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Perturbative derivation of the determinant of massless fermions in two dimensional space-time[1])

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Abstract. The functional determinant of massless fermions in two dimensional space-time submitted to a non-abelian gauge potential is defined as the vacuum-to-vacuum amplitude and obtained from its perturbation series. The result is expressed in terms of integrals involving an auxiliary variable. It is an extension to a situation where space-time cannot be compactified of an expression given by Polyakov and Wiegmann [6]. Our treatment takes into account the fermion pairs created by the external potential. Matrix elements of the induced current are displayed.

1. Introduction

In a gauge theory with fermions, the summation over fermion loops, that is the evaluation of the fermion determinant is a first step towards the construction of the theory. The lagrangian being bilinear in the fermion fields, the integration over fermions is a standard and well defined problem, once a regularization scheme has been fixed. If there are no fermion sources, the euclidean fermion determinant can be obtained from the solution of the Dirac equation in an external gauge potential A_μ. In the case of two dimensional theories with massless fermions, systematic investigations started a few yers ago [1]. For QCD_2, quantum chromodynamics in two dimensions, the use of the decoupling gauge invented by Roskies [2] combined with Fujikawa's method [3] is particularly well suited. The tool of this method is a one parameter family of changes of fermion variables based on chiral transformations. The determinant is related to the jacobians of these changes of variables; it is given by an integral over the parameter which labels them [4].

In the case of QED_2, this integral is easily computed and the fermion determinant produces the mass term of the free boson which is the only physical particle of that model [5].

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In QCD_2, the integral cannot be performed explicitly and the result remained rather obscure until a contribution due to Polyakov and Wiegmann [6] clarified the situation. These authors wrote down an expression for the fermion determinant that is compatible with the form of the current induced by the external potential. In the context of the present work it is convenient to present this important expression in the following way. Consider a non abelian gauge potential in two-dimensional Minkowski space $\mathbb{M}^2$ given, in the matrix notation, by

$$A_\pm(x) = -(i/g)(\partial_\pm V_\mp \cdot V_\mp^{-1})(x) \tag{1.1}$$

where $A_\pm$ are light-cone components, $A_\pm = A_0 \pm A_1$, $\partial_\pm = \partial_0 \pm \partial_1$, and $V_\pm$ are x-dependent unitary matrices in the fundamental representation of the gauge group which we choose to be $SU(N)$. These matrices are such that $(V_\pm - \mathbb{1})$ have compact support. Let t be an auxiliary variable ($t \in [0, 1]$) and let $V_\pm(x, t)$ be extensions of $V_\pm(x)$ to $\mathbb{M}^2 \times [0, 1]$, $(V_\pm(x, t) - \mathbb{1})$ has compact support in x; $V_\pm(x, 1) = V_\pm(x)$, $V_\pm(x, 0) = \mathbb{1}$. Write the determinant of a multiplet of fermions in the fundamental representation of $SU(N)$ submitted to the background potential (1.1) as $\exp(iW[A])$. The effective action $W[A]$ is:

$$W[A] = I[V_+] + I[V_-^{-1}] + \frac{1}{4\pi}\int (dx)^2 \operatorname{Tr}(A_+A_-)(x), \tag{1.2}$$

with:

$$\begin{aligned} I[V] = {} & \frac{1}{8\pi}\int (dx)^2 \operatorname{Tr}(\partial_+ V \cdot V^{-1} \cdot \partial_- V \cdot V^{-1})(x) \\ & - \frac{1}{8\pi}\int_0^1 dt \int (dx)^2 \operatorname{Tr}(\partial_t V \cdot V^{-1}[\partial_+ V \cdot V^{-1}, \partial_- V \cdot V^{-1}])(x, t). \end{aligned} \tag{1.3}$$

This last functional is proportional to the effective action obtained by Witten [7] in his non abelian fermion bosonization. The three dimensional integral in (1.3) is the analogue of the Wess–Zumino functional [8]. Because of the properties of $V_\pm$, $\mathbb{M}^2$ can be compactified and $\mathbb{M}^2 \times [0, 1]$ can be transformed into a hemisphere of S^3, its boundary being the $t = 1$ compactified $\mathbb{M}^2$. These transformations lead to the original formulation of Polyakov and Wiegmann, the second integral in (1.3) being extended to the hemisphere.

The euclidean version of (1.2) has been identified with the determinants produced by Fujikawa's method [9] and by an extension of a method due to O. Alvarez [10]. Furthermore, (1.2) has been used for the construction of effective bosonic actions for two dimensional gauge theories [11].

One may have the feeling that the limitation to a situation allowing compactification of space-time is very restrictive. The infinite past and infinite future get identified and it seems that any true evolution has been excluded. Whereas compactification of an euclidean space with its elliptic operators is unproblematic, this is not obvious for a Minkowski space and the associated hyperbolic operators. Furthermore, before integration over the gauge potential, one is dealing with an external field problem and there is no direct connection between the Minkowski space and euclidean space versions of that problem, no well defined way of translating an external potential on one space into a potential on the other space. Therefore, there is no indication that the compactifications of both spaces are on the same footing. These questions are at the origin of the present investigation.

A potential of the form (1.1) has a compact support but this is not sufficient for this representation to hold, with $(V_\pm - \mathbb{1})$ having compact supports. A potential A_μ with compact support has such a representation iff:

$$P \exp\left[\frac{ig}{2}\int_{-\infty}^{+\infty} dx^\pm A_\pm(x^+, x^-)\right] = 1 \tag{1.4}$$

for all x^-, resp. x^+; $x^\pm = x^0 \pm x^1$, the symbol P indicates path ordering. We shall always work with potentials with compact support and call such potentials *localized.* Each of them is the representative of a class of gauge equivalent potentials which are pure gauges outside a compact domain. A localized potential fulfilling (1.4), which is a very restrictive condition, will be called *strictly localized.*

Our purpose is to compute the fermion determinant of a localized potential and determine how far it differs from (1.2). Given the potential A_μ, equation (1.1) is a differential equation for $V_\pm$; in terms of arbitrary matrices $T_\pm$, this equation is:

$$\partial_\mp T_\pm(x) = igA_\mp(x)T_\pm(x). \tag{1.5}$$

Its solution is not unique; in the case of a strictly localized potential, it has the particular solution $T_\pm = V_\pm$ specified by the boundary condition $T_\pm \to \mathbb{1}$ for $x \to \infty$. This boundary condition is impossible in the case of an arbitrary localized potential. One may have the suspicion that $iW[A]$ given by (1.2) is the logarithm of the fermion determinant in the case of a localized potential if $V_\pm$ are replaced by extensions in the t variable of suitable solutions of (1.5). Our experience of field theory in Minkowski space suggests that these solutions are defined by Feynman–Stueckelberg boundary conditions: $(T_\pm - \mathbb{1})$ have to have only negative (positive) frequencies for $x^0 \to +\infty$ $(-\infty)$. (Our convention is that $e^{i\omega x^0}$ has a positive (negative) frequency if $\omega > 0$ $(\omega < 0)$.) Matrices $T_\pm$ verifying these boundary conditions will be called causal solutions of (1.5). We observe that contrary to $V_\pm$, they are no longer unitary matrices. We shall call causal form of the functional $W[A]$ the quantity obtained by substituting $T_\pm$ to $V_\pm$ in (1.2).

Our main result will be that these conjectures are correct if the fermion determinant is defined through the fermion vacuum-to-vacuum transition amplitude $(\Omega_{\text{out}}, \Omega_{\text{in}})$ in the presence of the external potential A_μ. Furthermore we shall find that if this potential is strictly localized, the causal solutions $T_\pm$ coincide with $V_\pm$; this provides a proof of (1.2). In the general case, the causal $T_\pm$ don't have a uniform limit for $x \to \infty$ and space-time cannot be compactified. This has to be related to the fact that an external potential induces pair creations in the incoming vacuum; unitarity implies that the effective action $W[A]$ has a non vanishing positive imaginary part. It turns out that pair creation is completely suppressed if the potential is strictly localized; the S matrix of the in-out transition becomes a pure phase factor equal to the fermion determinant. The evolution induced by a strictly localized potential is thus trivial, up to a phase; this explains why space-time can be compactified in that case.

Our derivation of the causal form of $W[A]$ is based on a straightforward application of ordinary Minkowski space perturbation theory. We find it of importance to show that (1.2), especially its Wess–Zumino term, is by no means a non-perturbative result. The three dimensional integral and its auxiliary variable enter naturally into the game in the process of summing the relevant perturbation series. The fact that the fermion determinant can be obtained perturbatively was noticed by Nielsen, Rothe and Schroer [1]. Our treatment of the perturbation series is non-conventional in one respect: the normal ordering of time ordered products is done directly in terms of fermion currents. This is possible because of the free field nature of the current of free massless fermions in two dimensions.

Once the fermion determinant is known, its functional derivatives give the in-out matrix elements $(\Omega_{\text{out}}, T[j_{\mu_1}(x_1)\cdots]\Omega_{\text{in}})$ of the current $j_\mu(x)$ induced by the external potential. In the model at hand, the operator solution of the external field problem is easily obtained and the current can be defined by means of the point-splitting technique. We have thus two ways of computing matrix elements of the current and the results can be compared. As may be expected, they coincide up to local singular terms.

This paper is organized as follows. In Section 2, we develop a simple illustration of our techniques in the abelian case. The perturbative evaluation of the non-abelian fermion determinant is presented in Section 3. Various topics are collected in Section 4: equivalence of the result of Section 3 with the causal version of (1.2), generalization to arbitrary extensions in the t variable, gauge invariance of the determinant and specialization to strictly localized potentials. Section 5 deals with the properties of the current generated by an external gauge potential, and our conclusions are presented in Section 6. Technicalities are discussed in two Appendices.

A brief account of this work is being published elsewhere [12].

2. The fermion determinant in the abelian case

The discussion of this case shows in a simple way how to construct a massless fermion determinant with the help of the causal solutions of (1.5). Before we start, we should remember that, in spite of its simplicity, the two dimensional abelian case has non trivial topological features which are absent in the non-abelian case. Abelian potentials which are pure gauges at infinitey are characterized by a winding number. Only those potentials with zero winding number can be gauge transformed into a non-singular potential which vanishes at infinity. As we restrict ourselves to localized potentials, we treat the winding number zero case. The effects of a non zero winding number have been discussed by Hortaçsu, Rothe and Schroer [1].

Here, we compute the vacuum-to-vacuum amplitude in a localized potential and compare the result with the abelian causal version of (1.2). This computation is well known [13], we do it again here in order to illustrate our method based on the free field nature of the massless fermion current in two dimensions. The definition of the determinant through the vacuum-to-vacuum amplitude is:

$$\det[A] = \exp(iW[A]) = (\Omega_{\text{out}}, \Omega_{\text{in}}) \tag{2.1}$$

Using the decoupling of left- and right-goers one gets:

$$W[A] = W_+[A_-] + W_-[A_+] \tag{2.2}$$

$$\exp(iW_\pm[A_\mp]) = \left(\Omega_{\text{in}}, T\exp\left(\frac{ig}{2}\int (dx)^2 A_\mp(x) j_\pm^{(\text{in})}(x)\right)\Omega_{\text{in}}\right)$$

where $j_\pm^{(\text{in})}$ are the light-cone components of the free incoming current. The index 'in' will be dropped in this and the next section. The positive and negative frequency parts, i.e. creation and annihilation parts, of $j_\pm$ have the following commutation relations:

$$[j_\pm^{(-)}(x), j_\pm^{(+)}(y)] = -\frac{1}{\pi^2}\frac{1}{((x-y)^\pm - i\varepsilon)^2}. \tag{2.3}$$

As $j_\pm^{(-)}$ annihilates the vacuum Ω, this gives:

$$iW_\pm[A] = \frac{g^2}{8\pi^2}\int (dx)^2(dy)^2 A_\mp(x)\,\Delta_\pm(x-y)A_\mp(y) \tag{2.4}$$

where:

$$\Delta_\pm(x) = \theta(x^0)\frac{1}{(x^\pm - i\varepsilon)^2} + \theta(-x^0)\frac{1}{(x^\pm + i\varepsilon)^2}. \tag{2.5}$$

As shown in Appendix A, this is a well defined distribution which can be written as:

$$\Delta_\pm(x) = i\pi[-D'_\pm(x) + \delta^{(2)}(x)], \qquad D'_\pm(x) = \frac{\partial}{\partial x^\pm} D_\pm(x) \tag{2.6}$$

the propagators $D_\pm$ being defined in equation (2.10) below. The δ-function gives a contribution to W which is not Lorentz invariant. Furthermore, the W produced by (2.4) and (2.6) fails to be gauge invariant. These defects are cured by a redefinition of W: the δ-function in (2.6) has to be dropped and an integral over A_+A_- has to be added. The final result is:

$$W[A] = \hat{W}_+[A_-] + \hat{W}_-[A_+] + \frac{g^2}{4\pi}\int (dx)^2 A_+(x)A_-(x) \tag{2.7}$$

with:

$$\hat{W}_\pm[A] = -\frac{g^2}{8\pi}\int (dx)^2(dy)^2 A(x)D'_\pm(x-y)A(y). \tag{2.8}$$

We see that the evaluation of the vacuum-to-vacuum amplitude in terms of the free currents avoids the infinite renormalization of the fermion loop. A finite renormalization is still needed to secure Lorentz and gauge invariance. This affects only the propagator's singularity at the origin. Gauge invariance enforces a coupling of the two light-cone components through the last term in (2.7).

The expression (2.7) has to be compared with the abelian version of (1.2) obtained by dropping the Wess–Zumino terms, ignoring the traces and replacing $V_\pm$ by $T_\pm$. The causal solutions of (1.5) are constructed by means of causal propagators $D_\pm(x)$ specified by the differential equations:

$$(\partial/\partial x^\mp)D_\pm(x) = \delta^{(2)}(x) \tag{2.9}$$

and Feynman boundary conditions. It is easily established (Appendix A) that:

$$D_\pm(x) = -\frac{i}{\pi}\frac{1}{x^\pm - ix^\mp\varepsilon}. \tag{2.10}$$

The solutions of (1.5) we are looking for are

$$T_\pm(x) = \exp\left[i\frac{g}{2}\int (dy)^2 D_\pm(x-y)A_\mp(y)\right]. \tag{2.11}$$

Substituting these solutions to $V_\pm$ in the abelian form of (1.2) we obtain an expression coinciding exactly with the perturbative result (2.7).[3])

Inspection of (2.8) leads to the following observations.

(i) The effective action $W[A]$ is complex, its positive imaginary part being:

$$\operatorname{Im} W[A] = g^2\int (dp)^2\,|p^\mu\tilde{A}_\mu(p)|^2\,\delta(p^2). \tag{2.12}$$

($\tilde{A}_\mu$ is the Fourier transform of A_μ). This means that the probability of the transition $\Omega_{\text{in}} \to \Omega_{\text{out}}$ is less than one, a consequence of the fermion pairs created by the external potential. Pair creation cannot be ignored without mutilating the physical processes induced by this potential. This is precisely what would happen if, instead of (2.11), one would use solutions of (1.5) belonging to $U(1)$. For instance, the retarded solutions

$$R_\pm(x) = \exp\left[i\frac{g}{2}\int_{-\infty}^{x^\mp} dy^\mp\begin{Bmatrix}A_-(x^+, y^-)\\ A_+(y^+, x^-)\end{Bmatrix}\right] \tag{2.13}$$

lead to a real effective action, equal to the real part of (2.7).

(ii) Equation (2.12) shows that it is only the restriction of $\tilde{A}_\mu$ to the light-cone which contributes to $\operatorname{Im} W[A]$; the pairs created by the external field are either left- or right-goers, they form massless kinematical bound states.

[3]) The resulting expression for $W[A]$ is particularily simple if it is written in terms of Fourier transforms:

$$W[A] = (g^2/\pi)\int (dp)^2\tilde{A}^*_\mu(p)\tilde{A}_\nu(p)[g^{\mu\nu} - (p^\mu p^\nu/(p^2 + i\varepsilon))].$$

(iii) Equation (2.12) tells us that $\operatorname{Im} W[A]=0$, there is no pair creation, iff:

$$p_-\tilde{A}_+(p)=p_+\tilde{A}_-(p)=0. \tag{2.14}$$

On the other hand, condition (1.4) characterizing the strictly localized potentials becomes, in the abelian case:

$$(g/2)\int_{-\infty}^{+\infty} dx^{\mp} A_{\mp}(x^+, x^-)=2n_{\mp}(x^{\pm})\pi, \qquad n_{\mp}\in\mathbb{Z}. \tag{2.15}$$

In order to secure a bounded field $F_{\mu\nu}$, we have to assume $A_\pm$ continuous in $x^{\mp}$; this implies that $n_\pm$ are continuous and, in fact, identically zero. But then (2.15) is the same contraint as (2.14); the strictly localized potentials are exactly those for which pair creation is completely suppressed. Clearly $R_\pm$ defined in (2.13) coincides with $V_\pm$ if the potential is strictly localized; furthermore $T_\pm=V_\pm$ as will be established in Section 4.

3. Evaluation of the non abelian fermion determinant

As in the previous Section, we compute $W[A]$ via the vacuum-to-vacuum amplitude; needless to say, much more work has to be done here. The right- and leftgoers are still decoupled and the decomposition (2.2) holds before renormalization. The single product $A_{\mp}j_{\pm}$ in the definition of the determinant becomes a sum $A^a_{\mp}j^a_{\pm}$ over the color index a. The commutation relations of the creation and annihilation parts of the free currents $j^a_\pm$ are found to be:

$$\begin{aligned}[][j^{a(-)}_{\pm}(x), j^{b(+)}_{\pm}(y)] &= \frac{1}{2\pi}\frac{f^{abc}}{(x-y)^{\pm}-i\varepsilon}(j^{c(-)}_{\pm}(x)+j^{c(+)}_{\pm}(y))\\ &\quad -\frac{1}{8\pi^2}\frac{\delta^{ab}}{((x-y)^{\pm}-i\varepsilon)^2},\\ [j^{a(-)}_{\pm}(x), j^{b(-)}_{\pm}(y)] &= -\frac{1}{4\pi}\frac{f^{abc}}{(x-y)^{\pm}}(j^{c(-)}_{\pm}(x)-j^{c(-)}_{\pm}(y)),\end{aligned} \tag{3.1}$$

with a similar expression for the commutator of two positive frequency parts (f^{abc} are the structure constants of the gauge group). These relations represent the non-abelian extension of (2.3). A remarkable feature of the non-abelian currents is that two negative (positive) frequency parts have non vanishing commutators; notice that these commutators are regular at $x=y$. We write:

$$W_\pm[A_\mp]=-i\int (dx)^2 A^a_{\mp}(x)H^a_{\pm}(x). \tag{3.2}$$

One way of obtaining the quantity $H^a_{\mp}(x)$, which is a functional of A_μ, is through a $y\to x$ limit of the fermion propagator $S[A; x, y]$ in the presence of the background potential A_μ [14]. We shall follow a different route here and work with $H_\pm^a$ expressed in terms of the free currents $j_\pm^a$ without explicit use of their fermionic content. The perturbation expansion of $H_\pm^a$, which is also an expansion in powers of the external potential, has the form:

$$H_\pm^a(x) = \sum_{n=2}^{\infty} \frac{(ig)^n}{n!} H_{\pm,n}^a(x) \tag{3.3}$$

where:

$$H_{\pm,n}^a(x) = \int (dx_1)^2 \cdots (dx_{n-1})^2 \langle 0| \, T(j_\pm^a(x) j_\pm^{a_1}(x_1) \cdots j_\pm^{a_{n-1}}(x_{n-1})) \, |0\rangle_c$$
$$A_\mp^{a_1}(x_1) \cdots A_\mp^{a_{n-1}}(x_{n-1}), \tag{3.4}$$

$\langle 0| \, T(\ ) \, |0\rangle_c$ denoting the connected part of the vacuum expectation value of a time ordered product.

In the abelian case, the sum (3.2) reduces to its first term ($n = 2$). In the non-abelian case, there are no second order contributions coming from the terms proportional to f^{abc} in (3.1). Consequently, the second order term of $W_\pm$ has the same structure in the abelian and non-abelian cases. Furthermore, the non-abelian $H_\pm^a$ reduces to its second order term if the external potential happens to be abelian (A_μ parallel to a fixed direction in group space). As the resulting W has to be invariant under Lorentz transformations and those gauge transformations which leave the potential abelian, the same finite renormalization has to be performed as in Section 2. The higher orders don't require renormalization and $W[A]$ has the final form:

$$W[A] = -i \int (dx)^2 \Big(A_-^a H_+^a + A_+^a H_-^a + i \frac{g^2}{8\pi} A_+^a A_-^a \Big)(x) \tag{3.5}$$

where $H_\pm^a$ is given by (3.3) with $H_{\pm,n}^a$ defined in (3.4) as long as $n \geqslant 3$. For $n = 2$ one has to use

$$H_{\pm,2}^a(x) = \frac{i}{8\pi} \int (dy)^2 D'_\pm(x-y) A_\mp^a(y). \tag{3.6}$$

The higher orders can be constructed recurrently from these second order terms. The underlying recurrence scheme is based on the commutation relations (3.1) and will be explained for H_+^a. The following relations among T-functions are obtained from (3.1):

$$\begin{aligned}
&\langle 0| \, T(j_+^{a_1}(x_1) \cdots j_+^{a_n}(x_n)) \, |0\rangle = \\
&= \frac{i}{4} \sum_{k=2}^{n} D_+(x_1 - x_k) f^{a_1 a_k c} \langle 0| \, T(j_+^{a_2}(x_2) \cdots j_+^{c}(x_k) \cdots j_+^{a_n}(x_n)) \, |0\rangle \\
&+ \frac{i}{8\pi} \sum_{k=2}^{n} D'_+(x_1 - x_k) \delta^{a_1 a_k} \langle 0| \, T(j_+^{a_2}(x_2) \cdots \cancel{j_+^{a_k}(x_k)} \cdots j_+^{a_n}(x_n)) \, |0\rangle.
\end{aligned} \tag{3.7}$$

The second sum generates disconnected parts of the n-point T-function. If we disregard it, we get a correct recurrence relation for the connected T-function if

$n \geqslant 3$.[4]) It leads to a recurrence relation for $H^a_{+,n}$:

$$H^a_{+,n}(x) = (n-1)\frac{i}{4}\int (dy)^2 D_+(x-y) f^{abc} A^b_-(y) H^c_{+,n-1}(y) \tag{3.8}$$

for $n \geqslant 3$. One would like to get information on the sum (3.3) by converting the relation (3.8) into an integral equation for H^a_+. This would work straightaway if instead of the combinatorial factor $(n-1)$ there would be a factor equal to n. The factor $(n-1)$ is the correct one: it comes from the $(n-1)$ equivalent pairs (x, x_k), $k = 1, \ldots, n-1$, one can form in the right-hand side of the definition (3.4). To circumvent this unlucky $(n-1)$, one replaces a factor $(1/n)$ in (3.3) by an integral over an auxiliary parameter $t: (1/n) = \int_0^1 dt\, t^{n-1}$. Interchanging summation over n and integration over t, one gets:

$$H^a_+(x) = ig\int_0^1 dt H^a_+(x, t), \qquad H^a_+(x, t) = \sum_{n=2}^{\infty} \frac{(igt)^{n-1}}{(n-1)!} H^a_{+,n}(x). \tag{3.9}$$

The recurrence relation (3.8) implies an integral equation for the x- and t-dependent quantity $H^a_+(x, t)$:

$$H_+(x, t) = igtH_{+,2}(x) + \frac{igt}{2}\int (dy)^2 D_+(x-y)[A_-(y), H_+(x, t)]. \tag{3.10}$$

For convenience, we have switched to the matrix notation: $H_+ = \frac{1}{2}\lambda^a H^a_+$, λ^a = generators of the gauge group. We have succeeded in our search of an integral equation at the price of an additional variable. In our approach, this is the origin of the 3-dimensional integral in (1.3). A similar device has been used by Sorensen and Thomas in their evaluation of the fermion propagator at coinciding points [14].

To find the solution of equation (3.10) we differentiate this equation with respect to x^- and use (2.9), (3.6) and (2.6). This gives the differential equation:

$$\partial_- H_+(x, t) = -\frac{g}{8\pi}\partial_+ A_-(x, t) + ig[A_-(x, t), H_+(x, t)] \tag{3.11}$$

where $A_-(x, t) := tA_-(x)$. One gets a similar equation for H_- and it is easy to check that the solutions of these equations have the form

$$H_\pm(x, t) = \frac{i}{8\pi}(\partial_\pm T_\pm \cdot T_\pm^{-1})(x, t) \tag{3.12}$$

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[4]) It is instructive to see how our relation (3.7) based on the properties of the current reproduces the 3-point function written in terms of fermion propagators. Dropping all irrelevant factors and indices, (3.7) gives, with $n = 3$:

$$[(x_1 - x_2)^{-1} - (x_1 - x_3)^{-1}](x_2 - x_3)^{-2}.$$

This is identical to $-(x_1 - x_2)^{-1}(x_2 - x_3)^{-1}(x_3 - x_1)^{-1}$; this is precisely the product of fermion propagators of the triangular graph.

with matrices $T_\pm(x, t)$ verifying:

$$\partial_\mp T_\pm(x, t) = igA_\mp(x, t)T_\pm(x, t). \tag{3.13}$$

Because of the presence of the causal propagator D_+ in the integral equation (3.10), H_+ given by (3.12) is the solution of (3.10) if T_+ is the causal solution of (3.13). This means that T_+, and analogously T_-, are defined by the integral equations:

$$T_\pm(x, t) = \mathbb{1} + \frac{ig}{2}\int (dy)^2 D_\pm(x-y)A_\mp(y, t)T_\pm(y, t). \tag{3.14}$$

With these results, equation (3.5) can be rewritten:

$$W[A] = K_+[T_+] + K_-[T_-] + \frac{g^2}{4\pi}\int (dx)^2 \operatorname{Tr}(A_+A_-)(x) \tag{3.15}$$

with:

$$K_\pm[T] = \frac{1}{4\pi}\int_0^1 dt \int (dx)^2 \operatorname{Tr}(\partial_\pm T \cdot T^{-1}\partial_t(\partial_\mp T \cdot T^{-1}))(x, t). \tag{3.16}$$

We summarize the results of this section in:

Proposition 3.1. *The logarithm $iW[A]$ of the determinant of massless fermions coupled to a localized non-abelian gauge potential, defined via the vacuum-to-vacuum amplitude, is given by formulae* (3.15) *and* (3.16), *the causal matrices $T_\pm$ being the solutions of the integral equations* (3.14).

We close this section with two comments.

(i) In our derivation of Proposition 3.1, the existence of invertible solutions $T_\pm$ of the integral equations (3.14) has been taken for granted. This existence can be established as follows. Write the Ansatz:

$$T_\pm(x) = R_\pm(x)Q_\pm(x^\pm) \tag{3.17}$$

where $R_\pm$ is the unitary matrix given by the path ordered version of (2.13). As $T_\pm$ and $R_\pm$ are solutions of the same differential equations (3.13), they differ multiplicatively by matrices $Q_\pm$ depending on a single variable: x^+, resp. x^-. The existence of $R_\pm$ is secured if $A_\pm$ are integrable [15]. The integral equations (3.14) are equivalent to singular integral equations for $Q_\pm$. Applying standard methods [16] one finds that these equations have unique invertible solutions if $R_\pm$ exist.

(ii) We have absorbed the auxiliary variable t in a t-dependent potential $A_\pm(x, t) = tA_\pm(x)$ which is a particular extension of $A_\pm(x)$ to $\mathbb{M}^2 \times [0, 1]$. Inspection of (3.10) shows that t could also have been combined with g. The t-integral would then become an integral over a variable coupling constant $g' = gt$ and one would get the so-called Pauli representation noticed by Nielsen, Rothe and Schroer [1]. As will be seen in the next section, our procedure is more flexible as it can be generalized to extensions $A_\pm(x, t)$ where the x- and t-dependences do not factorize.

4. Properties of the non-abelian fermion determinant

This section is technical and has a fourfold purpose:

A: first we establish the identity of $W[A]$ given by (3.15) with the causal version of (1.2) for a special extension in the t-variable.

B: next we show that the definition (3.14) of the matrices $T_\pm$ can be generalized to other extensions without changing $W[A]$; this leads to the general form of the causal version of (1.2).

C: this point deals with the invariance of $W[A]$ under localized non-abelian gauge transformations.

D: finally we show that (1.2) coincides as it stands with its causal version if the potential is strictly localized.

A. The matrices $T_\pm(x, t)$ defined by (3.14) are extensions of solutions of (1.5) from $\mathbb{M}^2$ to the three-dimensional slice $\mathbb{M}^2 \times [0, 1]$, they reduce to the identity matrix on the bottom plane $t = 0$. Therefore, our W contains the ingredients appearing in the causal form of (1.2), however its expression (3.15) is not yet identical to it. The equality of these expressions can be obtained through integrations by parts, in so far as the $x = \infty$ boundary terms vanish. As $T_\pm$ are not constant matrices outside the support of A_μ, this is not automatically the case. Nevertheless, these boundary terms happen to be zero, precisely because $T_\pm$ are causal. To show this, we define the integrals over x as the $L \to \infty$ limits of integrals extended to the square $|x^\pm| < L$. Integrating $K_+[T_+]$ by parts, one meets two types of boundary terms:

$$B_1(\pm L) = \int_0^1 dt \int_{-L}^{+L} dx^- \operatorname{Tr}(\partial_t T_+ \partial_- T_+^{-1})(x^+ = \pm L, x^-, t),$$
$$B_2(\pm L) = \int_0^1 dt \int_{-L}^{+L} dx^+ \operatorname{Tr}(\partial_t T_+ \partial_+ T_+^{-1})(x^+, x^- = \pm L, t). \tag{4.1}$$

Lemma 4.1. *If $T_\pm$ are the solutions of the integral equations* (3.14), *the integrals $B_{1,2}[\pm L]$ tend to zero when $L \to \infty$.*

Proof. If the support of A_μ is contained in $|x^\pm| < l$, T_+ becomes a function of x^+ and t alone, once $|x^+| > l$. This is an immediate consequence of (3.14): if $|x^+| > l$, $D_+(x - y)$ can be replaced there by $(-i/\pi)(x^+ - y^+)^{-1}$. Therefore $\partial_- T_+^{-1}(x^+ = \pm L, x^-, t) = 0$ if $L > l$ and $B_1 = 0$.

Concerning B_2, we observe that $D_+(x - y)$ can be replaced by $(-i/\pi)(x^+ - y^+ - i\varepsilon)^{-1}$ in (3.14) if $x^- > l$. This implies that $T_+(x^+, L, t)$ is an analytic function of x^+ which is regular in the lower half plane, $\operatorname{Im} x^+ < 0$, and behaves there as $(\mathbb{1} + 0(1/x^+))$ for large x^+. The same is true for T_+^{-1}, which is the solution of

$$T_+^{-1}(x, t) = \mathbb{1} - \frac{ig}{2} \int (dy)^2 D_+(x - y) T_+^{-1}(y, t) A_-(y, t). \tag{4.2}$$

Consequently the integrand of $B_2(+L)$ is an analytic function of x^+, regular and decreasing like $(1/x^+)^3$ in $\mathrm{Im}\, x^+ < 0$. The path of integration over x^+ can be deformed into the lower half of the circle $|x^+| = L$ and one gets $\lim_{L\to\infty} B_2(L) = 0$. A similar reasoning holds for $B_2(-L)$, the lower half x^+ plane being replaced by the upper one.

We have thus established that integrals like (3.16) can be transformed by partial integrations without appearance of $x = \infty$ boundary terms. The result of a rather lengthy computation is that (3.15) is indeed identical to (1.2) with $V_\pm$ replaced by the solutions $T_\pm$ of the equations (3.14).

Lemma 4.1 leads to another compact expression for $W[A]$ which will be used in the discussion of point C:

$$W[A] = \frac{1}{4\pi} \int_0^1 dt \int (dx)^2 (\partial_- T \cdot T^{-1}) \partial_t (\partial_+ T \cdot T^{-1})(x, t) \tag{4.3}$$

with

$$T(x, t) = T_+^{-1}(x, t) T_-(x, t). \tag{4.4}$$

B. The integral equations (3.14) use a special interpolation of the potential between zero and its actual value $A_\mu(x): A_\mu(x, t) = tA_\mu(x)$. We show now that $W[A]$ is unchanged if we replace this particular interpolation by an arbitrary one. Let $T_\pm[A_\mp]$ be the solutions of equations (3.14) with an arbitrary smooth interpolation $A_\pm(x, t)(A_\pm(x, 0) = 0,\ A_\pm(x, 1) = A_\pm(x))$. We want to establish the following result.

Proposition 4.1. *The value of $W[A]$ obtained from* (1.2) *by replacing $V_\pm$ by causal matrices $T_\pm[A_\mp]$ is independent of the choice of the interpolation $A_\pm(x, t)$, as long as it has a compact support in x.*

Proof. We have to compare the values of the functionals $T_\pm[A_\mp]$ for different interpolations. This is done by means of a lemma that is proven in Appendix B:

Lemma 4.2. *Let $A_\pm^{(0)}$ and $A_\pm^{(1)}$ be two potentials defined on $\mathbb{M}^2 \times [0, 1]: T_\pm^{(0)}$ and $T_\pm^{(1)}$, the corresponding solutions of equations* (3.14), *are related by*:

$$T_\pm^{(1)} = T_\pm^{(0)} T_\pm[(T_\pm^{(0)})^{-1}(A_\mp^{(1)} - A_\mp^{(0)}) T^{(0)}], \qquad T_\pm^{(i)} = T_\pm[A_\mp^{(i)}]. \tag{4.5}$$

Here we have to apply this Lemma to a case where $A_\pm^{(1)}(x, 1)$ is equal to $A_\pm^{(0)}(x, 1)$. The form of (1.2) implies that Proposition 4.1 is true if the Wess–Zumino terms $Z[T_\pm]$ are independent of the interpolation:

$$Z[T_\pm^{(1)}] = Z[T_\pm^{(0)}]. \tag{4.6}$$

Our strategy will be to consider a one-parameter family of interpolations:

$$A_\pm^{(\lambda)}(x, t) = A_\pm^{(0)}(x, t) + \lambda a_\pm(x, t), \qquad a_\pm(x, t) = (A_\pm^{(1)} - A_\pm^{(0)})(x, t) \tag{4.7}$$

and prove that the corresponding Wess–Zumino terms are independent of λ.

Keeping $A_{\pm}^{(0)}$ and $a_{\pm}$ fixed, $T_{\pm}$ and $Z[T_{\pm}]$ become functions of $\lambda: T_{\pm}(\lambda)$ and $Z_{\pm}(\lambda)$. Equations (4.7) and (4.6) give:

$$T_{\pm}(\lambda) = T_{\pm}(0)T_{\pm}[\lambda\hat{a}_{\mp}], \qquad \hat{a}_{\pm}(x, t) = (T_{\mp}(0)^{-1}a_{\pm}T_{\mp}(0))(x, t). \tag{4.8}$$

This shows that $Z_{\pm}(\lambda)$ can be expressed as the Wess–Zumino functional evaluated for a product of two causal matrices. One can derive from Lemma 4.1 that, whenever $T'_{\pm}$ and $T''_{\pm}$ are such matrices:

$$Z[T'_{\pm}T''_{\pm}] = Z[T'_{\pm}] + Z[T''_{\pm}] + \text{boundary terms}, \tag{4.9}$$

the boundary terms coming from the top and bottom planes $t = 1$ and $t = 0$. In our case, they vanish because $\hat{a}_{\pm}$ is zero on these planes. Combining (4.9) and (4.8), we get:

$$Z_{\pm}(\lambda) - Z_{\pm}(0) = Z[T_{\pm}[\lambda\hat{a}_{\mp}]]. \tag{4.10}$$

We evaluate the variation of the right-hand side of this equation with respect to λ. Using Lemma 4.2 once again, we see that:

$$T_{\pm}[(\lambda + \delta\lambda)\hat{a}_{\mp}] = T_{\pm}[\lambda\hat{a}_{\mp}]T_{\pm}[\delta\lambda(T_{\pm}[\lambda\hat{a}_{\mp}])^{-1}\hat{a}_{\mp}T_{\pm}[\lambda\hat{a}_{\mp}]]. \tag{4.11}$$

According to (3.14), the second factor of the product in the right hand side has the form $(\mathbb{1} + 0(\delta\lambda))$. We apply (4.9) to this product; there are still no boundary terms and we find:

$$Z[T_{\pm}[(\lambda + \delta\lambda)\hat{a}_{\mp}]] - Z[T_{\pm}[\lambda\hat{a}_{\mp}]] = Z[\mathbb{1} + 0(\delta\lambda)]. \tag{4.12}$$

The integrand of the Wess–Zumino term containing three derivatives $Z[\mathbb{1} + 0(\delta\lambda)]$ is $0(\delta\lambda^3)$. Consequently (4.12) and (4.11) imply $(\partial/\partial\lambda)Z_{\pm}(\lambda) = 0$. This establishes (4.6).

We observe that the extensions in the t variable needed in the evaluation of $W[A]$ are determined by the choice of $A_{\pm}(x, t)$, the extension of the potential $A_{\pm}(x)$; $A_{\pm}(x, t)$ defines $T_{\pm}(x, t)$ uniquely. Any smooth extension $A_{\pm}(x, t)$ can be deformed continuously into any other one. There are no topologically inequivalent extensions and, consequently, no possibility of ambiguities in the value of $W[A]$. This has also been noticed in the euclidean context in [9].

The results of parts A and B of this section are collected in the following proposition:

Proposition 4.2. *Under the conditions stated in Proposition* 3.1, *the effective action* $W[A]$ *is given by*:

$$W[A] = I[T_+] + I[T_-^{-1}] + \frac{1}{4\pi}\int (dx)^2 \operatorname{Tr}(A_+A_-)(x), \tag{4.13}$$

the functional I *is defined in equation* (1.3) *and* $T_{\pm}(x, t)$ *are the solutions of the integral equations* (3.14) *with an arbitrary smooth localized extension* $A_{\pm}(x, t)$ *of the localized potential* $A_{\pm}(x)$.

C. In Section 3, $W[A]$ has been defined so that it is invariant under abelian gauge transformations if A_μ is abelian. We turn now to the non abelian gauge transformations

$$A_\mu(x) \to A'_\mu(x) = S(x)A_\mu(x)S^{-1}(x) - (i/g)\partial_\mu S(x)S^{-1}(x) \tag{4.14}$$

and show that $W[A] = W[A']$ if the gauge transformation is localized, i.e. if the unitary matrices $S(x)$ in the fundamental representation of the gauge group are such that the support of $(S(x) - \mathbb{1})$ is compact. To define the effect of (4.14) on W, we have to extend this transformation law to the interpolations $A_\mu(x, t)$ and $A'_\mu(x, t)$. We do this by extending $S(x)$ to $S(x, t)$ defined on $\mathbb{M}^2 \times [0, 1]$ in such a way that $S(x, 0) = \mathbb{1}$, $S(x, t) = S(x)$ (we may choose $S(x, t) = \exp[it\theta^a(x)\lambda^a]$ if $S(x) = \exp[i\theta^a(x)\lambda^a]$). Substituting $S(x, t)$ and $A_\mu(x, t)$ to $S(x)$ and $A_\mu(x)$ in (4.14), we get an interpolation $A'_\mu(x, t)$ for $A'_\mu(x)$.

As $T_\pm[A_\pm]$ are solutions of the differential equations (3.13), it is readily seen that $ST_\pm[A_\mp]$ are solutions of the same equations with $A_\mp$ replaced by $A'_\mp$. It is not immediate that $ST_\pm[A_\mp]$ are causal, i.e. that they are the solutions of the integral equations (3.14) with $A_\mp \to A'_\mp$. If this is true:

$$(T_\pm[A'_\mp])(x, t) = (ST_\pm[A_\mp])(x, t) \tag{4.15}$$

and this establishes the gauge invariance of $W[A]$. To see this one observes that: (i) $A'_\mu(x, t)$ is an interpolation of $A'_\mu(x)$ with compact support and Proposition 4.1 applies; (ii) according to (4.3)–(4.4), $W[A']$ is obtained from $T' = T_+[A'_-]^{-1}T_-[A'_+]$; (iii) equation (4.15) implies the equality $T' = T$.

Thus we have to prove (4.15). The extended transformation law (4.14) defines A'_μ as a sum of two terms and Lemma 4.2 gives:

$$T_\pm[A'_\mp] = U_\pm T[U_\pm^{-1}SA_\mp S^{-1}U_\pm] \tag{4.16}$$

with $U_\pm = T_\pm[-(i/g)\partial_\mp S \cdot S^{-1}]$. As a consequence of Proposition 4.4, which will be stated under point D, U_+ and U_- turn out to be both equal to S. This turns (4.16) into (4.15) and we have:

Proposition 4.3. *The determinant* $\exp(iW[A])$ *is invariant under localized non-abelian gauge transformations.*

D. The following proposition is proved in Appendix B:

Proposition 4.4. *Let the potential* $A_\pm$ *and its interpolation be strictly localized*:

$$A_\pm(x, t) = -(i/g)(\partial_\pm V_\mp \cdot V_\mp^{-1})(x, t) \tag{4.17}$$

where $V_\pm(x, t)$ *are unitary matrices in the fundamental representation of* $SU(N)$ *such that* $(V_\pm - \mathbb{1})$ *have compact support. The corresponding causal matrices coincide with* $V_\pm$:

$$(T_\pm[A_\mp])(x, t) = V_\pm(x, t) \tag{4.18}$$

With this result, we recover (1.2) in the case of a strictly localized potential. The matrices $V_\pm$ being unitary, the effective action is real. Therefore pair creation is completely suppressed in a strictly localized potential in the non-abelian case as well as in the abelian one. At the end of Section 2, we could show that in the abelian case there is absence of pair creation only if the potential is strictly localized; we didn't succeed in extending this result to the non-abelian situation.

5. Fermionic currents in an external potential

As mentioned in the introduction, we have two ways of determining properties of the fermion current in the presence of a localized external potential: from functional derivatives of the fermion determinant and from the operator solution of the external field problem. It is instructive to check whether both methods give the same result. We display a detailed discussion of the in–out matrix elements of the current:

$$\langle j^a_\pm(x)\rangle := (\Omega_{\text{out}}, j^a_\pm(x)\Omega_{\text{in}}). \tag{5.1}$$

It is related to the first order functional derivative of $W[A]$:

$$\langle j^a_\pm(x)\rangle = -(2i/g)\frac{\delta}{\delta A^a_\mp(x)}\exp(iW[A]). \tag{5.2}$$

The evaluation of this derivative requires the knowledge of the first variation of $T_\pm[A_\mp]$ which can be obtained from Lemma 4.2. One finds:

$$\delta T_\pm = igT_\pm(x, t)\,\delta\phi_\pm(x, t) \tag{5.3}$$

where:

$$\delta\phi_\pm(x) = \frac{1}{2}\int (dy)^2 D_\pm(x-y)(T^{-1}_\pm\,\delta A_\mp T_\pm)(y, t). \tag{5.4}$$

Here and in the rest of this section $T_\pm$ stands for $T_\pm[A_\mp]$.

Proposition 4.1 tells us that the variation of W depends only on $\delta A_\pm(x) = \delta A_\pm(x, 1)$ if the variation $\delta A_\pm(x, t)$ has compact support in x and $\delta A_\pm(x, 0) = 0$. Using (5.3) and, for instance, the expression (3.15), (5.2) becomes:

$$\langle j_\pm(x)\rangle = \frac{i}{4\pi}e^{iW[A]}(\partial_\pm T_\pm\cdot T^{-1}_\pm - \partial_\pm T_\mp\cdot T^{-1}_\mp)(x). \tag{5.5}$$

This result shows that, in general, $\langle j_\pm(x)\rangle$ has no compact support. Outside the support of A_-, $\langle j_+(x)\rangle$ is locally a function of x^+ alone; globally one has two distinct functions of this variable (Fig. 1a).

We turn now to the determination of $\langle j_+(x)\rangle$ through the operator solution of the external field problem. The solution of the Dirac equation is:

$$\psi_\pm(x) = R_\pm(x)\psi^{(\text{in})}_\pm(x^\pm) \tag{5.6}$$

where $R_\pm$ is the retarded solution of (3.13), that is the path ordered version of

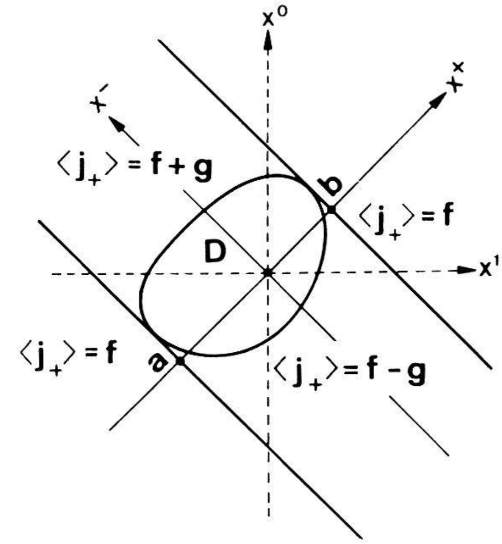

Figure 1.a
Properties of the in–out matrix element $\langle j_+(x)\rangle$ of the current defined in equation (5.5). Outside the support D of the potential, this quantity is locally a left-mover, a function of x^+ alone. It has different values in the past and future left moving causal shadow of D. One may write $\langle j_+\rangle$ in terms of two functions $f(x^+)$ and $g(x^+)$, as indicated in the figure (supp $g = (a, b)$ where (a, b) is the x^+-projection of D, this domain is assumed to be connected for simplicity).

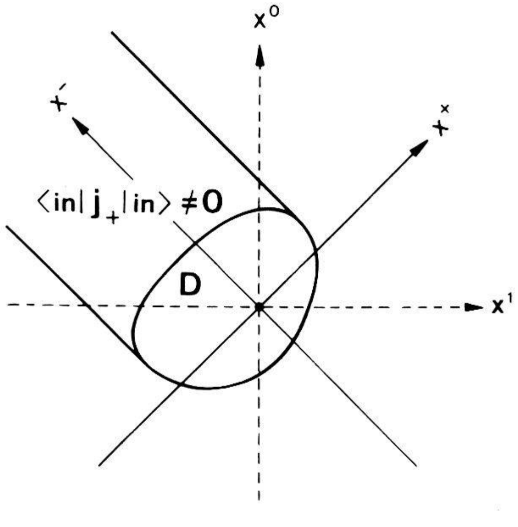

Figure 1.b
The in–in matrix element $\langle \text{in}\, |j_+(x)|\, \text{in}\rangle$ of the current. It is zero outside D, except in the future left moving causal shadow of D, where it signals the pairs created by the external potential.

(2.13) and $\psi^{(\text{in})}$ is the quantized canonical free ingoing fermion field. The point-split definition of the fermion current is:

$$j^a_\mu(x) = \frac{1}{2}\lim_{\varepsilon\to 0}\,[\bar\psi(x)\gamma_\mu \tfrac{1}{2}\lambda^a \exp(-igA_\nu(x)\varepsilon^\nu)\psi(x+\varepsilon) + (\varepsilon \to -\varepsilon)]. \tag{5.7}$$

Insertion of (5.6) gives, in matrix notation [17]:

$$j_\pm(x) = J_\pm(x) + R_\pm(x) j^{(\text{in})}_\pm(x) R^{-1}_\pm(x), \tag{5.8}$$

$j_{\pm}^{(\mathrm{in})}$ is the current of the ingoing field and $J_{\pm}$ is a c-number:

$$J_{\pm}(x) = \frac{i}{4\pi}(\partial_{\pm}R_{\pm}\cdot R_{\pm}^{-1} - \partial_{\mp}R_{\pm}\cdot R_{\pm}^{-1})(x). \tag{5.9}$$

It is equal to the mean value of the current induced in the incoming vacuum:

$$(\Omega_{\mathrm{in}}, j_{\pm}(x)\Omega_{\mathrm{in}}) = J_{\pm}(x). \tag{5.10}$$

We notice that the incoming fields $\psi_{\pm}^{(\mathrm{in})}$, $j_{\pm}^{(\mathrm{in})}$ are invariant under the gauge transformations (4.13). The fields ψ and $j_{\pm}$ transform correctly because the transformation law of $R_{\pm}$ is $R_{\pm}(x) \to S(x)R_{\pm}(x)$.

The definition (5.9) implies that $J_{\pm}(x)$ has the support properties imposed by its identification (5.10) as in–in matrix element of the current. It is non zero only in the future causal shadow of the external potential, more precisely in the left-, resp. right-moving shadow (Fig. 1b). this is in contrast with $\langle j_{\pm}(x)\rangle$; this quantity is not a mean value in a given state, it is complex and causality does not restrict its localization.

Comparing (5.5) with (5.10) and (5.9), we discover that, if (5.2) is correct, the in–out matrix element of $j_{\pm}$ is obtained from its in–in mean value through multiplication by $\exp(iW)$ and replacement of $R_{\pm}$ by $T_{\pm}$. Equation (5.8) shows that this substitution has to be an effect of the in–out matrix element of the ingoing current. This equation gives:

$$\langle j_{\pm}(x)\rangle = e^{iW}J_{\pm}(x) + R_{\pm}(x)\langle j_{\pm}^{(\mathrm{in})}(x)\rangle R_{\pm}^{-1}(x). \tag{5.11}$$

To establish the identity of (5.5) with this point-splitting result, one has to evaluate $\langle j_{\pm}^{(\mathrm{in})}\rangle$ in the operator solution approach. This requires the construction of Ω_{out} in terms of in-states by means of the $\psi_{\mathrm{out}} - \psi_{\mathrm{in}}$ relationship defined by (5.6). We restrict ourselves to an easier consistency check based on the following observation. The free current $j_{+}^{(\mathrm{in})}(x)$ is a function of x^{+} alone; we do not change its value if we replace x by any other point x' such that $x'^{+} = x^{+}$. If x' is chosen outside the future of the support D of the potential, $\psi^{(\mathrm{in})}(x') = \psi(x')$, $A_{-}(x') = 0$, and, consequently $\langle j_{+}^{(\mathrm{in})}(x)\rangle = \langle j_{+}(x')\rangle$. We may use the value of the latter quantity given by (5.5) and insert the resulting expression of $\langle j_{+}^{(\mathrm{in})}(x)\rangle$ into (5.11). This leads to the following test: the point-split definition of the current is consistent with its definition via derivatives of the determinant, at least at the level of single current matrix elements, if:

$$J_{\pm}(x) = e^{-iW}[\langle j_{\pm}(x)\rangle - R_{\pm}(x)\langle j_{\pm}(x')\rangle R_{\pm}^{-1}(x)] \tag{5.12}$$

for $x' < D$ or $x' \sim D$, $x'^{+} = x^{+}$, resp. $x'^{-} = x^{-}$, $\langle j_{\pm}\rangle$ in the right hand side being given by (5.5). To prove (5.12), we use the representation (3.17) of $T_{\pm}$. As $R_{\pm}(x') = \mathbb{1}$, it shows that $Q_{\pm}(x') = T_{\pm}(x')$ and another way of writing (3.17) is:

$$T_{\pm}(x) = R_{\pm}(x)T(x'). \tag{5.13}$$

Inserting this into (5.5), we find that the right hand side of (5.12) is indeed identical to the expression (5.9) of $J_{\pm}$.

The previous discussion holds for the general case of a localized potential. If the potential is strictly localized, $T_\pm = V_\pm$ according to Proposition (4.4); furthermore (1.4) implies that the retarded matrices $R_\pm$ coincide with $V_\pm$. As a consequence the outgoing fermion field $\psi^{(\text{out})}$, obtained from (5.6) if x is in the future of D, is identical to the ingoing field. Therefore, the S-matrix implementing the automorphism $\psi^{(\text{in})} \to \psi^{(\text{out})}$ reduces to a pure phase factor equal to $\exp(iW)$, as announced in the introduction. Furthermore, $\langle j_\pm \rangle$ becomes proportional to the mean value in the incoming vacuum and it has the same compact support D as the potential. All these facts are in accordance with the complete absence of pair creation.

In the remainder of this section, we indicate how the previous results extend to the two-point functions of the induced current. In terms of the determinant's derivatives, we have:

$$(\Omega_{\text{out}}, T(j^a_\pm(x) j^b_\pm(y))\Omega_{\text{in}}) = -\frac{4}{g^2} \frac{\delta^2}{\delta A_\mp(x)\, \delta A_\mp(y)} e^{iW[A]}. \tag{5.14}$$

This gives:

$$\begin{aligned}(\Omega_{\text{out}}, &\, T(j^a_+(x) j^b_+(y))\Omega_{\text{in}}) \\ &= \langle j^a_+(x)\rangle \langle j^b_+(y)\rangle e^{-iW} + \frac{i}{16\pi} \operatorname{Tr}(T_+^{-1}(x)\lambda^a T_+(x) T_+^{-1}(y)\lambda^b T_+(y)).\end{aligned} \tag{5.15}$$

An argument following the same lines as above shows that this result is consistent with the corresponding two-point function obtained from the point-split definition (5.7) of the current. As already noticed by several authors [18] this consistency is no longer complete for the two-point functions of the two distinct light-cone components of the current. On one hand (5.14) gives:

$$(\Omega_{\text{out}}, T(j^a_+(x) j^b_-(y))\Omega_{\text{in}}) = \langle j^a_+(x)\rangle \langle j^b_-(y)\rangle e^{-iW} - \frac{i}{2\pi} \delta^{ab}\, \delta^2(x-y) e^{iW}. \tag{5.16}$$

On the other hand the components j^a_+ and j^b_- given by (5.8) are totally uncorrelated, commuting operators. Therefore, their two-point function reduces to the first product in the right-hand side of (5.16). There is a discrepancy coming from the second term in this right-hand side. Its origin is the integral over $\operatorname{Tr}(A_+A_-)$ in the expression (1.2) of $W[A]$ whose presence was imposed by gauge invariance. Thus we have two distinct gauge invariant schemes which define the same currents, up to some connected two-point functions at coinciding points.

6. Conclusions

The central objective of the present work was the determination of the determinant of massless fermions in two space-time dimensions from its perturbation series. This goal has been achieved in the case of a non-abelian localized gauge potential, i.e. a potential A_μ with compact support D in Minkowski space

$\mathbb{M}^2$. The logarithm $iW[A]$ of the determinant can be written in various equivalent ways as a sum of integrals over $\mathbb{M}^2$ and, for some of them, an auxiliary variable t. The building blocks of the integrands are matrices $T_\pm$, (x, t)-dependent functionals of the background potential extended in the t variable. These matrices are specified by causal Feynman–Stueckelberg boundary conditions; they are not unitary matrices, they are not constant outside the support D and they have no uniform limit for $x \to \infty$. These asymptotic properties do not show up in the integrals giving $W[A]$; all their integrands contain a factor A_+ or A_- and the integals extend in fact to D. However, the asymptotic properties of $T_\pm$ prevent a complete formulation of our results in terms of functions which are non-singular on a compactified space-time. This is our answer to a question raised in the introduction.

A compactified space-time can be used without limitations only if the potential is strictly localized, i.e. if it has compact support and fulfills the constraint (1.4). This is a very limited class of potentials in which all created pairs are annihilated before the potential is switched off. For a strictly localized potential, the matrices $T_\pm$ are equal to the identity outside D. What is more important is that they are unitary; $W[A]$ can be identified as a term of the action of a non-linear σ-model through a change of variables $A_\pm \to \phi_\mp$, with $T_\pm = \exp(i\phi_\pm)$ [11]. It must be stressed that the identification of the complete gauge theory to such a model in Minkowski space is legitimate only if it makes sense to restrict the functional integration over the gauge potential to the class of strictly localized ones. To our knowledge, this is an open question. In any case, our work shows how to compute the fermion determinant unambiguously in the more general case of a localized potential.

After completion of this work our attention has been drawn on a paper by H. Arodz [19] where the relevance of Feynman–Stueckelberg boundary conditions in the Minkowsi space computation of fermion determinants is emphasized.

One of the authors (G. W.) had the benefit of an instructive and encouraging discussion with H. Leutwyler.

Appendix A. Propagators

We observe that the function

$$D_+(x) = -\frac{i}{\pi}\left(\theta(x^0)\frac{1}{x^+ - i\varepsilon} + \theta(-x^0)\frac{1}{x^+ + i\varepsilon}\right) \tag{A.1}$$

fulfills the Feynman–Stueckelberg boundary conditions. If $x^0 > 0$, it reduces to $(-i/\pi)(x^+ - i\varepsilon)^{-1}$, a function which has only negative frequencies. Similarily, $D_+(x)$ reduces to a function with only positive frequencies if $x^0 < 0$. Another form of definition (A.1) is

$$D_+(x) = -\frac{i}{\pi} P\frac{1}{x^+} + \operatorname{sgn} x^0\, \delta(x^+). \tag{A.2}$$

Because of the delta function, $\operatorname{sgn} x^0$ can be replaced by $\operatorname{sgn} x^-$; it is evident from (A.2) that $D_+(x)$ is a solution of equation (2.9). The expression (2.10) is obtained by transforming (A.2) into:

$$D_+(x) = -\frac{i}{\pi}(x^+ - i\varepsilon \operatorname{sgn} x^-)^{-1} \tag{A.3}$$

and replacing $\varepsilon \operatorname{sgn} x^-$ by εx^-.

To derive (2.6), one transforms the definition (2.5) of Δ_+ as follows:

$$\begin{aligned}\Delta_+(x) &= -\,\theta(x^0)\frac{\partial}{\partial x^+}\left(P\frac{1}{x^+} + i\pi\,\delta(x^+)\right) - \theta(-x^0)\Big(\quad\Big)\\ &= -\frac{\partial}{\partial x^+}P\frac{1}{x^+} - i\pi \operatorname{sgn} x^0 \frac{\partial}{\partial x^+}\delta(x^+)\\ &= -\frac{\partial}{\partial x^+}\left(P\frac{1}{x^+} - i\pi \operatorname{sgn} x^0\,\delta(x^+)\right) + i\pi\frac{\partial}{\partial x^+}\operatorname{sgn} x^0\,\delta(x^+).\end{aligned} \tag{A.4}$$

The last expression coincides with (2.6).

Appendix B. Proving a lemma and a proposition of Section 4

Proof of Lemma 4.2. We establish equation (4.5) for T_+ and drop the indices $\pm$. We write

$$T^{(1)} = T^{(0)}Q \tag{B.1}$$

and determine the matrix Q. As $T^{(0)}$ and $T^{(1)}$ are solutions of the differential equation (3.13), we find that Q too is a solution of such an equation:

$$\partial_- Q(x, t) = igB(x, t)Q(x, t) \tag{B.2}$$

where $B(x, t) = ((T^{(0)})^{-1}(A^{(1)} - A^{(0)})T^{(0)})(x, t)$. The solutions of (B.2) being determined up to right multiplication by an arbitrary matrix depending only on x^+ and t, Q is related to the causal solution $T[B]$ by:

$$Q(x, t) = (T[B])(x, t)Q_+(x^+, t). \tag{B.3}$$

Eliminating Q from this equation by means of its definition (B.1), we see that Q_+ is the product of 3 causal matrices:

$$Q_+(x^+, t) = ((T[B])^{-1}(T^{(0)})^{-1}T^{(1)})(x, t). \tag{B.4}$$

The properties of causal matrices used in the proof of Lemma 4.1 imply that the right-hand side of equation (B.4) is a function of x^+ and t if $x^- > l$ $(x^- < -l)$ which is regular and tends asymptotically to $\mathbb{1}$ in $\operatorname{Im} x^+ < 0$ $(\operatorname{Im} x^+ > 0)$. Therefore $Q_+(x^+, t)$ is bounded and regular in the shole x^+-plane; it has to be identical to its value at infinity, i.e. $Q_+(x^+, t) = \mathbb{1}$. This result, equations (B.3) and (B.1) give equation (4.5).

Proof of Proposition 4.4. We check that V_+ is the solution of (3.14) if A_- is given by (4.15). Inserting (4.15) into the right-hand side of (3.14) and replacing T_+ by V_+, we get:

$$\mathbb{1} + \int (dy)^2 D_+(x-y)(\partial/\partial y^-)V_+(y, t). \tag{B.5}$$

Integrating by parts, using $V(y^+, \pm\infty) = \mathbb{1}$ and the explicit form (2.10) of D_+, we find that the boundary terms cancel the identity matrix and we are left with:

$$\int (dy)^2 (\partial/\partial x^-)D_+(x-y)V_+(y, t). \tag{B.6}$$

According to (2.9), this is identical to $V_+(x, t)$.

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