

Zeitschrift: Helvetica Physica Acta
Band: 60 (1987)
Heft: 4

Artikel: Time-delay operator for a class of singular potentials
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DOI: <https://doi.org/10.5169/seals-115860>

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Time-delay operator for a class of singular potentials

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(1. IX. 1986)

Abstract. We prove the existence of the time-delay operator defined by taking the large space limit of the approximate sojourn times for a class of singular potentials: $V = V_1 + V_2$, where V_1 is a smooth short range potential and V_2 and $x \cdot \nabla V_2$ are both bounded from H^2 to $L^{2,2+\varepsilon_0}$ for some $\varepsilon_0 > 0$.

1. Introduction

In [8], we proved the finiteness of time-delay defined by taking the space limit of sojourn times and established its equivalence with Eisenbud–Wigner’s time-delay in scattering theory for smooth short range potentials. In this work we will show that our method developed there can be also applied to a class of singular potentials.

Let $H_0 = -\Delta$ and $H = H_0 + V$ in $L^2(\mathbb{R}^n)$. We suppose that the short range potential V can be decomposed as: $V(x) = V_1(x) + V_2(x)$ where V_1 is C^∞ on \mathbb{R}^n and for some $\varepsilon_0 > 0$

$$|\partial_x^\alpha V_1(x)| \leq c_\alpha \langle x \rangle^{-1-\varepsilon_0-|\alpha|} \quad (1.1)$$

and the multiplication by V_2 is bounded as operator from H^2 to $L^{2,2+\varepsilon_0}$ and so is the distributional derivative $x \nabla_x \cdot V_2$. Here $\langle x \rangle = (1 + |x|^2)^{1/2}$ and H^s is the usual Sobolev space of order s ; $L^{2,s}$ is the weighted L^2 space with the norm: $\|f\|_s = \|\langle x \rangle^s f\|$. This assumption will be made throughout this work. Under this condition on V , it is well known that the wave operators W_\pm defined by:

$$W_\pm = s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0} \quad \text{in } L^2$$

exist and are complete. Let \tilde{P}_R denote the multiplication by the characteristic function for the ball $\{|x| < R\}$. Then the local time-delay of f in $\{|x| < R\}$ is defined as the difference of the sojourn times:

$$\langle f, T_R f \rangle = \int_{-\infty}^{\infty} (\|\tilde{P}_R e^{-itH} W_- f\|^2 - \|\tilde{P}_R e^{-itH_0} f\|^2) dt \quad (1.2)$$

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Notice that (1.2) is well defined for $f \in L^2$ such that the Fourier transform \hat{f} has suitable compact support in $\mathbb{R}^n \setminus \{0\}$. Finally the time-delay operator T is defined by:

$$\langle f, Tf \rangle = \lim_{R \rightarrow +\infty} \langle f, T_R f \rangle \quad (1.3)$$

whenever the limit exists. Surely the existence of time-delay operator depends on how much such f 's we can find. As in [8], we consider here a similar question. Let P_R be the multiplication by $P(x/R)$, where $P(\cdot)$ is a smooth, spherically symmetrical function such that $P(x) = 1$ for $|x| \leq 1$ and $P(x) = 0$ for $|x| \geq 2$. It is clear that P_R can be regarded as an approximation of \tilde{P}_R . In the following we denote still T_R the operator defined by (1.2) with \tilde{P}_R replaced by P_R .

Then for smooth short range potentials we proved in [8] that the limit (1.3) exists for a dense subset in L^2 and in the spectral representation of H_0 , the time-delay operator T is given by a family of operators $T(\lambda)$, $\lambda > 0$, where

$$T(\lambda) = -iS(\lambda)^* \frac{d}{d\lambda} S(\lambda) \quad (1.4)$$

$S(\lambda)$ being the scattering matrix. (1.4) is the Eisenbud–Wigner's formula for time-delay. It reveals that the method and techniques used in [8] are powerful enough. It can be applied to treat time-delay in other scattering theories (see [6]) and to include a class of singular potentials.

Let $A = -i(x \cdot \nabla_x + \nabla_x x)/2$ be the generator of dilation group. We define the set \mathcal{D} by:

$$\mathcal{D} = \{f \in L^2; f \in D(\langle x \rangle) \cap D(A^2) \text{ and } \exists \chi \in C_0^\infty(\mathbb{R}_+ \setminus \sigma_p(H)), \chi(H_0)f = f\} \quad (1.5)$$

In this work we want to prove the following result.

Theorem 1. *Under the above assumption on V , the limit (1.3) exists for T_R defined by (1.2) with \tilde{P}_R replaced by P_R and for $f \in \mathcal{D}$. We have:*

$$\langle f, Tf \rangle = \langle f, -S^*[A, S]f_1 \rangle$$

where f_1 is determined by $2H_0f_1 = f$ and $S = W_+^*W_-$ is scattering operator. The time-delay operator T is essentially selfadjoint with core \mathcal{D} and the Eisenbud–Wigner formula (1.4) is true for $\lambda \in \mathbb{R}_+ \setminus \sigma_p(H)$.

The proof of this result consists in regarding V_2 as a perturbation of the Hamiltonian $H_1 = H_0 + V_1$. In §2, we give some technical preparations, which were mostly proved in [8]. In §3, we achieve the main step of the proof, reducing the existence of the limit (1.3) to that of $\lim_{R \rightarrow +\infty} \int_0^\infty \langle U_0(t)f, S^*P_R S - P_R \rangle f \rangle dt$. We finish the proof of Theorem 1 in §4 by the method of [8]. Very recently Nakamura ([12]) considered the similar problem by a different method. His proof is in the spirit of Lavine [4], while ours is in that of Martin [5].

2. Some preparations

Let $U(t)$ (resp. $U_0(t)$, $U_1(t)$) denote the unitary group associated to H (resp. H_0 , H_1). Let $E_{ac}(H_1)$ denote the spectral projector onto the absolute continuous space of H_1 . The wave operators W_{\pm}^1 , W_{\pm}^2 are defined by:

$$W_{\pm}^1 = s\text{-}\lim_{t \rightarrow \pm\infty} U_1(t)^* U_0(t)$$

$$W_{\pm}^2 = s\text{-}\lim_{t \rightarrow \pm\infty} U(t)^* U_1(t) E_{ac}(H_1)$$

in $L^2(\mathbb{R}^n)$. By chain rule, $W_{\pm} = W_{\pm}^2 W_{\pm}^1$ (see [1]). Put: $\mathbb{R}_+ =]0, +\infty[$.

Lemma 2.1. *Let $f \in C_0^\infty(\mathbb{R}_+/\sigma_p(H))$. Then for every $0 \leq \mu \leq 1$, one has:*

$$\|\langle A \rangle^{-\mu} f(H) U(t) W_{\pm} \langle A \rangle^{-\mu}\| \leq C(1 + |t|)^{-\mu} \quad (2.1)$$

for $t \in \mathbb{R}$. For every $\mu > 1$, there exists $\rho > 1$ such that

$$\|\langle A \rangle^{-\mu} f(H) U(t) W_{\pm} \langle A \rangle^{-\mu}\| \leq c(1 + |t|)^{-\rho} \quad \text{for } t \in \mathbb{R} \quad (2.2)$$

Note that this result is proved in [8] for $V = V_1(V_2 = 0)$. But the proof can be carried over, because we used only the short range properties of V and $x \cdot \nabla V$.

Recall that if $V = V_1 + V_2$ with V_1 satisfying (1.1) and V_2 bounded from H^2 to $L^{2,2+\varepsilon_0}$ it is proved in [3] that $S(\lambda)$ is continuously differentiable in $\mathcal{L}(L^2(S^{n-1}))$ for $\lambda \in \mathbb{R}_+/\sigma_p(H)$. Since under the assumptions of Theorem 1, $x \cdot \nabla V$ satisfies still the above conditions, it should be clear that by exterior scaling method, we can easily prove that $S(\lambda)$ is two times differentiable for $\lambda \in \mathbb{R}_+/\sigma_p(H)$. For reader's convenience we give the details of the proof.

Let $\lambda \in \mathbb{R}_+/\sigma_p(H)$. Then we have the following representation for the scattering matrix $S(\lambda)$ ($= S(\lambda, V)$) ([11]):

$$S(\lambda, V) = 1 - i\pi \mathcal{F}(\lambda)(V - VR(\lambda + i0, V)V)\mathcal{F}(\lambda)^*$$

where $R(\lambda \pm i0, V)$ is the boundary values of the resolvent $(H_0 + V - z)^{-1}$ and $\mathcal{F}(\cdot): L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}_+, L^2(S^{n-1}))$ is a spectral representation for the free Hamiltonian H_0 . Take $a > 0$ to be sufficiently small. Put: $I =]-a, a[$. We can prove that:

$$S(e^{2\theta}\lambda, V) = S(\lambda, V(\theta)) \quad \text{for } \theta \in I \quad (S)$$

where $V(\theta) = e^{-2\theta}U(\theta)^* V U(\theta)$ and $U(\theta)$ is the unitary group generated by A . Now we check the derivability of $V(\theta)$ and $R(\lambda + i0, V(\theta))$ for $\theta \in I$. Let $H^{s,m}$ denote the weighted Sobolev space with the norm $\|\langle x \rangle^m (1 - \Delta)^{s/2} f\|$. Put: $\rho = 1 + \varepsilon_0 > 1$. Then the assumptions on V say that $i[A, V_2]$ defines a bounded operator from $H^{s,r}$ to $H^{s-2, r+\rho+1}$ for $0 \leq s \leq 2$ and $r \in \mathbb{R}$. From this we derive that $A[A, V_2]$ and $[A, V_2]A$ are both bounded from $H^{s,r}$ to $H^{s-4, r+\rho}$ for $0 \leq s \leq 3$ and $r \in \mathbb{R}$. Since V_1 satisfies (1.1), we conclude easily from the above remarks that the operator valued function $\theta \mapsto V(\theta)$ is in the class

$$C^1(I; \mathcal{L}(H^{s,r}; H^{s-2, r+\rho+1})) \cap C^2(I; \mathcal{L}(H^{s+1, r}; H^{s-3, r+\rho}))$$

Since $R(\lambda \pm i0, V(\theta))$ is in $\mathcal{L}(H^{0,r}; H^{2,-r})$ if $r > \frac{1}{2}$ ([10]), we can also prove that the map $\theta \mapsto R(\lambda + i0, V(\theta))$ belongs to the class:

$$C^1(I; \mathcal{L}(H^{0,r}; H^{2,-r})) \cap C^2(I; \mathcal{L}(H^{0,r}; H^{0,-r}))$$

for $r > \frac{1}{2}$. This means that the map

$$I \ni \theta \mapsto \langle H_0 \rangle^{-2} V(\theta) (1 - R(\lambda + i0, V(\theta)) V(\theta)) \langle H_0 \rangle^{-2}$$

is in $C^2(I; \mathcal{L}(L^{2,-s}; L^{2,s}))$ for $s \in]\frac{1}{2}, \rho/2[$. Making use of the relation: $\mathcal{F}(\lambda) \langle H_0 \rangle^{-1} = \langle \lambda \rangle^{-1} \mathcal{F}(\lambda)$, we derive from (S) that $S(\lambda, V)$ is twice continuously differentiable in $\mathcal{L}(L^2(S^{n-1}))$ for $\lambda \in \mathbb{R}_+ / \sigma_p(H)$. This proves our assertion.

Since in the spectral representation of H_0 , A is given by a family of operators $A(\lambda) = i(\lambda d/d\lambda + d/d\lambda \cdot \lambda)$, we conclude that the domain of A^2 is invariant by $Sf(H_0)$ for $f \in C_0^\infty(\mathbb{R}_+ / \sigma_p(H))$. Now we can easily prove the following lemma which is important in this work.

Lemma 2.2. *Let \mathcal{D} be defined by (1.5). Then \mathcal{D} is invariant by S . In particular if f belongs to \mathcal{D} , $\langle x \rangle Sf$ and $A^2 Sf$ are both in $L^2(\mathbb{R}^n)$.*

Proof. It remains to show that $\langle x \rangle Sf$ is in L^2 . We can use the same commutator method as in the proof of Prop. 4.2 in [8]. The details are omitted here.

In order to regard V_2 as a perturbation to H_1 , we need some continuity of wave operators W_\pm^1 .

Lemma 2.3. *Let $f \in C_0^\infty(\mathbb{R}_+)$. Under the condition (1.1) on V_1 , the four operators $A^2 W_\pm^1 f(H_0) \langle A \rangle^{-2}$ and $A^2 W_\pm^{1*} f(H_1) \langle A \rangle^{-2}$ are all bounded on L^2 .*

Proof. We prove only the result for $A^2 W_+^{1*} f(H_1) \langle A \rangle^{-2}$. The other cases can be treated in the same way. Put $W(t) = U_0(t)^* U_1(t)$. We can write, as forms on $D(A) \times D(A)$,

$$\begin{aligned} AW(t)f(H_1) &= W(t)f(H_1)A + U_0(t)^*[A, f(H_1)]U_1(t) \\ &\quad + 2tU_0(t)V_1U_1(t)f(H_1) + W(t) \int_0^t U_1(s)^* f(H_1) \bar{V} U_1(s) ds \end{aligned} \quad (2.3)$$

where $\bar{V} = i[A, V_1] - 2V_1$. Notice that $i[A, f(H_1)] = 2f'(H_1) + Q$ with Q bounded from L^2 to $L^{2,1+\varepsilon_0}$ (see [8]). Now we need the following result due to Jensen et al.:

$$\|\langle A \rangle^{-r} f(H_1) U_1(t) \langle A \rangle^{-r}\| \leq c(1 + |t|)^{-r+\varepsilon} \quad t \in \mathbb{R} \quad (2.4)$$

for every $r > 0$ and $0 < \varepsilon \ll r$. Take $g \in C_0^\infty(\mathbb{R}_+)$ such that $g = 1$ on $\text{supp } f$. Multiplying (2.3) by $g(H_1) \langle A \rangle^{-2}$ and taking the limit $t \rightarrow +\infty$, applying (2.4), we

get:

$$\begin{aligned} AW_+^{1*}f(H_1)\langle A \rangle^{-2} &= W_+^{1*}f(H_1)Ag(H_1)\langle A \rangle^{-2} + W_+^{1*}h(H_1)\langle A \rangle^{-2} \\ &\quad + W_+^{1*} \int_0^{+\infty} f(H_1)U(-s)\tilde{V}U(s)g(H_1)\langle A \rangle^{-2} dt \end{aligned} \quad (2.5)$$

where $h = 2if'g$. Since $AW_+^{1*}f(H_1)\langle A \rangle^{-1}$ is bounded on L^2 , in order to prove the desired result by (2.5), it is sufficient to show that

$$\int_0^{+\infty} Af(H_1)U_1(-s)\tilde{V}U_1(s)g(H_1)\langle A \rangle^{-2} ds \quad (2.6)$$

is bounded on L^2 . To simplify notations, we denote f, g the operators $f(H_1), g(H_1)$ respectively. We have the following relation:

$$\begin{aligned} [A, U_1(-s)g\tilde{V}gU_1(s)] &= -2sU_1(-s)g[H_0, \tilde{V}]gU_1(s) \\ &\quad + U_1(-s)[A, g\tilde{V}g]U_1(s) \\ &\quad - \int_0^s U_1(t-s)\tilde{V}gU_1(-t)\tilde{V}U_1(s)g dt \\ &\quad + \int_0^s U_1(-s)g\tilde{V}U_1(s-t)g\tilde{V}U_1(t) dt \end{aligned} \quad (2.7)$$

Since $g[H_0, \tilde{V}]$ is continuous from $L^{2,r}$ to $L^{2,r+2+\varepsilon_0}$, we can prove as in [8] (Prop. 4.2) that $\int_0^{+\infty} 2sfU_1(-s)[H_0, \tilde{V}]U_1(s)g\langle A \rangle^{-1} ds$ is bounded on $L^2(\mathbb{R}^n)$. Since $[A, g\tilde{V}g]$ is bounded as operator from $L^{2,r}$ to $L^{2,r+1+\varepsilon_0}$, it follows from (2.4) that:

$$\|[A, g\tilde{V}g]U_1(s)g_1\langle A \rangle^{-2}\| \leq C(1 + |s|)^{-1-\varepsilon_0/2}$$

for $s \in \mathbb{R}$. Here $g_1 = g_1(H_1)$ is chosen so that $g_1g = g$. Therefore the integral $\int_0^\infty fU_1(-s)[A, g\tilde{V}g]U_1(s)g_1\langle A \rangle^{-2} ds$ defines a bounded operator on L^2 . To treat the last two terms in (2.7), we use the local H_1 -smoothness of $\langle x \rangle^{-1/2-\varepsilon}$, which implies that the operator $\int_0^s fU_1(t-s)\tilde{V}gU_1(-t)\tilde{V}U_1(s)g dt$ is uniformly bounded with respect to $s \in \mathbb{R}$. Applying (2.4), we get the estimate over the third term in (2.7):

$$\left\| \int_0^s fU_1(t-s)\tilde{V}gU_1(-t)\tilde{V}U_1(s)g_1\langle A \rangle^{-2} dt \right\| \leq C(1 + |s|)^{-1-\varepsilon_0/2}$$

The last term in (2.7) can be estimated in the same way. Since the commutator $[A, f]$ is bounded, we derive from (2.7) that (2.6) is a bounded operator on L^2 . This proves that $A^2W_+^{1*}f(H_1)\langle A \rangle^{-2}$ is bounded. The lemma is proved.

3. Reduction of the problem

In the proof of the finiteness of time-delay, an important step is to show that

$$\lim_{R \rightarrow +\infty} \left(\langle f, T_R f \rangle - \int_0^{+\infty} \langle U_0(t)f, (S^*P_R S - P_R)U_0(t)f \rangle dt \right) = 0 \quad (3.1)$$

(3.1) makes also clear the close relationship between time-delay operator T and scattering operator S . In this section we will prove the following result which implies (3.1).

Theorem 3.1. *Let $f \in \mathcal{D}$. Put $g = Sf$. Then we have:*

$$\lim_{R \rightarrow +\infty} \int_{-\infty}^0 (\langle U_0(t)f, (W_-^* P_R W_- - P_R) U_0(t)f \rangle) dt = 0 \quad (3.2)$$

$$\lim_{R \rightarrow +\infty} \int_0^{\infty} (\langle U_0(t)g, (W_+^* P_R W_+ - P_R) U_0(t)g \rangle) dt = 0 \quad (3.3)$$

Proof. Since the set \mathcal{D} is invariant by S , it suffices to prove (3.2). Put $h = W_-^1 f$, which is in $D(A^2)$ by Lemma 2.3. The integrand in (3.2) can be written as: $\langle U_1(t)h, (W_-^{2*} P_R W_-^2 - P_R) U_1(t)h \rangle + \langle U_0(t)f, (W_-^{1*} P_R W_-^1 - P_R) U_0(t)f \rangle$. It is proved in [8] by method of pseudo-differential operators that for $f \in \mathcal{D}$, we have:

$$\lim_{R \rightarrow +\infty} \int_{-\infty}^0 \langle U_0(t)f, (W_-^{1*} P_R W_-^1 - P_R) U_0(t)f \rangle dt = 0$$

Therefore we have to prove

$$\lim_{R \rightarrow +\infty} \int_{-\infty}^0 \langle U_1(t)h, (W_-^{2*} P_R W_-^2 - P_R) U_1(t)h \rangle dt = 0 \quad (3.4)$$

The integrand in (3.4) can be written as:

$$\begin{aligned} & \langle P_R U(t) W_- f, (U(t) W_-^2 - U_1(t))h \rangle \\ & + \langle (U(t) W_-^2 - U_1(t))h, P_R U_1(t)h \rangle \end{aligned} \quad (3.5)$$

Take $\chi \in C_0^\infty(\mathbb{R}_+ / \sigma_p(H))$ such that $\chi(H_0)f = f$. We get:

$$\begin{aligned} (U(t) W_-^2 - U_1(t))h &= - \int_{-\infty}^t \chi(H) U(t-s) V_2 U_1(s) \chi(H_1) h ds \\ &+ (\chi(H) - \chi(H_1)) U_1(t)h \end{aligned}$$

By the assumption, $V_2 \chi(H_1)$ is continuous from $L^{2, -\varepsilon_0}$ to $L^{2, 2}$. We can easily prove that $\chi(H) - \chi(H_1)$ is continuous from $L^{2, -1-\varepsilon_0}$ to L^2 . Thus the first term in (3.5) can be estimated as:

$$\begin{aligned} & |\langle P_R U(t) W_- f, (U(t) W_-^2 - U_1(t))h \rangle| \\ & \leq c \int_{-\infty}^t (1 + |s|)^{-2+\varepsilon} \|\langle x \rangle^{-\varepsilon_0} \chi(H) U(s-t) P_R U(t) \chi(H) W_- f\| \|\langle A \rangle^2 f\| ds \\ & + c(1 + |t|)^{-1-\varepsilon_0} \|\langle A \rangle^2 f\|^2, \quad \text{for any } \varepsilon > 0 \end{aligned} \quad (3.6)$$

Here we have used (2.4) and Lemma 2.3. Before going on with the proof of Theorem 3.1, we need still a lemma.

Lemma 3.2. For every $0 \leq \mu \leq 1$, we have:

$$\|\langle x \rangle^{-\mu} \chi(H) U(s-t) P_R U(t) W_- f\| \leq C(1+|s|)^{-\mu} \|\langle A \rangle W_- f\| \quad (3.7)$$

uniformly in $t \in \mathbb{R}$ and $R \geq 1$.

Proof. Observe first that $|\partial_x^\alpha P_R(x)| \leq c \langle x \rangle^{-|\alpha|}$ uniformly in $R \geq 1$. By the arguments used in the proof of Lemma 2.3, we can show that:

$$\|A \chi(H) U(-t) P_R U(t) \chi(H) \langle A \rangle^{-1}\| \leq C$$

uniformly in $t \in \mathbb{R}$ and $R \geq 1$. (3.7) follows from (2.1) and the fact that $\langle A \rangle \chi(H) \langle x \rangle^{-1}$ is bounded on L^2 .

Now return to the proof of Theorem 3.1. By (3.6) and (3.7) we obtain, for $\varepsilon > 0$ sufficiently small and for $t < 0$,

$$|\langle P_R U(t) W_- f, (U(t) W_-^2 - U_1(t)) h \rangle| \leq c(1+|t|)^{-1-\varepsilon/2} \|\langle A \rangle^2 f\|^2$$

uniformly in $R \geq 1$. The same estimate is also true for the second term in (3.5). Since the integrand in (3.4) tends to 0 as R tends to $+\infty$, by the dominated convergence theorem, (3.4) is proved. This finishes the proof of Theorem 3.1.

We remark that the various constants c appeared in the proof of Theorem 3.1 depend on the function χ , but not $f \in \mathcal{D}$ such that $\chi(H_0)f = f$.

4. Existence of time-delay operator

In this section we prove Theorem 1. Let $\chi \in C_0^\infty(\mathbb{R}_+/\sigma_p(H))$. We put:

$$\mathcal{D}_\chi = \{f \in \mathcal{D}; \chi(H_0)f = f\}$$

Lemma 4.1. Let $f \in \mathcal{D}_\chi$. Then,

$$\int_{R^{5/2}}^{+\infty} |\langle U_0(t)f, (S^* P_R S - P_R) U_0(t)f \rangle| dt \leq C R^{-1/2} \|\langle A \rangle f\|^2$$

Proof. It follows easily from the estimate:

$$\|P_R^{1/2} U_0(t)f\| \leq C_\chi R(1+|t|)^{-1} \|\langle A \rangle f\|$$

for $t \in \mathbb{R}$, $R \geq 1$ and $f \in \mathcal{D}_\chi$.

Lemma 4.2. For $f \in \mathcal{D}_\chi$, we have the asymptotic expansion:

$$\begin{aligned} \int_0^{R^{5/2}} \langle U_0(t)f, P_R U_0(t)f \rangle dt &= R \langle f, a(D)f \rangle \\ &\quad - \langle f, b^w(x, D; \chi)f \rangle + O(R^{-1/2}) \end{aligned} \quad (4.1)$$

where $a(\xi) = c_0 |\xi|^{-1} \chi(|\xi|^2)$ with $c_0 = \frac{1}{2} \int_0^\infty P(s) ds$, $b^w(x, D; \chi)$ is a Weyl pseudo-differential operator with symbol $\frac{1}{2} x \xi |\xi|^{-2} \chi(|\xi|^2)$. The remainder $O(R^{-1/2})$ can be

estimated by

$$|O(R^{-1/2})| \leq C_\chi R^{-1/2} (\|\langle x \rangle f\| + \|\langle A \rangle^2 f\|)^2 \quad R \geq 1$$

Proof. We use the fact that $U_0(-t)P_R U_0(t)$ is a Weyl pseudo-differential operator with symbol $P((x + 2t\xi)/R)$ and develop the symbol around $2t\xi/R$. Since this result is proved in [8] (Proposition 4.3) for $f \in D(\langle x \rangle) \cap D(\langle A \rangle^3)$, we indicate only the difference and omit the details. Checking the proof of Proposition 4.3([8]), we see that the condition $f \in D(\langle A \rangle^3)$ is only used to get the estimate (see (4.14)[8]):

$$\frac{t}{R^3} \|Q^w(\tau x/R, D)U_0(t/\tau)f\| \leq CR^{-1/2}t^{-3/2} \|\langle A \rangle^3 f\|$$

for $t \geq 1$, $\tau \in]0, 1]$ and $R \geq 1$, where $Q(x, \xi) = P_1(x)\chi(|\xi|^2)$, $P_1(x)$ is some derivative of $P(x)$. Hence it is supported in $\{1 \leq |x| \leq 2\}$. But this term can be equally estimated as follows: Since all the derivatives of $\tau^2 R^{-2} Q(\tau x/R, \xi)$ are bounded by a constant times $\langle x \rangle^{-2}$ uniformly with respect to $R \geq 1$ and $\tau \in]0, 1]$, we have, by continuity result of pseudo-differential operators ([2]),

$$\begin{aligned} \frac{t}{R^3} \|Q^w(\tau x/R, D)U_0(t/\tau)f\| \\ \leq cR^{-1}\tau^{-2}t \|\langle x \rangle^{-2}U_0(t/\tau)f\| \leq c_\chi R^{-1}t^{-1} \|\langle A \rangle^2 f\| \end{aligned} \quad (4.2)$$

for $t \geq 1$, $R \geq 1$ and $\tau \in]0, 1]$. Integrating (4.2) over $[1, R^{5/2}]$, we get:

$$\int_0^{R^{5/2}} tR^{-3} \|Q^w(\tau x/R, D)U_0(t/\tau)f\| dt \leq C_\chi R^{-1/2} \|\langle A \rangle^2 f\|$$

This gives the desired result. The lemma is proved.

Now we are able to give the proof of Theorem 1.

Proof of Theorem 1. Let f be in \mathcal{D} . Then Sf is also in \mathcal{D} . Take $\chi \in C_0^\infty(\mathbb{R}_+/\sigma_p(H))$ such that $\chi(H_0)f = f$. Since S commutes with $a(D)$, we derive from Lemmas 4.1 and 4.2 that

$$\begin{aligned} \left| \int_0^\infty \langle U_0(t)f, S^*[P_R, S]U_0(t)f \rangle dt + \langle f, S^*[b^w(x, D; \chi), S]f \rangle \right| \\ \leq C_f R^{-1/2} \end{aligned} \quad (4.3)$$

By a simple calculus of Weyl pseudo-differential operators ([2]), we get:

$$\langle f, S^*[b^w(x, D; \chi), S]f \rangle = \langle f_1, S^*[A, S]f \rangle \quad (4.4)$$

where $f_1 = (2H_0)^{-1}\chi(H_0)f$. Theorem 1 is a consequence of (4.3) and (4.4). See also [8].

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