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# Time-delay operator for a class of singular potentials

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Abstract. We prove the existence of the time-delay operator defined by taking the large space limit of the approximate sojourn times for a class of singular potentials:  $V = V_1 + V_2$ , where  $V_1$  is a smooth short range potential and  $V_2$  and  $x \cdot \nabla V_2$  are both bounded from  $H^2$  to  $L^{2,2+\epsilon_0}$  for some  $\epsilon_0 > 0$ .

## 1. Introduction

In [8], we proved the finiteness of time-delay defined by taking the space limit of sojourn times and established its equivalence with Eisenbud-Wigner's time-delay in scattering theory for smooth short range potentials. In this work we will show that our method developed there can be also applied to a class of singular potentials.

Let  $H_0 = -\Delta$  and  $H = H_0 + V$  in  $L^2(\mathbb{R}^n)$ . We suppose that the short range potential V can be decomposed as:  $V(x) = V_1(x) + V_2(x)$  where  $V_1$  is  $C^{\infty}$  on  $\mathbb{R}^n$  and for some  $\varepsilon_0 > 0$ 

$$\left|\partial_x^{\alpha} V_1(x)\right| \le c_{\alpha} \langle x \rangle^{-1 - \varepsilon_0 - |\alpha|} \tag{1.1}$$

and the multiplication by  $V_2$  is bounded as operator from  $H^2$  to  $L^{2,2+\varepsilon_0}$  and so is the distributional derivative  $x\nabla_x \cdot V_2$ . Here  $\langle x \rangle = (1+|x|)$  and  $H^s$  is the usual Sobolev space of order s;  $L^{2,s}$  is the weighted  $L^2$  space with the norm:  $||f||_s = ||\langle x \rangle^s f||$ . This assumption will be made throughout this work. Under this condition on V, it is well known that the wave operators  $W_\pm$  defined by:

$$W_{\pm} = s - \lim_{t \to \pm \infty} e^{itH} e^{-itH_0} \quad \text{in } L^2$$

exist and are complete. Let  $\tilde{P}_R$  denote the multiplication by the characteristic function for the ball  $\{|x| < R\}$ . Then the local time-delay of f in  $\{|x| < R\}$  is defined as the difference of the sojourn times:

$$\langle f, T_R f \rangle = \int_{-\infty}^{\infty} (\|\tilde{P}_R e^{-itH} W_- f\|^2 - \|\tilde{P}_R e^{-itH_0} f\|^2) dt$$
 (1.2)

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Notice that (1.2) is well defined for  $f \in L^2$  such that the Fourier transform  $\hat{f}$  has suitable compact support in  $\mathbb{R}^n|_{\{0\}}$ . Finally the time-delay operator T is defined by:

$$\langle f, Tf \rangle = \lim_{R \to +\infty} \langle f, T_R f \rangle$$
 (1.3)

whenever the limit exists. Surely the existence of time-delay operator depends on how much such f's we can find. As in [8], we consider here a similar question. Let  $P_R$  be the multiplication by P(x/R), where P(.) is a smooth, spherically symmetrical function such that P(x) = 1 for  $|x| \le 1$  and P(x) = 0 for  $|x| \ge 2$ . It is clear that  $P_R$  can be regarded as an approximation of  $\tilde{P}_R$ . In the following we denote still  $T_R$  the operator defined by (1.2) with  $\tilde{P}_R$  replaced by  $P_R$ .

Then for smooth short range potentials we proved in [8] that the limit (1.3) exists for a dense subset in  $L^2$  and in the spectral representation of  $H_0$ , the time-delay operator T is given by a family of operators  $T(\lambda)$ ,  $\lambda > 0$ , where

$$T(\lambda) = -iS(\lambda)^* \frac{d}{d\lambda} S(\lambda)$$
 (1.4)

 $S(\lambda)$  being the scattering matrix. (1.4) is the Eisenbud-Wigner's formula for time-delay. It reveals that the method and techniques used in [8] are powerful enough. It can be applied to treat time-delay in other scattering theories (see [6]) and to include a class of singular potentials.

Let  $A = -i(x \cdot \nabla_x + \nabla_{\dot{x}}x)/2$  be the generator of dilation group. We define the set  $\mathcal{D}$  by:

$$\mathcal{D} = \{ f \in L^2; f \in D(\langle x \rangle) \cap D(A^2) \text{ and } \exists \chi \in C_0^{\infty}(\mathbb{R}_+ \backslash \sigma_p(H)), \chi(H_0)f = f \}$$

$$\tag{1.5}$$

In this work we want to prove the following result.

**Theorem 1.** Under the above assumption on V, the limit (1.3) exists for  $T_R$  defined by (1.2) with  $\tilde{P}_R$  replaced by  $P_R$  and for  $f \in \mathcal{D}$ . We have:

$$\langle f, Tf \rangle = \langle f, -S^*[A, S]f_1 \rangle$$

where  $f_1$  is determined by  $2H_0f_1 = f$  and  $S = W_+^*W_-$  is scattering operator. The time-delay operator T is essentially selfadjoint with core  $\mathcal{D}$  and the Eisenbud–Wigner formula (1.4) is true for  $\lambda \in \mathbb{R}_+/\sigma_p(H)$ .

The proof of this result consists in regarding  $V_2$  as a perturbation of the Hamiltonian  $H_1 = H_0 + V_1$ . In §2, we give some technical preparations, which were mostly proved in [8]. In §3, we achieve the main step of the proof, reducing the existence of the limit (1.3) to that of  $\lim_{R\to +\infty} \int_0^\infty \langle U_0(t)f, S^*P_RS - P_R)f \rangle dt$ . We finish the proof of Theorem 1 in §4 by the method of [8]. Very recently Nakamura ([12]) considered the similar problem by a different method. His proof is in the spirit of Lavine [4], while ours is in that of Martin [5].

## 2. Some preparations

Let U(t) (resp.  $U_0(t)$ ,  $U_1(t)$ ) denote the unitary group associated to H (resp.  $H_0$ ,  $H_1$ ). Let  $E_{ac}(H_1)$  denote the spectral projector onto the absolute continuous space of  $H_1$ . The wave operators  $W_{\pm}^1$ ,  $W_{\pm}^2$  are defined by:

$$W_{\pm}^{1} = s - \lim_{t \to \pm \infty} U_{1}(t)^{*} U_{0}(t)$$

$$W_{\pm}^2 = s - \lim_{t \to +\infty} U(t)^* U_1(t) E_{ac}(H_1)$$

in  $L^2(\mathbb{R}^n)$ . By chain rule,  $W_{\pm} = W_{\pm}^2 W_{\pm}^1$  (see [1]). Put:  $\mathbb{R}_+ = ]0, +\infty[$ .

**Lemma 2.1.** Let  $f \in C_0^{\infty}(\mathbb{R}_+/\sigma_p(H))$ . Then for every  $0 \le \mu \le 1$ , one has:

$$\|\langle A \rangle^{-\mu} f(H) U(t) W_{\pm} \langle A \rangle^{-\mu} \| \le C (1+|t|)^{-\mu} \tag{2.1}$$

for  $t \in \mathbb{R}$ . For every  $\mu > 1$ , there exists  $\rho > 1$  such that

$$\|\langle A \rangle^{-\mu} f(H) U(t) W_{\pm} \langle A \rangle^{-\mu} \| \le c (1+|t|)^{-\rho} \quad \text{for } t \in \mathbb{R}$$
 (2.2)

Note that this result is proved in [8] for  $V = V_1(V_2 = 0)$ . But the proof can be carried over, because we used only the short range properties of V and  $x \cdot \nabla V$ .

Recall that if  $V = V_1 + V_2$  with  $V_1$  satisfying (1.1) and  $V_2$  bounded from  $H^2$  to  $L^{2,2+\varepsilon_0}$  it is proved in [3] that  $S(\lambda)$  is continuously differentiable in  $\mathcal{L}(L^2(S^{n-1}))$  for  $\lambda \in \mathbb{R}_+/\sigma_p(H)$ . Since under the assumptions of Theorem 1,  $x \cdot \nabla V$  satisfies still the above conditions, it should be clear that by exterior scaling method, we can easily prove that  $S(\lambda)$  is two times differentiable for  $\lambda \in \mathbb{R}_+/\sigma_p(H)$ . For reader's convenience we give the details of the proof.

Let  $\lambda \in \mathbb{R}_+/\sigma_p(H)$ . Then we have the following representation for the scattering matrix  $S(\lambda)$  (=  $S(\lambda, V)$ ) ([11]):

$$S(\lambda, V) = 1 - i\pi \mathcal{F}(\lambda)(V - VR(\lambda + i0, V)V)\mathcal{F}(\lambda)^*$$

where  $R(\lambda \pm i0, V)$  is the boundary values of the resolvent  $(H_0 + V - z)^{-1}$  and  $\mathcal{F}(.): L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}_+, L^2(S^{n-1}))$  is a spectral representation for the free Hamiltonian  $H_0$ . Take a > 0 to be sufficiently small. Put: I = ]-a, a[. We can prove that:

$$S(e^{2\theta}\lambda, V) = S(\lambda, V(\theta)) \text{ for } \theta \in I$$
 (S)

where  $V(\theta) = e^{-2\theta}U(\theta)^*VU(\theta)$  and  $U(\theta)$  is the unitary group generated by A. Now we check the derivability of  $V(\theta)$  and  $R(\lambda+i0,V(\theta))$  for  $\theta \in I$ . Let  $H^{s,m}$  denote the weighted Sobolev space with the norm  $\|\langle x \rangle^m (1-\Delta)^{s/2} f\|$ . Put:  $\rho = 1 + \varepsilon_0 > 1$ . Then the assumptions on V say that  $i[A, V_2]$  defines a bounded operator from  $H^{s,r}$  to  $H^{s-2,r+\rho+1}$  for  $0 \le s \le 2$  and  $r \in \mathbb{R}$ . From this we derive that  $A[A, V_2]$  and  $[A, V_2]A$  are both bounded from  $H^{s,r}$  to  $H^{s-4,r+\rho}$  for  $0 \le s \le 3$  and  $r \in \mathbb{R}$ . Since  $V_1$  satisfies (1.1), we conclude easily from the above remarks that the operator valued function  $\theta \mapsto V(\theta)$  is in the class

$$C^{1}(I; \mathcal{L}(H^{s,r}; H^{s-2,r+\rho+1})) \cap C^{2}(I; \mathcal{L}(H^{s+1,r}; H^{s-3,r+\rho}))$$

Since  $R(\lambda \pm i0, V(\theta))$  is in  $\mathcal{L}(H^{0,r}; H^{2,-r})$  if  $r > \frac{1}{2}$  ([10]), we can also prove that the map  $\theta \mapsto R(\lambda + i0, V(\theta))$  belongs to the class:

$$C^{1}(I; \mathcal{L}(H^{0,r}; H^{2,-r})) \cap C^{2}(I; \mathcal{L}(H^{0,r}; H^{0,-r}))$$

for  $r > \frac{1}{2}$ . This means that the map

$$I \ni \theta \mapsto \langle H_0 \rangle^{-2} V(\theta) (1 - R(\lambda + i0, V(\theta)) V(\theta)) \langle H_0 \rangle^{-2}$$

is in  $C^2(I; \mathcal{L}(L^{2,-s}; L^{2,s}))$  for  $s \in ]\frac{1}{2}$ ,  $\rho/2[$ . Making use of the relation:  $\mathcal{F}(\lambda)\langle H_0\rangle^{-1} = \langle \lambda \rangle^{-1}\mathcal{F}(\lambda)$ , we derive from (S) that  $S(\lambda, V)$  is twice continuously differentiable in  $\mathcal{L}(L^2(S^{n-1}))$  for  $\lambda \in \mathbb{R}_+/\sigma_p(H)$ . This proves our assertion.

Since in the spectral representation of  $H_0$ , A is given by a family of operators  $A(\lambda) = i(\lambda d/d\lambda + d/d\lambda \cdot \lambda)$ , we conclude that the domain of  $A^2$  is invariant by  $Sf(H_0)$  for  $f \in C_0^{\infty}(\mathbb{R}_+/\sigma_p(H))$ . Now we can easily prove the following lemma which is important in this work.

**Lemma 2.2.** Let  $\mathcal{D}$  be defined by (1.5). Then  $\mathcal{D}$  is invariant by S. In particular if f belongs to  $\mathcal{D}$ ,  $\langle x \rangle Sf$  and  $A^2Sf$  are both in  $L^2(\mathbb{R}^n)$ .

*Proof.* It remains to show that  $\langle x \rangle Sf$  is in  $L^2$ . We can use the same commutator method as in the proof of Prop. 4.2 in [8]. The details are omitted here.

In order to regard  $V_2$  as a perturbation to  $H_1$ , we need some continuity of wave operators  $W_{\pm}^1$ .

**Lemma 2.3.** Let  $f \in C_0^{\infty}(\mathbb{R}_+)$ . Under the condition (1.1) on  $V_1$ , the four operators  $A^2W_{\pm}^1f(H_0)\langle A \rangle^{-2}$  and  $A^2W_{\pm}^{1*}f(H_1)\langle A \rangle^{-2}$  are all bounded on  $L^2$ .

*Proof.* We prove only the result for  $A^2W_+^{1*}f(H_1)\langle A\rangle^{-2}$ . The other cases can be treated in the same way. Put  $W(t) = U_0(t)^*U_1(t)$ . We can write, as forms on  $D(A) \times D(A)$ ,

$$AW(t)f(H_1) = W(t)f(H_1)A + U_0(t)^*[A, f(H_1)]U_1(t)$$

$$+ 2tU_0(t)V_1U_1(t)f(H_1) + W(t) \int_0^t U_1(s)^*f(H_1)\tilde{V}U_1(s) ds \qquad (2.3)$$

where  $\tilde{V} = i[A, V_1] - 2V_1$ . Notice that  $i[A, f(H_1)] = 2f'(H_1) + Q$  with Q bounded from  $L^2$  to  $L^{2,1+\varepsilon_0}$  (see [8]). Now we need the following result due to Jensen et al.:

$$\|\langle A \rangle^{-r} f(H_1) U_1(t) \langle A \rangle^{-r} \| \le c (1+|t|)^{-r+\varepsilon} \quad t \in \mathbb{R}$$
 (2.4)

for every r > 0 and  $0 < \varepsilon \ll r$ . Take  $g \in C_0^{\infty}(\mathbb{R}_+)$  such that g = 1 on supp f. Multiplying (2.3) by  $g(H_1)\langle A \rangle^{-2}$  and taking the limit  $t \to +\infty$ , applying (2.4), we

get:

$$AW_{+}^{1*}f(H_{1})\langle A \rangle^{-2} = W_{+}^{1*}f(H_{1})Ag(H_{1})\langle A \rangle^{-2} + W_{+}^{1*}h(H_{1})\langle A \rangle^{-2} + W_{+}^{1*}\int_{0}^{+\infty} f(H_{1})U(-s)\tilde{V}U(s)g(H_{1})\langle A \rangle^{-2} dt$$
(2.5)

where h = 2if'g. Since  $AW_+^{1*}f(H_1)\langle A\rangle^{-1}$  is bounded on  $L^2$ , in order to prove the desired result by (2.5), it is sufficient to show that

$$\int_0^{+\infty} Af(H_1)U_1(-s)\tilde{V}U_1(s)g(H_1)\langle A \rangle^{-2} ds \tag{2.6}$$

is bounded on  $L^2$ . To simplify notations, we denote f, g the operators  $f(H_1)$ ,  $g(H_1)$  respectively. We have the following relation:

$$[A, U_{1}(-s)g\tilde{V}gU_{1}(s)] = -2sU_{1}(-s)g[H_{0}, \tilde{V}]gU_{1}(s) + U_{1}(-s)[A, g\tilde{V}g]U_{1}(s) - \int_{0}^{s} U_{1}(t-s)\tilde{V}gU_{1}(-t)\tilde{V}U_{1}(s)g dt + \int_{0}^{s} U_{1}(-s)g\tilde{V}U_{1}(s-t)g\tilde{V}U_{1}(t) dt$$
(2.7)

Since  $g[H_0, \tilde{V}]$  is continuous from  $L^{2,r}$  to  $L^{2,r+2+\varepsilon_0}$ , we can prove as in [8] (Prop. 4.2) that  $\int_0^{+\infty} 2sfU_1(-s)[H_0, \tilde{V}]U_1(s)g\langle A\rangle^{-1}ds$  is bounded on  $L^2(\mathbb{R}^n)$ . Since  $[A, g\tilde{V}g]$  is bounded as operator from  $L^{2,r}$  to  $L^{2,r+1+\varepsilon_0}$ , it follows from (2.4) that:

$$||[A, g\tilde{V}g]U_1(s)g_1\langle A\rangle^{-2}|| \leq C(1+|s|)^{-1-\varepsilon_0/2}$$

for  $s \in \mathbb{R}$ . Here  $g_1 = g_1(H_1)$  is chosen so that  $g_1g = g$ . Therefore the integral  $\int_0^\infty f U_1(-s)[A, g\tilde{V}g]U_1(s)g_1\langle A\rangle^{-2}ds$  defines a bounded operator on  $L^2$ . To treat the last two terms in (2.7), we use the local  $H_1$ -smoothness of  $\langle x \rangle^{-1/2-\varepsilon}$ , which implies that the operator  $\int_0^s f U_1(t)\tilde{V}U_1(-t)g\,dt$  is uniformly bounded with respect to  $s \in \mathbb{R}$ . Applying (2.4), we get the estimate over the third term in (2.7):

$$\left\| \int_0^s f U_1(t-s) \tilde{V} g U_1(-t) \tilde{V} U_1(s) g_1 \langle A \rangle^{-2} dt \right\| \le C (1+|s|)^{-1-\varepsilon_0/2}$$

The last term in (2.7) can be estimated in the same way. Since the commutator [A, f] is bounded, we derive from (2.7) that (2.6) is a bounded operator on  $L^2$ . This proves that  $A^2W_+^{1*}f(H_1)\langle A\rangle^{-2}$  is bounded. The lemma is proved.

# 3. Reduction of the problem

In the proof of the finiteness of time-delay, an important step is to show that

$$\lim_{R \to +\infty} \left( \langle f, T_R f \rangle - \int_0^{+\infty} \langle U_0(t) f, (S^* P_R S - P_R) U_0(t) f \rangle dt \right) = 0 \tag{3.1}$$

(3.1) makes also clear the close relationship between time-delay operator T and scattering operator S. In this section we will prove the following result which implies (3.1).

**Theorem 3.1.** Let  $f \in \mathcal{D}$ . Put g = Sf. Then we have:

$$\lim_{R \to +\infty} \int_{-\infty}^{0} (\langle U_0(t)f, (W_-^* P_R W_- - P_R) U_0(t)f \rangle) dt = 0$$
(3.2)

$$\lim_{R \to +\infty} \int_0^\infty (\langle U_0(t)g, (W_+^* P_R W_+ - P_R) U_0(t)g \rangle) dt = 0$$
(3.3)

*Proof.* Since the set  $\mathcal{D}$  is invariant by S, it suffices to prove (3.2). Put  $h = W_{-}^{1}f$ , which is in  $D(A^{2})$  by Lemma 2.3. The integrand in (3.2) can be written as:  $\langle U_{1}(t)h, (W_{-}^{2*}P_{R}W_{-}^{2} - P_{R})U_{1}(t)h \rangle + \langle U_{0}(t)f, (W_{-}^{1*}P_{R}W_{-}^{1} - P_{R})U_{0}(t)f \rangle$ . It is proved in [8] by method of pseudo-differential operators that for  $f \in \mathcal{D}$ , we have:

$$\lim_{R \to +\infty} \int_{-\infty}^{0} \langle U_0(t)f, (W_{-}^{1*}P_RW_{-}^{1} - P_R)U_0(t)f \rangle dt = 0$$

Therefore we have to prove

$$\lim_{R \to +\infty} \int_{-\infty}^{0} \langle U_1(t)h, (W_-^{2*} P_R W_-^2 - P_R) U_1(t)h \rangle dt = 0$$
(3.4)

The integrand in (3.4) can be written as:

$$\langle P_R U(t) W_- f, (U(t) W_-^2 - U_1(t)) h \rangle + \langle (U(t) W_-^2 - U_1(t)) h, P_R U_1(t) h \rangle$$
 (3.5)

Take  $\chi \in C_0^{\infty}(\mathbb{R}_+/\sigma_p(H))$  such that  $\chi(H_0)f = f$ . We get:

$$(U(t)W_{-}^{2} - U_{1}(t))h = -\int_{-\infty}^{t} \chi(H)U(t-s)V_{2}U_{1}(s)\chi(H_{1})h ds$$
$$+ (\chi(H) - \chi(H_{1}))U_{1}(t)h$$

By the assumption,  $V_2\chi(H_1)$  is continuous from  $L^{2,-\epsilon_0}$  to  $L^{2,2}$ . We can easily prove that  $\chi(H) - \chi(H_1)$  is continuous from  $L^{2,-1-\epsilon_0}$  to  $L^2$ . Thus the first term in (3.5) can be estimated as:

$$\begin{aligned} \left| \left\langle P_{R}U(t)W_{-}f, \left( U(t)W_{-}^{2} - U_{1}(t) \right) h \right\rangle \right| \\ &\leq c \int_{-\infty}^{t} (1 + |s|)^{-2 + \varepsilon} \left\| \left\langle x \right\rangle^{-\varepsilon_{0}} \chi(H)U(s - t) P_{R}U(t) \chi(H) W_{-}f \right\| \left\| \left\langle A \right\rangle^{2} f \right\| ds \\ &+ c (1 + |t|)^{-1 - \varepsilon_{0}} \left\| \left\langle A \right\rangle^{2} f \right\|^{2}, \quad \text{for any} \quad \varepsilon > 0 \end{aligned}$$

$$(3.6)$$

Here we have used (2.4) and Lemma 2.3. Before going on with the proof of Theorem 3.1, we need still a lemma.

**Lemma 3.2.** For every  $0 \le \mu \le 1$ , we have:

$$\|\langle x \rangle^{-\mu} \chi(H) U(s-t) P_R U(t) W_- f\| \le C (1+|s|)^{-\mu} \|\langle A \rangle W_- f\|$$
(3.7)

uniformly in  $t \in \mathbb{R}$  and  $R \ge 1$ .

*Proof.* Observe first that  $|\partial_x^{\alpha} P_R(x)| \le c \langle x \rangle^{-|\alpha|}$  uniformly in  $R \ge 1$ . By the arguments used in the proof of Lemma 2.3, we can show that:

$$||A\chi(H)U(-t)P_RU(t)\chi(H)\langle A\rangle^{-1}|| \leq C$$

uniformly in  $t \in \mathbb{R}$  and  $R \ge 1$ . (3.7) follows from (2.1) and the fact that  $\langle A \rangle \chi(H) \langle x \rangle^{-1}$  is bounded on  $L^2$ .

Now return to the proof of Theorem 3.1. By (3.6) and (3.7) we obtain, for  $\varepsilon > 0$  sufficiently small and for t < 0,

$$|\langle P_R U(t) W_- f, (U(t) W_-^2 - U_1(t)) h \rangle| \le c (1 + |t|)^{-1 - \varepsilon_0/2} ||\langle A \rangle^2 f||^2$$

uniformly in  $R \ge 1$ . The same estimate is also true for the second term in (3.5). Since the integrand in (3.4) tends to 0 as R tends to  $+\infty$ , by the dominated convergence theorem, (3.4) is proved. This finishes the proof of Theorem 3.1.

We remark that the various constants c appeared in the proof of Theorem 3.1 depend on the function  $\chi$ , but not  $f \in \mathcal{D}$  such that  $\chi(H_0)f = f$ .

# 4. Existence of time-delay operator

In this section we prove Theorem 1. Let  $\chi \in C_0^{\infty}(\mathbb{R}_+/\sigma_p(H))$ . We put:

$$\mathcal{D}_{\chi} = \{ f \in \mathcal{D}; \chi(H_0)f = f \}$$

**Lemma 4.1.** Let  $f \in \mathcal{D}_{\chi}$ . Then,

$$\int_{R^{5/2}}^{+\infty} |\langle U_0(t)f, (S^*P_RS - P_R)U_0(t)f \rangle| dt \le CR^{-1/2} \|\langle A \rangle f\|^2$$

Proof. It follows easily from the estimate:

$$||P_R^{1/2}U_0(t)f|| \le C_{\chi}R(1+|t|)^{-1}||\langle A\rangle f||$$

for  $t \in \mathbb{R}$ ,  $R \ge 1$  and  $f \in \mathcal{D}_{\alpha}$ .

**Lemma 4.2.** For  $f \in \mathcal{D}_{\chi}$ , we have the asymptotic expansion:

$$\int_{0}^{R^{5/2}} \langle U_0(t)f, P_R U_0(t)f \rangle dt = R \langle f, a(D)f \rangle$$

$$- \langle f, b^w(x, D; \chi)f \rangle + O(R^{-1/2})$$
(4.1)

where  $a(\xi) = c_0 |\xi|^{-1} \chi(|\xi|^2)$  with  $c_0 = \frac{1}{2} \int_0^\infty P(s) ds$ ,  $b^w(x, D; \chi)$  is a Weyl pseudo-differential operator with symbol  $\frac{1}{2} x \xi |\xi|^{-2} \chi(|\xi|^2)$ . The remainder  $O(R^{-1/2})$  can be

estimated by

$$|O(R^{-1/2})| \le C_{\chi} R^{-1/2} (\|\langle x \rangle f\| + \|\langle A \rangle^2 f\|)^2 \quad R \ge 1$$

*Proof.* We use the fact that  $U_0(-t)P_RU_0(t)$  is a Weyl pseudo-differential operator with symbol  $P((x+2t\xi)/R)$  and develop the symbol around  $2t\xi/R$ . Since this result is proved in [8] (Proposition 4.3) for  $f \in D(\langle x \rangle) \cap D(\langle A \rangle^3)$ , we indicate only the difference and omit the details. Checking the proof of Proposition 4.3([8]), we see that the condition  $f \in D(\langle A \rangle^3)$  is only used to get the estimate (see (4.14)[8]):

$$\frac{t}{R^3} \| Q^w(\tau x/R, D) U_0(t/\tau) f \| \le C R^{-1/2} t^{-3/2} \| \langle A \rangle^3 f \|$$

for  $t \ge 1$ ,  $\tau \in ]0, 1]$  and  $R \ge 1$ , where  $Q(x, \xi) = P_1(x)\chi(|\xi|^2)$ ,  $P_1(x)$  is some derivative of P(x). Hence it is supported in  $\{1 \le |x| \le 2\}$ . But this term can be equally estimated as follows: Since all the derivatives of  $\tau^2 R^{-2} Q(\tau x/R, \xi)$  are bounded by a constant times  $\langle x \rangle^{-2}$  uniformly with respect to  $R \ge 1$  and  $\tau \in ]0, 1]$ , we have, by continuity result of pseudo-differential operators ([2]),

$$\frac{t}{R^{3}} \|Q^{w}(\tau x/R, D)U_{0}(t/\tau)f\| 
\leq cR^{-1}\tau^{-2}t \|\langle x \rangle^{-2}U_{0}(t/\tau)f\| \leq c_{\chi}R^{-1}t^{-1} \|\langle A \rangle^{2}f\|$$
(4.2)

for  $t \ge 1$ ,  $R \ge 1$  and  $\tau \in ]0, 1]$ . Integrating (4.2) over  $[1, R^{5/2}]$ , we get:

$$\int_0^{R^{5/2}} tR^{-3} \| Q^w(\tau x/R, D) U_0(t/\tau) f \| dt \le C_{\chi} R^{-1/2} \| \langle A \rangle^2 f \|$$

This gives the desired result. The lemma is proved.

Now we are able to give the proof of Theorem 1.

**Proof** of Theorem 1. Let f be in  $\mathcal{D}$ . Then Sf is also in  $\mathcal{D}$ . Take  $\chi \in C_0^{\infty}(\mathbb{R}_+/\sigma_p(H))$  such that  $\chi(H_0)f = f$ . Since S commutes with a(D), we derive from Lemmas 4.1 and 4.2 that

$$\left| \int_0^\infty \langle U_0(t)f, S^*[P_R, S]U_0(t)f \rangle dt + \langle f, S^*[b^w(x, D; \chi), S]f \rangle \right|$$

$$\leq C_f R^{-1/2} \tag{4.3}$$

By a simple calculus of Weyl pseudo-differential operators ([2]), we get:

$$\langle f, S^*[b^w(x, D; \chi), S]f \rangle = \langle f, S^*[A, S]f \rangle \tag{4.4}$$

where  $f_1 = (2H_0)^{-1}\chi(H_0)f$ . Theorem 1 is a consequence of (4.3) and (4.4). See also [8].

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