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Autor: Jetzer, P. / Anishetty, R.

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Vacuum decay in gauge theories

By P. Jetzer[1]) and R. Anishetty[2])

Theoretical Physics, ETH-Hönggerberg, CH-8093 Zürich

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Abstract. Using the semiclassical method of instantons, developed by Callan and Coleman, we investigate the vacuum decay rate at zero temperature in theories containing scalar and gauge bosons. In particular we analyse a $U(1)$ gauge theory with a complex scalar and an $SU(N)$ gauge theory with scalar fields in the adjoint representation. Some of the results we obtain are of general validity and some are deduced in the thin-wall approximation.

1. Introduction

Grand unified [1] theories may describe the evolution of the early universe [2]. These theories have, in general, several ground states in which the original symmetry group is broken into various subgroups with different properties. It is usually assumed that the universe began in a symmetric phase at high temperature and cooled by expansion. At a certain temperature, a less symmetric ground state became energetically more favorable, i.e. had a lower energy density. However due to the inertia in the system, i.e. lack of instantaneous communications the system remained in the symmetric phase. But locally, little bubbles of less symmetric phase began to form; these grew to fill all of space [3]. At the point when the first bubbles start to form, the energy density difference Δ between their outside and inside (i.e. between the two corresponding minima of the model) is small; this is the situation in which one can use the thin-wall approximation to compute the bubble production rate.

Callan and Coleman [4] have developed a semiclassical approximation for describing the density of bubbles of a certain radius R; the method closely resembles the work of Langer [5]. The procedure simplifies in the thin-wall limit, which could be of relevance in estimating the time for which the unbroken phase underwent supercooling. In fact, if the nucleation rate for this phase transition turns out to be low, one expects the universe, as it expands, to supercool in the unbroken phase (high temperature phase). This is a crucial point for the inflationary models to work [3].

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[1]) Present address: Dépt. de Physique théorique, Université de Genève, 1211 Genève 4, Switzerland.

[2]) Present address: 5, Kundenbagh, Hyderabad A.P. 500 016 India

In the semiclassical approximation the decay rate per unit volume Γ/V of the false vacuum (unbroken phase) is given by [4]:

$$\frac{\Gamma}{V} = Ke^{-S_B/\hbar} \tag{1.1}$$

where K is the imaginary part of a ratio of determinants and S_B is the classical action of the bounce (classical instanton). As far as we know, this formula has been computed taking into account only the scalar fields [7–9] of a grand unified theory. Already in this case, computing K and S_B is a difficult task, feasible often only numerically. In estimates, K is usually approximated by M^4, where M is a typical mass involved in the problem, for instance the grand unification mass.

In Ref. [6] the factor K has been studied in detail for pure scalar theory in thin-wall approximation. It was realized that in this approximation there are many zero mode fluctuations about the equilibrium bubble configuration which give an important contribution to K.

In the present paper we extend the analysis of the factor K to theories containing scalar and gauge bosons. One might expect that, due to the inclusion of the vector bosons, there are additional zero modes, such as the Goldstone mode in global theories. Our analysis shows however that these zero modes are eliminated by the Higgs mechanism. It turns out that there are also negative and positive modes, which are unphysical, because, due to gauge fixing, they do not contribute to the factor K. The analysis we undertook is in Lorentz gauge but it is expected to hold in other gauges as well. Due to the complexity of the bounce equations, we solve them by making an ansatz, which turns out to be the solution with lowest action.

We consider two different cases: one with an $U(1)$ gauge theory with a complex scalar and one with an $SU(N)$ gauge theory with the scalar fields in the adjoint representation. The paper is organized in the following way: in section two we discuss the local $U(1)$ theory and in section three the global $U(1)$ case. In section four we treat the local $SU(N)$ gauge theory. In five appendices we explain in detail some results used in the main text.

2. Local $U(1)$ gauge theory

The Lagrangian, in Euclidean space, has the form:

$$\mathscr{L}_E = \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\mathscr{D}_\mu\phi)(\mathscr{D}^\mu\phi)^* + V(\phi, \phi^*) \tag{2.1}$$

It includes a gauge boson A_μ (with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$) and a complex scalar field ϕ. We take the Lorentz gauge $\partial^\mu A_\mu = 0$. The scalar potential $V(\phi, \phi^*)$ is such that it has two minima, a local one at $\phi = 0$ and a global one at $|\phi| = \phi_-$ with:

$$\begin{aligned} &V(|\phi| = 0) = 0 \\ &V(|\phi| = \phi_-) = -\Delta, \qquad \Delta > 0 \end{aligned} \tag{2.2}$$

(see Fig. 1)

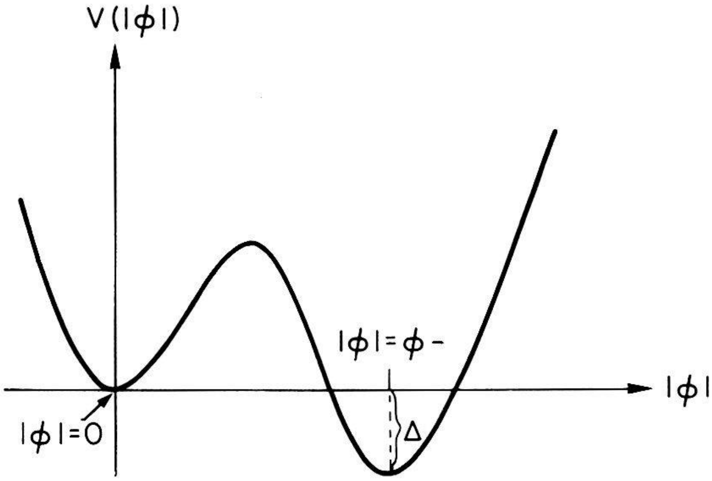

Figure 1
Typical shape of the potential $V(|\phi|)$.

For a $U(1)$ invariant potential V to have such a form requires the inclusion of nonrenormalizable terms or two or more scalar fields. For our purpose here we shall ignore this deficiency. The semiclassical analysis that we perform up to one loop, can be done for general potentials.

We consider the case where initially the system is in a $U(1)$ invariant metastable vacuum ($\phi = 0$). We expect that after some time, due to quantum fluctuations, it decays into the vacuum with lower energy density which breaks the $U(1)$ symmetry.

For the following computations it is more convenient to introduce two real scalar fields instead of one complex scalar field. We have then:

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$$
$$\phi^* = \frac{\phi_1 - i\phi_2}{\sqrt{2}} \tag{2.3a}$$

and

$$\mathscr{L}_E = \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}(\partial_\mu\phi_1)^2 + \tfrac{1}{2}(\partial_\mu\phi_2)^2 + \tfrac{1}{2}e^2A_\mu A^\mu(\phi_1^2 + \phi_2^2) + eA_\mu(\phi_2\,\partial^\mu\phi_1 - \phi_1\,\partial^\mu\phi_2) + V(\phi_1^2 + \phi_2^2) \tag{2.3b}$$

We want to compute the decay rate, per unit space volume, of the metastable vacuum, which is given, following Coleman's semiclassical method [4], by

$$\frac{\Gamma}{V} = Ke^{-S_B/\hbar} \tag{2.4}$$

where S_B is the classical action of the bounce. The factor K is the imaginary part of a ratio of two determinants, which are obtained by evaluating the functional integral

$$Z = \int \mathscr{D}A_\mu \mathscr{D}\phi_1 \mathscr{D}\phi_2 \prod_x \delta(\partial_\mu A^\mu) e^{-\int d^4x \mathscr{L}_E} \tag{2.5}$$

in the steepest-descent approximation, once with the bounce as background field and once without background field (corresponding to the false vacuum).

2.1. *The bounce equations*

The bounce satisfies the Euler–Lagrange equations:

$$\begin{cases} -\Delta\phi_1 - 2eA_u\, \partial^\mu\phi_2 + e^2A^2\phi_1 + \dfrac{\partial V}{\partial\phi_1} = 0 \\ -\Delta\phi_2 + 2eA_\mu\, \partial^\mu\phi_1 + e^2A^2\phi_2 + \dfrac{\partial V}{\partial\phi_2} = 0 \\ -\Delta A_\mu = e(\phi_2\, \partial_\mu\phi_1 - \phi_1\, \partial_\mu\phi_2) + e^2A_\mu(\phi_1^2 + \phi_2^2) \end{cases} \tag{2.6}$$

(Δ is the Laplace-operator in 4 dimensions) with boundary conditions:

$$\begin{aligned} &\lim_{x_4 \to \pm\infty} \phi_i(\vec{x}, x_4) = 0 \qquad i = 1, 2 \\ &\lim_{x_4 \to \pm\infty} A_\mu(\vec{x}, x_4) = 0. \end{aligned} \tag{2.7}$$

In order for the bounce action to be finite we require:

$$\begin{aligned} &\lim_{|\vec{x}| \to \infty} \phi_i(\vec{x}, x_4) = 0 \qquad i = 1, 2 \\ &\lim_{|\vec{x}| \to \infty} A_\mu(\vec{x}, x_4) = 0 \end{aligned} \tag{2.8}$$

A typical bounce in this theory can in general involve non-trivial ϕ_i ($i = 1, 2$) and A_μ fields. However not all such bounces are relevant in computing the decay rate of the metastable vacuum. Some of these bounces may not give any contribution to the factor K. In fact by scaling argument (Appendix A) we can demonstrate that there may exist bounces (or background fields) such that $A_\mu \neq 0$, which are completely stable. In other words, in gauge theories we may have stable solitons in any dimensions. For our purpose here we shall be considering the following ansatz:

$$A_\mu(\vec{x}, x_4) \equiv 0, \qquad \phi_2(\vec{x}, x_4) \equiv 0 \tag{2.9}$$

This renders our bubbles chargeless. Consequently we do not expect long range forces between bubbles. For such configurations, the scaling argument of Appendix A demonstrates the existence of a negative mode. Furthermore the ansatz is the solution to the Euler–Lagrange equations of motion with minimum action [15].

Then equation (2.6) reduces to:

$$-\Delta\phi_1 + \frac{\partial V}{\partial \phi_1} = 0 \tag{2.10}$$

with boundary conditions:

$$\begin{cases} \lim\limits_{x_4 \to \pm\infty} \phi_1(\vec{x}, x_4) = 0 \\ \lim\limits_{|\vec{x}| \to \infty} \phi_i(\vec{x}, x_4) = 0 \end{cases} \tag{2.11}$$

Note that $\partial V/\partial\phi_2|_{\phi_2=0} = 0$ is automatically satisfied, since V is a function of

$$y = \phi\phi^* = \frac{\phi_1^2 + \phi_2^2}{2} \tag{2.12}$$

alone. In fact y is the only $U(1)$ invariant quantity one can form. Then:

$$\left.\frac{\partial V}{\partial \phi_2}\right|_{\phi_2=0} = \left.\frac{\partial V(y)}{\partial y}\phi_2\right|_{\phi_2=0} = 0.$$

Equation (2.10) with the boundary conditions (2.11) is the same as one would get in the case of a Lagrangian with only one real scalar field. It is known that in this latter case the solution of equation (2.10), which leads to the minimal action, is $O(4)$ invariant [10]. The solution of equation (2.10), ϕ^B, can thus be taken as a function of r (the radial distance in 4 dimensions) alone. The boundary conditions, equation (2.11), are then:

$$\begin{cases} \left.\dfrac{d\phi^B}{dr}\right|_{r=0} = 0 \\ \lim\limits_{r \to \infty} \phi^B(r) = 0 \end{cases}$$

2.2. *The fluctuation operator*

The functional integral (2.5) is approximated by the Gaussian fluctuations around the bounce ($\phi_1 = \phi^B$, $\phi_2 = 0$, $A_\mu = 0$). The fluctuation operator $\mathscr{L}''_B$ is given by (see appendix B):

$$\mathscr{L}''_B = \frac{1}{2}\begin{bmatrix} (-\Delta + e^2(\phi^B)^2)\,\delta_{\mu\nu} & 2e(\partial_\nu\phi^B) & 0 \\ 2e(\partial_\mu\phi^B) & -\Delta + \left.\dfrac{\partial^2 V}{\partial\phi_2^2}\right|_{\substack{\phi_1=\phi^B \\ \phi_2=0}} & 0 \\ 0 & 0 & -\Delta + \left.\dfrac{\partial^2 V}{\partial\phi_1^2}\right|_{\substack{\phi_1=\phi^B \\ \phi_2=0}} \end{bmatrix} \tag{2.13}$$

We see from equation (2.13) that $\mathscr{L}_B''$ factorizes into two pieces:

$$A_B = \frac{1}{2}\left(-\Delta + \frac{\partial^2 V}{\partial \phi_1^2}\bigg|_{\substack{\phi_1=\phi^B \\ \phi_2=0}}\right) \tag{2.14}$$

and

$$B_B = \frac{1}{2}\begin{bmatrix} (-\Delta + e^2(\phi^B)^2)\,\delta_{\mu\nu} & 2e(\partial_\nu \phi^B) \\ 2e(\partial_\mu \phi^B) & -\Delta + \frac{\partial^2 V}{\partial \phi_2^2}\bigg|_{\substack{\phi_1=\phi^B \\ \phi_2=0}} \end{bmatrix} \tag{2.15}$$

The operator A_B has 4 zero modes, because the system is translationally invariant [4]. Differentiating equation (2.10) with respect to x_μ one gets:

$$-\Delta\, \partial_\mu \phi^B + \frac{\partial^2 V}{\partial \phi_1^2}\bigg|_{\substack{\phi_1=\phi^B \\ \phi_2=0}} \partial_\mu \phi^B + \frac{\partial^2 V}{\partial \phi_1\, \partial \phi_2}\bigg|_{\substack{\phi_1=\phi^B \\ \phi_2=0}} \partial_\mu \phi_2 = 0$$

The last term vanishes. (In fact, $\partial_\mu \phi_2$ vanishes and also

$$\frac{\partial^2 V}{\partial \phi_1\, \partial \phi_2} = \frac{\partial^2 V}{\partial y^2}\, \phi_1 \phi_2 \bigg|_{\substack{\phi_1=\phi^B \\ \phi_2=0}} = 0\Bigg).$$

Thus:

$$\left(-\Delta + \frac{\partial^2 V}{\partial \phi_1^2}\bigg|_{\substack{\phi_1=\phi^B \\ \phi_2=0}}\right) \partial_\mu \phi^B = 0 \tag{2.16}$$

which means that A_B has 4 eigenfunctions with zero eigenvalue. Due to the fact that ϕ^B depends only on r also

$$\frac{\partial^2 V}{\partial \phi_1^2}\bigg|_{\substack{\phi_1=\phi^B \\ \phi_2=0}}$$

is a spherically symmetric function. One can therefore decompose the operator A_B and its eigenfunctions into radial and angular pieces. It is then clear that the eigenfunctions $\partial_\mu \phi^B = (x_{\mu/r})(\partial_r \phi^B)$ have angular momentum $j = 1$. This implies that A_B has at least one eigenfunction with $j = 0$ and negative eigenvalue, which contributes then to the imaginary part of the path integral (2.5).

Moreover, since $A_\mu = 0$, $\phi_2 = 0$, the remaining Euler–Lagrange equation (equation (2.10)) is the same as one would obtain in the case of a Lagrangian with one single real scalar field. For this case it has been shown [10] that there is only one negative eigenvalue.

Thus the operator A_B has one negative eigenvalue. With the bounce as background field, the functional integral equation (2.5) is:

$$Z_B \cong e^{-S_B} \int \mathscr{D}\phi_1 c^{-\int d^4x(\phi_1,\, A_B \phi_1)} \int \mathscr{D}A_\mu \mathscr{D}\phi_2 \prod_x \delta(\partial_\mu A^\mu) e^{-\int d^4x[(A_{\mu_1},\, \phi_2)^+ B_B(A_\mu,\, \phi_2)]} \tag{2.17}$$

where S_B is the classical action of the bounce, and ϕ_1, ϕ_2, A_μ are fluctuations around it.

Before we continue to evaluate equation (2.17) further, we discuss the functional integral equation (2.15) in the false vacuum ($\phi^B = 0$), in order to compute then the quantity K defined in equation (2.4). In the false vacuum the functional integral equation (2.5) is approximated by the Gaussian fluctuations around $\phi_1 = 0$, $\phi_2 = 0$, $A_\mu = 0$. The fluctuation operator $\mathscr{L}_0''$ is given by:

$$\mathscr{L}_0'' = \frac{1}{2}\begin{bmatrix} -\Delta\,\delta_{\mu\nu} & 0 & 0 \\ 0 & -\Delta + \omega_2^2 & 0 \\ 0 & 0 & -\Delta + \omega_1^2 \end{bmatrix} \tag{2.18}$$

with

$$\omega_2^2 = \left.\frac{\partial^2 V}{\partial \phi_2^2}\right|_{\phi_1=\phi_2=0}$$

$$\omega_1^2 = \left.\frac{\partial^2 V}{\partial \phi_1^2}\right|_{\phi_1=\phi_2=0}.$$

ω_1^2 and ω_2^2 are positive quantities, since the false vacuum is a minimum of V; ω_1^2 and ω_2^2 are the values of the curvature of the potential at $\phi_1 = \phi_2 = 0$.

$\mathscr{L}_0''$ also factorizes; we write the factors in the following way:

$$A_0 = \tfrac{1}{2}(-\Delta + \omega_1^2) \tag{2.19}$$

and

$$B_0 = \frac{1}{2}\begin{bmatrix} -\Delta\,\delta_{\mu\nu} & 0 \\ 0 & -\Delta + \omega_2^2 \end{bmatrix} \tag{2.20}$$

The corresponding functional integral Z_0 becomes:

$$Z_0 \cong \int \mathscr{D}\phi_1 e^{-\int d^4x(\phi_1, A_0\phi_1)} \int \mathscr{D}A_\mu \mathscr{D}\phi_2 \prod_x \delta(\partial_\mu A^\mu) e^{-\int d^4x[(A_\mu,\phi_2)^+ B_0(A_\mu,\phi_2)]} \tag{2.21}$$

We have:

$$\int \mathscr{D}\phi_1 e^{-\int d^4x(\phi_1, A_0\phi_1)} = N(\det A_0)^{-1/2} \tag{2.22}$$

and in the presence of the bounce (equation (2.17)):

$$\int \mathscr{D}\phi_1 e^{-\int d^4x(\phi_1, A_B\phi_1)} = NV \prod_{\mu=0}^{3} \left(\int \frac{d^4x}{2\pi\hbar} \frac{(\partial_\mu \phi^B)^2}{4}\right)^{1/2} (\det{}' A_B)^{-1/2} \tag{2.23}$$

where det′ means that the zero modes must be omitted. Since A_B has one negative eigenvalue, $(\det{}' A_B)^{1/2}$ is purely imaginary. N is a constant, which depends on the normalization one chooses. The factor

$$V \prod_{\mu=0}^{3} \left(\int \frac{d^4x}{2\pi\hbar} \tfrac{1}{4}(\partial_\mu \phi^B)^2\right)^{1/2}$$

in (2.23) is due to the normalization of the eigenfunctions with zero eigenvalue, and V is the volume of the system.

With equations (2.17), (2.21), (2.22) and (2.23) we obtain for the decay rate per unit volume:

$$\frac{\Gamma}{V}=\prod_{\mu=0}^{3}\left(\int\frac{d^4x}{2\pi\hbar}\tfrac{1}{4}(\partial_\mu\phi^B)^2\right)^{1/2}\left|\frac{\det A_0}{\det A_B}\right|^{1/2}e^{-S_B/\hbar}K_g \tag{2.24}$$

with

$$K_g=\frac{\int\mathscr{D}A_\mu\mathscr{D}\phi_2\prod_x\delta(\partial_\mu A^\mu)e^{-\int d^4x[(A_\mu,\,\phi_2)^+B_B(A_\mu,\,\phi_2)]}}{\int\mathscr{D}A_\mu\mathscr{D}\phi_2\prod_x\delta(\partial_\mu A^\mu)e^{-\int d^4x[(A_\mu,\,\phi_2)^+B_0(A_\mu,\,\phi_2)]}} \tag{2.25}$$

We will not analyse in detail the factor:

$$\left|\frac{\det A_0}{\det' A_B}\right|^{1/2}=\left|\frac{\det(-\Delta+\omega_1^2)}{\det'\left(-\Delta+\left.\dfrac{\partial^2V}{\partial\phi_1^2}\right|_{\substack{\phi_1=\phi^B\\ \phi_2=0}}\right)}\right|^{1/2} \tag{2.26}$$

It is the same as one would obtain considering a scalar theory with one real scalar field. It has been investigated and computed numerically by some authors [4, 11]. It has also been pointed out that it may have infrared problems if one tries to evaluate it in the thin-wall approximation [6].

2.3. *The factor K_g*

We turn now to the factor K_g (equation (2.25)). Using

$$\prod_x\delta(\partial_\mu A^\mu)=\lim_{\alpha\to 0}\prod_x\frac{1}{\sqrt{2\pi\alpha}}e^{-(1/2\alpha)\int d^4x(\partial_\mu A^\mu)^2} \tag{2.27}$$

we get:

$$\int\mathscr{D}A_\mu\mathscr{D}\phi_2\prod_x\delta(\partial^\mu A_\mu)e^{-\int d^4x[(A_\mu,\,\phi_2)^+B_B(A_\mu,\,\phi_2)]}$$

$$=\lim_{\alpha\to 0}\int\mathscr{D}A_\mu\mathscr{D}\phi_2\prod_x\frac{1}{\sqrt{2\pi\alpha}}e^{-(1/2\alpha)\int d^4x(\partial^\mu A_\mu)^2-\int d^4x[(A_\mu,\,\phi_2)^+B_B(A_\mu,\,\phi_2)]} \tag{2.28}$$

The same applies when the operator B_B (equation (2.15)) is replaced by the operator B_0 (equation (2.20)). In the following we denote (A_μ, ϕ_2) by χ.

Since B_B and B_0 are hermitean operators, they possess a complete set of eigenfunctions. We consider first the operator B_B and expand $(A_\mu, \phi_2)=\chi$ in its eigenfunctions $\chi_n=(\chi_{\mu_n}, \chi_{5_n})$. We denote by χ_i^P the eigenfunctions for which $\partial^\mu\chi_{\mu_i}^P=0$. The corresponding eigenvalues are λ_i^P. χ_j^t are the remaining eigenfunctions, with eigenvalue λ_j^t. Then we get:

$$\chi=\sum_i c_i\chi_i^P+\sum_j b_j\chi_j^t \tag{2.29}$$

The measure

$$\left[\mathcal{D}A_\mu \mathcal{D}\phi_2 \prod_x \frac{1}{\sqrt{2\pi\alpha}}\right]$$

is then

$$\left[\prod_i \frac{dc_i}{\sqrt{2\pi}} \prod_j \frac{db_j}{\sqrt{2\pi\alpha}}\right] \tag{2.30}$$

Inserting equation (2.29) into equation (2.28) and using the orthogonality of the eigenfunctions, we obtain:

$$\lim_{\alpha\to 0} \int \prod_i \frac{dc_i}{\sqrt{2\pi}} \prod_j \frac{db_j}{\sqrt{2\pi\alpha}} \times \exp\left[-\frac{1}{2\alpha}\sum_{j,j'} \bar{b}_j b_{j'} \int d^4x (\partial^\mu \bar{\chi}^t_{\mu j})(\partial^\nu \chi^t_{\nu j'}) - \sum_i |c_i|^2 \lambda^P_i - \sum_j |b_j|^2 \lambda^t_j\right] \tag{2.31}$$

with

$$\int (\partial^\mu \bar{\chi}^t_{\mu j})(\partial^\nu \chi^t_{\nu j'})\, d^4x \equiv 2a_{jj'}, \tag{2.32}$$

$$\lim_{\alpha\to 0} \prod_i (\lambda^P_i)^{-1/2} \int \prod_j \frac{db_j}{\sqrt{2\pi\alpha}} \exp\left(-\sum_{j,j'} \bar{b}_j b_{j'} \frac{1}{\alpha}(a_{jj'} + \alpha\lambda^t_j \delta_{jj'})\right) \tag{2.33}$$

Finally we get, up to some overall constant, which cancels when computing K_g:

$$\lim_{\alpha\to 0} \prod_i (\lambda^P_i)^{-1/2} \left(\prod_j \frac{1}{\sqrt{\alpha}}\right)\left\{\mathrm{Det}\left[\frac{a_{jj}' + \alpha\lambda^t_j \delta_{jj'}}{\alpha}\right]\right\}^{-1/2} \tag{2.34}$$

The factor $1/\alpha$ can be taken out of the determinant, and cancels with the term $(\prod_j 1/\sqrt{\alpha})$.

We may thus perform the lim $\alpha \to 0$ and the final result is:

$$\prod_i (\lambda^P_i)^{-1/2} \cdot [\mathrm{Det}\,(a_{jj'})]^{-1/2} \tag{2.35}$$

One can evaluate in the same way the functional integral with the operator B_0. One expands then $(A_\mu, \phi_2) = \chi$ according to the eigenfunctions of B_0. We denote them with ψ^P_n and ψ^t_m and the corresponding eigenvalues with λ^P_{0n} and λ^t_{0m}.

Instead of equation (2.32) we get:

$$2a^0_{mm'} \equiv \int d^4x (\partial^\mu \bar{\psi}^t_{\mu m})(\partial^\nu \psi^t_{\nu m'}) \tag{2.36}$$

The factor K_g is then given by:

$$K_g = \frac{\prod_n (\lambda^P_{0n})^{1/2} (\mathrm{Det}\, a^0_{mm'})^{1/2}}{\prod_i (\lambda^P_i)^{1/2} (\mathrm{Det}\, a_{jj'})^{1/2}} \tag{2.37}$$

In appendix C we show that the eigenvalues λ_{0n}^P and λ_i^P are positive. We notice further that $\det(a_{jj'})$ and $\det(a^0_{mm'})$ are positive quantities. $a_{jj'}$ is an hermitean matrix and can thus be diagonalized by a suitable unitary matrix U:

$$U_{ij}a_{jj'}U^+_{j'k} = \tilde{a}_i\,\delta_{ik}$$

or, with (2.32):

$$\tilde{a}_i\,\delta_{ik} = \frac{1}{2}\int d^4x U_{ij}(\partial^\mu \bar{\chi}^t_{\mu j})\bar{U}_{kj'}(\partial^\nu \chi^t_{\nu j'}) = \frac{1}{2}\int d^4x f_i \bar{f}_k$$

$$f_i \equiv U_{ij}(\partial^\mu \bar{\chi}^t_{\mu j})$$

We see clearly that the elements $\tilde{a}_i$ are positive and thus $\det a_{jj'}$ and similarly $\det a^0_{mm'}$. We conclude therefore that the factor K_g is positive.

If the dimension of the spaces spanned by the eigenfunctions χ^t_j and ψ^t_m are the same, then there is a unitary transformation, which allows to write the χ^t_j in terms of the ψ^t_m. The matrix $a_{jj'}$ can then be obtained from the matrix $a^0_{mm'}$ by a unitary transformation; the determinants are then the same, and thus cancel in the factor K_g. In appendix D we give an estimate of the ratio:

$$\frac{\prod_n (\lambda^P_{0n})^{1/2}}{\prod_i (\lambda^P_i)^{1/2}}$$

With equations (2.24), (2.26) and (2.37) the decay rate per unit volume is:

$$\frac{\Gamma}{V} = \prod_{\mu=0}^{3}\left(\int \frac{d^4x}{2\pi\hbar}\,\tfrac{1}{4}(\partial_\mu \phi^B)^2\right)^{1/2} \left| \frac{\det(-\Delta + \omega_1^2)}{\det'\left(-\Delta + \left.\frac{\partial^2 V}{\partial \phi_1^2}\right|_{\substack{\phi_1=\phi^B\\ \phi_2=0}}\right)}\right|^{1/2}$$

$$\times \frac{\prod_n (\lambda^P_{0n})^{1/2}(\mathrm{Det}\, a^0_{mm'})^{1/2}}{\prod_i (\lambda^P_i)^{1/2}(\mathrm{Det}\, a_{jj'})^{1/2}}\, e^{-S_B/\hbar}. \tag{2.38}$$

Computation of equation (2.38) is a rather difficult task, feasible only numerically with computers. From the above analysis we see that the ansatz we made in order to solve the bounce equations (2.6), (2.7), is consistent. Only the operator A_B (equation (2.14)) has a negative eigenvalue, so that $(\det' A_B)^{1/2}$ is a purely imaginary quantity. All the remaining pieces are real and positive. Thus the decay rate, which is proportional to the imaginary part of the functional integral equation (2.5), is nonvanishing.

3. Global $U(1)$ theory

This case is obtained from the previous one dropping the terms coming from the gauge sector. The bounce ansatz is the same ($\phi_1 = \phi^B$, $\phi_2 = 0$). The fluctuation operator $\mathscr{L}''_B$ (equation (2.13)) is then

$$\begin{bmatrix} -\Delta + \left.\dfrac{\partial^2 V}{\partial \phi_2^2}\right|_{\substack{\phi_1=\phi^B\\ \phi_2=0}} & 0 \\ 0 & -\Delta + \left.\dfrac{\partial^2 V}{\partial \phi_1^2}\right|_{\substack{\phi_1=\phi^B\\ \phi_2=0}} \end{bmatrix} \tag{3.1}$$

The operator

$$A_B = -\Delta + \left.\frac{\partial^2 V}{\partial \phi_1^2}\right|_{\substack{\phi_1=\phi^B,\\ \phi_2=0}}$$

as we know from section 2.2, has 4 zero modes and one negative mode. The operator

$$C = -\Delta + \left.\frac{\partial^2 V}{\partial \phi_2^2}\right|_{\substack{\phi_1=\phi^B\\ \phi_2=0}} \tag{3.2}$$

has a zero mode. In fact by making a choice of a bounce, as given by the Euler-Lagrange equations, we break the global $U(1)$ symmetry. Consequently, from Goldstone's theorem we expect a zero mode [12]. Using equation (2.12)

$$\left.\frac{\partial^2 V}{\partial \phi_2^2}\right|_{\substack{\phi_1=\phi^B\\ \phi_2=0}} = \left.\left(\frac{\partial^2 V}{\partial y^2}\phi_2^2 + \frac{\partial V}{\partial y}\right)\right|_{\substack{\phi_1=\phi^B\\ \phi_2=0}} = \left.\frac{\partial V}{\partial y}\right|_{\substack{\phi_1=\phi^B\\ \phi_2=0}} \tag{3.3}$$

and with

$$\left.\frac{\partial V}{\partial \phi_1}\right|_{\substack{\phi_1=\phi^B\\ \phi_2=0}} = \frac{\partial V}{\partial y}\phi^B \tag{3.4}$$

one sees that ϕ_B is an eigenfunction with eigenvalue $\lambda = 0$ of the operator C. In fact the eigenvalue equation

$$-\Delta\phi^B + \frac{\partial V}{\partial y}\phi^B = 0 \tag{3.5}$$

corresponds, due to equation (3.4), to the bounce equation (2.10).

We expect the eigenvalue $\lambda = 0$ to be the lowest one of the operator C, since the bounce ($\phi_1 = \phi^B$, $\phi_2 = 0$) corresponds to the solution of minimum action [15], and thus there is only one negative eigenvalue. We know that the opeator A_B has a negative eigenvalue, thus the operator C can have only positive ones.

In computing the functional integral equation (2.5) in the global $U(1)$ case one has to take care of the zero mode of the operator C [12]. One has to integrate over all possible phases of the bounce, like for the translational zero-modes where one had to integrate over all locations of the bounce (collective coordinates). The decay rate per unit volume in the global $U(1)$ case is then given by:

$$\frac{\Gamma}{V}=\prod_{\mu=0}^{3}\left(\int\frac{d^4x}{2\pi\hbar}\tfrac{1}{4}(\partial_\mu\phi^B)^2\right)^{1/2}\left(\int\frac{d^4x}{2\pi\hbar}(\phi^B)^2\right)^{1/2}e^{-S_B/\hbar}$$
$$\times\left|\frac{\det(-\Delta+\omega_1^2)}{\det'\left(-\Delta+\left.\frac{\partial^2V}{\partial\phi_1^2}\right|_{\substack{\phi_1=\phi^B\\ \phi_2=0}}\right)}\right|^{1/2}\cdot\left|\frac{\det(-\Delta+\omega_2^2)}{\det'\left(-\Delta+\left.\frac{\partial^2V}{\partial\phi_2^2}\right|_{\substack{\phi_1=\phi^B\\ \phi_2=0}}\right)}\right|^{1/2} \tag{3.6}$$

where det′ means that one has to omit the zero modes.

We evaluate equation (3.6) in the thin-wall approximation [4], i.e. when the energy density difference Δ between metastable and stable vacuum is small. We follow the analysis of Ref. [6], where using Rayleigh–Ritz approximation the first term of the ratio of determinants has been estimated

$$\left|\frac{\det(-\Delta+\omega_1^2)}{\det'\left(-\Delta+\left.\frac{\partial^2V}{\partial\phi_1^2}\right|_{\substack{\phi_1=\phi^B\\ \phi_2=0}}\right)}\right|^{1/2}\geqslant e^{+(m_1R)^3/9} \tag{3.7}$$

where m_1 is a typical eigenvalue of the operator $(-\Delta+\omega_1^2)$. In the same way we can estimate the other ratio of determinants by Rayleigh–Ritz approximation. In detail, we decompose the operator C (equation (3.2)) into radial components, yielding the following eigenvalue problem for each angular momentum j

$$\left[-\frac{1}{r^3}\partial_r r^3\,\partial_r+\frac{j(j+2)}{r^2}+\left.\frac{\partial^2V(r)}{\partial\phi_2^2}\right|_{\substack{\phi_1=\phi^B\\ \phi_2=0}}\right]\psi(r)=\lambda_j\psi(r) \tag{3.8}$$

Furthermore for $j=0$, we know one exact eigenfunction ϕ^B, whose eigenvalue is zero. Using this as trial function we can estimate the lowest eigenvalue $\tilde{\lambda}_j$ for any j, namely

$$0<\tilde{\lambda}_j\leqslant j(j+2)\frac{\int dr\,r^3\frac{1}{r^2}(\phi^B)^2}{\int drr^3(\phi^B)^2} \tag{3.9}$$

$$0<\tilde{\lambda}_j\leqslant j(j+2)\frac{\int_0^R dr\,r}{\int_0^R dr\,r^3}=\frac{2j(j+2)}{R^2}. \tag{3.10}$$

In (3.10) we used the fact that ϕ^B in thin-wall approximation is nonvanishing and approximately constant in the region between $(0, R)$ and vanishes beyond. (This qualitative feature is explained in Ref. [4].)

The ratio of interest is then given to be

$$\prod_{j=1}^{m_2R}\left[\frac{2j(j+2)}{m_2^2R^2}\right]^{-(j+1)^2/2} \cong \exp\left(-\int_1^{m_2R} dj\,\frac{(j+1)^2}{2}\ln\left[\frac{2j(j+2)}{m_2^2R^2}\right]\right)$$
$$\cong \exp\left(-(m_2R)^3\left(\frac{\ln 2}{6}-\frac{1}{9}\right)\right). \tag{3.11}$$

m_2 is a typical eigenvalue of the operator $(-\Delta+\omega_2^2)$. Thus

$$\left|\frac{\det(-\Delta+\omega_2^2)}{\det'\left(-\Delta+\left.\frac{\partial^2 V}{\partial\phi_2^2}\right|_{\substack{\phi_1=\phi^B\\ \phi_2=0}}\right)}\right|^{1/2} \geqslant e^{-(m_2R)^3(\ln 2/6-1/9)}. \tag{3.12}$$

Putting it all together, and using the fact that

$$\prod_{\mu=0}^{3}\left(\int\frac{d^4x}{2\pi\hbar}\tfrac{1}{4}(\partial_\mu\phi^B)^2\right)^{1/2}=\frac{S_B^2}{(2\pi\hbar)^2}\quad [4]$$

and in thin-wall approximation

$$\left(\int\frac{d^4x}{2\pi\hbar}(\phi^B)^2\right)^{1/2}=\phi_-R^2\frac{1}{2}\left(\frac{\pi}{\hbar}\right)^{1/2},$$

(3.6) is given by

$$\Gamma/V \geqslant \frac{\phi_-R^2S_B^2}{8\pi^{3/2}\hbar^{5/2}}\,e^{R^3[m_1^3/9-m_2^3(\ln 2/6-1/9)]}e^{-S_B/\hbar}. \tag{3.13}$$

4. Local $SU(N)$ gauge theory

In this section we consider an $SU(N)$ invariant Lagrangian, in Euclidean space, of the form:

$$\mathcal{L}_E=\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+|\mathcal{D}_\mu\phi|^2+V(\phi) \tag{4.1}$$

which includes gauge bosons A_μ^a $(a=1,\ldots,N^2-1)$. $F_{\mu\nu}$ is the field strength tensor for the nonabelian case. We take the scalar field ϕ in the adjoint representation. This choice is motivated by the fact that in grand unified theories one often breaks the symmetry of the gauge group with a scalar field in the adjoint representation.

We write ϕ as a vector ϕ_a $(a=1,\ldots,N^2-1)$ with N^2-1 real components. For each ϕ_a there is a generator λ_a of the group. $(\mathcal{D}_\mu\phi)$ is then a vector:

$$(\mathcal{D}_\mu\phi)_a=\partial_\mu\phi_a+gf_{abc}A_{\mu b}\phi_c \tag{4.2}$$

where summation over repeated indices is understood. f_{abc} is a totally antisymmetric tensor defined by the commutation relations between the N^2-1 generators λ_a of the group:

$$[\lambda_i,\lambda_j]=2if_{ijn}\lambda_n \tag{4.3}$$

The λ_a are normalized as follows

$$\mathrm{tr}\,(\lambda_i\lambda_j) = 2\delta_{ij}$$

They satisfy also the anticommutation relation:

$$\{\lambda_i, \lambda_j\} = \frac{4}{N}\delta_{ij}\mathbb{1} + 2d_{ijm}\lambda_m \tag{4.4}$$

d_{ijm} is a totally symmetric tensor. We choose the Lorentz-gauge:

$$\partial_\mu A_a^\mu = 0 \qquad a = 1, \ldots, N^2 - 1. \tag{4.5}$$

Then, in the Lagrangian we also have to add a ghost term

$$\bar{\eta}_a(\partial_\mu \mathscr{D}^\mu)_{ab}\eta_b \tag{4.6}$$

where η_a $(a = 1, \ldots, N^2 - 1)$ is the ghost field. Like in the $U(1)$ case the scalar potential $V(\phi)$ is taken to have minima for $\phi_a = 0$ $(a = 1, \ldots, N^2 - 1)$ with $V = 0$ and for $\phi_{N^2-1} = \phi_-$, $\phi_i = 0$ $i = 1, \ldots, N^2 - 2$ with $V = -\Delta$, $\Delta > 0$.

With the adjoint representation one can achieve the following symmetry breaking patterns [13], which are global minima:

$$SU(N) \to SU(N-1) \times U(1)$$

or

$$SU(N) \to SU\left(N - \left[\frac{N}{2}\right]\right) \times SU\left(\left[\frac{N}{2}\right]\right) \times U(1) \tag{4.7}$$

where [ ] means the integer part.

The generator corresponding to the component ϕ_{N^2-1} is λ_{N^2-1}. If we write ϕ as a matrix $(\phi = \phi_a\lambda_a)$, then the vacuum expectation value of ϕ is: $\langle\phi\rangle = \phi_-\lambda_{N^2-1}(\langle\phi_{N^2-1}\rangle = \phi_-)$. For instance for $SU(5)$ breaking to $SU(4) \times U(1)$ λ_{24} would be the diagonal generator:

$$\frac{1}{\sqrt{10}}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -4 \end{pmatrix} \tag{4.8}$$

and for the breaking to $SU(3) \times SU(2) \times U(1)$:

$$\frac{2}{\sqrt{15}}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -\frac{3}{2} & \\ & & & & -\frac{3}{2} \end{pmatrix} \tag{4.9}$$

4.1. *The bounce equations*

We proceed in the same way as we did for the $U(1)$ case. We compute first the bounce which satisfies the Euler–Lagrange equations. Like in the $U(1)$ case, in order to simplify the problem we make the ansatz for the bounce: $A_a^\mu(x^\nu) \equiv 0$ $(a = 1, \ldots, N^2 - 1)$. Then the Euler–Lagrange equations for the scalar fields ϕ_a become:

$$-\Delta\phi_a + \frac{\partial V}{\partial \phi_a} = 0 \qquad a = 1, \ldots, N^2 - 1 \tag{4.10}$$

with the boundary conditions

$$\lim_{x_4 \to \pm\infty} \phi_a(\vec{x}, x_4) = 0 \qquad a = 1, \ldots, N^2 - 1 \tag{4.11}$$

and

$$\lim_{|\vec{x}| \to \infty} \phi_a(\vec{x}, x_4) = 0 \qquad a = 1, \ldots, N^2 - 1 \tag{4.12}$$

in order for the bounce action to be finite.

We can simplify further the problem taking $\phi_i \equiv 0$ for $i = 1, \ldots, N^2 - 2$, then equation (4.10) becomes:

$$-\Delta\phi_{N^2-1} + \left.\frac{\partial V}{\partial \phi_{N^2-1}}\right|_{\phi_i = 0} = 0 \tag{4.13}$$

We expect that the solution of equation (4.13), $\phi_{N^2-1} = \phi^B$, has an $O(4)$ invariance and thus is a function of r alone. ϕ^B has then to satisfy the boundary conditions: $d\phi^B/dr(r=0) = 0$, $\phi^B(r \to \infty) = 0$. The above ansatz is consistent if

$$\left.\frac{\partial V}{\partial \phi_i}\right|_{\substack{\phi_i = 0 \\ \phi_{N^2-1} = \phi^b}} \quad (i = 1, \ldots, N^2 - 2) \quad \text{vanishes.}$$

The now renormalizable potential $V(\phi)$ is a function of the quantities: $\operatorname{tr}(\phi_a\lambda_a)^2$, $\operatorname{tr}(\phi_a\lambda_a)^3$ and $\operatorname{tr}(\phi_a\lambda_a)^4$. We have

$$\left.\frac{\partial}{\partial \phi_i} \operatorname{tr}(\phi_a\lambda_a)^2\right|_{\substack{\phi_1 = \cdots = \phi_{N^2-2} = 0 \\ \phi_{N^2-1} = \phi^B}} = 4\phi^B\, \delta_{i,N^2-1} \tag{4.14}$$

$$\left.\frac{\partial}{\partial \phi_i} \operatorname{tr}(\phi_a\lambda_a)^3\right|_{\substack{\phi_1 = \cdots = \phi_{N^2-2} = 0 \\ \phi_{N^2-1} = \phi^B}} = 6(\phi^B)^2\, d_{i,N^2-1,N^2-1} \tag{4.15}$$

$$\left.\frac{\partial}{\partial \phi_i} \operatorname{tr}(\phi_a\lambda_a)^4\right|_{\substack{\phi_1 = \cdots = \phi_{N^2-2} = 0 \\ \phi_{N^2-1} = \phi^B}}$$

$$= 4(\phi^B)^3\left[\frac{4}{N}\delta_{i,N^2-1} + 2\sum_k d_{i,N^2-1,k}\, d_{k,N^2-1,N^2-1}\right] \tag{4.16}$$

Since λ_{N^2-1} is a diagonal generator of a form as in equation (4.8) or equation (4.9), $d_{N^2-1,N^2-1,k}$ vanishes for $k \neq N^2 - 1$, which can be easily checked using equation (4.4). Inserting this result into (4.14), (4.15) and (4.16), one sees that in fact

$$\left.\frac{\partial V}{\partial \phi_i}\right|_{\substack{\phi_i=0\\ \phi_{N^2-1}=\phi^B}} \quad (i=1,\ldots,N^2-2) \quad \text{vanishes.}$$

4.2. *The fluctuation operator*

To compute the decay rate per unit volume we have to determine the factor K, defined in equation (2.4). As in the $U(1)$ case we obtain the determinants by evaluating the functional integral in the steepest-descent approximation;

$$Z=\int \prod_{a=1}^{N^2-1} [\mathscr{D}A_\mu^a \mathscr{D}\phi^a \prod_x \delta(\partial^\mu A_\mu^a)] \operatorname{Det} M e^{-\int d^4x \mathscr{L}_E} \tag{4.17}$$

once with the bounce as background field and once in the false vacuum. Det M is the ghost determinant, which depends also on A_μ^a. In our approximation we insert for A_μ^a its "bounce" value: $A_\mu^a \equiv 0$. The ghost determinant can thus be taken out of the integral. In the expression for K it cancels, since in this approximation we have then the same determinant in the denominator and in the numerator. The functional integral (4.17) is then approximated by the Gaussian fluctuations around the bounce ($A_\mu^a \equiv 0$, $a=1,\ldots,N^2-1$; $\phi_i=0$, $i=1,\ldots,N^2-2$; $\phi_{N^2-1}=\phi^B$).

The fluctuation operator $\mathscr{L}_B''$ is

$$\mathscr{L}_B''=\frac{1}{2}\begin{bmatrix} \left[-\Delta\,\delta_{b\bar{b}}+g^2\sum_{a=1}^{N^2-1} f_{abN^2-1}f_{a\bar{b}N^2-1}(\phi^B)^2\right]\delta_{\mu\nu} & 2gf_{N^2-1bc}\partial_\mu\phi^B \\ 2gf_{N^2-1\bar{b}\bar{c}}\partial_\nu\phi^B & -\Delta\,\delta_{c\bar{c}}+\left.\dfrac{\partial^2 V}{\partial\phi_c\,\partial\phi_{\bar{c}^2}}\right|_{\phi^B} \end{bmatrix} \tag{4.18}$$

The indices b, $\bar{b}$, c, $\bar{c}$ run from 1 to N^2-1.

As a next step we will analyse the structure of the operator $\mathscr{L}_B''$ and see that det $\mathscr{L}_B''$ factorizes into pieces which have the same structure as the one we found in the $U(1)$ case. The fluctuation operator $\mathscr{L}_0''$, which one gets when the functional integral is evaluated in the false vacuum, can be obtained from the above one by setting $\phi_B=0$.

4.3. *The matrix* $\left.\dfrac{\partial^2 V}{\partial\phi_c\,\partial\phi_{\bar{c}}}\right|_{\phi_B}$

We first study the structure of the matrix $\partial^2 V/\partial\phi_c\,\partial\phi_{\bar{c}}|_{\phi^B}$, which appears in equation (4.18). We will see that it is diagonal; only the terms $\partial^2 V/\partial\phi_c^2|_{\phi_B}$ are nonvanishing. We have:

$$\frac{\partial^2}{\partial\phi_c\,\partial\phi_{\tilde c}}\operatorname{tr}(\lambda_a\phi_a)^2\bigg|_{\phi_B} = 4\,\delta_{c\tilde c} \tag{4.19}$$

$$\frac{\partial^2}{\partial\phi_c\,\partial\phi_{\tilde c}}\operatorname{tr}(\lambda_a\phi_a)^3\bigg|_{\phi_B} = 12\phi_B\,d_{c\tilde c N^2-1} \tag{4.20}$$

$$\begin{aligned}\frac{\partial^2}{\partial\phi_c\,\partial\phi_{\tilde c}}\operatorname{tr}(\lambda_a\phi_a)^4\bigg|_{\phi_B} = \phi_B^2\bigg\{&\frac{4}{N}[4\,\delta_{c,\tilde c}+8\,\delta_{c,N^2-1}\,\delta_{\tilde c,N^2-1}]\\ &+8\sum_e d_{c\tilde c e}\,d_{eN^2-1N^2-1}\\ &+16\sum_e d_{cN^2-1e}\,d_{\tilde c N^2-1e}\bigg\}\end{aligned} \tag{4.21}$$

$\partial^2 V/\partial\phi_c\,\partial\phi_{\tilde c}|_{\phi_B}$ contains terms like those of equation (4.14), (4.15), (4.16), (4.19), (4.20) and (4.21). In particular we see that it is a function only of the totally symmetric tensor d_{abN^2-1}; one of the three indices being N^2-1. d_{abN^2-1} is a symmetric matrix in the indices a and b. By an appropriate choice of the generators λ_i $(i\neq N^2-1)d_{abN^2-1}$ can be brought into diagonal form such that $d_{abN^2-1}\neq 0$ only for $a=b$. If a (or b) $=N^2-1$ then we know already that d_{N^2-1,b,N^2-1} vanishes for b (or a) $\neq N^2-1$. With d_{abN^2-1} in diagonal form one can easily check that the above equations (4.19), (4.20), (4.21) are nonvanishing only if $c=\tilde c$. Thus:

$$\frac{\partial^2 V}{\partial\phi_c\,\partial\phi_{\tilde c}}\bigg|_{\phi_B} = \delta_{c\tilde c}\frac{\partial^2 V}{\partial\phi_c^2}\bigg|_{\phi_B} \tag{4.22}$$

With the fact that $f_{N^2-1N^2-1N^2-1}=0$ we see that the operator

$$A_B = -\Delta + \frac{\partial^2 V}{\partial\phi_{N^2-1}^2}\bigg|_{\phi_B} \tag{4.23}$$

factorizes from det $\mathscr{L}''_B$.

The operator A_B has four zero modes, since the system is translationally invariant [4]. Differentiating the bounce equation (4.13) with respect to x_μ, one obtains (using equation (4.22)):

$$\partial_\mu\bigg(-\Delta\phi_B + \frac{\partial V}{\partial\phi_{N^2-1}}\bigg|_{\phi_B}\bigg) = \bigg(-\Delta + \frac{\partial^2 V}{\partial\phi_{N^2-1}^2}\bigg|_{\phi_B}\bigg)\,\partial_\mu\phi_B = 0 \tag{4.24}$$

A_B has thus four eigenfunctions with zero eigenvalue. For the same reasons we discussed in the $U(1)$ case (see section 2.2), we expect that the operator A_B has also one negative eigenvalue.

4.4. *The SU(N) symmetry of the matrix* $\left.\dfrac{\partial^2 V}{\partial\phi_c\,\partial\phi_{\bar c}}\right|_{\phi_B}$

We will now exploit the $SU(N)$ symmetry of the potential to show that some of the elements of the matrix $\delta_{c\bar c}(\partial^2 V/\partial\phi_c^2)|_{\phi_B}$ are related to $\partial V/\partial\phi_{N^2-1}|_{\phi_B}$.

Under an $SU(N)$ transformation the potential V remains invariant, that means:

$$\sum_{n=1}^{N^2-1}\frac{\partial V}{\partial\phi_n}\,\delta\phi_n=0 \tag{4.25}$$

with

$$\delta\phi_n=2i\sum_{j,k=1}^{N^2-1} f_{njk}\xi_j\phi_k \tag{4.26}$$

where ξ_j is an arbitrary parameter.

Inserting (4.26) into (4.25) and differentiating once more with respect to ϕ_m we obtain (summation over repeated indices is understood):

$$\left[\left.\frac{\partial^2 V}{\partial\phi_n\,\partial\phi_m}\right|_{\phi_B} f_{njk}\phi_k+\left.\frac{\partial V}{\partial\phi_n}\right|_{\phi_B} f_{njm}\right]\xi_j=0\quad\forall\xi_j \tag{4.27}$$

with equation (4.22) and $\phi_k=0$, $\partial V/\partial\phi_k|_{\phi_B}=0$ for $k\neq N^2-1$, equation (4.27) becomes:

$$\left.\partial^2 V/\partial\phi_n^2\right|_{\phi_B} f_{njN^2-1}\phi_B=\left.\frac{\partial V}{\partial\phi_{N^2-1}}\right|_{\phi_B} f_{mjN^2-1}. \tag{4.28}$$

We notice that f_{njN^2-1} is an antisymmetric matrix in the indices n, j ($n, j=1,\dots,N^2-2$). Thus it can be brought into the following block-diagonal form by a suitable choice of the generators λ_i ($i=1,\dots,N^2-2$):

$$\begin{bmatrix} a_1\begin{pmatrix}0&1\\-1&0\end{pmatrix} & & & & 0\\ & a_2\begin{pmatrix}0&1\\-1&0\end{pmatrix} & & & \\ & & \ddots & & \\ & & & 0 & \\ 0 & & & & \ddots \\ & & & & 0\end{bmatrix} \tag{4.29}$$

Not all elements in the diagonal are filled with blocks of the form $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, but some vanish, since the corresponding a_i are zero. The one that vanish are related to the unbroken generators of the group, whose gauge bosons remain massless. The number of unbroken generators depends on the symmetry breaking pattern.

With the above result we can also analyse the structure of the operator:

$$\left(-\Delta\,\delta_{b\tilde{b}}+g^2\sum_{a=1}^{N^2-1} f_{abN^2-1}f_{a\tilde{b}N^2-1}\phi_B^2\right) \tag{4.30}$$

which occurs in $\mathscr{L}''_B$ equation (4.18). Due to equation (4.29) it follows that (4.30) vanishes for $b\neq\tilde{b}$. For $b=\tilde{b}$ we have

$$\left(-\Delta+g^2\sum_{a=1}^{N^2-1} f^2_{abN^2-1}\phi_B^2\right)\delta_{b\tilde{b}} \tag{4.31}$$

For a given value of $b: f_{abN^2-1}$ is nonvanishing only for $a=b-1$ or $a=b+1$. If the generators λ_a, λ_b are unbroken ones, the corresponding gauge bosons have to be massless, which implies that f_{abN^2-1} vanishes in that case.

With equation (4.29) one sees that for the nonvanishing (broken generators) f_{abN^2-1} equation (4.28) becomes (j is fixed and n is then equal to m):

$$\left.\frac{\partial^2 V}{\partial\phi_n^2}\right|_{\phi_B}\phi_B=\left.\frac{\partial V}{\partial\phi_{N^2-1}}\right|_{\phi_B}. \tag{4.32}$$

We consider now the operators $-\Delta+(\partial^2 V/\partial\phi_c^2)|_{\phi_B}$ $(c=1,\ldots,N^2-1)$ which appear in $\mathscr{L}''_B$ equation (4.18). We can easily verify that for the operators, whose index c corresponds to that of a broken generator λ_c, ϕ_B is an eigenfunction with zero eigenvalue. In fact $-\Delta\phi_B+(\partial^2 V/\partial\phi_c^2)|_{\phi_B}\phi_B$ is, using equation (4.32), equal to the bounce equation (4.13):

$$-\Delta\phi_B+\left.\frac{\partial V}{\partial\phi_{N^2-1}}\right|_{\phi_B}=0.$$

These zero eigenvalues are the Goldstone modes, which are present if the $SU(N)$ symmetry is global.

In the local $SU(N)$ symmetry case, however, the operators $-\Delta+(\partial^2V/\partial\phi_c^2)|_{\phi_B}$, related to the broken generators, do not factorize out of det $\mathscr{L}''_B$. This is like in the $U(1)$ case, where $-\Delta+(\partial^2 V/\partial\phi_2^2)|_{\phi_B}$ did not factorize out of det B_B (equation 2.15). For the above analysis we used the fact that we could "diagonalize" the matrices d_{abN^2-1} and f_{abN^2-1}. However for consistency we have to check that they commute, because only then we can diagonalize both matrices simultaneously. This can be easily proven using the Jacobian identities for the generators [14].

4.5. *The factorization of the determinant of the fluctuation operator*

Let us denote by $n=1,\ldots,\alpha$ the broken generators λ_n and by $n=\alpha+1,\ldots,N^2-1$ the unbroken ones. Notice that the generator λ_{N^2-1} is an unbroken one. With the previous results we see that the determinant of $\mathscr{L}''_B$

factorizes into the following pieces:

$$\begin{bmatrix} (-\Delta + g^2 f^2_{N^2-1,n,n+1}\phi_B^2)\,\delta_{\mu\nu} & 2gf_{N^2-1,n,n+1}\,\partial_\mu\phi_B \\ 2gf_{N^2-1,n,n+1}\,\partial_\nu\phi_B & -\Delta + \left.\dfrac{\partial^2 V}{\partial\phi_n^2}\right|_{\phi_B} \end{bmatrix} \tag{4.33}$$

for $n = 1, \ldots, \alpha$

$$\left(-\Delta + \left.\frac{\partial^2 V}{\partial\phi_n^2}\right|_{\phi_B}\right) \quad \text{for} \quad n = \alpha+1, \ldots, N^2-1 \tag{4.34}$$

and

$$(-\Delta\,\delta_{\mu\nu}) \quad \text{for} \quad n = \alpha+1, \ldots, N^2-1 \tag{4.35}$$

Equation (4.33) is the same as the operator B_B (equation (2.15)) in the $U(1)$ case. Therefore we can apply to each operator ($n = 1, \ldots, \alpha$) the same analysis we did for B_B, and also the results of the appendices C and D hold. Each of these operators will give a contribution to the decay rate per unit volume similar to the factor K_g of equation (2.37), which is real and positive.

The fluctuation operator $\mathscr{L}_0''$ around the false vacuum does also factorize, and can be obtained from the above equation (4.33), (4.34) and (4.35) by setting $\phi_B = 0$. One sees then that the terms $(-\Delta\,\delta_{\mu\nu})$ (equation (4.35)) cancel when computing the decay rate, since they are present in the numerator and in the denominator. The operator $-\Delta + (\partial^2 V/\partial\phi^2_{N^2-1})|_{\phi_B}$ of equation (4.34) contains four zero modes and one negative mode; thus $\det'(-\Delta + (\partial^2 V/\partial\phi^2_{N^2-1})|_{\phi_B})^{1/2}$ is imaginary and therefore the quantity K (equation (2.4)) is nonvanishing.

We expect that the other operators of equation (4.34):

$$-\Delta + \left.\frac{\partial^2 V}{\partial\phi_n^2}\right|_{\phi_B} \quad \text{with} \quad n = \alpha+1, \ldots, N^2-2$$

have only positive eigenvalues, and their determinants to be real. This is clearly the case if our ansatz for the bounce corresponds to the solution with lowest action. Although we have no proof, we believe that this is the case. Nevertheless in the thin-wall approximation we can give an argument which shows that the above operators are indeed positive. We discuss this in appendix E.

Finally, the decay rate per unit volume is:

$$\frac{\Gamma}{V} = \prod_{\mu=0}^{3}\left(\int \frac{d^4x}{2\pi\hbar}\,\tfrac{1}{4}(\partial_\mu\phi_B)^2\right)^{1/2} e^{-S_B/\hbar}\left|\frac{\det(-\Delta + \omega^2_{N^2-1})}{\det'\left(-\Delta + \left.\dfrac{\partial^2 V}{\partial\phi^2_{N^2-1}}\right|_{\phi_B}\right)}\right|^{1/2}$$
$$\times \prod_{n=\alpha+1}^{N^2-2}\left[\frac{\det(-\Delta + \omega_n^2)}{\det\left(-\Delta + \left.\dfrac{\partial^2 V}{\partial\phi_n^2}\right|_{\phi_B}\right)}\right]^{1/2} \times \prod_{n=1}^{\alpha}(K_g)_n \tag{4.36}$$

with

$$\omega_i^2 = \left.\frac{\partial^2 V}{\partial\phi_i^2}\right|_{\phi_B=0}$$

and

$$(K_g)_n = \frac{\prod_e (\lambda^P_{0e,n})^{1/2} (\text{Det}\,(a^0_n)_{mm'})^{1/2}}{\prod_i (\lambda^P_{i,n})^{1/2} (\text{Det}\,(a_n)_{jj'})^{1/2}}$$

We see that essentially the $SU(N)$ case with the scalar fields in the adjoint representation can be reduced to a set of $U(1)$ cases, one for each broken generator.

5. Conclusions

Using the semiclassical method of instantons we computed the vacuum decay rate in models with scalar and gauge bosons. All the computations we have done are valid at zero temperature. In particular we considered a $U(1)$ gauge theory with a complex scalar field and an $SU(N)$ gauge theory with the scalar fields in the adjoint representation. Although the $U(1)$ case has no direct applications for realistic models, the analysis of it is important, since it contains all the new features which arise due to the inclusion of gauge bosons. The $SU(N)$ case can in fact essentially be reduced to a set of $U(1)$ cases, one for each broken generator. We made an ansatz for the solution of the bounce equations, which is consistent in the sense that it satisfies the scaling argument and the fluctuation operator has only one negative eigenvalue, thus the decay rate per unit volume is nonvanishing. The final formula for the decay rate (equation (2.38)) is computable only numerically. In the thin-wall limit there are no additional infrared problems to the one which have already been pointed out in the scalar case [6].

For the $SU(N)$ case, using the symmetry of the potential and the properties of the structure constants d_{abc} and f_{abc}, we could show that det $\mathscr{L}''_B$, the fluctuation operator, factorizes. For each broken generator we get a factor similar to K_g (equation (2.37)) of the $U(1)$ case. The unbroken generators do not contribute to the decay rate. From our analysis we can also compute the decay rate in the case of a scalar theory with a global $U(1)$ or $SU(N)$ symmetry.

In the previous chapters we did not consider the factor $\exp(-S_B)$ which enters in the formula for the decay rate. With our solution for the bounce it is the same as for a pure scalar theory. One can either compute it numerically or estimate it in the thin-wall approximation. In the latter case it is given by [4]:

$$S_B = 27\pi^2 S_1^4/2\Delta^3, \quad \text{and} \quad \frac{3S_1}{\Delta} = R$$

where R is the critical radius of the bubble, and Δ is the energy density difference between the two minima of the potential.

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Appendix A

Consider the action of a non-abelian gauge boson coupled to a scalar, given by the Euclidean action (in d dimensions)

$$S_E = \int d^dx \{\tfrac{1}{4}F^2 + |\mathscr{D}_\mu \phi|^2 + V(\phi)\} \tag{A.1}$$

In this appendix we would like to investigate the existence of negative modes when fluctuations about a non-trivial bounce are considered.

Construct S_λ which is gotten by replacing $\phi(x) \to \phi(x/\lambda)$ and $A_\mu(x) \to A_\mu(x/\lambda)$ in (A.1). By trivial rescaling the integration variable x, we have

$$S_E(\lambda) = \int d^dx \{\lambda^{d-2}[\tfrac{1}{4}F^2(e\lambda) + |\mathscr{D}_\mu(e\lambda)\phi|^2] + \lambda^d V(\phi)\} \tag{A.2}$$

where e is the gauge coupling constant. Since we are interested in the bounce (A^B_μ, ϕ^B) which minimizes the action, we have

$$0 = \frac{dS_E(\lambda)}{d\lambda}\bigg|_{\lambda=1} = \int d^dx \left\{\left((d-2) + e\frac{\partial}{\partial e}\right)[\tfrac{1}{4}F^2 + |\mathscr{D}_\mu \phi^B|^2] + dV(\phi^B)\right\} \tag{A.3}$$

where the partial derivative $\partial/\partial e$ acts only on the explicit e dependence (not on the implicit dependence coming through the bounce fields). Evaluating the scaling fluctuations and using (A.3), we obtain

$$\frac{d^2S_E(\lambda)}{d\lambda^2}\bigg|_{\lambda=1} = \int d^dx \left\{\left(e\frac{\partial}{\partial e}\right)^2 + (d-3)e\frac{\partial}{\partial e} - 2(d-2)\right\}[\tfrac{1}{4}F^2 + |\mathscr{D}_\mu \phi^B|^2] \tag{A.4}$$

From (A.4) we notice that the right hand side is not always negative. However, if $e=0$ or $A^B_\mu = 0$, then it is certainly negative for $d>2$, thus justifying that the ansatz in consideration in sections 2 and 4 does possess the required negative mode.

Appendix B

To compute the fluctuation operator $\mathscr{L}''_B$ we consider the Euclidean action given in equation (2.3b) and make the following substitution

$$\begin{aligned} A_\mu &\to A^B_\mu + \delta A_\mu \\ \phi_1 &\to \phi^B_1 + \delta\phi_1 \\ \phi_2 &\to \phi^B_2 + \delta\phi_2 \end{aligned} \tag{B.1}$$

With our ansatz for the bounce we have $A^B_\mu = 0$ and $\phi^B_2 = 0$. We expand and consider only terms which are quadratic in the fluctuations $(\delta A_\mu, \delta\phi_1, \delta\phi_2)$.

After some partial integration and with $\partial_\mu A^\mu = 0$ also $\partial_\mu\, \delta A^\mu = 0$ we get:

$$\Delta S_E = \int d^4x \bigg[2 \cdot \frac{1}{2} \frac{\partial^2 V}{\partial\phi_1\, \partial\phi_2}\bigg|_{\phi_B} \delta\phi_1\, \delta\phi_2 + \frac{1}{2}\frac{\partial^2 V}{\partial\phi_1^2}\bigg|_{\phi_B} (\delta\phi_1)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial\phi_2^2}\bigg|_{\phi_B} (\delta\phi_2)^2$$
$$+ 2e\, \delta A_\mu\, \delta\phi_2\, \partial_\mu\phi_B - \tfrac{1}{2}\delta\phi_1\, \Delta(\delta\phi_1) - \tfrac{1}{2}\delta\phi_2\, \Delta(\delta\phi_2)$$
$$+ \frac{e^2}{2}(\phi_B)^2\, \delta A_\mu\, \delta A^\mu - \tfrac{1}{2}\delta A^\mu\, \Delta(\delta A_\mu) \bigg] \tag{B.2}$$

We can write (B.2) also in matrix form ($\partial^2 V/\partial\phi_1\, \partial\phi_2|_{\phi_B}$ vanishes):

$$\Delta S_E = \frac{1}{2}\int d^4x [\delta A_\mu, \delta\phi_2, \delta\phi_1] \begin{bmatrix} (-\Delta + e^2\phi_B^2)\, \delta_{\mu\nu} & 2e\, \partial_\nu\phi_B & 0 \\ 2e\, \partial_\mu\phi_B & -\Delta + \dfrac{\partial^2 V}{\partial\phi_2^2}\bigg|_{\phi_B} & 0 \\ 0 & 0 & -\Delta + \dfrac{\partial^2 V}{\partial\phi_1^2}\bigg|_{\phi_B} \end{bmatrix} \begin{bmatrix} \delta A_\nu \\ \delta\phi_2 \\ \delta\phi_1 \end{bmatrix} \tag{B.3}$$

The fluctuation operator $\mathscr{L}''_B$ (equation (2.13)) is just the above matrix. In the same way we obtain the fluctuation operator in the $SU(N)$ case (equation (4.18)).

Appendix C

In this appendix we show that the eigenvalues λ_i^P and λ_{0i}^P are positive. The eigenvalues λ_{0i}^P of the operator

$$B_0 = \tfrac{1}{2}\begin{bmatrix} -\Delta\, \delta_{\mu\nu} & 0 \\ 0 & -\Delta + \omega_2^2 \end{bmatrix} \tag{C.1}$$

are clearly positive, since the essential spectrum of the Laplace operator $-\Delta$ is $[0, \infty)$. For the λ_i^P we have to show that the operator

$$B_B = \frac{1}{2}\begin{bmatrix} (-\Delta + e^2\phi_B^2)\, \delta_{\mu\nu} & 2e\, \partial_\nu\phi_B \\ 2e\, \partial_\mu\phi_B & -\Delta + \dfrac{\partial^2 V}{\partial\phi_2^2}\bigg|_{\phi_B} \end{bmatrix} \tag{C.2}$$

has positive eigenvalues for the eigenfunctions $\chi_i^P = (\chi_{\mu_i}^P, \chi_{5i}^P)$ which satisfy the condition $\partial^\mu\chi_{\mu_i}^P = 0$. Since $-\Delta\phi_B + (\partial^2 V/\partial\phi_2^2)|_{\phi_B}\phi_B = 0$ (equation (3.5)), we can write $\partial^2 V/\partial\phi_2^2|_{\phi_B}$ as $\Delta\phi_B/\phi_B$.

The eigenfunctions χ_i^P satisfy the equation:

$$\tfrac{1}{2}\begin{bmatrix} (-\Delta + e^2\phi_B^2)\, \delta_{\mu\nu} & 2e\, \partial_\nu\phi_B \\ 2e\, \partial_\mu\phi_B & -\Delta + \dfrac{\Delta\phi_B}{\phi_B} \end{bmatrix} \begin{bmatrix} \chi_{\nu_i}^P \\ \chi_{5i}^P \end{bmatrix} = \lambda_i^P \begin{bmatrix} \chi_{\mu_i}^P \\ \chi_{5i}^P \end{bmatrix} \tag{C.3}$$

or

$$\begin{cases} -\frac{1}{2}\Delta\chi^P_{\mu_i} + \frac{1}{2}e^2\phi_B^2\chi^P_{\mu_i} + e(\partial_\mu\phi_B)\chi^P_{5i} = \lambda_i^P\chi^P_{\mu_i} & \text{(C.4)} \\ e(\partial_\nu\phi_B)\chi^P_{\nu_i} - \frac{1}{2}\Delta\chi^P_{5i} + \frac{1}{2}\frac{\Delta\phi_B}{\phi_B}\chi^P_{5i} = \lambda_i^P\chi^P_{5i} & \text{(C.5)} \end{cases}$$

Differentiating (C.4) with respect to x^μ and using the fact that $\partial^\mu\chi^P_{\mu_i} = 0$ we obtain

$$\tfrac{1}{2}e^2(\partial_\mu(\phi_B^2))\chi^P_{\mu_i} + e\,\partial_\mu((\partial_\mu\phi_B)\chi^P_{5i}) = 0$$

or

$$e\phi_B(\partial_\mu\phi_B)\chi^P_{\mu_i} = -(\Delta\phi_B)\chi^P_{5i} - (\partial_\mu\phi_B)(\partial_\mu\chi^P_{5i}) \tag{C.6}$$

Multiplying (C.5) with ϕ_B and using (C.6) we get:

$$-(\Delta\phi_B)\chi^P_{5i} - (\partial_\mu\phi_B)(\partial_\mu\chi^P_{5i}) - \phi_B\,\tfrac{1}{2}\Delta\chi^P_{5i} + \tfrac{1}{2}(\Delta\phi_B)\chi^P_{5i} = \lambda_i^P\phi_B\chi^P_{5i} \tag{C.7}$$

with $\psi_i^P = \phi_B\chi^P_{5i}$ (C.7) becomes:

$$-\Delta\psi_i^P = 2\lambda_i^P\psi_i^P \tag{C.8}$$

Multiplying (C.8) with $\bar{\psi}_i^P$ and after a partial integration, we obtain for λ_i^P:

$$\lambda_i^P = \frac{1}{2}\frac{\int|\partial_\mu\psi_i^P|^2\,d^4x}{\int|\psi_i^P|^2\,d^4x} \tag{C.9}$$

which shows clearly that λ_i^P is positive.

The case $\lambda_0^P = 0$ can be excluded, because this would imply: $\psi_0^P = \text{constant}$ or $\chi^P_{50} \sim 1/\phi_B$ but $\phi_B \to 0$ for $r \to \infty$, thus χ^P_{50} is not in the Hilbert-space.

The above is an example of Higgs mechanism. Namely by breaking the continuous $U(1)$ symmetry we expect Goldstone massless modes but these modes do not persist in the physical sector. Furthermore naively in four dimension $\chi^P_{\mu_i}$ has four degrees of freedom but the gauge condition freezes one and yet another is frozen because of the condition (C.6) thus resulting in really two degrees of freedom. We notice that among the unphysical eigenvalues λ_j^t there are also negative ones.

Appendix D

In this appendix we estimate the ratio

$$\frac{\prod_n (\lambda^P_{0n})^{1/2}}{\prod_i (\lambda_i^P)^{1/2}} \tag{D.1}$$

which appears in the factor K_g (equation (2.37)). To do that we have to take a suitable basis for the space spanned by the functions $\chi_i^P = (\chi_{\mu i}^P, \chi_{5i}^P)$ with $\partial^\mu \chi_{\mu i}^P = 0$.

We choose the normalized eigenfunctions ψ_n^P of the operator B_0. For this choice we have to assume that the dimension of the spaces spanned by the eigenfunctions χ_i^P and ψ_n^P are the same.

We do first the computations in a finite volume V, and we let then V go to ∞. Since:

$$\int d^4x \bar{\psi}_n^P B_0 \psi_n^P = \lambda_{0n}^P$$

we get:

$$\prod_n (\lambda_{0n}^P) = \text{Det}\left(\int d^4x \bar{\psi}_n^P B_0 \psi_n^P\right) \tag{D.2}$$

For $\prod_i (\lambda_i^P)$ we obtain in this approximation:

$$\prod_i (\lambda_i^P) \cong \text{Det} \int d^4x \bar{\psi}_i^P B_B \psi_j^P) \tag{D.3}$$

Since we know that all the eigenvalues λ_i^P of the operator B_B are positive (see Appendix C), we can use the following inequality:

$$\text{Det}\left(\int \bar{\psi}_i^P B_B \psi_j^P \, d^4x\right) \leqq \text{Det}\left(\int \bar{\psi}_i^P B_B \psi_i^P \, d^4x\right) \tag{D.4}$$

and thus consider only the diagonal elements. We get:

$$\prod_i \left(\frac{\lambda_i^P}{\lambda_{0i}^P}\right) \leqq \frac{\text{Det}\left(\int \bar{\psi}_i^P B_B \psi_i^P \, d^4x\right)}{\text{Det}\left(\int \bar{\psi}_i^P B_0 \psi_i^P \, d^4x\right)} = \text{Det}\left[\frac{\int \bar{\psi}_i^P B_B \psi_i^P \, d^4x}{\lambda_{0i}^P}\right] \tag{D.5}$$

or

$$\prod_i \left(\frac{\lambda_i^P}{\lambda_{0i}^P}\right) \leqq \exp\left\{\sum_i \ln\left(\frac{\int \bar{\psi}_i^P B_B \psi_i^P \, d^4x}{\lambda_{0i}^P}\right)\right\} \tag{D.6}$$

We notice that $\int d^4x \bar{\psi}_i^P B_B \psi_i^P$ can be written in the following way, using equations (2.15), (2.20) and $\psi_i^P = (\psi_{\mu i}^P, \psi_{5i}^P)$

$$\begin{aligned}\int \bar{\psi}_i^P B_B \psi_i^P \, d^4x = &\int \bar{\psi}_i^P B_0 \psi_i^P \, d^4x + e^2 \int \bar{\psi}_{\mu i}^P \phi_B^2 \psi_{\theta i}^P \, d^4x \\ &+ 2e \int \bar{\psi}_{\mu i}^P (\partial_\mu \phi_B) \psi_{5i}^P \, d^4x + 2e \int \bar{\psi}_{5i}^P (\partial_\mu \phi_B) \psi_{\mu i}^P \, d^4x \\ &\int \bar{\psi}_{5i}^P \left(\left.\frac{\partial^2 V}{\partial \phi_2^2}\right|_{\phi_B} - \omega_2^2\right) \psi_{5i}^P \, d^4x\end{aligned} \tag{D.7}$$

with $\int \bar{\psi}_i^P B_0 \psi_i^P \, d^4x = \lambda_{0i}^P$.

Since the operator B_0 (equation (2.20)) factorizes, we can take for the eigenfunctions ψ_i^P the following functions:

$(i \in (i_1, i_2, i_3, i_4, i_5)(\nu = 1, 2, 3, 4))$

$$\frac{1}{\sqrt{V}}\begin{bmatrix} e^{ik_\nu^{i_1}x^\nu} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{with } k_1^{i_1} = 0, \text{ such that } \partial^\mu \psi_{\mu_i}^P = o, \text{ and } \lambda_{0i_1}^P = (k_\nu^{i_1})^2;$$

$$\frac{1}{\sqrt{V}}\begin{bmatrix} 0 \\ e^{ik_\nu^{i_2}x^\nu} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{with } k_2^{i_2} = 0, \text{ and } \lambda_{0i_2}^P = (k_\nu^{i_2})^2;$$

$$\frac{1}{\sqrt{V}}\begin{bmatrix} 0 \\ 0 \\ e^{ik_\nu^{i_3}x^\nu} \\ 0 \\ 0 \end{bmatrix} \quad \text{with } k_3^{i_3} = 0, \text{ and } \lambda_{0i_3}^P = (k_\nu^{i_3})^2;$$

$$\frac{1}{\sqrt{V}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ e^{ik_\nu^{i_4}x^\nu} \\ 0 \end{bmatrix} \quad \text{with } k_4^{i_4} = 0, \text{ and } \lambda_{0i_4}^P = (k_\nu^{i_4})^2;$$

$$\frac{1}{\sqrt{V}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ e^{ik_\nu^{i_5}x^\nu} \end{bmatrix} \quad \text{with } \lambda_{0i_5}^P = (k_\nu^{i_5})^2 + \omega_2^2. \tag{D.8}$$

With (D.8) we obtain for (D.7):

$$\int \bar{\psi}_i^P B_B \psi_i^P \, d^4x = \lambda_{0i}^P + \begin{cases} \dfrac{1}{V} e^2 \displaystyle\int d^4x \phi_B^2 & \text{for } i \in (i_1, i_2, i_3, i_4) \\ \dfrac{1}{V} \displaystyle\int d^4x \left(\dfrac{\partial^2 V}{\partial \phi_2^2}\bigg|_{\phi_B} - \omega_2^2 \right) & \text{for } i \in i_5. \end{cases} \tag{D.9}$$

Equation (D.6) becomes:

$$\exp\left\{\sum_{n=1}^{4}\sum_{i_n}\ln\left(1+\frac{1}{V}\frac{e^2\int\phi_B^2\,d^4x}{\lambda_{0i_n}^P}\right)+\sum_{i_5}\ln\left(1+\frac{1}{V}\frac{\int d^4x\left(\frac{\partial^2V}{\partial\phi_2^2}\Big|_{\phi_B}-\omega_2^2\right)}{\lambda_{0i_5}^P}\right)\right\} \tag{D.10}$$

We make the following approximation:

$$\ln\left(1+\frac{1}{V}\frac{e^2\int d^4x\phi_B^2}{\lambda_{0i_n}^P}\right)\cong\frac{1}{V}\frac{e^2\int d^4x\phi_B^2}{\lambda_{0i_n}^P}$$

and

$$\sum_{i_n}\to\frac{V}{(2\pi)^4}\int d^4k\,\delta(k_n)$$

For the last term we write:

$$\ln\left(\frac{\lambda_{0i_5}^P+\frac{1}{V}\int d^4x\left(\frac{\partial^2V}{\partial\phi_2^2}\Big|_{\phi_B}-\omega_2^2\right)}{\lambda_{0i_5}^P}\right)$$

$$=\ln\left[\frac{\frac{1}{V}\int d^4xe^{-ik\frac{i_5}{v}x^v}\left(-\Delta+\frac{\partial^2V}{\partial\phi_2^2}\Big|_{\phi_B}\right)e^{ik\frac{i_5}{v}x^v}}{\lambda_{0i_5}^P}\right]\equiv\ln\left(\frac{\lambda_{i_5}^P}{\lambda_{0i_5}^P}\right)$$

Equation (D.10) becomes:

$$\exp\left\{\frac{1}{(2\pi)^4}\sum_{n=1}^{4}e^2\int d^4x\phi_B^2\int\frac{d^4k\,\delta(k_n)}{k^2}\right\}\times\prod_{i_5}\left(\frac{\lambda_{i_5}^P}{\lambda_{0i_5}^P}\right) \tag{D.11}$$

We evaluate (D.11) further in the thin-wall approximation. To give a meaning to equation (D.11) one has to regularize the integral $\int d^4k\,\delta(k_n)/k^2$, which is otherwise ultraviolet divergent.

Using the dimensional regularization technique, one finds that after renormalization the integral gives zero contribution.

The last term in equation (D.11) corresponds to the ratio:

$$\left|\frac{\det(-\Delta+\omega_2^2)}{\det'\left(-\Delta+\frac{\partial^2V}{\partial\phi_2^2}\Big|_{\substack{\phi_1=\phi^B\\ \phi_2=0}}\right)}\right|^{-1}$$

putting it all together, and using equation (3.11) we find

$$\frac{\prod_n(\lambda_{0n}^P)^{1/2}}{\prod_i(\lambda_i^P)^{1/2}}\geqslant e^{-(m_2R)^3(\ln 2/6-1/9)} \tag{D.12}$$

Appendix E

In this appendix we give an argument which shows that in thin-wall approximation the operators

$$-\Delta + \frac{\partial^2 V}{\partial \phi_n^2}\bigg|_{\phi_B} \qquad (u = \alpha + 1, \ldots, N^2 - 2)$$

corresponding to the unbroken generators, have only positive eigenvalues. In this approximation the bounce has the form

$$\phi_B(r) = \phi_- \theta(R - r) \tag{E.1}$$

with

$$\theta(R - r) = \begin{cases} 1 & r \leqslant R \\ 0 & r > R \end{cases}$$

Then $\partial^2 V/\partial\phi_n^2|_{\phi_B}$, which is a polynomial in ϕ_B can be written as

$$\frac{\partial^2 V}{\partial \phi_n^2}\bigg|_{\phi_B} = a + b\theta(R - r) \tag{E.2}$$

since powers of $\theta(R-r)$ are still equal to $\theta(R-r)$. We know that $\partial^2 V/\partial\phi_n^2|_{\phi_B}(r = 0) > 0$ and $\partial^2 V/\partial\phi_n^2|_{\phi_B}(r \to \infty) > 0$ since for $\phi_B = \phi_-(r = 0)$ and for $\phi_B = 0$ $(r \to \infty)$ V has minima (a global and a local one). These two conditions imply that $a + b > 0$ and $a > 0$. Therefore with equation (E.2) it follows that $\partial^2 V/\partial\phi_n^2|_{\phi_B} > 0$ $\forall r$, and thus the operators $-\Delta + \partial^2 V/\partial\phi_n^2|_{\phi_B}$ have only positive eigenvalues.

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