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Helvetica Physica Acta, Vol. 60 (1987) 501–509
0018–0238/87/040501–09$1.50 + 0.20/0

Time-delay operator for a class of singular potentials

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(1. IX. 1986)

Abstract. We prove the existence of the time-delay operator defined by taking the large space limit of the approximate sojourn times for a class of singular potentials: $V = V_1 + V_2$, where V_1 is a smooth short range potential and V_2 and $x \cdot \nabla V_2$ are both bounded from H^2 to $L^{2,2+\varepsilon_0}$ for some $\varepsilon_0 > 0$.

1. Introduction

In [8], we proved the finiteness of time-delay defined by taking the space limit of sojourn times and established its equivalence with Eisenbud–Wigner's time-delay in scattering theory for smooth short range potentials. In this work we will show that our method developed there can be also applied to a class of singular potentials.

Let $H_0 = -\Delta$ and $H = H_0 + V$ in $L^2(\mathbb{R}^n)$. We suppose that the short range potential V can be decomposed as: $V(x) = V_1(x) + V_2(x)$ where V_1 is C^∞ on $\mathbb{R}^n$ and for some $\varepsilon_0 > 0$

$$|\partial_x^\alpha V_1(x)| \leqslant c_\alpha \langle x \rangle^{-1-\varepsilon_0-|\alpha|} \tag{1.1}$$

and the multiplication by V_2 is bounded as operator from H^2 to $L^{2,2+\varepsilon_0}$ and so is the distributional derivative $x\nabla_x \cdot V_2$. Here $\langle x \rangle = (1 + |x|)$ and H^s is the usual Sobolev space of order s; $L^{2,s}$ is the weighted L^2 space with the norm: $\|f\|_s = \|\langle x \rangle^s f\|$. This assumption will be made throughout this work. Under this condition on V, it is well known that the wave operators $W_\pm$ defined by:

$$W_\pm = s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0} \quad \text{in } L^2$$

exist and are complete. Let $\tilde{P}_R$ denote the multiplication by the characteristic function for the ball $\{|x| < R\}$. Then the local time-delay of f in $\{|x| < R\}$ is defined as the difference of the sojourn times:

$$\langle f, T_R f \rangle = \int_{-\infty}^{\infty} (\|\tilde{P}_R e^{-itH} W_- f\|^2 - \|\tilde{P}_R e^{-itH_0} f\|^2)\, dt \tag{1.2}$$

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Notice that (1.2) is well defined for $f \in L^2$ such that the Fourier transform $\hat{f}$ has suitable compact support in $\mathbb{R}^n|_{\{0\}}$. Finally the time-delay operator T is defined by:

$$\langle f, Tf \rangle = \lim_{R \to +\infty} \langle f, T_R f \rangle \tag{1.3}$$

whenever the limit exists. Surely the existence of time-delay operator depends on how much such f's we can find. As in [8], we consider here a similar question. Let P_R be the multiplication by $P(x/R)$, where $P(\,.\,)$ is a smooth, spherically symmetrical function such that $P(x) = 1$ for $|x| \leqslant 1$ and $P(x) = 0$ for $|x| \geqslant 2$. It is clear that P_R can be regarded as an approximation of $\tilde{P}_R$. In the following we denote still T_R the operator defined by (1.2) with $\tilde{P}_R$ replaced by P_R.

Then for smooth short range potentials we proved in [8] that the limit (1.3) exists for a dense subset in L^2 and in the spectral representation of H_0, the time-delay operator T is given by a family of operators $T(\lambda)$, $\lambda > 0$, where

$$T(\lambda) = -iS(\lambda)^* \frac{d}{d\lambda} S(\lambda) \tag{1.4}$$

$S(\lambda)$ being the scattering matrix. (1.4) is the Eisenbud–Wigner's formula for time-delay. It reveals that the method and techniques used in [8] are powerful enough. It can be applied to treat time-delay in other scattering theories (see [6]) and to include a class of singular potentials.

Let $A = -i(x \cdot \nabla_x + \nabla_x x)/2$ be the generator of dilation group. We define the set $\mathscr{D}$ by:

$$\mathscr{D} = \{f \in L^2; f \in D(\langle x \rangle) \cap D(A^2) \text{ and } \exists \chi \in C_0^\infty(\mathbb{R}_+ \backslash \sigma_p(H)),\ \chi(H_0)f = f\} \tag{1.5}$$

In this work we want to prove the following result.

Theorem 1. *Under the above assumption on V, the limit* (1.3) *exists for T_R defined by* (1.2) *with $\tilde{P}_R$ replaced by P_R and for $f \in \mathscr{D}$. We have:*

$$\langle f, Tf \rangle = \langle f, -S^*[A, S]f_1 \rangle$$

where f_1 is determined by $2H_0 f_1 = f$ and $S = W_+^* W_-$ is scattering operator. The time-delay operator T is essentially selfadjoint with core $\mathscr{D}$ and the Eisenbud–Wigner formula (1.4) *is true for $\lambda \in \mathbb{R}_+ / \sigma_p(H)$.*

The proof of this result consists in regarding V_2 as a perturbation of the Hamiltonian $H_1 = H_0 + V_1$. In §2, we give some technical preparations, which were mostly proved in [8]. In §3, we achieve the main step of the proof, reducing the existence of the limit (1.3) to that of $\lim_{R \to +\infty} \int_0^\infty \langle U_0(t)f, S^* P_R S - P_R)f \rangle \, dt$. We finish the proof of Theorem 1 in §4 by the method of [8]. Very recently Nakamura ([12]) considered the similar problem by a different method. His proof is in the spirit of Lavine [4], while ours is in that of Martin [5].

2. Some preparations

Let $U(t)$ (resp. $U_0(t)$, $U_1(t)$) denote the unitary group associated to H (resp. H_0, H_1). Let $E_{ac}(H_1)$ denote the spectral projector onto the absolute continuous space of H_1. The wave operators $W^1_\pm$, $W^2_\pm$ are defined by:

$$W^1_\pm = s-\lim_{t\to\pm\infty} U_1(t)^* U_0(t)$$

$$W^2_\pm = s-\lim_{t\to\pm\infty} U(t)^* U_1(t) E_{ac}(H_1)$$

in $L^2(\mathbb{R}^n)$. By chain rule, $W_\pm = W^2_\pm W^1_\pm$ (see [1]). Put: $\mathbb{R}_+ =]0, +\infty[$.

Lemma 2.1. *Let $f \in C_0^\infty(\mathbb{R}_+/\sigma_p(H))$. Then for every $0 \leq \mu \leq 1$, one has:*

$$\|\langle A\rangle^{-\mu} f(H) U(t) W_\pm \langle A\rangle^{-\mu}\| \leq C(1+|t|)^{-\mu} \tag{2.1}$$

for $t \in \mathbb{R}$. For every $\mu > 1$, there exists $\rho > 1$ such that

$$\|\langle A\rangle^{-\mu} f(H) U(t) W_\pm \langle A\rangle^{-\mu}\| \leq c(1+|t|)^{-\rho} \quad \textit{for } t \in \mathbb{R} \tag{2.2}$$

Note that this result is proved in [8] for $V = V_1 (V_2 = 0)$. But the proof can be carried over, because we used only the short range properties of V and $x \cdot \nabla V$.

Recall that if $V = V_1 + V_2$ with V_1 satisfying (1.1) and V_2 bounded from H^2 to $L^{2,2+\varepsilon_0}$ it is proved in [3] that $S(\lambda)$ is continuously differentiable in $\mathscr{L}(L^2(S^{n-1}))$ for $\lambda \in \mathbb{R}_+/\sigma_p(H)$. Since under the assumptions of Theorem 1, $x \cdot \nabla V$ satisfies still the above conditions, it should be clear that by exterior scaling method, we can easily prove that $S(\lambda)$ is two times differentiable for $\lambda \in \mathbb{R}_+/\sigma_p(H)$. For reader's convenience we give the details of the proof.

Let $\lambda \in \mathbb{R}_+/\sigma_p(H)$. Then we have the following representation for the scattering matrix $S(\lambda)$ $(= S(\lambda, V))$ ([11]):

$$S(\lambda, V) = 1 - i\pi \mathscr{F}(\lambda)(V - VR(\lambda + i0, V)V)\mathscr{F}(\lambda)^*$$

where $R(\lambda \pm i0, V)$ is the boundary values of the resolvent $(H_0 + V - z)^{-1}$ and $\mathscr{F}(\,.\,): L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}_+, L^2(S^{n-1}))$ is a spectral representation for the free Hamiltonian H_0. Take $a > 0$ to be sufficiently small. Put: $I =]-a, a[$. We can prove that:

$$S(e^{2\theta}\lambda, V) = S(\lambda, V(\theta)) \quad \text{for } \theta \in I \tag{S}$$

where $V(\theta) = e^{-2\theta} U(\theta)^* V U(\theta)$ and $U(\theta)$ is the unitary group generated by A. Now we check the derivability of $V(\theta)$ and $R(\lambda + i0, V(\theta))$ for $\theta \in I$. Let $H^{s,m}$ denote the weighted Sobolev space with the norm $\|\langle x\rangle^m (1-\Delta)^{s/2} f\|$. Put: $\rho = 1 + \varepsilon_0 > 1$. Then the assumptions on V say that $i[A, V_2]$ defines a bounded operator from $H^{s,r}$ to $H^{s-2,r+\rho+1}$ for $0 \leq s \leq 2$ and $r \in \mathbb{R}$. From this we derive that $A[A, V_2]$ and $[A, V_2]A$ are both bounded from $H^{s,r}$ to $H^{s-4,r+\rho}$ for $0 \leq s \leq 3$ and $r \in \mathbb{R}$. Since V_1 satisfies (1.1), we conclude easily from the above remarks that the operator valued function $\theta \mapsto V(\theta)$ is in the class

$$C^1(I; \mathscr{L}(H^{s,r}; H^{s-2,r+\rho+1})) \cap C^2(I; \mathscr{L}(H^{s+1,r}; H^{s-3,r+\rho}))$$

Since $R(\lambda \pm i0, V(\theta))$ is in $\mathscr{L}(H^{0,r}; H^{2,-r})$ if $r > \frac{1}{2}$ ([10]), we can also prove that the map $\theta \mapsto R(\lambda + i0, V(\theta))$ belongs to the class:

$$C^1(I; \mathscr{L}(H^{0,r}; H^{2,-r})) \cap C^2(I; \mathscr{L}(H^{0,r}; H^{0,-r}))$$

for $r > \frac{1}{2}$. This means that the map

$$I \ni \theta \mapsto \langle H_0 \rangle^{-2} V(\theta)(1 - R(\lambda + i0, V(\theta))V(\theta))\langle H_0 \rangle^{-2}$$

is in $C^2(I; \mathscr{L}(L^{2,-s}; L^{2,s}))$ for $s \in]\frac{1}{2}, \rho/2[$. Making use of the relation: $\mathscr{F}(\lambda)\langle H_0 \rangle^{-1} = \langle \lambda \rangle^{-1} \mathscr{F}(\lambda)$, we derive from (S) that $S(\lambda, V)$ is twice continuously differentiable in $\mathscr{L}(L^2(S^{n-1}))$ for $\lambda \in \mathbb{R}_+ / \sigma_p(H)$. This proves our assertion.

Since in the spectral representation of H_0, A is given by a family of operators $A(\lambda) = i(\lambda\, d/d\lambda + d/d\lambda \cdot \lambda)$, we conclude that the domain of A^2 is invariant by $Sf(H_0)$ for $f \in C_0^\infty(\mathbb{R}_+/\sigma_p(H))$. Now we can easily prove the following lemma which is important in this work.

Lemma 2.2. *Let $\mathscr{D}$ be defined by* (1.5). *Then $\mathscr{D}$ is invariant by S. In particular if f belongs to $\mathscr{D}$, $\langle x \rangle Sf$ and A^2Sf are both in $L^2(\mathbb{R}^n)$.*

Proof. It remains to show that $\langle x \rangle Sf$ is in L^2. We can use the same commutator method as in the proof of Prop. 4.2 in [8]. The details are omitted here.

In order to regard V_2 as a perturbation to H_1, we need some continuity of wave operators $W_\pm^1$.

Lemma 2.3. *Let $f \in C_0^\infty(\mathbb{R}_+)$. Under the condition* (1.1) *on V_1, the four operators $A^2 W_\pm^1 f(H_0)\langle A \rangle^{-2}$ and $A^2 W_\pm^{1*} f(H_1)\langle A \rangle^{-2}$ are all bounded on L^2.*

Proof. We prove only the result for $A^2 W_+^{1*} f(H_1)\langle A \rangle^{-2}$. The other cases can be treated in the same way. Put $W(t) = U_0(t)^* U_1(t)$. We can write, as forms on $D(A) \times D(A)$,

$$\begin{aligned} AW(t)f(H_1) &= W(t)f(H_1)A + U_0(t)^*[A, f(H_1)]U_1(t) \\ &\quad + 2tU_0(t)V_1U_1(t)f(H_1) + W(t)\int_0^t U_1(s)^* f(H_1)\tilde{V}U_1(s)\, ds \end{aligned} \tag{2.3}$$

where $\tilde{V} = i[A, V_1] - 2V_1$. Notice that $i[A, f(H_1)] = 2f'(H_1) + Q$ with Q bounded from L^2 to $L^{2,1+\varepsilon_0}$ (see [8]). Now we need the following result due to Jensen et al.:

$$\|\langle A \rangle^{-r} f(H_1)U_1(t)\langle A \rangle^{-r}\| \le c(1 + |t|)^{-r+\varepsilon} \quad t \in \mathbb{R} \tag{2.4}$$

for every $r > 0$ and $0 < \varepsilon \ll r$. Take $g \in C_0^\infty(\mathbb{R}_+)$ such that $g = 1$ on $\operatorname{supp} f$. Multiplying (2.3) by $g(H_1)\langle A \rangle^{-2}$ and taking the limit $t \to +\infty$, applying (2.4), we get:

$$AW_+^{1*}f(H_1)\langle A\rangle^{-2} = W_+^{1*}f(H_1)Ag(H_1)\langle A\rangle^{-2} + W^{1*}h(H_1)\langle A\rangle^{-2} + W_+^{1*}\int_0^{+\infty} f(H_1)U(-s)\tilde{V}U(s)g(H_1)\langle A\rangle^{-2}\,dt \tag{2.5}$$

where $h = 2if'g$. Since $AW_+^{1*}f(H_1)\langle A\rangle^{-1}$ is bounded on L^2, in order to prove the desired result by (2.5), it is sufficient to show that

$$\int_0^{+\infty} Af(H_1)U_1(-s)\tilde{V}U_1(s)g(H_1)\langle A\rangle^{-2}\,ds \tag{2.6}$$

is bounded on L^2. To simplify notations, we denote f, g the operators $f(H_1)$, $g(H_1)$ respectively. We have the following relation:

$$\begin{aligned}[A, U_1(-s)g\tilde{V}gU_1(s)] &= -2sU_1(-s)g[H_0, \tilde{V}]gU_1(s)\\ &\quad + U_1(-s)[A, g\tilde{V}g]U_1(s)\\ &\quad - \int_0^s U_1(t-s)\tilde{V}gU_1(-t)\tilde{V}U_1(s)g\,dt\\ &\quad + \int_0^s U_1(-s)g\tilde{V}U_1(s-t)g\tilde{V}U_1(t)\,dt \end{aligned} \tag{2.7}$$

Since $g[H_0, \tilde{V}]$ is continuous from $L^{2,r}$ to $L^{2,r+2+\varepsilon_0}$, we can prove as in [8] (Prop. 4.2) that $\int_0^{+\infty} 2sfU_1(-s)[H_0, \tilde{V}]U_1(s)g\langle A\rangle^{-1}\,ds$ is bounded on $L^2(\mathbb{R}^n)$. Since $[A, g\tilde{V}g]$ is bounded as operator from $L^{2,r}$ to $L^{2,r+1+\varepsilon_0}$, it follows from (2.4) that:

$$\|[A, g\tilde{V}g]U_1(s)g_1\langle A\rangle^{-2}\| \le C(1+|s|)^{-1-\varepsilon_0/2}$$

for $s \in \mathbb{R}$. Here $g_1 = g_1(H_1)$ is chosen so that $g_1g = g$. Therefore the integral $\int_0^\infty fU_1(-s)[A, g\tilde{V}g]U_1(s)g_1\langle A\rangle^{-2}\,ds$ defines a bounded operator on L^2. To treat the last two terms in (2.7), we use the local H_1-smoothness of $\langle x\rangle^{-1/2-\varepsilon}$, which implies that the operator $\int_0^s fU_1(t)\tilde{V}U_1(-t)g\,dt$ is uniformly bounded with respect to $s \in \mathbb{R}$. Applying (2.4), we get the estimate over the third term in (2.7):

$$\left\|\int_0^s fU_1(t-s)\tilde{V}gU_1(-t)\tilde{V}U_1(s)g_1\langle A\rangle^{-2}\,dt\right\| \le C(1+|s|)^{-1-\varepsilon_0/2}$$

The last term in (2.7) can be estimated in the same way. Since the commutator $[A, f]$ is bounded, we derive from (2.7) that (2.6) is a bounded operator on L^2. This proves that $A^2W_+^{1*}f(H_1)\langle A\rangle^{-2}$ is bounded. The lemma is proved.

3. Reduction of the problem

In the proof of the finiteness of time-delay, an important step is to show that

$$\lim_{R\to+\infty}\left(\langle f, T_Rf\rangle - \int_0^{+\infty}\langle U_0(t)f, (S^*P_RS - P_R)U_0(t)f\rangle\,dt\right) = 0 \tag{3.1}$$

(3.1) makes also clear the close relationship between time-delay operator T and scattering operator S. In this section we will prove the following result which implies (3.1).

Theorem 3.1. *Let $f \in \mathcal{D}$. Put $g = Sf$. Then we have:*

$$\lim_{R\to+\infty}\int_{-\infty}^{0} (\langle U_0(t)f, (W_-^* P_R W_- - P_R)U_0(t)f\rangle)\, dt = 0 \tag{3.2}$$

$$\lim_{R\to+\infty}\int_{0}^{\infty} (\langle U_0(t)g, (W_+^* P_R W_+ - P_R)U_0(t)g\rangle)\, dt = 0 \tag{3.3}$$

Proof. Since the set $\mathcal{D}$ is invariant by S, it suffices to prove (3.2). Put $h = W_-^1 f$, which is in $D(A^2)$ by Lemma 2.3. The integrand in (3.2) can be written as: $\langle U_1(t)h, (W_-^{2*}P_R W_-^2 - P_R)U_1(t)h\rangle + \langle U_0(t)f, (W_-^{1*}P_R W_-^1 - P_R)U_0(t)f\rangle$. It is proved in [8] by method of pseudo-differential operators that for $f \in \mathcal{D}$, we have:

$$\lim_{R\to+\infty}\int_{-\infty}^{0} \langle U_0(t)f, (W_-^{1*}P_R W_-^1 - P_R)U_0(t)f\rangle\, dt = 0$$

Therefore we have to prove

$$\lim_{R\to+\infty}\int_{-\infty}^{0} \langle U_1(t)h, (W_-^{2*}P_R W_-^2 - P_R)U_1(t)h\rangle\, dt = 0 \tag{3.4}$$

The integrand in (3.4) can be written as:

$$\langle P_R U(t)W_- f, (U(t)W_-^2 - U_1(t))h\rangle + \langle (U(t)W_-^2 - U_1(t))h, P_R U_1(t)h\rangle \tag{3.5}$$

Take $\chi \in C_0^\infty(\mathbb{R}_+/\sigma_p(H))$ such that $\chi(H_0)f = f$. We get:

$$(U(t)W_-^2 - U_1(t))h = -\int_{-\infty}^{t} \chi(H)U(t-s)V_2 U_1(s)\chi(H_1)h\, ds + (\chi(H) - \chi(H_1))U_1(t)h$$

By the assumption, $V_2\chi(H_1)$ is continuous from $L^{2,-\varepsilon_0}$ to $L^{2,2}$. We can easily prove that $\chi(H) - \chi(H_1)$ is continuous from $L^{2,-1-\varepsilon_0}$ to L^2. Thus the first term in (3.5) can be estimated as:

$$\begin{aligned}&|\langle P_R U(t)W_- f, (U(t)W_-^2 - U_1(t))h\rangle|\\ &\quad \leqslant c\int_{-\infty}^{t} (1+|s|)^{-2+\varepsilon}\, \|\langle x\rangle^{-\varepsilon_0}\chi(H)U(s-t)P_R U(t)\chi(H)W_- f\|\, \|\langle A\rangle^2 f\|\, ds\\ &\quad + c(1+|t|)^{-1-\varepsilon_0}\, \|\langle A\rangle^2 f\|^2, \quad \text{for any} \quad \varepsilon > 0\end{aligned} \tag{3.6}$$

Here we have used (2.4) and Lemma 2.3. Before going on with the proof of Theorem 3.1, we need still a lemma.

Lemma 3.2. *For every $0 \leq \mu \leq 1$, we have:*

$$\|\langle x\rangle^{-\mu}\chi(H)U(s-t)P_R U(t)W_- f\| \leq C(1+|s|)^{-\mu}\,\|\langle A\rangle W_- f\| \tag{3.7}$$

uniformly in $t \in \mathbb{R}$ and $R \geq 1$.

Proof. Observe first that $|\partial_x^\alpha P_R(x)| \leq c\langle x\rangle^{-|\alpha|}$ uniformly in $R \geq 1$. By the arguments used in the proof of Lemma 2.3, we can show that:

$$\|A\chi(H)U(-t)P_R U(t)\chi(H)\langle A\rangle^{-1}\| \leq C$$

uniformly in $t \in \mathbb{R}$ and $R \geq 1$. (3.7) follows from (2.1) and the fact that $\langle A\rangle\chi(H)\langle x\rangle^{-1}$ is bounded on L^2.

Now return to the proof of Theorem 3.1. By (3.6) and (3.7) we obtain, for $\varepsilon > 0$ sufficiently small and for $t < 0$,

$$|\langle P_R U(t)W_- f, (U(t)W_-^2 - U_1(t))h\rangle| \leq c(1+|t|)^{-1-\varepsilon_0/2}\,\|\langle A\rangle^2 f\|^2$$

uniformly in $R \geq 1$. The same estimate is also true for the second term in (3.5). Since the integrand in (3.4) tends to 0 as R tends to $+\infty$, by the dominated convergence theorem, (3.4) is proved. This finishes the proof of Theorem 3.1.

We remark that the various constants c appeared in the proof of Theorem 3.1 depend on the function χ, but not $f \in \mathcal{D}$ such that $\chi(H_0)f = f$.

4. Existence of time-delay operator

In this section we prove Theorem 1. Let $\chi \in C_0^\infty(\mathbb{R}_+/\sigma_p(H))$. We put:

$$\mathcal{D}_\chi = \{f \in \mathcal{D};\ \chi(H_0)f = f\}$$

Lemma 4.1. *Let $f \in \mathcal{D}_\chi$. Then,*

$$\int_{R^{5/2}}^{+\infty} |\langle U_0(t)f, (S^*P_R S - P_R)U_0(t)f\rangle|\,dt \leq CR^{-1/2}\,\|\langle A\rangle f\|^2$$

Proof. It follows easily from the estimate:

$$\|P_R^{1/2}U_0(t)f\| \leq C_\chi R(1+|t|)^{-1}\,\|\langle A\rangle f\|$$

for $t \in \mathbb{R}$, $R \geq 1$ and $f \in \mathcal{D}_\chi$.

Lemma 4.2. *For $f \in \mathcal{D}_\chi$, we have the asymptotic expansion:*

$$\int_0^{R^{5/2}} \langle U_0(t)f, P_R U_0(t)f\rangle\,dt = R\langle f, a(D)f\rangle - \langle f, b^w(x, D;\chi)f\rangle + O(R^{-1/2}) \tag{4.1}$$

where $a(\xi) = c_0|\xi|^{-1}\chi(|\xi|^2)$ with $c_0 = \frac{1}{2}\int_0^\infty P(s)\,ds$, $b^w(x, D;\chi)$ is a Weyl pseudo-differential operator with symbol $\frac{1}{2}x\xi\,|\xi|^{-2}\chi(|\xi|^2)$. The remainder $O(R^{-1/2})$ can be *estimated by*

$$|O(R^{-1/2})| \leq C_\chi R^{-1/2}(\|\langle x\rangle f\| + \|\langle A\rangle^2 f\|)^2 \quad R \geq 1$$

Proof. We use the fact that $U_0(-t)P_R U_0(t)$ is a Weyl pseudo-differential operator with symbol $P((x+2t\xi)/R)$ and develop the symbol around $2t\xi/R$. Since this result is proved in [8] (Proposition 4.3) for $f \in D(\langle x\rangle) \cap D(\langle A\rangle^3)$, we indicate only the difference and omit the details. Checking the proof of Proposition 4.3([8]), we see that the condition $f \in D(\langle A\rangle^3)$ is only used to get the estimate (see (4.14)[8]):

$$\frac{t}{R^3}\|Q^w(\tau x/R, D)U_0(t/\tau)f\| \leq CR^{-1/2}t^{-3/2}\|\langle A\rangle^3 f\|$$

for $t \geq 1$, $\tau \in]0, 1]$ and $R \geq 1$, where $Q(x, \xi) = P_1(x)\chi(|\xi|^2)$, $P_1(x)$ is some derivative of $P(x)$. Hence it is supported in $\{1 \leq |x| \leq 2\}$. But this term can be equally estimated as follows: Since all the derivatives of $\tau^2R^{-2}Q(\tau x/R, \xi)$ are bounded by a constant times $\langle x\rangle^{-2}$ uniformly with respect to $R \geq 1$ and $\tau \in]0, 1]$, we have, by continuity result of pseudo-differential operators ([2]),

$$\frac{t}{R^3}\|Q^w(\tau x/R, D)U_0(t/\tau)f\| \leq cR^{-1}\tau^{-2}t\,\|\langle x\rangle^{-2}U_0(t/\tau)f\| \leq c_\chi R^{-1}t^{-1}\|\langle A\rangle^2 f\| \tag{4.2}$$

for $t \geq 1$, $R \geq 1$ and $\tau \in]0, 1]$. Integrating (4.2) over $[1, R^{5/2}]$, we get:

$$\int_0^{R^{5/2}} tR^{-3}\|Q^w(\tau x/R, D)U_0(t/\tau)f\|\,dt \leq C_\chi R^{-1/2}\|\langle A\rangle^2 f\|$$

This gives the desired result. The lemma is proved.

Now we are able to give the proof of Theorem 1.

Proof of Theorem 1. Let f be in $\mathscr{D}$. Then Sf is also in $\mathscr{D}$. Take $\chi \in C_0^\infty(\mathbb{R}_+/\sigma_p(H))$ such that $\chi(H_0)f = f$. Since S commutes with $a(D)$, we derive from Lemmas 4.1 and 4.2 that

$$\left|\int_0^\infty \langle U_0(t)f, S^*[P_R, S]U_0(t)f\rangle\,dt + \langle f, S^*[b^w(x, D; \chi), S]f\rangle\right| \leq C_f R^{-1/2} \tag{4.3}$$

By a simple calculus of Weyl pseudo-differential operators ([2]), we get:

$$\langle f, S^*[b^w(x, D; \chi), S]f\rangle = \langle f_1, S^*[A, S]f\rangle \tag{4.4}$$

where $f_1 = (2H_0)^{-1}\chi(H_0)f$. Theorem 1 is a consequence of (4.3) and (4.4). See also [8].

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