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Autor: Collet, Pierre / Eckmann, Jean-Pierre

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The stability of modulated fronts

By Pierre Collet

Ecole Polytechnique, Laboratoire de Physique Théorique, F-91128 Palaiseau Cedex

and Jean-Pierre Eckmann

Université de Genève, Physique Théorique, CH-1211 Genève 4

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Abstract. We study the stability of fourth order differential equations which are known to exhibit modulated front solutions. We analyze these equations in the moving frame, and find a criterion for marginal stability. For stationary (periodic) solutions we discuss the appearance of an Eckhaus instability.

1. Introduction

The purpose of this paper is a stability analysis of oscillating front solutions for the equation

$$\partial_t U = (\epsilon - (1 + \partial_x^2)^2)U - \epsilon U^3, \tag{1.1}$$

where U is a function of x and t, and $\epsilon > 0$. In the paper [CE] we showed that equation (1.1) has propagating front solutions of the form

$$U(x, t) = W(x, \eta x - c\eta^2 t), \tag{1.2}$$

where $\eta = \epsilon^{1/2} > 0$, and

$$W(x_1, x_2) = \sum_{n \in \mathbf{Z}} e^{in\omega x_1} W_n(x_2), \tag{1.3}$$

with

$$\lim_{x \to \infty} W_n(x) = 0, \tag{1.4}$$

$$\lim_{x \to -\infty} W_n(x) = S_n. \tag{1.5}$$

Here, we have $S_1 \approx 3^{-1/2}$, so that (in particular) W_1 is a nontrivial function. In fact, for all practical purposes, we may think of W as

$$W(x_1, x_2) \approx \frac{1}{3^{1/2}} (l(x_2)e^{ix_1} + l(x_2)e^{-ix_1}), \tag{1.6}$$

where l is the real solution of the *amplitude equation*

$$4l'' + cl' + l - l^3 = 0, \tag{1.7}$$

with the conditions $l(-\infty) = 1$, $l(+\infty) = 0$, $l(x) > 0$ for all x. The wavenumber ω is approximately equal to 1, and the speed of the front is ηc where c is close to but greater than 4.

In this paper, we study the *stability* of these solutions under infinitesimal perturbations. Before we can start with this analysis, we need to explain in detail what kind of stability we are considering. There is, in fact, an abundant literature on this and related problems, and the word stability may have slightly different meaning in different contexts. Our considerations come closest to, and overlap in part, those of Sattinger [S], and of Dee and Langer [DL]. On the other hand, what is new here is the rigorous treatment of the stability problem for a front which leaves a pattern in the laboratory frame as it passes by.

The first observation is that a stability analysis for moving fronts is done always with respect to some *frame of reference* moving at some speed c' which may be different from the speed c with which the front itself propagates. In this paper, we study stability in two frames:
- The frame moving with the front, i.e., $c' = c$ (Sections 1, 2),
- The laboratory frame, i.e., $c' = 0$ (Section 3).

The argument by which one can see that stability depends on c' is based on the fact that one may 'outrun' a growing instability by moving the point of observation away from it fast enough. For example, if a solution grows, in the laboratory frame, like

$$e^{\Omega t - \gamma x}, \tag{1.8}$$

with $\Omega > 0$ then in a frame moving with speed c to the right, we will see an exponential of the form

$$e^{\Omega t - \gamma(x + ct)} \tag{1.9}$$

and this function tends, for fixed x, to zero as $t \to \infty$ if $c > \Omega/\gamma$. Thus the function (1.8) looks unstable in any frame moving with speed slower than Ω/γ, and it looks stable in any frame for which $c > \Omega/\gamma$.

A second aspect of instability (or stability) is the choice of the space of allowed perturbations. For example, if we choose an empty space of perturbations, the front is stable, and, more generally, a small space of perturbations leads to a more stable situation than a large one. We want to look at this problem in more detail, by studying the stability in the frame moving with the front. Although this problem has already been discussed in detail in [S] and in [DL], we repeat its discussion here, because it is one of the central issues in any stability analysis. The confusing fact is that in a sufficiently large function space the front solutions are *always* unstable, no matter how 'small' the perturbation is. We illustrate this paradox with the equation of Kolmogoroff, Petrovsky and Piscounoff,

$$\partial_t u = \partial_x^2 u + u - u^3. \tag{1.10}$$

It is well known, see e.g. [AW] for an exposition, that for all $c \geq 2$, the equation (1.10) has a travelling wave solution of the form $u(x, t) = v_c(x - ct)$, with $v_c(+\infty) = 0$, $v_c(-\infty) = 1$, and $v_c > 0$. These solutions behave (for $c > 2$) near $+\infty$ like

$$v_c(z) \approx e^{-\Omega(c)z}, \tag{1.11}$$

where

$$\Omega(c) = \frac{c - (c^2 - 4)^{1/2}}{2}. \tag{1.12}$$

Consider now the front v_c for a given c. Then for every c' (close to c), $v_c - v_{c'}$ is a function which goes to zero at infinity. However, the difference

$$u_c(x, t) - u_{c'}(x, t)$$

(where $u_c(x, t) = v_c(x - ct)$ and $u_{c'}(x, t) = v_{c'}(x - c't)$) *grows* as a function of time. This is easily seen, since, in the frame $y = x + ct$ moving with speed c,

$$u_c(y, t) - u_{c'}(y, t) = v_c(x) - v_{c'}(x + (c - c')t).$$

This quantity does not tend to zero as $t \to +\infty$. In other words, very small perturbations of the travelling wave solution lead to an instability: *All* solutions are unstable in this sense.

The preceding discussion shows that stability is not a natural requirement for the analysis of front solutions, since they are all unstable. Thus, another kind of stability criterion is needed, and it is based on the observation that the instability is caused by a change of speed of the front. In fact, it is this instability which seems to be relevant from a physical point of view, and we want to minimize it. It follows from the previous discussion that this kind of instability will be weakest when the change of speed caused by a perturbation will be smallest. We thus claim that the relevant quantity for stability analysis is the derivative

$$\frac{dc}{d\Omega} = \frac{1}{\dfrac{d\Omega(c)}{dc}}.$$

In our example, we get

$$\frac{d\Omega(c)}{dc} = \frac{1}{2} - \frac{c}{(c^2 - 4)^{1/2}},$$

and this tends to $-\infty$ when $c \to 2$. Therefore, $dc/d\Omega$ tends to zero and thus the front for $c = 2$ is *least* unstable to variations in the exponential decay rate. In accordance with common usage, we call $c = 2$ the speed for *marginal instability*.

In the example of the equation (1.10) discussed above, we see that the stability analysis is described by the function $\Omega(c)$, which relates the decay of the front to the speed c. We shall find a similar function Q in the case of oscillating front solutions. The relation between this function and the spectrum of the perturbations will be more delicate, because Q is not real, and this problem will be discussed in detail in Section 2. We study the equation (1.1) in a coordinate system moving with speed ηc, i.e., with the speed of the front. For this purpose, it is useful to change coordinates in a way which differs from equation (1.2), namely

$$U(x, t) = V(\eta x - \eta^2 ct, t). \tag{1.13}$$

In the new frame, equation (1.1) is transformed to

$$\partial_2 V = P(-i\eta\partial_1)V - \epsilon V^3, \tag{1.14}$$

where ∂_1 and ∂_2 are the derivatives with respect to the first and second argument of V, respectively, and

$$P(q) = \epsilon - (1 - q^2)^2 + i\eta cq. \tag{1.15}$$

The front solution equations (1.2)–(1.3) in the 'V'-frame will be called V_0 and it takes the form

$$V_0 = W\left(\frac{\zeta}{\eta} + \eta ct, \zeta\right) = \sum_{n\in\mathbf{Z}} W_n(\zeta)e^{i\omega n((\zeta/\eta)+\eta ct)}. \tag{1.16}$$

The equation for the evolution of an infinitesimal perturbation of V_0 is then obtained by linearizing equation (1.14) at this solution. The corresponding equation for such a perturbation α is therefore

$$\partial_2\alpha(\zeta, t) = P(-i\eta\partial_1)\alpha(\zeta, t) - 3\epsilon\alpha(\zeta, t)\left(\sum_{n\in\mathbf{Z}} W_n(\zeta)e^{i\omega n((\zeta/\eta)+\eta ct)}\right)^2. \tag{1.17}$$

This is a linear differential equation with periodic potential in t. We are interested in the time evolution of α, but to eliminate the time dependence, we proceed with the following (well-known) manipulations. The equation (1.17) is of the form

$$\partial_t\alpha_t = A_t\alpha_t, \tag{1.18}$$

where

$$\alpha_t(\zeta) = \alpha(\zeta, t), \tag{1.19}$$

and

$$A_t = P(-i\eta\partial_1) - 3\epsilon\left(\sum_{n\in\mathbf{Z}} W_n(\zeta)e^{i\omega n((\zeta/\eta)+\eta ct)}\right)^2. \tag{1.20}$$

Clearly,

$$A_t = A_{t+mT}, \tag{1.21}$$

for all $m \in \mathbf{Z}$, with $T = 2\pi/(\omega\eta c)$. We denote by $\Omega(t, t')$ the operator solution of equation (1.18), i.e., for any $t' \in \mathbf{R}$ and any $t > t'$, we find for given $\alpha_{t'}$ that α_t equals

$$\alpha_t = \Omega(t, t')\alpha_{t'}. \tag{1.22}$$

By equation (1.21), we find

$$\Omega(t + kT, t' + kT) = \Omega(t, t'), \tag{1.23}$$

for all $k \in \mathbf{Z}$, and

$$\Omega(t+kT, t') = \Omega(t+T, t)^k \Omega(t, t'), \tag{1.24}$$

for all $k \geq 0$, $k \in \mathbf{Z}$ and $t \geq t'$. Assuming $\Omega(t, t')$ is uniformly bounded for $0 \leq t - t' \leq T$, we see that *if the spectral radius of* $\Omega(T, 0)$ *is less than one, then*

$$\lim_{t \to \infty} \alpha_t = 0 \tag{1.25}$$

in norm. In practice, this will mean that we shall study the Fourier transform in time of α, by which the time dependence is eliminated.

We now informally describe the kind of perturbations we consider. We shall study the behavior of α's which are bounded (and approximately constant) in the bulk behind the front (i.e., in the region $\zeta < 0$) and which decay like $e^{-s\zeta}$ for positive ζ. For each $s > 0$, we study the spectrum of the evolution operator A_t, when acting on functions of this type. Because of the intrinsic instability of propagating fronts, as explained before, we expect, and find, that the spectrum lies in the closed left half-plane, and in fact contains at least one point of the imaginary axis. The *marginal* stability is now defined as follows: We shall find a function $Q(q, s, c)$ which describes the location of the spectrum of A_t acting on perturbations decaying like e^{-sx} and oscillating like e^{iqx}. Q is really nothing else than the linear differential operator in (1.17), for a bulk with frequency ω:

$$Q(q, s, c) = P(i\eta s + q) \pm i\omega\eta c,$$

where P was defined in equation (1.15). (The spectrum is really contained in the union of the sets described by the two choices of sign.) Marginal stability amounts to saying that the spectrum lies in the closed left half-plane. This means that the range of Q must be contained in the closed left half-plane. Thus, for every point q_c, γ_c for which

$$\mathrm{Re}\, Q(q_c, \gamma_c, c) = 0, \tag{1.26}$$

we must have

$$\partial_q \mathrm{Re}\, Q(q_c, \gamma_c, c) = 0, \tag{1.27}$$

and

$$\partial_q^2 \mathrm{Re}\, Q(q_c, \gamma_c, c) \leq 0. \tag{1.28}$$

We now impose the additional condition that the space in which we consider the perturbations corresponds to the front solution. This means that the decay rate which we allow is exactly that of the front solution, i.e., q_c and γ_c are exactly the parameters describing a front solution. In other words, q_c, γ_c, c is a zero of Q:

$$\mathrm{Im}\, Q(q_c, \gamma_c, c) = 0. \tag{1.29}$$

But we also want to express the condition of marginality: It says that, for the front itself, the variation of the decay rate as a function of c is maximal (in fact, infinite). Since we have $\mathrm{Re}\, Q(q_c, \gamma_c, c) = 0$, we find by differentiation,

$$\partial_q \mathrm{Re}\, Q(q_c, \gamma_c, c)\partial_c q_c + \partial_\gamma \mathrm{Re}\, Q(q_c, \gamma_c, c)\partial_c \gamma_c + \partial_c \mathrm{Re}\, Q(q_c, \gamma_c, c) = 0.$$

The first term is zero by (1.26). Assuming that $\partial_c \operatorname{Re} Q(q_c, \gamma_c, c) \neq 0$, we see that a necessary condition for $\partial_c \gamma = \pm\infty$ is

$$\partial_\gamma \operatorname{Re} Q(q_c, \gamma_c, c) = 0. \tag{1.30}$$

We shall therefore say that marginality follows from (1.26)–(1.30). (In fact, the condition (1.30) is seen to be also sufficient, by explicit calculation.)

In Section 2 we fill in the mathematical details of the above discussion. In Section 3, we deal with a related subject, namely the stability of the *stationary* solutions to (1.1) in the laboratory frame. The stability analysis for this problem is rather straightforward, we include it here because it has never been done rigorously, and because we want to be sure that the propagating front (1.1) leads to a *stable or marginal* bulk. In perturbation theory, the spectrum of periodic excitations can be studied very easily, and we will do this in the calculations leading to (3.67). One sees the emergence of an instability for certain wavelengths of the perturbation: This is known as the Eckhaus instability [E]. The hard, and we believe, new part of the argument is that we have to show the applicability of perturbation theory. In fact, while perturbation theory works fine for isolated eigenvalues, we have in the problem (1.1) *two* eigenvalues close to each other, namely at a distance of order $\epsilon^{1/2}$. We give *a priori* bounds on the position, and hence the separation of these eigenvalues, and this allows us to prove the validity of perturbation theory.

2. Stability analysis of the front solution

As explained in the introduction, we study now the Fourier transform in time of the perturbation α. We write

$$\alpha(\zeta, t) = \sum_{n \in \mathbf{Z}} \hat{\alpha}_n(\zeta) e^{i\omega n \eta c t}, \tag{2.1}$$

and then we get

$$i\omega n \eta c \hat{\alpha}_n = P(-i\eta\partial_\zeta)\hat{\alpha}_n - 3\epsilon \sum_{p+q+r=n} W_p(\zeta) W_q(\zeta) e^{i\omega(p+q)\zeta/\eta} \hat{\alpha}_r. \tag{2.2}$$

To eliminate the phases, we set now

$$\beta_n(\zeta) = \hat{\alpha}_n(\zeta) e^{-i\omega n \zeta/\eta}, \tag{2.3}$$

for $n \in \mathbf{Z}$. The equation (2.2) becomes now, for each $n \in \mathbf{Z}$,

$$i\omega n \eta c \beta_n = P(-i\eta\partial)\beta_n - 3\eta^2 \sum_{p+q+r=n} W_p W_q \beta_r. \tag{2.4}$$

The system in equation (2.4) is coupled, and we consider first the decoupled problem which is obtained by replacing the sum in equation (2.4) by a sum over $p = -q$. Thus we study the operators

$$X_n = P(-i\eta\partial + \omega n) - F_n(x) - i\omega n \eta c, \tag{2.5}$$

where the multiplication operator F_n is given by

$$F_n(x) = 3\eta^2 \sum_{p \in \mathbf{Z}} |W_p(z)|^2, \tag{2.6}$$

since W_{-p} is the complex conjugate of W_p. For further use, we note that F_n is independent of n and is a positive, bounded function. We thus define

$$F(x) = 3\eta^2 \sum_{p \in \mathbf{Z}} |W_p(z)|^2, \tag{2.7}$$

and we note from [CE], equations 3.4, 2.7, that

$$\lim_{x \to -\infty} F(x) = 2\eta^2(1 + o(\eta)), \tag{2.8}$$

and

$$\lim_{x \to \infty} F(x) = 0. \tag{2.9}$$

In the introduction, we have argued that we want to analyze the stability of the solution relative to various rates of decay of the perturbation. We now specify the function spaces with which we describe these decays. These spaces should contain all functions which decay like the front itself, i.e., a possible choice would be

$$L^2\left(\mathbf{R}, \left(\sum_{p \in \mathbf{Z}} |W_p(z)|^2\right)^{-1} dz\right). \tag{2.10}$$

In the forthcoming calculations, we choose an equivalent formulation, based on the observation that there is a γ such that, up to polynomial corrections,

$$\left(\sum_{p \in \mathbf{Z}} |W_p(z)|^2\right)^{1/2} = \mathcal{O}(e^{-\gamma x}), \tag{2.11}$$

as $x \to +\infty$, while near $-\infty$ the above sum tends to a constant. Thus we define a function $A \in \mathscr{C}^\infty$ as follows:

$$A(x) = \begin{cases} 0, & \text{for } x < -S, \\ x, & \text{for } x > S, \end{cases} \tag{2.12}$$

where S is a (large but) finite constant, which will be chosen later. We also require that $A(x)$ be a monotone function of x. The spaces on which we consider X_n are

$$H_s \equiv L^2(\mathbf{R}, \Phi_s^{-2}(x)\, dx), \tag{2.13}$$

where

$$\Phi_s(x) = e^{-sA(x)}. \tag{2.14}$$

It is now easy to see that the operator X_n acting on H_s is conjugate to an operator $Y_{n,s}$ on $L^2(\mathbf{R}, dx)$, where

$$Y_{n,s} = \Phi_s^{-1} X_n \Phi_s = P(-i\eta\partial + \omega n + i\eta s A'(x)) - F(x) - i\omega n \eta c. \tag{2.15}$$

We want to study the spectrum of $Y_{n,s}$. Since $Y_{n,s}$ does not have constant coefficients, we use methods from pseudo-differential calculus (see e.g. [H]) to compute this spectrum. We thus define

$$D = -i\eta\partial, \tag{2.16}$$

and we consider the functions

$$\mathbf{P}_n(q, x) = P(q + \omega n + i\eta s A'(x)) - F(x) - i\omega n\eta c. \tag{2.17}$$

With this definition, we have

$$Y_{n,s} = \mathbf{P}_n(D, x), \tag{2.18}$$

with the convention

$$(D + g(x))^k = \underbrace{(D + g(x)) \cdots (D + g(x))}_{k \text{ times}}. \tag{2.19}$$

If $\mathbf{P}_n(D, x)$ had constant coefficients, i.e., $\mathbf{P}_n(D, x) = \mathbf{P}_n(D, 0)$, then the *spectrum* of $\mathbf{P}_n(D, x)$ would be a subset of the *range* of $\mathbf{P}_n(q, 0)$ when q varies in $\mathbf{R}$. Our strategy consists in viewing $\mathbf{P}_n(q, x)$ as a perturbation of the sum of *two* operators which have essentially constant coefficients. One of them is constant at $+\infty$ and describes the stability ahead of the front and the other is constant at $-\infty$ and is thus related to the stability in the bulk. We shall see that the stability is first lost in the front, so that the overall stability of the problem is determined by the analysis in the front. This will make precise arguments such as the one presented in [DL].

We now define

$$\psi(x) = (A'(x))^{1/2}. \tag{2.20}$$

Note that by the definition (2.12) of A, we find that

$$\psi(x) = \begin{cases} 0, & \text{for } x < -S, \\ 1, & \text{for } x > S. \end{cases} \tag{2.21}$$

Therefore ψ is a $\mathscr{C}^\infty$ version of a θ-function. Note that, for all $m \geq 1$, $\psi^m(x) - \psi(x)$ is a $\mathscr{C}^\infty$-function which is bounded by 1 and has support in $[-S, S]$. It follows that for all $m \geq 1$, $m \in \mathbf{N}$,

$$(A'(x))^m = \psi(x) + \phi_m(x), \tag{2.22}$$

where ϕ_m is a $\mathscr{C}^\infty$-function with support in $[-S, S]$. Consider now the operator $(D + i\eta s A'(x))^m$ for $m \geq 1$.

Lemma 2.1. *The operators* $(D + i\eta s A'(x))^m$ *have a decomposition*

$$(D + i\eta s A'(x))^m = \psi(D + i\eta s)^m \psi + \tau D^m \tau + Q_{m,s}, \tag{2.23}$$

where

$$\tau(x) = (1 - \psi(x)^2)^{1/2}, \tag{2.24}$$

and $Q_{m,s}$ is of the form

$$Q_{m,s} = \sum_{j=0}^{m-1} q_{m,j,s}(x)D^j, \tag{2.25}$$

with $q_{m,j,s}$ a $\mathscr{C}^\infty$-function with support in $[-S, S]$.

Proof. The proof of equation (2.23) is by induction. We first prove an auxiliary identity, equations (2.26), (2.28) below. We start with $m = 1$. Then we have, by equation (2.20),

$$D - i\eta sA' = D + i\eta s\psi^2 = \psi^2(D - i\eta s) + (1 - \psi^2)D. \tag{2.26}$$

We shall define recursively operators $R_{m,s}$, and we set $R_{0,s} = 0$. For $m \geq 1$, we use inductively equation (2.26) or (2.28). Then we can write

$$\begin{aligned}(D + i\eta sA')^m &= (D + i\eta s\psi^2)(D + i\eta s\psi^2)^{m-1}\\ &= (D + i\eta s\psi^2)(\psi^2(D + i\eta s)^{m-1} + (1 - \psi^2)D^{m-1} + R_{m-1,s})\\ &= \psi^2 D(D + i\eta s)^{m-1} + [D, \psi^2](D + i\eta s)^{m-1}\\ &\quad + (1 - \psi^2)DD^{m-1} + [D, 1 - \psi^2]D^{m-1} + DR_{m-1,s}\\ &\quad + i\eta s\psi^2(D + i\eta s)^{m-1} + i\eta s(\psi^4 - \psi^2)(D + i\eta s)^{m-1}\\ &\quad + i\eta s\psi^2(1 - \psi^2)D^{m-1} + i\eta s\psi^2 R_{m-1,s}.\end{aligned} \tag{2.27}$$

Here, $[\cdot, \cdot]$ denotes the commutator. We combine all terms except the first, third and sixth in equation (2.27) into $R_{m,s}$. It is easy to see inductively that they form a differential operator of order $m - 1$ whose coefficients are $\mathscr{C}^\infty$ with support in $[-S, S]$. Combining the three special terms, we find

$$(D + i\eta sA')^m = \psi^2(D + i\eta s)^m + (1 - \psi^2)D^m + R_{m,s}. \tag{2.28}$$

Note now that ψ^2 and $1 - \psi^2$ are nonnegative, so that we can define their (nonnegative) square roots. This leads to

$$\begin{aligned}(D + i\eta sA')^m &= \psi(D + i\eta s)^m\psi + (1 - \psi^2)^{1/2}D^m(1 - \psi^2)^{1/2} + R_{m,s}\\ &\quad + \psi[\psi, (D + i\eta s)^m] + (1 - \psi^2)^{1/2}[(1 - \psi^2)^{1/2}, D^m]\\ &= \psi(D + i\eta s)^m\psi + (1 - \psi^2)^{1/2}D^m(1 - \psi^2)^{1/2} + Q_{m,s},\end{aligned} \tag{2.29}$$

and we see that $Q_{m,s}$ has the required properties. The proof of Lemma 2.1 is complete.

We can now consider the operators $\mathbf{P}_n$.

Proposition 2.2. *The operators $\mathbf{P}_n(D, x)$ have a decomposition*

$$\begin{aligned}\mathbf{P}_n(D, x) &= \psi(x)(P(D + \omega n + i\eta s) - F(x))\psi(x)\\ &\quad + \tau(x)(P(D + \omega n) - F(x))\tau(x)\\ &\quad + K_{n,s} - i\omega n\eta c,\end{aligned} \tag{2.30}$$

where $K_{n,s}$ is a differential operator of order 3 with coefficients in $\mathscr{C}^\infty[-S, S]$.

Proof. This is immediate from (2.17) and from Lemma 2.1.

We next view $K_{n,s} - i\omega n\eta c$ as a perturbation and we study the two other terms in equation (2.30). We study their spectrum as a function of their numerical range, cf. Kato [K]. The numerical range Θ of an operator A on a domain $D(A)$ in Hilbert space is defined by

$$\Theta(A) = \{(u, Au) \mid u \in D(A),\ \|u\| = 1\}.$$

The following facts which are an immediate consequence of the definition will be useful later.

(a) The numerical range of any operator is a convex set ([S], p. 131).
(b) $\Theta(A + B) \subseteq \Theta(A) + \Theta(B)$.
(c) If ψ is (the operator of multiplication by) a non-negative function, and $0 \le \psi \le 1$ then $\Theta(\psi A \psi) \subseteq \bigcup_{0 \le \lambda \le 1} \lambda\Theta(A)$. The addition and multiplication above are for sets.
(d) If P is a polynomial, then $\Theta(P(-i\partial_x)) \subseteq$ convex hull $(P(\mathbf{R}))$. (Use Fourier transform.)
(e) If F is a real function and $\hat{F}$ the corresponding multiplication operator, then

$$\Theta(\hat{F}) = [\inf F, \sup F].$$

We begin by considering the operators

$$Z_{n,s} = \psi P(D + \omega n + i\eta s)\psi - \psi F \psi. \tag{2.31}$$

Using the simple facts about Θ, we get

$$\Theta(P(D + \omega n + i\eta s)) = \text{convex hull } \{P(q + \omega n + i\eta s) \in \mathbf{C} \mid q \in \mathbf{R}\}. \tag{2.32}$$

Thus,

$$\Theta(Z_{n,s}) \subseteq \text{convex hull} \bigcup_{0 \le \lambda \le 1} \lambda P(\mathbf{R} + i\eta s) + \Theta(-\psi(F + i\omega n\eta c)\psi) \equiv D_s. \tag{2.33}$$

Clearly, the set D_s is independent of n, but not of s. We define b_s:

$$b_s = \sup_{z \in D_s} \operatorname{Re} z. \tag{2.34}$$

Lemma 2.3. *We have*

$$b_s = \max(0, \eta^2 + (4s - c)\eta^2 s + 8\eta^4 s^4). \tag{2.35}$$

Before we prove Lemma 2.3, we state the analogous result for the second term in equation (2.30). This term is equal to

$$Z = \tau P(D + \omega n)\tau - \tau F \tau, \tag{2.36}$$

where $\tau = (1 - \psi^2)^{1/2}$. We observe that

$$\Theta(Z) \subseteq \bigcup_{0 \le \lambda \le 1} \text{convex hull } \lambda P(\mathbf{R}) + \Theta(-\tau F \tau) \equiv D. \tag{2.37}$$

We define

$$b = \sup_{z \in D} \operatorname{Re} z, \tag{2.38}$$

and the result corresponding to Lemma 2.3 is

Lemma 2.4. *We have*

$$b = \eta^2 - 2\eta^2 + C_S, \tag{2.39}$$

where C_S is a positive number which tends to zero as $S \to \infty$.

Proof of Lemma 2.3 and Lemma 2.4. If we consider the contributions from F to b_s or b, then we see that they are given by

$$\max_{x \geq S} - F(x) = 0,$$

$$\max_{x \leq -S} - F(x) = -F(-\infty) + \max_{x \leq -S} (F(-\infty) - F(x)). \tag{2.40}$$

We next analyze the common polynomial part

$$x_s = \sup_{q \in \mathbf{R}} \operatorname{Re} P(q + \omega n + i\eta s). \tag{2.41}$$

as a function of s. Clearly, the result does not depend on ωn. Recall the definition

$$P(z) = \eta^2 - (1 - z^2)^2 + i\eta cz. \tag{2.42}$$

A simple calculation shows that

$$x_s = \sup_{t \in \mathbf{R}} \{-(1 - t + \eta^2 s^2)^2 + 4\eta^2 s^2 t^2 - \eta^2 cs\}. \tag{2.43}$$

The extremum of this quantity lies at

$$t^2 = 1 + 3\eta^2 s^2, \tag{2.44}$$

so that the extremum x_s is given by

$$x_s = \eta^2 + (4s - c)\eta^2 s + 8\eta^4 s^4. \tag{2.45}$$

The assertion of the lemmas follows now from the properties equations (2.8), (2.9) of F.

By choosing S sufficiently large, we see that Lemma 2.3 and Lemma 2.4 imply

Lemma 2.5. *For every $\eta > 0$ and every $s \geq 0$ we have*

$$b < b_s, \tag{2.46}$$

if S is sufficiently large.

This means that the instability of the front is stronger than that of the bulk.

Since the operators $Z_{n,s}$ and Z are densely defined, closeable, and have a common dense domain, (being fourth order ordinary differential operators with constant coefficients plus a potential) we see that the sum $Z_{n,s} + Z$ is densely defined and

$$\Theta(Z_{n,s} + Z) \subseteq \Theta(Z_{n,s}) + \Theta(Z) \subseteq D_s + D. \tag{2.47}$$

By Lemma 2.5, we see that for sufficiently large S,

$$\sup_{z \in \Theta(Z_{n,s}+Z)} \operatorname{Re} z < b_s. \tag{2.48}$$

We now relate b_s to the spectrum of $Z_{n,s} + Z$. Some care is needed in the discussion of these matters, since different authors use different conventions. We follow here the conventions of Kato [K].

Definition 2.6. *The essential spectrum Σ_e of an operator T is the set of $\zeta \in \mathbf{C}$ for which either*

$R(T - \zeta)$ *is not closed,*

or

$R(T - \zeta)$ *is closed but* $\operatorname{nul}(T - \zeta) = \operatorname{def}(T - \zeta) = \infty$.

Here, R denotes the range, nul is the dimension of the null space, def is the dimension of the null space of the adjoint. According to Problem 3.6 in Section V.3.2 of Kato [K], the essential spectrum of an operator A is a subset of the closure of $\Theta(A)$. The preceding discussion is not optimal when $b_s = 0$ but $u_s \equiv \eta^2 + (4s - c)\eta^2 s + 8\eta^4 s^4$ is negative, because (2.35) is not optimal. By subtracting u_s from $Y_{n,s}$, and going through the proof, we find

Proposition 2.7. *For every $s \geq 0$, the essential spectrum of $Z_{n,s} + Z - i\omega n\eta c$ is contained in the halfplane* $\operatorname{Re} z \leq u_s$.

Note now that u_s can be computed from the polynomial

$$\begin{aligned} Q(z) &= P(z) - i\omega n\eta c \\ &= \eta^2 - (1 - z^2)^2 + i\eta c z - i\omega n\eta c. \end{aligned} \tag{2.49}$$

The points of maximal $\operatorname{Re} u_s$ are given by the extrema of $\operatorname{Re} Q$, cf. (2.41)–(2.45). We can now easily determine the conditions for marginal stability, which can be viewed as the point where the variation of u_s is minimal. As we have explained in the introduction, the point of marginality z_0 is determined by the following conditions:

$$\operatorname{Re} Q(z_0) = 0, \tag{2.50}$$

$$\partial_{z_0} \operatorname{Re} Q(z_0) = 0, \tag{2.51}$$

with the additional condition

$$\partial_{z_0}^2 \operatorname{Re} Q(z_0) < 0. \tag{2.52}$$

But we want to choose also the imaginary part of z_0, i.e., s, in such a way that the front solution decays like e^{-sx}. In other words, z_0 must be chosen in such a way that not only (2.50) holds, but also

$$\operatorname{Im} Q(z_0) = 0. \tag{2.53}$$

Note now that z_0 should really be written as $q_0 + i\eta s_0 \equiv q_0 + i\eta\gamma$, where γ is the decay rate of the front described earlier. We now sketch the main steps of the simple calculation to determine z_0. If we fix γ, then we see from (2.51) that q_0 must satisfy either $q_0 = 0$, or

$$q_0^2 = \frac{3\epsilon\gamma^2 + 1}{\epsilon}. \tag{2.54}$$

The first solution leads to a critical point whose real part is more negative than that of the second, and thus marginality is dictated by (2.54). Next we see that (2.50) leads to

$$1 - c\gamma + 8\eta^2\gamma^4 + 4\gamma^2 = 0, \tag{2.55}$$

while (2.53) leads to

$$-c\eta\omega + c\eta^2 q_0 - 8\eta^4\gamma^4 q_0 = 0 \tag{2.56}$$

i.e.,

$$q_0 = \frac{\omega}{\eta(1 - 8\eta^2\gamma^4/c)}. \tag{2.57}$$

The derivative (2.51), with respect to γ leads to

$$-c + 8\gamma + 32\eta^2\gamma^3 = 0. \tag{2.58}$$

Solving these equations perturbatively, we find the solutions (including the first non-trivial corrections in η),

$$\begin{aligned} \omega^2 &= 1 + \eta^2/4, \\ c &= 4 + \eta^2, \\ \gamma &= \tfrac{1}{2} - 3\eta^2/8. \end{aligned} \tag{2.59}$$

We next discuss the spectrum of $Y_{n,s}$. The reader should realize that up to now, we have only analyzed the essential spectrum of $Z_{n,s} + Z - i\omega n\eta c$. Intuitively, the numerical range of an operator corresponds to its continuous spectrum. Furthermore, a perturbation, if it is relatively compact, (in our case $K_{n,s}$ cf. (2.30)) should only change the discrete spectrum, which can only accumulate at the continuous spectrum. We now make this argument more precise.

Definition 2.8. We denote by $\Delta(T)$ the complement of the closure of $\Theta(T)$.

Theorem V.3.2 in [K] says that if T is closable then for $\zeta \in \Delta(T)$, $T - \zeta$ has closed range, and $\operatorname{nul}(T - \zeta) = 0$. Assume now $\zeta \in \Delta(T), \cap \overline{\Delta(T^*)}$, where overbars denote complex conjugates. Then $\operatorname{def}(T - \zeta) = \operatorname{nul}(T^* - \bar{\zeta}) = 0$, since $\bar{\zeta} \in \Delta(T^*)$. By the second half of Theorem V.3.2, this implies $\Delta(T) \cap \overline{\Delta(T^*)}$ is in the resolvent set of T and T has no spectrum there.

We can apply this discussion to the operators $Z_{s,n} + Z$ and we see that this sum has no spectrum in the open right-hand plane if $u_s \leq 0$. In fact, this is an 'if and only if' statement since it is easy to construct approximate eigenvectors for any value with real part below u_s.

Our next task is to analyze the spectrum of $Y_{n,s}$. Note that the operator $Z_{s,n} + Z$ has a constant fourth order term (since $\psi^2 + \tau^2 = 1$). By construction, the perturbing operator $(K_{n,s} - i\omega n\eta c)$ is the sum of a third order differential operator whose coefficients have compact support, and of a constant piece, which is purely imaginary. Therefore, we can apply Theorem IV.5.35 in [K], which says that the essential spectrum is converved under relatively compact perturbations. The imaginary constant just pushes the spectrum in the imaginary direction. Hence we have shown the following

Theorem 2.9. *The operators $X_{n,s}$ have, for s close to γ_c, an essential spectrum in the closed half-space* $\operatorname{Re} z \leq u_s$.

The above bound is saturated, i.e., one can find spectrum arbitrarily close to the line $\operatorname{Re} z = u_s$.

We finally have to deal with the problem that there are several sectors, i.e., the operator acts on the direct sum of spaces indexed by n. This operator is almost, but not quite, a direct sum. The only nondiagonal terms are of the form (cf. (2.4))

$$(N_n\beta)_n = -3\eta^2 \sum_{\substack{p+q+r=n \\ r\neq n}} W_p W_q \beta_r. \tag{2.60}$$

These terms go faster to zero at $+\infty$ than the term $F(x)$ of (2.7), and at $-\infty$ they tend to a constant (very fast) which is of order $o(\eta^2)$. If we equip the space with a norm,

$$\|\alpha\| = \sum_{n=-\infty}^{n=\infty} \epsilon^{-\rho|n|}\left(\int |\alpha_n(x)|^2 \Phi_s^{-2}(x)\,dx\right)^{1/2}$$

then we see that the discussion goes through as before. Hence, Theorem 2.9 implies the

Theorem 2.10. *For s sufficiently close to γ_c, the essential spectrum of $\{X_n\}_{n\in\mathbf{N}}$ in the space H_s lies in the set*

$$\operatorname{Re} z \leq \eta^2 + (4s - c)\eta^2 s + 8\eta^4 s^4.$$

This set is saturated, i.e., there is a point in the spectrum for which equality holds.

Furthermore, up to terms of higher order, the spectrum is marginal in the sense of our earlier discussion, for

$$\begin{aligned} \omega^2 &= 1 + \eta^2/4, \\ c &= 4 + \eta^2, \\ s &= \tfrac{1}{2} - 3\eta^2/8. \end{aligned} \tag{2.61}$$

Here, s is equal to the decay rate of the front.

3. Stability of stationary solutions

In this section, we study the stability of stationary solutions. A stationary solution is a solution U of equation (1.1) which is independent of t. In [CE] we have shown that there are non-trivial stationary solutions: Given $K > 0$, there is, for all $\epsilon > 0$, and all ω satisfying

$$0 \le (1-\omega^2)^2 \le K\epsilon \tag{3.1}$$

a solution $U(x, t) = S(x)$ of equation (1.1) which is of the form

$$S(x) = \sum_{n\in\mathbf{Z}} S_n e^{in\omega x}, \quad \text{with } S_{-n} = S_n \in \mathbf{R}. \tag{3.2}$$

The S_n depend on ω. We have

$$S_1 = \Gamma/2 + \mathcal{O}(\epsilon^{4/5}), \tag{3.3}$$

with Γ satisfying

$$(1-\omega^2)^2 = \epsilon(1 - 3\Gamma^2/4), \tag{3.4}$$

$$S_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \mathcal{O}(\epsilon) & \text{if } |n| = 3, \\ \mathcal{O}(\epsilon^2) & \text{if } |n| = 5, \\ \mathcal{O}(\epsilon^{|n|/5}) & \text{if } n \text{ is odd and } |n| \ge 7. \end{cases} \tag{3.5}$$

Remark. In [CE], we have proved less than the properties stated above. In fact we restricted $(1-\omega^2)^2 \le \mathcal{O}(\epsilon^2)$, and we gave bounds which are less good than equation (3.5). It is, however, easy to see that straightforward modifications of the proof lead to these improved bounds (which we did not need in [CE], because of restrictions on ω having to do with the existence problem for the front).

The perturbations of the stationary solutions will be considered in the *laboratory frame*. Thus, the equation for their evolution takes the form

$$\partial_t \alpha(x, t) = \epsilon\alpha(x, t) - (1 + \partial_x^2)^2 \alpha(x, t) - 3\epsilon S(x)^2 \alpha(x, t). \tag{3.6}$$

This is again a problem with periodic potential, but this time the periodicity is in x, not in t as in equation (1.17). According to the theory of Bloch waves, (see e.g. [RS, Vol IV]), we have to look for eigenfunctions of the r.h.s. operator in equation (3.5) which are of the form

$$\alpha(x) = \beta(x)e^{i\kappa x}, \tag{3.7}$$

with β a $2\pi/\omega$-periodic function and $-\omega/2 \leq \kappa \leq \omega/2$. This leads to the study of the spectrum of the operators A_κ, given by

$$A_\kappa\beta(x) = \epsilon\beta(x) - (1 + (i\kappa + \partial_x)^2)^2\beta(x) - 3\epsilon S_\omega(x)^2\beta(x). \tag{3.8}$$

and acting on $\beta \in L^2([0, 2\pi/\omega], dx)$.

Theorem 3.1. *For every ω satisfying*

$$|\omega^2 - 1| < (\epsilon/3)^{1/2}, \tag{3.9}$$

and for every κ satisfying $-\omega/2 \leq \kappa \leq \omega/2$ the spectrum of A_κ lies in the closed left half-plane. On the other hand, if the inequality (3.9) *is reversed, the spectrum of A_κ intersects the right half-plane for some κ, (and sufficiently small ϵ), i.e., the stationary solution exhibits a so-called Eckhaus instability* [E].

Proof. The proof will be in two parts. We begin by showing the *absence* of spectrum in certain regions of the κ, ω-plane (Theorem 3.2). In particular, we shall see that for small κ there are two isolated eigenvalues close to zero. We shall then use perturbation theory to show that these two eigenvalues are negative.

We want to solve the equation

$$(A_\kappa - \lambda)\beta = u \tag{3.10}$$

when $u \in L_2([0, 2\pi/\omega], dx)$, i.e., we want to show that equation (3.10) has a unique solution $\beta \in L_2([0, 2\pi/\omega], dx)$. We decompose β and u into their Fourier components,

$$\beta(x) = \sum_{n \in \mathbf{Z}} \beta_n e^{i\omega n x}. \tag{3.11}$$

The equation (3.10) is then equivalent to the system of equations, defined for $n \in \mathbf{Z}$,

$$\epsilon\beta_n - (1 + (i\kappa + i\omega n)^2)^2\beta_n - \lambda\beta_n - 3\epsilon \sum_{p+q=0} S_p S_q \beta_n = 3\epsilon = \sum_{\substack{p+q\neq 0\\ p+q+r=n}} S_p S_q \beta_r + u_n. \tag{3.12}$$

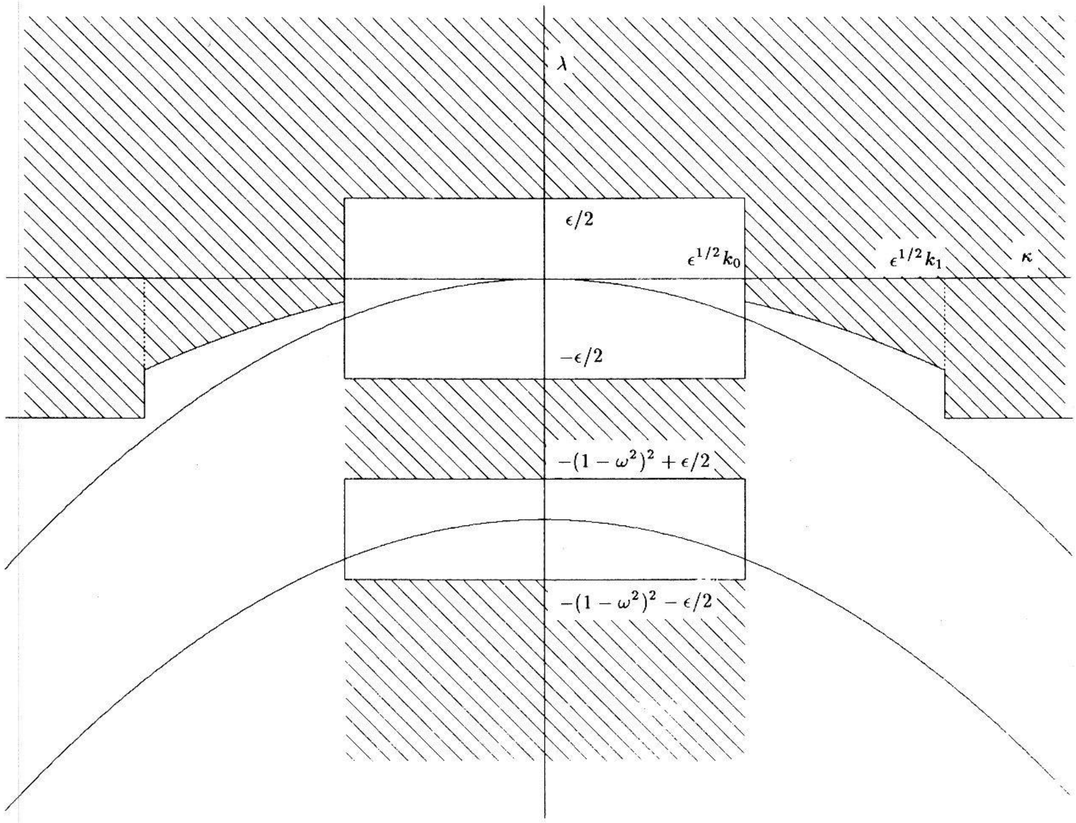
Since $S_p = 0$ for even p, the system equation (3.12) decouples over the even and odd subspaces in n.

It is useful to introduce the notation

$$T_m = 3 \sum_{p+q=m} S_p S_q. \tag{3.13}$$

Note that $T_m = T_{-m}$, and that $T_m = 0$ when m is odd. Furthermore, by equation (3.5),

$$T_0 \approx 3 \cdot 2(\Gamma/2)^2 \approx 2 \tag{3.14}$$



and

$$T_2 = T_{-2} \approx 1. \tag{3.15}$$

Theorem 3.2. *Let $a < 3^{-1/2}$, and assume that $|\omega^2 - 1| \le a\epsilon^{1/2}$. For sufficiently small ϵ (depending on a), the operator $A_\omega - \lambda$ has a bounded inverse on l_2 for λ in the region* **D** *of Fig.* 1.

Proof. Our basic strategy is to invert the operator on the l.h.s. of equation (3.12) and to view the problem equation (3.10) as a contraction problem. Several modifications of this basic strategy will be needed, and in particular, we need to consider with special care the sectors $n = +1, -1$ and $n = 0, \pm 2$.

We define the multiplication operators K_n by

$$K_n = \epsilon - (1 - (\kappa + \omega n)^2)^2 - \lambda - \epsilon T_0, \tag{3.16}$$

and the (convolution) operators $B_{n,j}$, by

$$\beta \to B_{n,j}(\beta) = 3 \sum_{\substack{|p+q| \ge j \\ p+q+r=n}} S_p S_q \beta_r. \tag{3.17}$$

With these notations, equation (3.12) takes the form

$$K_n \beta_n = \epsilon B_{n,1}(\beta) + u_n. \tag{3.18}$$

We begin by giving bounds on $B_{n,j}$. There is a constant L such that for all $j \ge 0$, one has

$$|B_{n,j}(\beta)| \le L \, \|\beta\|_2. \tag{3.19}$$

This follows from

$$
\begin{aligned}
|B_{n,j}(\beta)| &\leq \mathcal{O}(1) \sum_r |\beta_r| \sum_{p+q=n-r} \epsilon^{\max(|p|+|q|-2,0)/5} \\
&\leq \mathcal{O}(1) \sum_r |\beta_r|\, e^{|n-r|/5} \\
&\leq \mathcal{O}(1)\, \|\beta\|_2.
\end{aligned}
\tag{3.20}
$$

Equation (3.5) implies that if $j \geq 3$, then

$$|B_{n,j}(\beta)| \leq L\epsilon\, \|\beta\|_2. \tag{3.21}$$

We begin now the detailed study of equation (3.18).

Case $|n| \geq 3$. In this case, we write equation (3.18) as

$$\beta_n = \epsilon K_n^{-1} B_{n,1}(\beta) + K_n^{-1} u_n, \tag{3.22}$$

and we define the map U_n by

$$U_n(\beta) = \epsilon K_n^{-1} B_{n,1}(\beta) + K_n^{-1} u_n. \tag{3.23}$$

Since $|\kappa| \leq \omega/2$ and $T_0 > 0$, we see that the operator $-K_n$ is bounded below by

$$-K_n \geq Cn^4,$$

for all $\lambda \geq -1$. Hence,

$$|K_n^{-1} u_n| \leq C^{-1} n^{-4}\, \|u\|_2, \tag{3.24}$$

and

$$|\epsilon K_n^{-1} B_{n,1}(\beta)| \leq \mathcal{O}(\epsilon n^{-4})\, \|\beta\|_2, \tag{3.25}$$

so that

$$\left(\sum_{|n|\geq 3} |\epsilon K_n^{-1} B_{n,1}(\beta)|^2 \right)^{1/2} \leq \mathcal{O}(\epsilon)\, \|\beta\|_2. \tag{3.26}$$

Thus, the homogeneous part of U_n is a contraction when $|n| \geq 3$.

Case $|n| = 2$ **or** $n = 0$. For $|n| = 2$, resp. $n = 0$, the operators K_n take the form

$$K_{\pm 2} = \epsilon - (1 - (\kappa \pm 2\omega)^2)^2 - \epsilon T_0 - \lambda, \tag{3.27}$$

$$K_0 = \epsilon - (1 - \kappa^2)^2 - \epsilon T_0 - \lambda. \tag{3.28}$$

Since $|\kappa| \leq \omega/2$, we see that for $n = -2, 0, 2$, one has

$$|K_n^{-1} u_n| \leq \mathcal{O}(1)\, \|u\|_2. \tag{3.29}$$

Case $|n| = 1$. This is really the interesting case, because the linear operator has spectrum very close to zero. Here we study the coupled system of equations, which we first rewrite in the form

$$
\begin{aligned}
K_1 \beta_1 - \epsilon T_2 \beta_{-1} &= \epsilon T_{-2} \beta_3 + \epsilon B_{1,4}(\beta) + u_1, \\
-\epsilon T_{-2} \beta_1 + K_{-1} \beta_{-1} &= \epsilon T_2 \beta_{-3} + \epsilon B_{-1,4}(\beta) + u_{-1}.
\end{aligned}
\tag{3.30}
$$

In matrix notation, this is then written as

$$\mathbf{M}\begin{pmatrix}\beta_1\\ \beta_{-1}\end{pmatrix}=\epsilon\begin{pmatrix}T_{-2}\beta_3\\ T_2\beta_{-3}\end{pmatrix}+\epsilon\begin{pmatrix}B_{1,4}(\beta)\\ B_{-1,4}(\beta)\end{pmatrix}+\begin{pmatrix}u_1\\ u_{-1}\end{pmatrix},\tag{3.31}$$

with

$$\mathbf{M}=\begin{pmatrix}K_1 & \epsilon T_2\\ -\epsilon T_{-2} & K_{-1}\end{pmatrix}.\tag{3.32}$$

Instead of solving equation (3.31), we use the identity equation (3.22), and we formulate the problem as

$$\begin{aligned}\begin{pmatrix}\beta_1\\ \beta_{-1}\end{pmatrix}&=\epsilon\mathbf{M}^{-1}\begin{pmatrix}T_{-2}K_3^{-1}\epsilon B_{3,1}(\beta)\\ T_2K_{-3}^{-1}\epsilon B_{-3,1}(\beta)\end{pmatrix}+\epsilon\mathbf{M}^{-1}\begin{pmatrix}B_{1,4}(\beta)\\ B_{-1,4}(\beta)\end{pmatrix}\\ &\quad+\mathbf{M}^{-1}\begin{pmatrix}u_1+K_3^{-1}u_3\\ u_1+K_{-3}^{-1}u_{-3}\end{pmatrix}\\ &\equiv\epsilon\mathbf{M}_1(\beta)+\mathbf{M}_1'u.\end{aligned}\tag{3.33}$$

In this way, we shall gain a factor ϵ in the operator $\mathbf{M}_1$. We view the r.h.s. of equation (3.33) as a map from l_2 to itself. To control this map, we first study the inverse of $\mathbf{M}$. For sufficiently small κ, the eigenvalues of $\mathbf{M}$ are

$$\begin{aligned}&-\epsilon T_0+\epsilon-(1-\omega^2)^2-\lambda\pm\epsilon T_2+\mathcal{O}(\kappa^2)\\ &=-\epsilon\pm\epsilon-(1-\omega^2)^2-\lambda+\mathcal{O}(\kappa^2)+o(\epsilon)\equiv X.\end{aligned}\tag{3.34}$$

If we consider the '+' sign, then we see that $|X|>\epsilon/4$, provided

$$|\lambda+(1-\omega^2)^2|>\epsilon/2\quad\text{and}\quad|\kappa|\le\epsilon^{1/2}k_0,\tag{3.35}$$

with k_0 sufficiently small. If we consider the '$-$' sign, then we see that $|X|>\epsilon/4$, provided

$$|\lambda|>\epsilon/2\quad\text{and}\quad|\kappa|\le\epsilon^{1/2}k_0,\tag{3.36}$$

with k_0 sufficiently small. Thus, if (3.35) and (3.36) hold, then $\|\mathbf{M}^{-1}\|\le 2/\epsilon$. This means that equation (3.33) is defined in this case. Note now that the first two terms on the r.h.s. of equation (3.33) are maps whose norms are bounded (using (3.19) and (3.21)) by

$$\epsilon\cdot(2/\epsilon)\cdot\mathcal{O}(1)\epsilon L+\epsilon\cdot(2/\epsilon)\cdot\mathcal{O}(1)\epsilon L.\tag{3.37}$$

and this is smaller than 1 if ϵ is sufficiently small. Hence they are contractions and equation (3.33) has a unique solution in this case. We have therefore shown: There is no spectrum in the complement of the regions described by (3.35) and (3.36), with the additional restriction (made above) that $\lambda\ge-1$. (In fact, the next eigenvalue is near to -64.)

We next consider the case $\epsilon^{1/2}k_1>|\kappa|>\epsilon^{1/2}k_0$, where the constant k_1 will be fixed later. It is useful to parametrize ω and κ as follows:

$$\omega^2=1+\eta w,\qquad\kappa=\eta k/2.\tag{3.38}$$

with $\eta = \epsilon^{1/2}$. A straightforward calculation shows that

$$(1 - (\omega \pm \kappa)^2)^2 = \epsilon(w \pm k)^2 + \mathcal{O}(\eta^3). \tag{3.39}$$

Using $3\Gamma^2/4 = 1 - w^2$, we can rewrite the matrix $\mathbf{M} + \lambda\mathbf{1}$ as

$$\epsilon\begin{pmatrix} -1 - (w + k)^2 + 2w^2 & w^2 - 1 \\ w^2 - 1 & -1 - (w - k)^2 + 2w^2 \end{pmatrix} + \mathcal{O}(\eta^3). \tag{3.40}$$

The determinant of $\epsilon^{-1}(\mathbf{M} + \lambda\mathbf{1})$ is

$$k^4 - 2k^2(3w^2 - 1) = k^2(k^2 + 2(1 - 3w^2)), \tag{3.41}$$

and this quantity is positive for all κ if and only if

$$|w| \leq 3^{-1/2}. \tag{3.42}$$

This is the celebrated *Eckhaus instability*. If the inequality in (3.42) is strict, then both eigenvalues of $\epsilon^{-1}(\mathbf{M} + \lambda\mathbf{1})$ have the same sign (in fact, they are negative). From (3.41) we see then that the eigenvalue closer to zero satisfies

$$|\tau| \geq \text{const.}\, k^2(1 - 3w^2)$$

since the other eigenvalue is $\mathcal{O}(1)$ and their product is given by (3.41). In the Theorem 3.2, we *assume* that $|w| \leq a$ and therefore $\epsilon^{-1}(\mathbf{M} + \lambda\mathbf{1})$ has no spectrum for $\tau \geq -\mathcal{O}(1)k^2$, which means that $\mathbf{M}$ is invertible for $\lambda \geq -\mathcal{O}(1)k^2\epsilon$. For λ in this region the inverse of $\mathbf{M}$ exists for $|w| < 3^{-1/2}$, and is bounded by

$$\mathcal{O}(\epsilon^{-1}k_0^{-2}(1 - 3w^2)^{-1}). \tag{3.43}$$

It follows, as before in equation (3.37), that for sufficiently small ϵ *depending on* w, the operator occurring in equation (3.33) is a contraction, if $\lambda \geq -\text{const.}\, k^2\epsilon$. This shows that there is no spectrum in $\lambda \geq -k^2\epsilon$, when $\epsilon^{1/2}k_1 > |\kappa| > \epsilon^{1/2}k_0$.

We leave to the reader the details of the proof that if (3.42) is violated, then one can find an unstable state (for sufficiently small ϵ).

We finally deal with the case $|\kappa| > k_1\epsilon^{1/2}$, where we still may fix k_1 sufficiently large (and then ϵ sufficiently small). We again set $\kappa = k\eta/2$ and $\omega^2 = 1 + \eta w$, and we assume $|w| < 3^{-1/2}$, and without loss of generality, $w > 0$. We define v by $\omega = 1 + \eta v$. Then we have $|v| < 3^{-1/2}$, as well. Consider the function

$$g(x) = (1 - x^2)^2. \tag{3.44}$$

This function is monotone on $[1, \infty]$. We shall fix $k_1 > 3^{-1/2}$. If k_1 is sufficiently large then $\omega + \kappa > 1$ and by the above discussion we find

$$\begin{aligned}(1 - (\omega + \kappa)^2)^2 &\geq (1 - (1 + v\eta - k_1\eta)^2)^2 \\ &\geq \epsilon(k_1 - v)^2 \geq \epsilon(k_1 - 3^{-1/2})^2.\end{aligned} \tag{3.45}$$

We now set

$$k_2 = \tfrac{1}{2}(k_1 - 3^{-1/2})^2 \tag{3.46}$$

and assume $\lambda \geq -k_2\epsilon$. Then the diagonal elements of $\mathbf{M}$ cf. (3.32), satisfy

$$\epsilon - (1 - (\kappa \pm \omega)^2)^2 - \lambda - 3\epsilon T_0 \leq -\epsilon k_2 - \epsilon \tag{3.47}$$

since $\epsilon - 3\epsilon T_0 \approx -\epsilon$, by equation (3.14), $-\lambda \leq \epsilon k_2$, by assumption, and

$$-(1-(\kappa \pm \omega)^2)^2 \leq -2\epsilon k_2,$$

by (3.45) and (3.46). Hence, if k_1 is sufficiently large, the inverse of $\mathbf{M}$ is bounded by $\mathcal{O}(\epsilon^{-1}k_1^{-1})$ and equations (3.20) and (3.47) lead to a convergent iteration scheme, in l_2, for the solution β of

$$(A_\kappa - \lambda)\beta = u.$$

This shows that there is no spectrum in $\lambda \geq -k_2\epsilon$ when $|\kappa| > k_1\epsilon^{1/2}$. This completes the proof of Theorem 3.2.

We now show that in the region $|\kappa| < k_0\epsilon^{1/2}$, there is no spectrum above zero. This will then show the stability of the stationary solution and complete the proof of Theorem 3.1. The method of proof is easy, because by the results of Theorem 3.2, we can apply ordinary perturbation theory (cf. [K]).

By definition, we can expand A_κ in powers of κ, and we have

$$A_\kappa = \sum_{m=0}^{4} A^{(m)}\kappa^m, \tag{3.48}$$

and by analytic perturbation theory in κ we get for the smallest eigenvalue λ_κ an expansion

$$\lambda_\kappa = \sum_{m=0}^{\infty} \lambda^{(m)}\kappa^m. \tag{3.49}$$

If we denote by $e^{(j)}$ the coefficients of the expansion of the eigenvector, then the first few coefficients satisfy the equations

$$A^{(0)}e^{(0)} = \lambda^{(0)}e^{(0)}, \tag{3.50}$$

$$A^{(0)}e^{(1)} + A^{(1)}e^{(0)} = \lambda^{(0)}e^{(1)} + \lambda^{(1)}e^{(0)}, \tag{3.51}$$

$$A^{(0)}e^{(2)} + A^{(1)}e^{(1)} + A^{(2)}e^{(0)} = \lambda^{(0)}e^{(2)} + \lambda^{(1)}e^{(1)} + \lambda^{(2)}e^{(0)}. \tag{3.52}$$

Note now that $\lambda^{(0)} = 0$ and $e^{(0)} = \partial_x S(x)$. We have

$$\begin{aligned}\lambda^{(1)}(e^{(0)}, e^{(1)}) &= (e^{(0)}, A^{(0)}e^{(1)}) + (e^{(0)}, A^{(1)}e^{(0)}) \\ &= 0 + 0.\end{aligned} \tag{3.53}$$

The notation $(\cdot, \cdot)$ stands for the scalar product in L_2. The first zero above is a consequence of the symmetry of $A^{(0)}$, and the second follows by observing that $A^{(1)}$ has an odd number of derivatives and $S_n = S_{-n}$. Therefore, $\lambda^{(1)} = 0$, and we get

$$\lambda^{(2)} = \frac{1}{(e^{(0)}, e^{(0)})}((e^{(0)}, A^{(1)}e^{(1)}) + (e^{(0)}, A^{(2)}e^{(0)})). \tag{3.54}$$

The term $(e^{(0)}, A^{(0)}e^{(1)})$ is absent because $A^{(0)}$ is symmetric. A simple calculation shows

$$(e^{(0)}, A^{(2)}e^{(0)}) = 2(1-3w^2)S_1^2 + o(\epsilon), \tag{3.55}$$

and from

$$A^{(0)}e^{(1)} = -A^{(1)}e^{(0)} \tag{3.56}$$

we deduce

$$e^{(1)} = -(A^{(0)})^{-1}A^{(1)}e^{(0)} + \rho e^{(0)}, \tag{3.57}$$

where the value of ρ is irrelevant for what follows. We have to show that $A^{(0)}$ is invertible, but since it has a potentially degenerate eigenvalue, we only show that the equation

$$A^{(0)}\beta = u \tag{3.58}$$

has a unique solution provided we require $\beta_1 = \beta_{-1}$. Then the problem (3.58) is really a special case of our previous analysis: For $|n| \neq 1$, we view (3.58) as the fixed point problem (3.22),

$$\beta_n = \frac{\epsilon B_{n,1}(\beta) + u_n}{\epsilon - (1 - n^2\omega^2)^2 - \epsilon T_0}, \tag{3.59}$$

and the operator on the r.h.s. of (3.59) is a contraction. Now if $\beta_1 = \beta_{-1}$, then the matrix $\mathbf{M}$ of (3.32) becomes multiplication by

$$X = \epsilon - (1 - \omega^2)^2 - \epsilon T_0 - \epsilon T_2) \tag{3.60}$$

so that we can write the $n = 1$ component of (3.58) as

$$\begin{aligned} \beta_1 &= X^{-1}(\epsilon T_{-2}\beta_3 + \epsilon B_{1,4}(\beta) + u_1) \\ &= X^{-1}(\epsilon^2 T_{-2}K_3^{-1}B_{3,1}(\beta) + \epsilon B_{1,4}(\beta)) + X^{-1}(u_1 + K_3^{-1}u_3). \end{aligned} \tag{3.61}$$

Again, the operator on the r.h.s. is a contraction and thus the proof of invertibility of $A^{(0)}$ is complete.

We now compute the second derivative of λ_κ, showing that it has a negative sign. We have

$$(e^{(0)}, A^{(1)}e^{(1)}) = -(e^{(0)}, A^{(1)}(A^{(0)})^{-1}A^{(1)}e^{(0)}). \tag{3.62}$$

We use (3.58) with $u = A^{(1)}e^{(0)}$. We find, for this u,

$$u_1 = -4\eta w S_1 + o(\eta), \tag{3.63}$$

and for $|n| > 1$,

$$u_n = \mathcal{O}(n^4 S_n), \tag{3.64}$$

so that

$$\beta_1 \approx \frac{u_1\epsilon^{-1}}{1 - \omega^2 - 6S_1^2} \approx \frac{4\eta w\epsilon^{-1}S_1}{1 + w^2},$$

and finally,

$$\begin{aligned} (e^{(0)}, A^{(1)}e^{(1)}) &\approx -4\eta S_1\left(\frac{4\eta w\epsilon^{-1}S_1}{1 + w^2}\right) \cdot 2 \\ &\approx \frac{16S_1^2 w^2}{1 - w^2}. \end{aligned}$$

We now easily get the final identity:

$$\lambda^{(2)} \approx \frac{|S_1^2|}{(e^{(0)}, e^{(0)})}\left(-4 + \frac{16w^2}{1+\omega^2}\right). \tag{3.65}$$

This quantity is negative if $|w| < 3^{-1/2}$, and ϵ is sufficiently small. We see again the appearance of the Eckhaus instability.

We can now complete the proof of Theorem 3.1. Near $\kappa = 0$, we see from Theorem 3.2 that perturbation theory holds. Then (3.65) implies that for sufficiently small $|\kappa|$, the spectrum lies below zero. This dictates a (possibly smaller) choice of k_0 in the proof of Theorem 3.2. Now Theorem 3.2 applies for sufficiently small ϵ (depending on this k_0 and the value of ω). Since we have already shown that there is no spectrum in the region **D** of Fig. 1, the assertion of the theorem follows.

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Note added in proof

An interesting early paper is
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