathbb{N}$, $\{f_i\}_{i=1}^n \subset L^1(\mathbb{R}^4)$ such that $\{\tilde{f}_i\}_{i=1}^n \subset C^2(\mathbb{R}^4)$ and $|\tilde{f}_i(p)| \leq ce^{-|p|^{\beta_i}}$, for some $\beta_i > \alpha$ and some finite constant c when $|p|$ is sufficiently large and $i = 1, \ldots, n$. In fact, in such a case, for any $0 \leq m \leq n-1$,

$$W_{\Phi,\Psi}(f_1, \ldots, f_n) = \left\langle \left(\prod_{i=m}^{1} \varphi(f_i)^*\right)\Phi, \varphi(f_{m+1})\left(\prod_{j=m+2}^{n} \varphi(f_j)\right)\Psi \right\rangle,$$

by Proposition 3.3, so that by (3.6) and Proposition 3.3

$$|W_{\Phi,\Psi}(f_1, \ldots, f_n)| \leq c \, \|f_{m+1}\|_1, \tag{3.10}$$

for some finite constant c depending on Φ, Ψ and f_i, $1 \leq i \leq n$ and $i \neq m+1$. But the L^1-norm is continuous in the topology on $\mathscr{C}_\omega(\mathbb{R}^4)$ (and $\mathscr{S}(\mathbb{R}^4)$) for any indicatrix function ω. If $\beta > \alpha$, then under the stated conditions on φ, Ψ and Φ, $W_{\Phi,\Psi}(\cdot, \ldots, \cdot)$ is an element of $\mathscr{C}'_{\omega_\beta}(\mathbb{R}^4)$ in each variable singly. And since $\mathscr{C}_{\omega_\beta}(\mathbb{R}^k)$ is a nuclear space and $\mathscr{C}_{\omega_\beta}(\mathbb{R}^k) \otimes \mathscr{C}_{\omega_\beta}(\mathbb{R}^\ell) \cong \mathscr{C}_{\omega_\beta}(\mathbb{R}^{k+\ell})$ for any k, $\ell \in \mathbb{N}$ [27, 21], we have the following result.

Proposition 3.5. *If φ satisfies* (C1) *and $\Phi, \Psi \in D_H(\alpha)$, then for any $n \in \mathbb{N}$, $W_{\Phi,\Psi}(\cdot, \ldots, \cdot)$ determines a unique element of $\mathscr{C}'_{\omega_\beta}(\mathbb{R}^{4n})$ for any $\beta > \alpha$.*

By (3.10) we can, in fact, extend $W_{\Phi,\Psi}(f_1, \ldots, f_n)$ continuously in one variable to a continuous functional on $L^1(\mathbb{R}^4)$ (or to a continuous functional on $\mathscr{S}(\mathbb{R}^4)$.) But without further information, we cannot extend in all variables simultaneously past the point marked by Proposition 3.5, because we must know that the extension leaves $D_H(\alpha)$ invariant. We recall a useful result from [13] for immediate application.

Theorem 3.6. [13] *Let $f \to A(f)$ be a linear mapping of a complete, countably normed, linear topological space $\mathscr{F}$ into $\mathscr{B}(\mathscr{H})$ such that the mapping $f \to \langle \Phi, A(f)\Psi \rangle$ is in $\mathscr{F}'$ for all $\Phi \in \mathscr{M}$, $\Psi \in \mathscr{N}$, where $\mathscr{M}$ and $\mathscr{N}$ are dense subsets of $\mathscr{H}$. Then the mapping $f \to A(f)$ is continuous relative to the norm topology on $\mathscr{B}(\mathscr{H})$. Furthermore, there exists a norm $|\cdot|$ continuous in the topology of $\mathscr{F}$ such that $\|A(f)\| \leq |f|$, for all $f \in \mathscr{F}$.*

Since the ultradistribution test function spaces are complete and countably normed [21], we may apply (3.10), Theorem 3.6 and Proposition 3.4 to show the following.

Theorem 3.7. *Let φ satisfy condition* (C1), *and for any $f \in \mathscr{C}_{\omega_\alpha}(\mathbb{R}^4)$ let $\varphi(f)$ be the operator on $D_H(\alpha)$ obtained in Proposition* 3.3. *Then for any $\gamma > \alpha$ there exists a norm $|\cdot|_\gamma$ continuous in the topology on $\mathscr{C}_{\omega_\alpha}(\mathbb{R}^4)$ such that*

$$\|\varphi(f)e^{-\omega_\gamma}\| \leq |f|_\gamma, \qquad \forall f \in \mathscr{C}_{\omega_\alpha}(\mathbb{R}^4). \tag{3.11}$$

Proof. In light of (3.10), Proposition 3.4 and the fact that $e^{-\omega_\gamma}D_H(\alpha) \subseteq D_H(\alpha)$, the claim is an immediate consequence of Theorem 3.6 if one takes $\mathscr{M} = \mathscr{N} = D_H(\alpha)$ and $A(f) = \varphi(f)e^{-\omega_\gamma}$.

Note that Theorem 3.6 also entails that if $\varphi(\cdot)$ is a tempered quantum field satisfying the GHB (3.8), then there exists a norm $|\cdot|_\gamma$ continuous in the topology on $\mathscr{S}(\mathbb{R}^4)$ such that

$$\|\overline{\varphi(f)}e^{-\omega_\gamma}\| \leq |f|_\gamma, \forall f \in \mathscr{S}(\mathbb{R}^4). \tag{3.12}$$

We call H-bounds such as (3.11) $\mathscr{C}_{\omega_\alpha}$-continuous generalized H-bounds, those such as (3.12) $\mathscr{S}$-continuous GHB's and bounds such as

$$\|\overline{\varphi(f)}e^{-\omega_\gamma}\| \leq c\, \|f\|_1, \tag{3.13}$$

for all test functions f and with $c < \infty$ fixed, L^1-continuous GHB's. Due to the density of $\mathscr{C}_\omega(\mathbb{R}^4)$ in both $\mathscr{S}(\mathbb{R}^4)$ and $L^1(\mathbb{R}^4)$, an ultradistribution field $\varphi(x)$ satisfying an $\mathscr{S}$-continuous (resp. L^1-continuous) GHB can be continuously extended to define an operator-valued generalized function $\varphi(f)$ with domain $D_H(\gamma)$ for all $f \in \mathscr{S}(\mathbb{R}^4)$ (resp. $f \in L^1(\mathbb{R}^4)$). Of course, the domain $D_H(\gamma)$ will not necessarily be invariant for the field operators of this extension.

We have seen that sesquilinear forms satisfying (C1) determine quantum fields satisfying (3.11) and (3.7). To close this circle, we next show that quantum fields satisfying (3.11) and (3.7) determine sesquilinear forms satisfying (C1).

Theorem 3.8. *There is a one-to-one relation between sesquilinear forms φ satisfying a generalized H-bound of order α_0 and ultradistribution quantum fields $\varphi(x)$ in the sense of Definition* 2.1 *satisfying (for all $f \in \mathscr{C}_{\omega_{\alpha_0}}(\mathbb{R}^4)$)*

(i) $\|e^{-\omega_\alpha}\overline{\varphi(f)}e^{-\omega_\alpha}\| \leq c\, \|f\|_1,\ c < \infty,$

and

(ii) $\|\overline{\varphi(f)}e^{-\omega_\alpha}\| \leq |f\|_\alpha,$

for some norm $|\cdot|_\alpha$ continuous in the topology on $\mathscr{C}_{\omega_{\alpha_0}}(\mathbb{R}^4)$ and for each $\alpha > \alpha_0$. (The implicit uniqueness in this assertion refers to the restriction to $D_H(\alpha_0)$ for both the form and all field operators.)

Proof. That every sesquilinear form φ satisfying a generalized H-bound of order α_0 determines such a unique ultradistribution quantum field is an immediate consequence of the results already established in this section.

Let then $\varphi(\cdot)$ be an ultradistribution quantum field satisfying (i) and (ii). By [21] the "field at the origin" $\varphi(0)$ is a well-defined sesquilinear form on $D \times D$ (see Definition 2.1). If $f \in \mathscr{D}(\mathbb{R}^4)$ is chosen nonnegative and such that $\int f = 1$, and if one sets $f_n(x) \equiv n^4 f(nx)$ for each $n \in \mathbb{N}$, then for all $\Phi, \Psi \in D$,

$$\langle \Phi, \varphi(f_n)\Psi \rangle \xrightarrow[n\to\infty]{} \langle \Phi, \varphi(0)\Psi \rangle$$

(see [21]). Since (i) implies that $\|e^{-\omega_\alpha}\overline{\varphi(f_n)}e^{-\omega_\alpha}\| \leq c$ for each $n \in \mathbb{N}$, it follows that

$$\|e^{-\omega_\alpha}\varphi(0)e^{-\omega_\alpha}\| \leq c, \ \forall\ \alpha \geq \alpha_0.$$

The uniqueness is a consequence of [21].

We wish to emphasize that conditions (i) and (ii) are independent conditions for ultradistribution fields. To put this in the proper perspective, recall that for tempered distribution fields $\varphi(\cdot)$ it is known [16] that the following three conditions are equivalent:

(a) $\|(I+H)^{-m}\overline{\varphi(f)}(I+H)^{-m}\| \leq c\,|f|_m$, for some $m \in \mathbb{N}$,

(b) $\|(I+H)^{-n}\overline{\varphi(f)}(I+H)^{-n}\| \leq c'\,\|f\|_1$, for some $n \in \mathbb{N}$,

and

(c) $\|\overline{\varphi(f)}(I+H)^{-k}\| \leqq c''\,|f|_k$, for some $k \in \mathbb{N}$,

where $|\cdot|_m$ and $|\cdot|_k$ are norms continuous in the topology on $\mathscr{S}(\mathbb{R}^4)$. In the Appendix we produce examples of nontempered ultradistribution fields such that (a) and (c) hold for norms continuous in the topology on $\mathscr{C}_\omega(\mathbb{R}^4)$, but (b) is false. And we have seen that fields arising from generalized H-bounded sesquilinear forms must satisfy (the analog of) (b). Thus, for the examples of the Appendix, the sesquilinear form $\varphi(0)$ cannot satisfy a (polynomial) H-bound even though $\varphi(\cdot)$ satisfies (a) and (c). In this paper we restrict our attention to fields satisfying both (i) and (ii) of Theorem 3.8.

We wish to make a final point before closing this section. Let $F:\mathbb{R} \to (0, \infty)$ be an infinitely differentiable, monotone decreasing function and let φ be a sesquilinear form on $F(H)\mathscr{H} \times F(H)\mathscr{H}$ such that

$$\|F(H)\varphi F(H)\| < \infty.$$

It is easy to verify, using the arguments presented in this section, that for such sesquilinear forms, $\varphi(f)$ defined by (3.1) determines an operator with an invariant domain $D_c(H)$ for every $f \in \mathscr{K}(\mathbb{R}^4)$, where $\mathscr{K}(\mathbb{R}^4)$ is the space of infinitely differentiable functions whose Fourier transforms are in $\mathscr{D}(\mathbb{R}^4)$ supplied with the topology induced by the inverse Fourier transform and the topology on $\mathscr{D}(\mathbb{R}^4)$. Moreover, for any $\Phi, \Psi \in D_c(H)$, $n \in \mathbb{N}$, $W_{\Phi,\Psi}(f_1, \ldots, f_n)$ (see (3.9)) is well-defined and determines a unique element of $\mathscr{K}'(\mathbb{R}^{4n})$. But there are no nontrivial functions of compact support in $\mathscr{K}(\mathbb{R}^d)$. And if φ satisfies a GHB of order $\alpha_0 \geq 1$, then the Wightman functions given by Proposition 3.5 cannot necessarily be defined for any test functions with compact support. Because in the next section we are interested in quantum fields locally associated to a net of local algebras, we shall henceforth restrict our attention to GHB's of order $\alpha_0 < 1$.

IV. *H*-bounded sesquilinear forms as limits of local observables

We shall next discuss a topology on a net of local algebras $\{\mathscr{A}(\mathcal{O})\}$, in which the limit points of "collapsing" sequences $\{A_\nu\} \subset \mathscr{A}$ are sesquilinear forms satisfying condition (C1) of the previous section and thus determine ultradistribution or tempered distribution quantum fields satisfying (i) and (ii) of Theorem 3.8. It will be shown, in addition, that such sesquilinear forms and quantum fields are locally associated to the net in a manner to be specified below and that all generalized H-bounded sesquilinear forms and quantum fields that are locally associated to the net in this manner can be obtained as limit points of local, bounded operators in this topology.

To begin, we say that a sesquilinear form φ on $D_H(\alpha) \times D_H(\alpha)$ is locally associated with the net $\{\mathscr{A}(\mathcal{O})\}$ of local algebras if for each neighborhood $\mathcal{O}$ of the origin in $\mathbb{R}^4$ and for each $A \in \mathscr{A}(\mathcal{O})'$ such that $AD_H(\alpha) \subseteq D_H(\alpha)$ and $A^*D_H(\alpha) \subseteq D_H(\alpha)$,

$$\langle A^*\Phi, \varphi\Psi\rangle = \langle \Phi, \varphi A\Psi\rangle, \qquad \text{any } \Phi,\ \Psi \in D_H(\alpha). \tag{4.1}$$

Note that unless $\alpha < 1$, there will be no nontrivial local operators A, A^* leaving $D_H(\alpha)$ invariant. In the following lemma it is pointed out that the quantum field operators $\varphi(f)$ determined by such sesquilinear forms are affiliated with the appropriate local algebras of the net $\{\mathscr{A}(\mathcal{O})\}$.

Lemma 4.1. *Let φ satisfy condition* (C1) *with $\alpha < 1$, and for $f \in C_{\omega_\alpha}(\mathbb{R}^4)$ let $\varphi(f)$ be the operator on $D_H(\alpha)$ constructed in Section* 3. *Then if φ is locally associated to the net $\{\mathscr{A}(\mathcal{O})\}$ of local algebras, $\overline{\varphi(f)}$ is affiliated with $\mathscr{A}(\mathcal{O})$ for any $\mathcal{O} \subset \mathbb{R}^4$ such that* $\operatorname{supp}(f) \subset \mathcal{O}$.

Proof. The proof parallels that of Lemma 2.2 in [16], using Lemma 3.2 (and the remark following it) and the fact that for any open $\mathcal{O} \subset \mathbb{R}^4$ and $\beta < 1$ there exists an $f \in \mathscr{C}_{\omega_\beta}(\mathbb{R}^4)$ such that $\operatorname{supp}(f) \subset \mathcal{O}$ and $\int f(x)\, d^4x = 1$ [23], so the details will be suppressed.

From this it readily follows that all sesquilinear forms satisfying (C1) with $\alpha < 1$ and locally associated to the same net of local algebras determine ultradistribution (or tempered distribution) fields that are local and relatively local in the sense of [1, 2, 23].

Proposition 4.2. *φ is a sesquilinear form satisfying condition* (C1) *with $\alpha < 1$ and is locally associated to the net $\{\mathscr{A}(\mathcal{O})\}$ of local algebras if and only if $e^{-\omega_\alpha}\varphi e^{-\omega_\alpha} \in (e^{-\omega_\alpha}\mathscr{A}(\mathcal{O})e^{-\omega_\alpha})''$ for any neighborhood $\mathcal{O}$ of the origin in $\mathbb{R}^4$.*

Proof. Let $e^{-\omega_\alpha}\varphi e^{-\omega_\alpha}$ be contained in $(e^{-\omega_\alpha}\mathscr{A}(\mathcal{O})e^{-\omega_\alpha})''$ for any neighborhood $\mathcal{O}$ of the origin in $\mathbb{R}^4$. There thus exists a net $\{A_v\} \subset \mathscr{A}(\mathcal{O})$ such that $e^{-\omega_\alpha}A_v e^{-\omega_\alpha}$ converges weakly to $e^{-\omega_\alpha}\varphi e^{-\omega_\alpha}$. Let $B \in \mathscr{A}(\mathcal{O})'$ be such that $BD_H(\alpha) \subseteq D_H(\alpha)$ and $B^*D_H(\alpha) \subseteq D_H(\alpha)$. Then for any $\Phi, \Psi \in D_H(\alpha)$,

$$\langle B^*\Phi, \varphi\Psi\rangle = \lim_v \langle \Phi, BA_v\Psi\rangle = \lim_v \langle \Phi, A_v B\Psi\rangle = \langle \Phi, \varphi B\Psi\rangle.$$

To prove the other implication, let $\mathcal{O} \subset \mathbb{R}^4$ be a neighborhood of the origin and $f \in \mathscr{C}_{\omega_\alpha}(\mathbb{R}^4)$ satisfy $\mathrm{supp}(f) \subset \mathcal{O}$. Then $\overline{\varphi(f)}$ is affiliated with $\mathscr{A}(\mathcal{O})$ by Lemma 4.1. Moreover, define for $c > 0$

$$A_c(f) \equiv (1 + c^2 |\overline{\varphi(f)}|^2)^{-1}\overline{\varphi(f)}.$$

Then $A_c(f) \in \mathscr{A}(\mathcal{O})$, $\|(1 + c^2 |\overline{\varphi(f)}|^2)^{-1}\| \leq 1$, and

$$|\langle \Phi, (A_c(f) - \varphi(f))\Phi\rangle| \leq \frac{c}{2} \|\varphi(f)^*\Phi\| \cdot \|\varphi(f)\Phi\|,$$

for every $\Phi \in D_H(\alpha)$. Thus,

$$\{e^{-\omega_\alpha}(A_c(f) - \varphi(f))e^{-\omega_\alpha}\}_{c>0}$$

is uniformly bounded and converges weakly to zero on $D_H(\alpha)$ as $c \to 0$. Hence,

$$e^{-\omega_\alpha}A_c(f)e^{-\omega_\alpha} \xrightarrow[c\to 0]{w} e^{-\omega_\alpha}\varphi(f)e^{-\omega_\alpha}.$$

Again choose a nonnegative test function f with $\int f = 1$ and set $f_n(x) \equiv n^4 f(nx)$. Using (3.7) and the strong continuity of the representation of the translations, it follows that

$$e^{-\omega_\alpha}\varphi(f_n)e^{-\omega_\alpha} \xrightarrow{w} e^{-\omega_\alpha}\varphi e^{-\omega_\alpha}.$$

Thus, if $\{A_v\} \subset \mathscr{A}$ is a collapsing net (i.e. $A_v \in \mathscr{A}(\mathcal{O}_v)$ with $\mathcal{O}_{v_1} \subset \mathcal{O}_{v_2}$, whenever $v_1 > v_2$, and $\bigcap \mathcal{O}_v = \{0\}$) such that there exists an $\alpha \in [0, 1)$ with $\{e^{-\omega_\alpha}A_v e^{-\omega_\alpha}\}$ weakly Cauchy, then

$$\varphi \equiv e^{\omega_\alpha}(w - \lim e^{-\omega_\alpha}A_v e^{-\omega_\alpha})e^{\omega_\alpha}$$

defines a sesquilinear form on $D_H(\alpha) \times D_H(\alpha)$ that is locally associated to $\{\mathscr{A}(\mathcal{O})\}$. We now combine these results with those of the previous section to obtain the following theorems.

Theorem 4.3. *Let $\varphi(\cdot)$ be a hermitian, scalar quantum field satisfying for some $\alpha < 1$*

(i) $\|e^{-\omega_\alpha}\overline{\varphi(f)}e^{-\omega_\alpha}\| \leq c\, \|f\|_1$, $c < \infty$

and

(ii) $\|\overline{\varphi(f)}e^{-\omega_\alpha}\| \leq |f|_\alpha$

for all test functions f and for some norm $|\cdot|_\alpha$ *continuous in the topology on the test function space. And let* $\{\mathscr{A}(\mathcal{O})\}$ *be a net of local algebras transforming under the same representation of the translation group as* $\varphi(\cdot)$. *Then the following are equivalent*:

(1) *For any open double cone* $\mathcal{O} \subset \mathbb{R}^4$ *and test function f with* $\mathrm{supp}(f) \subset \mathcal{O}$, $\overline{\varphi(f) \restriction D}$ *is affiliated with* $\mathscr{A}(\mathcal{O})$.
(2) For any open $\mathcal{O} \subset \mathbb{R}^4$ *and test function f with* $\mathrm{supp}(f) \subset \mathcal{O}$, $\overline{\varphi(f) \restriction D}$ *is affiliated with* $\mathscr{A}(\mathcal{O})$.
(3) $e^{-(1+\epsilon)\omega_\alpha}\varphi(0)e^{-(1+\epsilon)\omega_\alpha} \in (e^{-(1+\epsilon)\omega_\alpha}\mathscr{A}(\mathcal{O})e^{-(1+\epsilon)\omega_\alpha})''$ *for every* $\epsilon > 0$ *and every neighborhood* $\mathcal{O} \subset \mathbb{R}^4$ *of the origin.*

Remark. Note that under the stated assumptions, Theorem 3.8 entails that $e^{-(1+\epsilon)\omega_\alpha}\varphi(0)e^{-(1+\epsilon)\omega_\alpha}$ is a well-defined bounded operator for all $\epsilon > 0$.

Proof. (3) implies (2) by Proposition 4.2, Lemma 4.1 (the factor $(1+\epsilon)$ before ω_α carries through in the obvious manner) and the fact that (ii) implies that D and $D_H(\alpha)$ are both cores for the same closed operator (Lemma 5.2 in [12]). And, of course, (2) implies (1) trivially.

Finally, if $\mathcal{O}$ is an open double cone centered at the origin and f is a nonnegative test function with $\mathrm{supp}(f) \subset \mathcal{O}$ and $\int f(x)\, d^4x = 1$, set $f_n(x) = n^4 f(nx)$ and $\mathcal{O}_n = \frac{1}{n}\mathcal{O}$. Assuming (1), it follows that

$$\langle A\Phi, \overline{\varphi(f_n)}\Psi\rangle = \langle \varphi(f_n)^*\Phi, A^*\Psi\rangle, \tag{4.2}$$

for all $\Phi, \Psi \in D_H(\alpha)$ and all $A \in \mathscr{A}(\mathcal{O}_n)'$. Let then $\mathcal{O}_0 \subset \mathbb{R}^4$ be any neighborhood of the origin and $B \in \mathscr{A}(\mathcal{O}_0)'$ be such that

$$BD_H(\alpha) \subseteq D_H(\alpha) \quad \text{and} \quad B^*D_H(\alpha) \subseteq D_H(\alpha). \tag{4.3}$$

Then by (4.2) there exists an $N \in \mathbb{N}$ such that for all $n \geq N$

$$\langle B\Phi, \overline{\varphi(f_n)}\Psi\rangle = \langle \Phi, \overline{\varphi(f_n)}B^*\Psi\rangle, \tag{4.4}$$

for all $\Phi, \Psi \in D_H(\alpha)$. By (4.3) and the definition of $D_H(\alpha)$, for any $\epsilon > 0$, $B\Phi$ and $B^*\Psi$ are contained in $D(e^{(1+\epsilon)\omega_\alpha})$, so that (4.4) entails

$$\begin{aligned}&\langle e^{(1+\epsilon)\omega_\alpha}B\Phi, e^{-(1+\epsilon)\omega_\alpha}\overline{\varphi(f_n)}e^{-(1+\epsilon)\omega_\alpha}e^{(1+\epsilon)\omega_\alpha}\Psi\rangle \\ &\qquad = \langle e^{(1+\epsilon)\omega_\alpha}\Phi, e^{-(1+\epsilon)\omega_\alpha}\overline{\varphi(f_n)}e^{-(1+\epsilon)\omega_\alpha}e^{(1+\epsilon)\omega_\alpha}B^*\Psi\rangle.\end{aligned} \tag{4.5}$$

By the proof of Theorem 3.8, $e^{-(1+\epsilon)\omega_\alpha}\overline{\varphi(f_n)}e^{-(1+\epsilon)\omega_\alpha}$ converges weakly to $e^{-(1+\epsilon)\omega_\alpha}\varphi(0)e^{-(1+\epsilon)\omega_\alpha}$. Thus, (4.5) implies

$$\langle B\Phi, \varphi(0)\Psi\rangle = \langle \Phi, \varphi(0)B^*\Psi\rangle$$

for all $\Phi, \Psi \in D_H(\alpha)$. Thus, $\varphi(0)$ is locally associated to the net $\{\mathscr{A}(\mathcal{O})\}$. (3) then follows from Proposition 4.2.

If the net $\{\mathscr{A}(\mathcal{O})\}$ of local algebras satisfies duality for the wedge and double cone algebras and both field and net transform under the same representation of the Poincaré group, then under the hypothesis of Theorem 4.3 it follows from [12] that assertion (1) above is equivalent to the following:

There exists a double cone $\mathcal{O} \subset \mathbb{R}^4$ and a test function f with $\mathrm{supp}(f) \subset \mathcal{O}$ such that $\tilde{f}(p) \neq 0$ for all $p \in \mathbb{R}^4$ and $\langle \varphi(f)\Phi, A\Psi \rangle = \langle A^*\Phi, \varphi(f)^*\Psi \rangle$, for all $\Phi, \Psi \in D$ and $A \in \mathcal{A}(\mathcal{O})'$.

The extension of the proofs in [12] to ultradistribution fields requires no modification of any proof.

Theorem 4.4. *Let $\{\mathcal{A}\{\mathcal{O}\}\}$ be a net of local algebras. $\varphi(\cdot)$ is a hermitian, scalar quantum field that transforms under the same representation of the translation group as $\{\mathcal{A}(\mathcal{O})\}$ and satisfies*

(i) *for any open $\mathcal{O} \subset \mathbb{R}^4$ and test function f with* $\mathrm{supp}(f) \subset \mathcal{O}$, $\overline{\varphi(f) \restriction D}$ *is affiliated with $\mathcal{A}(\mathcal{O})$*

(ii) *$\varphi(x)$ satisfies a generalized H-bound of order $\alpha < 1$,*

and

(iii)

$$\|e^{-(1+\epsilon)\omega_\alpha}\overline{\varphi(f)}e^{-(1+\epsilon)\omega_\alpha}\| \leq c_\alpha \|f\|_1, \tag{4.6}$$

for all test functions f and all $\epsilon > 0$,

if and only if

there exists a collapsing net $\{A_\nu\} \subset \mathcal{A}$ such that

$$w - \lim e^{-(1+\epsilon)\omega_\alpha} A_\nu e^{-(1+\epsilon)\omega_\alpha} \equiv e^{-(1+\epsilon)\omega_\alpha} \varphi e^{-(1+\epsilon)\omega_\alpha} \tag{4.7}$$

exists for every $\epsilon > 0$, and $\varphi(f) = \int d^4x f(x) U(x) \varphi U(x)^{-1}$ *as sesquilinear forms on $D_H(\alpha) \times D_H(\alpha)$, for every test function f.*

Proof. The implication ($\Rightarrow$) is an immediate consequence of Theorem 4.3. And the converse follows from Proposition 3.5 (and the development leading up to it), Lemma 4.1 and Proposition 4.2. (Note that (4.7) implies $\|e^{-(1+\epsilon)\omega_\alpha}\varphi e^{-(1+\epsilon)\omega_\alpha}\| < \infty$, for all $\epsilon > 0$ since $\mathcal{B}(\mathcal{H})$ is complete in the weak operator topology.)

We have, therefore, established that ultradistribution fields $\varphi(x)$ that satisfy the regularity conditions (ii) and (iii) of Theorem 4.4 are locally associated to a net $\{\mathcal{A}(\mathcal{O})\}$ of local algebras if and only if there exists a collapsing net $\{A_\nu\} \subset \mathcal{A}$ such that

$$e^{-(1+\epsilon)\omega_\alpha} A_\nu e^{-(1+\epsilon)\omega}$$

is weakly Cauchy and converges to $e^{-(1+\epsilon)\omega_\alpha}\varphi(0)e^{-(1+\epsilon)\omega_\alpha}$. All such fields are in the same Borchers class (on the domain $D_H(\alpha)$) – see [12].

Appendix. Examples of H-bounded nontempered ultradistribution quantum fields

We begin this Appendix by recalling an idea credited to Wollenberg in [21]. Let $\varphi(x)$ be a quantum field satisfying the axioms of [1, 2] and $\omega(t)$ be a Jaffe indicatrix function. Define

$$\exp(\omega_u(|p\cdot p|)) \equiv \exp(\omega(p\cdot p)) + \exp(\omega(-p\cdot p)) = 2\sum_{n=0}^{\infty} c_{2n}(p\cdot p)^{2n},$$

where $p\cdot p$ denotes the scalar product in Minkowski space and $p\in\mathbb{R}^4$. Let $\mathcal{F}:\mathcal{S}(\mathbb{R}^4)\to\mathcal{S}(\mathbb{R}^4)$ denote the Fourier transform and define

$$\varphi_u(f)\equiv\varphi(\mathcal{F}^{-1}(e^{\omega_u}\mathcal{F}f)), \tag{A.1}$$

for all $f\in\mathscr{C}_\omega(\mathbb{R}^4)$. It was shown in [21] that $\varphi_u(x)$ determines a quantum field satisfying the axioms of [23]. Moreover, Kern showed that if $\varphi(x)$ is the generalized free scalar field with mass distribution $d\rho(m)=m^s dm$, $s\in\mathbb{R}$, then there exists an $f_0\in\mathcal{S}(\mathbb{R}^4)$ such that

$$\langle\varphi_u(f_0)\Omega,\ \varphi_u(f_0)\Omega\rangle=\infty, \tag{A.2}$$

where Ω is the vacuum vector. However, for any quantum field $\varphi(x)$ satisfying the Wightman axioms, one has by the Källen–Lehmann representation (see, e.g. [28])

$$\|\varphi(f)\Omega\|^2=\int_0^\infty\int_{H_m}|\tilde{f}(\vec{p})|^2\frac{d^3\vec{p}}{\sqrt{m^2+|\vec{p}|^2}}\,d\rho(m),$$

where H_m is the mass hyperboloid of mass m and $\rho(m)$ is a tempered measure on $[0,\infty)$. Thus, the argument of [21] may be used to conclude (A.2) for *any* Wightman field $\varphi(x)$ such that

$$\int_2^\infty(e^{\omega(m^2)}+e^{\omega(-m^2)})^2 m^{-1}e^{-\omega(m^2)}\,d\rho(m)=\infty. \tag{A.3}$$

Since already the condition $\int m\,d\rho(m)<\infty$ is only expected to hold for theories with finite mass renormalization [29], (A.3) and hence (A.2) will typically be satisfied.

Now let $\varphi(x)$ be a Wightman field satisfying the generalized H-bound (3.8) for all $f\in\mathcal{S}(\mathbb{R}^4)$. Since for any $f\in\mathscr{C}_\omega(\mathbb{R}^4)$ $\mathcal{F}^{-1}(e^{\omega_u}\mathcal{F}f)$ is an element of $\mathcal{S}(\mathbb{R}^4)$, $\varphi_u(x)$ satisfies (3.8) for all $f\in\mathscr{C}_\omega(\mathbb{R}^4)$. Thus, Theorem 3.6 entails the existence of a norm $|\cdot|_\gamma$ continuous in the topology of $\mathscr{C}_\omega(\mathbb{R}^4)$ such that

$$\|\varphi_u(f)e^{-\omega_\gamma}\|\le|f|_\gamma, \tag{A.4}$$

for all $f\in\mathscr{C}_\omega(\mathbb{R}^4)$.

Proposition A.1. *If $\varphi_u(x)$ is a quantum field constructed as above and fulfills condition (A.3), and if it satisfies a polynomial H-bound, i.e. for some $k \in \mathbb{N}$*

$$\|\overline{\varphi_u(f)}(I+H)^{-k}\| < \infty \tag{A.5}$$

for all $f \in \mathscr{C}_\omega(\mathbb{R}^4)$, then for any $n \in \mathbb{N}$ the bound

$$\|(I+H)^{-n}\varphi_u(0)(I+H)^{-n}\| < \infty \tag{A.6}$$

is false, as is

$$\|(I+H)^{-n}\overline{\varphi_u(f)}(I+H)^{-n}\| \leq c_n \|f\|_1, \forall f \in \mathscr{C}_\omega(\mathbb{R}^4). \tag{A.7}$$

Proof. It is known from [16] and Section 3 that (A.6) and (A.7) are equivalent. By [16] and Theorem 3.6, (A.6) entails that

$$\|\overline{\varphi_u(f)}(I+H)^{-2n}\| \leqq |f|, \forall f \in \mathscr{C}_\omega(\mathbb{R}^4), \tag{A.8}$$

for a norm $|\cdot|$ continuous in the topology of $\mathscr{S}(\mathbb{R}^4)$, and $\overline{\varphi_u(f)}$ can thus be extended to an operator-valued tempered distribution with invariant domain $C^\infty(H)$. Therefore, $\langle \varphi_u(f)^*\Omega, \varphi_u(\cdot)\Omega\rangle$ and $\langle \varphi_u(\cdot)^*\Omega, \varphi_u(g)\Omega\rangle$ can be continuously extended to an element of $\mathscr{S}'(\mathbb{R}^4)$ for every $f \in \mathscr{S}(\mathbb{R}^4)$, $g \in \mathscr{S}(\mathbb{R}^4)$. By the nuclear theorem $\langle \varphi_u(\cdot)^*\Omega, \varphi_u(\cdot)\Omega\rangle$ determines a unique element of $\mathscr{S}'(\mathbb{R}^8)$, which contradicts (A.2).

It is not known whether the same negative result can be proven if the polynomial H-bounds are replaced by nonpolynomial generalized H-bounds, since in that case one cannot be sure that the extension of φ_u to $\mathscr{S}'(\mathbb{R}^4)$ leaves $D_H(\alpha)$ invariant. As an aside, taking $\varphi(x)$ to be the quantum field of the ϕ_3^4 model and constructing $\varphi_u(x)$ from it as above provides an example of a field satisfying the hypothesis of Proposition A.1. This answers in the negative a question left open in [21]. (See also the end of Section 3 of the present paper.)

Next we show that if $\varphi(x)$ is a Wightman field locally associated with a net of local algebras $\{\mathscr{A}(\mathcal{O})\}$, then $\varphi_u(x)$ constructed as above is also locally associated with the net $\{\mathscr{A}(\mathcal{O})\}$. Let D_0 be the standard domain for $\varphi(x)$ and D_0^u be that for $\varphi_u(x)$.

Proposition A.2. *Let $\varphi(x)$ be a hermitian, scalar quantum field fulfilling the axioms of* [1, 2] *and let $\{\mathscr{A}(\mathcal{O})\}$ be a net of local algebras such that for every open $\mathcal{O} \subset \mathbb{R}^4$ and every $f \in \mathscr{S}(\mathbb{R}^4)$ such that* $\mathrm{supp}(f) \subset \mathcal{O}$, *$\overline{\varphi_u(f) \restriction D_0}$ is affiliated with $\mathscr{A}(\mathcal{O})$. Then if $\varphi_u(x)$ is constructed as above, one has $\overline{\varphi_u(f) \restriction D_0^u}$ affiliated with $\mathscr{A}(\mathcal{O})$ for every open $\mathcal{O} \subset \mathbb{R}^4$ and every $f \in \mathscr{C}_\omega(\mathbb{R}^4)$ with* $\mathrm{supp}(f) \subset \mathcal{O}$.

Proof. Of course, for every $f \in \mathscr{C}_\omega(\mathbb{R}^4)$, $\mathscr{F}^{-1}(e^{\omega_u}\mathscr{F}f)$ is an element of $\mathscr{S}(\mathbb{R}^4)$. Let $\mathrm{supp}(f) \subset \mathcal{O}$ and define

$$\omega_u^N(p) \equiv \left(2\sum_{k=0}^{N} c_{2k}(p\cdot p)^{2k}\right).$$

Then $\omega_u^N(p)\mathscr{F}f$ converges in $\mathscr{S}(\mathbb{R}^4)$ to $e^{\omega_u}\mathscr{F}f$ as $N \to \infty$, and for each N,

$$\mathscr{F}^{-1}(\omega_u^N \mathscr{F}f)(x) = \left(\sum_{k=0}^{N} c_{2k}\Box^{2k}\right) f(x), \tag{A.9}$$

where $\Box$ is the d'Alembertian. The support of the function in (A.9) is contained in $\mathscr{O}$ for each $N \in \mathbb{N}$. Thus, $\text{supp}(\mathscr{F}^{-1}(e^{\omega_u}\mathscr{F}f)) \subset \mathscr{O}$. By a standard argument [30] using the density of $\mathscr{C}_\omega(\mathbb{R}^{4n})$ in $\mathscr{S}(\mathbb{R}^{4n})$, $\forall n \in \mathbb{N}$, one sees that

$$\overline{\varphi_u(f) \upharpoonright D_0^u} \equiv \overline{\varphi(\mathscr{F}^{-1}(e^{\omega_u}\mathscr{F}f)) \upharpoonright D_0^u} = \overline{\varphi(\mathscr{F}^{-1}(e^{\omega_u}\mathscr{F}f)) \upharpoonright D_0}$$

for every $f \in \mathscr{C}_\omega(\mathbb{R}^4)$. Hence $\overline{\varphi_u(f) \upharpoonright D_0^u}$ is affiliated with $\mathscr{A}(\mathscr{O})$.

Since the field $\varphi(x)$ of the ϕ_3^4 model satisfies the hypothesis of Theorem 4.8 of [12], we may conclude from Propositions A.1 and A.2 that there exists an example of a net of local algebras, generated by a polynomially H-bounded tempered distribution field, that has locally associated to it a polynomially H-bounded nontempered ultradistribution field whose H-bound is not L^1-continuous. In fact, from the discussion above, it is clear that this will be no isolated exception.

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