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Autor: Purice, R.
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Wigner's theorem and the asymptotic condition in scattering theory

By R. Purice

Department of Theoretical Physics, Central Institute of Physics, Bucharest, P.O. Box MG 6

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Abstract. The object of this paper is two-fold: first to give an elementary, analytic proof of Wigner's theorem by directly proving that the linear extension of an isometry between two projective spaces to the algebras of finite rank operators is multiplicative or antimultiplicative, and then to give a simpler proof of a theorem of Jauch, Misra and Gibson about the existence of wave operators and of the modified free evolution.

1. Introduction

To each quantum system without superselection rules the standard formalism of quantum mechanics ([8]) associates a complex Hilbert space $\mathscr{H}$ and an isomorphism from the lattice of propositions of the system onto the lattice of all orthogonal projections $\mathscr{L}(\mathscr{H})$ in $\mathscr{H}$. This isomorphism enables one to identify the set of pure states of the physical system, with $\mathbb{P}(\mathscr{H})$, the set of one-dimensional orthogonal projections in $\mathscr{H}$ (or, equivalently, the set of rays in $\mathscr{H}$).

As remarked by J. M. Jauch, B. Misra and A. G. Gibson [7], it is natural from the physical point of view, to consider on $\mathbb{P}(\mathscr{H})$ the following metric (where, $p, q \in \mathbb{P}(\mathscr{H})$ and Tr is the usual trace function in $\mathscr{H}$)

$$d(p, q) = \sup_{E \in \mathscr{L}(\mathscr{H})} |\mathrm{Tr}\,(pE) - \mathrm{Tr}\,(qE)|. \tag{1}$$

In fact, $\mathrm{Tr}\,(pE)$ is the mean value of the proposition E in the state p, hence $d(p, q)$ small means that the difference between the mean values of any proposition E in the states p and q is small, uniformly in E.

It is clear that $\mathbb{P}(\mathscr{H})$ can be identified with the projective space of $\mathscr{H}$, i.e. with the quotient of the unit sphere in $\mathscr{H}$:

$\mathscr{S}(\mathscr{H}) = \{\varphi \in \mathscr{H} \mid \|\varphi\| = 1\}$ by the action of the group

$U(1) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$.

For $\varphi \in \mathscr{S}(\mathscr{H})$ we denote by $\hat{\varphi}$ its image in $\mathbb{P}(\mathscr{H})$ (i.e. the orthogonal projection on the subspace generated by φ). It is easy to show that d is a metric on $\mathbb{P}(\mathscr{H})$, which becomes a complete metric space, and that we have the following explicit expression for any $\varphi, \psi \in \mathcal{S}(\mathcal{H})$

$$d(\hat{\varphi}, \hat{\psi}) = \sqrt{1 - |(\varphi, \psi)|^2}. \tag{2}$$

The following well-known theorem of Wigner completely describes the structure of the isometries of the above metric space.

Theorem (Wigner). *Let $\mathcal{H}$ and $\mathcal{K}$ be two complex Hilbert spaces and $S: \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{K})$ an isometry. Then there is a (unique, up to multiplication by a complex number of modulus one) linear or antilinear isometry: $U: \mathcal{H} \to \mathcal{K}$ such that: $S\hat{\varphi} = \widehat{(U\varphi)}$ for any $\varphi \in \mathcal{S}(\mathcal{H})$.*

In this paper we shall give an elementary, 'analytic' proof of this result. See V. Bargmann [2], G. Emch and C. Piron [4], U. Ulhorn [9], for other proofs, and W. Hunziker [5] for a related theorem. We hope that the analytic methods we shall use will provide deeper insights into the metric structure of the projective space $\mathbb{P}(\mathcal{H})$. In fact, this will also give us the tools for proving a theorem of Jauch, Misra and Gibson which is quite close to Wigner's theorem, as will be seen later.

The topology defined by the metric (1) has been used in [7] in order to formulate an asymptotic condition in the quantum theory of scattering with a much stronger physical motivation than the usual one (cf. [6]). The price paid for this naturality is that the existence of wave operators is no more evident. One can also prove that the wave operators can still be defined by the usual strong limits if one uses a modified free evolution. The proof of these results in [7] is quite involved. In the last section of this paper we shall give a simple proof of the following theorem which contains the main results of [7].

Theorem (Jauch–Misra–Gibson). *Let $\mathcal{H}$, $\mathcal{K}$ be two complex Hilbert spaces and $\{W_t\}_{t\geqq 0}$ a family of linear isometries $W_t: \mathcal{H} \to \mathcal{K}$. Let $\hat{W}_t: \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{K})$ be the induced isometries (i.e. $\hat{W}_t\hat{\varphi} = \widehat{W_t\varphi}$). Then the following statements are equivalent:*

(1) *For each $p \in \mathbb{P}(\mathcal{H})$ the limit $\lim_{t\to\infty} \hat{W}_t p \stackrel{d}{=} \omega(p)$ exists in $P(\mathcal{K})$.*
(2) *There is a family: $\{\xi_t\}_{t\geqq 0}$ of complex numbers of modulus one, i.e.: $\xi_t \in U(1)$, such that $\lim_{t\to\infty} W_t\xi_t\varphi \stackrel{d}{=} W\varphi$ exists in the strong topology of $\mathcal{K}$ for each $\varphi \in \mathcal{H}$.*

Under the above conditions, $W: \mathcal{H} \to \mathcal{K}$ is a linear isometry which induces ω, i.e.: $\omega(p) \equiv WpW^*$ (or $\omega(\hat{\varphi}) \equiv \widehat{W\varphi}$). Moreover, W is uniquely defined by this relation, up to a multiplicative complex factor of modulus one.

Let us remark that the existence of a linear isometry W such that $\omega(p) = WpW^*$ is assured by Wigner's theorem (W must be linear because the linear extension of ω to the algebra of finite-rank operators from $\mathcal{H}$ to $\mathcal{K}$, is multiplicative, see Lemma 4). However, a much simpler proof can be given in this case (see Section 4). Also Theorem 3 of [7] follows from Wigner's Theorem (see Lemma 7 for a simple proof).

The paper is organized as follows: in Section 2 we make some general remarks on $\mathbb{P}(\mathcal{H})$, which we consider of interest (even if they are not strictly necessary for the rest of the paper), Section 3 will contain the proof of Wigner's Theorem and Section 4 the proof of the Jauch–Misra–Gibson theorem.

2. Remarks on projective spaces

We first recall some facts about finite rank operators. Everything is elementary because it can be reduced to the finite dimensional case. If $\mathcal{H}$ is a complex Hilbert space we denote by $\mathcal{F}(\mathcal{H})$ the *-algebra of finite rank operators in $\mathcal{H}$. It is clear that $\mathbb{P}(\mathcal{H})$, the set of all orthogonal one-dimensional projections of $\mathcal{H}$, is a subset of $\mathcal{F}(H)$, which generates it as a complex vector space (because each self-adjoint operator in $\mathcal{F}(\mathcal{H})$ is a finite linear combination of elements in $\mathbb{P}(\mathcal{H})$). The following norms on $\mathcal{F}(\mathcal{H})$ will be of interest for us:

(a) operator norm: $\|A\|_\infty = \sup_{\varphi \in \mathcal{S}(\mathcal{H})} \|A\varphi\|$

(b) trace norm: $\|A\|_1 = \operatorname{Tr}|A| = \sup_{\substack{B \in \mathcal{F}(\mathcal{H}) \\ \|B\|_\infty \leqq 1}} |\operatorname{Tr}(AB)|$

(c) Hilbert–Schmidt norm: $\|A\|_2 = [\operatorname{Tr}(A^*A)]^{1/2}$.

(The second equality in (b) follows easily by taking $B = U^*$ where $A = |A|\, U$ is the polar decomposition of A).

We denote $\mathcal{F}_i(\mathcal{H})$, $i = \infty$, 1, 2, the space $\mathcal{F}(\mathcal{H})$ provided with the norm $\|\cdot\|_i$. Remark that $\mathcal{F}_2(\mathcal{H})$ is a prehilbert space, the scalar product being: $\langle A, B\rangle_2 = \operatorname{Tr}(A^*B)$. We shall need further the following lemma, which shows the physical meaning of the trace norm (cf. the introduction):

Lemma 1. *Let $A \in \mathcal{F}(\mathcal{H})$ be self-adjoint. Then*:

$$\sup_{E \in \mathcal{L}(\mathcal{H})} |\operatorname{Tr}(AE)| = \tfrac{1}{2}\operatorname{Tr}|A| + \tfrac{1}{2}|\operatorname{Tr} A|$$

where $\mathcal{L}(\mathcal{H})$ is the lattice of orthogonal projections in $\mathcal{H}$.

Proof. Let us denote F_+, $F_- \in \mathcal{L}(\mathcal{H})$ the orthogonal projections on the subspaces generated by the eigenvectors of A associated to positive (resp. negative) eigenvalues. Then: $F_+F_- = F_-F_+ = 0$; $A = AF_+ + AF_- = |A|\,F_+ - |A|\,F_-$; $|A| = AF_+ - AF_-$. Then for all $E \in \mathcal{L}(\mathcal{H})$:

$$\begin{aligned}|\operatorname{Tr}(AE)| &= |\operatorname{Tr}(|A|\,F_+E) - \operatorname{Tr}(|A|\,F_-E)| \leqq \max[\operatorname{Tr}(|A|\,F_+E), \operatorname{Tr}(|A|\,F_-E)] \\ &\leqq \max[\operatorname{Tr}(|A|\,F_+), \operatorname{Tr}(|A|\,F_-)] = \tfrac{1}{2}[\operatorname{Tr}|A|\,F_+ + \operatorname{Tr}|A|\,F_- \\ &\quad + |\operatorname{Tr}|A|\,F_+ - \operatorname{Tr}|A|\,F_-|] = \tfrac{1}{2}[\operatorname{Tr}|A| + |\operatorname{Tr} A|].\end{aligned}$$

In the third step we have used the inequality $|\operatorname{Tr}(BC)| \leqq \|C\|_\infty \operatorname{Tr}|B|$ and the fact that $\|E\|_\infty = 1$. The supremum is effectively reached by taking $E = F_+$, if $\operatorname{Tr}(|A|\,F_+) \geqq \operatorname{Tr}(|A|\,F_-)$ or $E = F_-$ if the opposite inequality is true. Q.E.D.

Now, besides the metric d defined by (1), we can consider on $\mathbb{P}(\mathcal{H})$ the metrics induced by the embeddings: $\mathbb{P}(\mathcal{H}) \subset \mathcal{F}_i(\mathcal{H})$, $i = 1, 2$. The lemma below will show the exact relation between them. On the other hand, as we said in the introduction, $\mathbb{P}(\mathcal{H})$ can also be identified with the quotient of $\mathcal{S}(\mathcal{H})$ through the equivalence relation induced by the action of $U(1)$, i.e. $\varphi \sim \psi$ if and only if there is $\lambda \in C$ with $|\lambda| = 1$ such that $\varphi = \lambda\psi$. Hence we get naturally a quotient metric on $\mathbb{P}(\mathcal{H})$, namely if $\varphi, \psi \in \mathcal{S}(\mathcal{H})$, then

$$d'(\hat{\varphi}, \hat{\psi}) = \inf_{\substack{\lambda,\mu\in\mathbb{C}\\|\lambda|=|\mu|=1}} \|\lambda\varphi - \mu\psi\|. \tag{3}$$

Lemma 2. *For any $\varphi, \psi \in \mathcal{S}(\mathcal{H})$ and $i = 1, 2, \infty$ we have*

$$d(\hat{\varphi}, \hat{\psi}) = 2^{-1/i} \|\hat{\varphi} - \hat{\psi}\|_i = [1 - |(\varphi, \psi)|^2]^{1/2},$$

$$d'(\hat{\varphi}, \hat{\psi}) = [2(1 - |(\varphi, \psi)|)]^{1/2}.$$

In particular:

$$\frac{1}{\sqrt{2}} d'(\hat{\varphi}, \hat{\psi}) \leqq d(\hat{\varphi}, \hat{\psi}) \leqq d'(\hat{\varphi}, \hat{\psi}).$$

Proof. If we take $A = \hat{\varphi} - \hat{\psi}$ in Lemma 1 we get $d(\hat{\varphi}, \hat{\psi}) = \frac{1}{2}\|\hat{\varphi} - \hat{\psi}\|_1$. The equalities: $\|\hat{\varphi} - \hat{\psi}\|_1 = 2\|\hat{\varphi} - \hat{\psi}\|_\infty = 2[1 - |(\varphi, \psi)|^2]^{1/2}$ are proved in [7] by an elementary calculation (the eignvalues of the operator $\hat{\varphi} - \hat{\psi}$ are 0 and $\pm[1 - |(\varphi, \psi)|^2]^{1/2}$. Then:

$$\|\hat{\varphi} - \hat{\psi}\|_2^2 = \operatorname{Tr}(\hat{\varphi} - \hat{\psi})^2 = 2 - 2\operatorname{Tr}(\hat{\varphi} \cdot \hat{\psi})$$
$$= 2 - 2|(\varphi, \psi)|^2 = 2\|\hat{\varphi} - \hat{\psi}\|_\infty^2.$$

Finally:

$$(d'(\hat{\varphi}, \hat{\psi}))^2 = \inf_{\substack{z\in\mathbb{C}\\|z|=1}} \|\varphi - z\psi\|^2 = \inf_{\substack{z\in\mathbb{C}\\|z|=1}} (2 - 2\operatorname{Re} z(\varphi, \psi))$$
$$= 2(1 - |(\varphi, \psi)|). \quad \text{Q.E.D.}$$

Corollary. *If $\mathcal{H}$ and $\mathcal{K}$ are complex Hilbert spaces and $S: \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{K})$, then S is an isometry for d if and only if it is an isometry for d' and if and only if for each $p, q \in \mathbb{P}(\mathcal{H})$*:

$$\operatorname{Tr}[(Sp) \cdot (Sq)] = \operatorname{Tr}(p \cdot q).$$

The relation between convergence in $\mathbb{P}(\mathcal{H})$ and that in $\mathcal{S}(\mathcal{H})$ is given by:

Lemma 3. *Let $\varphi_t, \varphi \in \mathcal{S}(\mathcal{H})$, $t \in \mathbb{R}_+$. Then $\lim_{t\to\infty} \hat{\varphi}_t = \hat{\varphi}$ in $\mathbb{P}(\mathcal{H})$ if and only if there is a sequence $\{\xi_t\}_{t\in\mathbb{R}_+}$ of numbers $\xi_t \in U(1)$ such that*: $s\text{-}\lim_{t\to\infty} \xi_t\varphi_t = \varphi$.

Proof. Since: $\widehat{(\xi_t\varphi_t)} = \hat{\varphi}_t$ and the projection: $\mathcal{S}(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$ is continuous, one implication is trivial. Reciprocally, let:

$$\xi_t = (\varphi_t, \varphi) \cdot |(\varphi_t, \varphi)|^{-1} \quad (\text{with } 0 \cdot 0^{-1} = 1 \text{ by definition}),$$

then $\xi_t \in U(1)$, and:

$$\|\xi_t\varphi_t - \varphi\|^2 = 2 - 2\,\mathrm{Re}\,\bar{\xi}_t(\varphi_t, \varphi) = (d'(\hat{\varphi}_t, \hat{\varphi}))^2 \to 0. \quad \text{Q.E.D.}$$

We shall discuss now the problem of lifting an isometry between the projective spaces of two Hilbert spaces to a linear or antilinear isometry between the corresponding Hilbert spaces. The procedure will consist in passing first to the finite rank operators and then proving that the isometric *-homomorphisms of $\mathscr{F}_2(\mathscr{H})$ are induced by linear isometries on $\mathscr{H}$.

Definition. We shall say that μ: $\mathscr{F}(\mathscr{H}) \to \mathscr{F}(\mathscr{K})$ is a *morphism* (resp. *antimorphism*) if it is linear, $\mu(A^*) \equiv \mu(A)^*$ and $\mu(A \cdot B) \equiv \mu(A)\mu(B)$ (resp. $\mu(A \cdot B) \equiv \mu(B)\mu(A)$).

Lemma 4. *Suppose* $\mu : \mathscr{F}_i(\mathscr{H}) \to \mathscr{F}_i(\mathscr{K})$ *is an isometric morphism (resp. antimorphism), for* $i = 1$ *or* 2. *Then* $\mu[\mathbb{P}(\mathscr{H})] \subset \mathbb{P}(\mathscr{K})$.

Proof. Let us take $p \in \mathbb{P}(\mathscr{H})$. Then, with the above definition we have:
$\mu(p) = \mu(p^*) = \mu(p)^*$,
$\mu(p) = \mu(p^2) = \mu(p)^2$.

Hence $\mu(p)$ is self-adjoint and idempotent, thus $\mu(p)$ is an orthogonal projection in $\mathscr{K}$. Now if $i = 1$, we have $\mathrm{Tr}\,\mu(p) = \mathrm{Tr}\,|\mu(p)| = \mathrm{Tr}\,|p| = \mathrm{Tr}\,p = 1$ and if $i = 2\ \mathrm{Tr}\,\mu(p) = \mathrm{Tr}\,(\mu(p)^*\mu(p)) = \mathrm{Tr}\,(p^*p) = \mathrm{Tr}\,p = 1$, hence in both cases $\mu(p)$ will be a one-dimensional orthogonal projection, and thus $\mu(p) \in \mathbb{P}(\mathscr{K})$. Q.E.D.

Lemma 5 (Cf. Ulhorn [9]). *Each isometry* ω: $\mathbb{P}(\mathscr{H}) \to \mathbb{P}(\mathscr{K})$ *has a unique extension to a linear isometry* $\tilde{\omega}$: $\mathscr{F}_2(\mathscr{H}) \to \mathscr{F}_2(\mathscr{K})$ *such that*: $\tilde{\omega}(A^*) \equiv \tilde{\omega}(A)^*$.

Proof. Clearly:

$$\left\|\sum_{i=1}^n \lambda_i \omega(p_i)\right\|_2^2 = \sum_{i,j=1}^n \bar{\lambda}_i \lambda_j < \omega(p_i), \omega(p_j) >_2$$
$$= \sum_{i,j=1}^n \bar{\lambda}_i \lambda_j \langle p_i, p_j \rangle_2 = \left\|\sum_{i=1}^n \lambda_i p_i\right\|_2^2.$$

Hence, if we denote:

$$\tilde{\omega}\left(\sum_{i=1}^n \lambda_i p_i\right) \stackrel{d}{=} \sum_{i=1}^n \lambda_i \omega(p_i),$$

the above relation defines an isometry from $\mathscr{F}_2(\mathscr{H})$ to $\mathscr{F}_2(\mathscr{K})$. Q.E.D.

Notation. For any φ, $\psi \in \mathscr{H}$ we denote by $|\varphi\rangle\langle\psi|$ the linear opeator $|\varphi\rangle\langle\psi| \in \mathscr{F}(\mathscr{H})$ given by: $\xi \to (\psi, \xi)\varphi$. In particular $|\varphi\rangle\langle\varphi| = \hat{\varphi}$ if $\varphi \in \mathscr{S}(\mathscr{H})$.

Lemma 6. *Let* $A \in \mathscr{F}(\mathscr{H})$. *If there are* φ, $\psi \in \mathscr{H}$ *such that* $A^*A = \|\varphi\|^2 |\psi\rangle\langle\psi|$ and $AA^* = \|\psi\|^2 |\varphi\rangle\langle\varphi|$, *then there is* $\lambda \in U(1)$ *such that*: $A = \lambda |\varphi\rangle\langle\psi|$.

Proof. We have for $\xi \in \mathscr{H}$: $\|A\xi\|^2 = (\xi, A^*A\xi) = \|\varphi\|^2 |(\psi, \xi)|^2$, so that $\{\xi \in \mathscr{H} \mid \xi \perp \psi\} \subset \operatorname{Ker} A$, and similarly:

$$\{\xi \in H \mid \xi \perp \varphi\} \subset \operatorname{Ker} A^* = [\operatorname{Im} A]^{\perp}.$$

Thus: $A\xi = \lambda(\xi)\varphi$, with $\lambda(\xi) = \lambda(\psi, \xi)$, $|\lambda| = 1$. Q.E.D.

Lemma 7. *Suppose we have two Hilbert spaces* $\mathscr{H}$ *and* $\mathscr{K}$, *and an isometric morphism (resp. antimorphism)* $\mu: \mathscr{F}_i(\mathscr{H}) \to \mathscr{F}_i(\mathscr{K})$ $(i = 1$ *or* $2)$. *Then there is a linear (resp. antilinear) isometry* $U: \mathscr{H} \to \mathscr{K}$, *unique up to a multiplicative complex factor of modulus one, such that* $\mu(\hat{\varphi}) = \widehat{U\varphi}$ *for each* $\varphi \in \mathscr{S}(\mathscr{H})$ *(or, equivalently,* $\mu(A) = UAU^*$ *for all* $A \in \mathscr{F}(\mathscr{H})$*).*

Proof. We begin with a preliminary remark. Given two arbitrary vectors, φ, $\psi \in \mathscr{H}$, we may use Lemma 4 in order to find φ', $\psi' \in \mathscr{K}$ such that $\|\varphi'\|_{\mathscr{K}} = \|\varphi\|_{\mathscr{H}}$, $\|\psi'\|_{\mathscr{K}} = \|\psi\|_{\mathscr{H}}$ and:

$$\mu(|\varphi\rangle\langle\varphi|) = |\varphi'\rangle\langle\varphi'|; \qquad \mu(|\psi\rangle\langle\psi|) = |\psi'\rangle\langle\psi'|$$

We show that there is $\lambda \in U(1)$ such that $\mu(|\varphi\rangle\langle\psi|) = \lambda |\varphi'\rangle\langle\psi'|$ if μ is a morphism or $\mu(|\varphi\rangle\langle\psi|) = \lambda |\psi'\rangle\langle\varphi'|$ if μ is an antimorphism. In fact this follows by an immediate application of Lemma 6 to the operator $A = \mu(|\varphi\rangle\langle\psi|)$.

Now let us pick two unit vectors $e \in \mathscr{S}(\mathscr{H})$ and $e' \in \mathscr{S}(\mathscr{K})$ such that $\mu(\hat{e}) = \hat{e}'$ (the existence of e' is assured by Lemma 4), which will be fixed from now on. Then for any $\psi \in \mathscr{H}$ we have $\psi = |\psi\rangle\langle e|\, e$. Assuming μ is a morphism, we define:

$$U\psi = \mu(|\psi\rangle\langle e|)e'. \tag{4}$$

Thus we have defined a map $U: \mathscr{H} \to \mathscr{K}$, which is linear because

$$|\alpha_1\psi_1 + \alpha_2\psi_2\rangle\langle e| = \alpha_1 |\psi_1\rangle\langle e| + \alpha_2 |\psi_2\rangle\langle e|.$$

That U is also isometric follows from the remark we made at the beginning of the proof: we can choose $\psi' \in \mathscr{K}$ such that $\|\psi'\|_{\mathscr{K}} = \|\psi\|_{\mathscr{H}}$ and $\mu(|\psi\rangle\langle e|) = |\psi'\rangle\langle e'|$, hence:

$$\begin{aligned}\|U\psi\|_{\mathscr{K}} &= \|\mu(|\psi\rangle\langle e|)e'\|_{\mathscr{K}} = \|\, |\psi'\rangle\langle e'|\, e'\|_{\mathscr{K}} \\ &= \|\psi'\|_{\mathscr{K}} = \|\psi\|_{\mathscr{H}}.\end{aligned}$$

If μ is an antimorphism, we define:

$$U\psi = \mu(|e\rangle\langle\psi|)e', \tag{5}$$

which gives us an antilinear isometry $U: \mathscr{H} \to \mathscr{K}$, because:

$$|e\rangle\langle\alpha_1\psi_1 + \alpha_2\psi_2| = \bar{\alpha}_1 |e\rangle\langle\psi_1| + \bar{\alpha}_2 |e\rangle\langle\psi_2|.$$

We prove now the essential uniqueness of U. Let U_1, U_2 be two isometries

$\mathscr{H} \to \mathscr{H}$ such that for all $\varphi \in \mathbb{P}(\mathscr{H}): \mu(\hat{\varphi}) = \widehat{U_i\varphi}$ $(i = 1, 2)$. U_i can be linear or antilinear. Remark that U_1 and U_2 will have the same range, namely the subspace generated by the vectors $\psi \in \mathscr{S}(\mathscr{H})$ such that $\hat{\psi} \in \mu(\mathbb{P}(\mathscr{H}))$. Hence $V = U_2^{-1}U_1 : \mathscr{H} \to \mathscr{H}$ is a bijective isometry (linear or antilinear) and clearly $\widehat{V\varphi} = \hat{\varphi}$ for all $\varphi \in \mathscr{S}(\mathscr{H})$. Hence for each $\varphi \in \mathscr{H}$ there is $\lambda(\varphi) \in U(1)$ such that $V\varphi = \lambda(\varphi)\varphi$. Let φ, ψ be linearly independent, then:

$$\lambda(\varphi + \psi)(\varphi + \psi) = V(\varphi + \psi) = V\varphi + V\psi = \lambda(\varphi)\varphi + \lambda(\psi)\psi,$$

hence $\lambda(\varphi) = \lambda(\psi)$ (being equal to $\lambda(\varphi + \psi)$), from which we get that $\lambda(\varphi)$ does not depend on φ. Q.E.D.

The essential fact proved in Section 3 is that under the conditions of Lemma 5, μ is either multiplicative or antimultiplicative, hence Wigner's theorem results from an application of Lemma 7.

In the next two lemmas we give a description of projective subspaces. What we want is a purely metric criterion for a subset of $\mathbb{P}(\mathscr{H})$ to be a projective subspace. This will be used later on in order to show that the range of an isometry is a projective subspace.

Notation. Suppose P, $Q \in \mathscr{L}(\mathscr{H})$. We denote by $P \vee Q$ the orthogonal projection on the closed subspace generated by $P\mathscr{H}$ and $Q\mathscr{H}$. We write $P \leqq Q$ if $P\mathscr{H} \subset Q\mathscr{H}$ and $P \perp Q$ if $P\mathscr{H} \perp Q\mathscr{H}$. Remark that for p, $q \in \mathbb{P}(\mathscr{H})$ $p \perp q$ is equivalent to $d(p, q) = 1$, hence it is a purely metric notion.

Definition. Suppose p, $q \in \mathbb{P}(\mathscr{H})$ are distinct. We define the *complex projective line* through p and q:

$$L(p, q) \stackrel{d}{=} \{r \in \mathbb{P}(\mathscr{H}) \mid r \leqq p \vee q\}.$$

Lemma 8. *Suppose p, $q \in \mathbb{P}(\mathscr{H})$, $p \perp q$ and let us choose $s \in L(p, q)$ with* $\mathrm{Tr}\,(p \cdot s) = \mathrm{Tr}\,(q \cdot s) = \frac{1}{2}$. *Consider the following subset of* $\mathbb{R}^2$

$$Q \stackrel{d}{=} \{(x, y) \in \mathbb{R}^2 \mid (x - \tfrac{1}{2})^2 + (y - \tfrac{1}{2})^2 \leqq \tfrac{1}{4}\},$$

and the application $\mathscr{T} : L(p, q) \to \mathbb{R}^2$ *given by* $\mathscr{T}(r) \stackrel{d}{=} (\mathrm{Tr}\,(p \cdot r), \mathrm{Tr}\,(s \cdot r))$. *Then*:

(i) $\mathrm{Im}\,\mathscr{T} \equiv \mathscr{T}[L(p, q)] = Q$.

(ii) *The inverse image of each point in* $\partial Q \equiv \{x, y) \in \mathbb{R}^2 \mid (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{4}\}$ *contains exactly one point and the inverse image of each point in the interior of* Q: $\mathring{Q} \equiv \{(x, y) \in \mathbb{R}^2 \mid (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 < \frac{1}{4}\}$ *contains exactly two points in* $L(p, q)$.

Proof. Let us choose $\varphi \in \mathscr{S}(\mathscr{H})$ such that $\hat{\varphi} = p$. Then there are two uniquely defined vectors ξ, $\psi \in \mathscr{S}(\mathscr{H})$ such that: $\hat{\xi} = s$, $\hat{\psi} = q$ and: $(\xi, \varphi) = 1/\sqrt{2}$, $(\xi, \psi) = 1/\sqrt{2}$. Remark that $\varphi \perp \psi$. Then there is a one-to-one correspondence between $D = \{\alpha \in \mathbb{C} \mid |\alpha| < 1$ or $\alpha = 1\}$ and $L(p, q)$ given by: $\alpha \to |\alpha\varphi + \sqrt{1 - |\alpha|^2}\psi\rangle\langle\alpha\varphi + \sqrt{1 - |\alpha|^2}\psi|$. In particular s corresponds to $\alpha = 1/\sqrt{2}$. Thus for any $r \in L(p, q)$, we can associate a unique $\alpha \in D$ and we have: $\mathrm{Tr}\,(p \cdot r) = |\alpha|^2$, $\mathrm{Tr}\,(s \cdot r) = \frac{1}{2}\,|\alpha + \sqrt{1 - |\alpha|^2}|^2 = \frac{1}{2} + \sqrt{1 - |\alpha|^2} \cdot \mathrm{Re}\,\alpha$. Hence we shall consider the application $\tilde{\mathcal{T}} : D \to \mathbb{R}^2$ given by:

$$\tilde{\mathcal{T}}(\alpha) = (|\alpha|^2, \tfrac{1}{2} + \sqrt{1 - |\alpha|^2} \cdot \mathrm{Re}\,\alpha) \equiv (x, y) \in \mathbb{R}^2.$$

Then $x \in [0, 1]$ and for a fixed x, $y \in [\frac{1}{2} - \sqrt{x(1-x)},\ \frac{1}{2} + \sqrt{x(1-x)}]$ or equivalently: $|y - \frac{1}{2}| \leqq \sqrt{x(1-x)}$. Thus the image of $\tilde{\mathcal{T}}$ is the closed disc: $\{(x, y) \in \mathbb{R}^2 \mid (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 \leqq \frac{1}{4}\} = Q$. We see that ∂Q is the image of $D \cap \mathbb{R}$ and that $\mathring{Q}$ is the image of $D \backslash \mathbb{R}$. The application $\tilde{\mathcal{T}}$ depends only on $|\alpha|$ and $\mathrm{Re}\,\alpha$ so that for $\mathrm{Im}\,\alpha \neq 0 : \tilde{\mathcal{T}}(\alpha) = \tilde{\mathcal{T}}(\bar{\alpha})$. The conclusion of the lemma follows easily. Q.E.D.

Remark. If we consider $p, q \in \mathbb{P}(\mathcal{H})$ with $p \perp q$, and $s \in L(p, q)$ with $\mathrm{Tr}\,(p \cdot s) = \mathrm{Tr}\,(q \cdot s) = \frac{1}{2}$, the inverse image of ∂Q defined in Lemma 8, may be given a geometrical meaning as a kind of "real projective line" passing through p, q and s. More precisely, we can define:

$$L_s^R(p, q) \stackrel{d}{=} \{r \in L(p, q) \mid \exists \mu, \nu, \lambda \in \mathbb{R} \text{ such that } r = \mu p + \nu q + \lambda s\}.$$

Then one can prove by some tedious but straightforward calculations that:

(i) $L_s^R(p, q) = \mathcal{T}^{-1}(\partial Q)$.
(ii) $L(p, q)$ is the set of projections $r \in P(\mathcal{H})$ such that there are μ, ν, λ, $\rho \in \mathbb{R}$ satisfying $r = \mu p + \nu q + \lambda s + i\rho \cdot \frac{1}{2}(psq - qsp)$ and $(\mu - \frac{1}{2})^2 + (\nu - \frac{1}{2})^2 = R^2$; $\lambda = 1 - \mu - \nu$; $\rho^2 = 2(1 - 2R^2)$; $0 \leqq R \leqq 1/\sqrt{2}$
(iii) $L_s^R(p, q)$ is a subset of $L(p, q)$ defined by the condition $\rho = 0$.

Lemma 9. *Let $\Sigma \subset \mathbb{P}(\mathcal{H})$ have the following three properties:*

(i) *Σ is closed in $\mathbb{P}(\mathcal{H})$.*
(ii) *For any $p, q \in \Sigma$, with $p \neq q$, there are two orthogonal projections p_0, $q_0 \in \Sigma$ such that: $p_0 \perp q_0$ and $p_0, q_0 \leqq p \vee q$.*
(iii) *For any $p, q \in \Sigma$ with $p \perp q$, if $r \in \mathbb{P}(\mathcal{H})$ satisfies:*

$$\mathrm{Tr}\,(p \cdot r) + \mathrm{Tr}\,(q \cdot r) = 1,$$

then $r \in \Sigma$.

Then there is a Hilbert subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that: $\Sigma = \mathbb{P}(\mathcal{H}_0)$.

Proof. Let us define

$$\mathcal{H}_0 = \left\{\psi \in \mathcal{H} \backslash \{0\} \;\middle|\; \frac{\hat{\psi}}{\|\psi\|} \in \Sigma\right\} \cup \{0\},$$

with the scalar product induced from $\mathcal{H}$. Then (i) implies $\mathcal{H}_0$ is closed in $\mathcal{H}$, the projection: $\mathcal{H} \backslash \{0\} \to \mathbb{P}(\mathcal{H})$ being continuous. It remains to prove that $\mathcal{H}_0$ is invariant under linear combinations. Clearly we have:

(a) $p \perp q \Rightarrow L(p, q) = \{r \in \mathbb{P}(\mathcal{H}) \mid \mathrm{Tr}\,(p \cdot r) + \mathrm{Tr}\,(q \cdot r) = 1\}$,
(b) $p', q' \in L(p, q),\ p' \neq q',\ p \neq q \Rightarrow L(p, q) = L(p', q')$.

Using now (ii) and (iii) the assertion of the lemma follows easily. Q.E.D.

3. Wigner's theorem

Suppose we have two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and an isometry $S: \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{K})$ (with the metrics discussed in Sections 1 and 2). In order to lift the above isometry to a linear or antilinear isometry from $\mathcal{H}$ to $\mathcal{K}$ we shall use Lemma 5 and Lemma 7 and the fact that the linear isometry: $\tilde{S}: \mathcal{F}_2(\mathcal{H}) \to \mathcal{F}_2(\mathcal{K})$ induced by S (as stated in Lemma 5) is either multiplicative or antimultiplicative. This last fact will be the main result of this section.

Lemma 10. *If $S: \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{K})$ is an isometry, then $\Sigma \stackrel{d}{=} S[\mathbb{P}(\mathcal{H})]$ satisfies the three requirements in Lemma* 9.

Proof. (i) is clearly satisfied S being an isometry. In order to prove (ii) let $p', q' \in \Sigma$ with $p' \neq q'$. Then there are $p, q \in \mathbb{P}(\mathcal{H})$ with: $p' = S(p)$, $q' = S(q)$. Now we can choose $p_0, q_0 \in \mathbb{P}(\mathcal{H})$ such that: $p_0, q_0 \leqq p \vee q$ and $p_0 \perp q_0$. Now defining: $p_0' = S(q_0)$, $q_0' = S(q_0)$ and taking into account that S is an isometry, by the corollary to Lemma 2 we have for any $r, s \in \mathbb{P}(\mathcal{H})$: $\mathrm{Tr}\,(S(r) \cdot S(s)) = \mathrm{Tr}\,(r \cdot s)$, so that $p_0' \perp q_0'$. We show $p_0', q_0' \leqq p' \vee q'$. This is a consequence of the fact that $S(L(p, q)) \subset L(p_0', q_0')$, because this implies $p', q' \in L(p_0', q_0')$ and $p' \neq q'$ gives $L(p', q') = L(p_0', q_0')$. But:

$$
\begin{aligned}
S[L(p, q)] &= S[L(p_0, q_0)] \\
&= \{r' \in P(\mathcal{K}) \mid \exists r \in P(\mathcal{H}), \quad r' = S(r) \quad \text{and} \quad \mathrm{Tr}\,(r \cdot p_0) + \mathrm{Tr}\,(r \cdot q_0) = 1\} \\
&\subset \{r' \in P(\mathcal{K}) \mid \mathrm{Tr}\,(t' \cdot p_0') + \mathrm{Tr}\,(r' \cdot q_0') = 1\} = L(p_0', q_0').
\end{aligned}
$$

Finally, let us prove (iii). Consider $p', q' \in S[\mathbb{P}(\mathcal{H})]$ with $p' \perp q'$, and $r' \in \mathbb{P}(\mathcal{K})$ such that: $\mathrm{Tr}\,(r' \cdot p') + \mathrm{Tr}\,(r' \cdot q') = 1$. There are $p, q \in \mathbb{P}(\mathcal{H})$ such that $p' = S(p)$, $q' = S(q)$. Choose $s \in \mathbb{P}(\mathcal{H})$ such that $\mathrm{Tr}\,(p \cdot s) = \mathrm{Tr}\,(q \cdot s) = \frac{1}{2}$ and define $s' = S(s)$. Then $\mathrm{Tr}\,(p' \cdot s') = \mathrm{Tr}\,(q' \cdot s') = \frac{1}{2}$, in particular $s' \in L(p', q')$. Clearly $r' \in L(p', q')$ also. By Lemma 8, if $(\mathrm{Tr}\,(p' \cdot r'), \mathrm{Tr}\,(s' \cdot r')) \in \mathring{Q}$ (resp. ∂Q), there are exactly two projectors $r_1, r_2 \in L(p, q)$ (resp. a unique $r \in L(p, q)$) such that $\mathrm{Tr}\,(r_i \cdot p) = \mathrm{Tr}\,(r' \cdot p')$ and $\mathrm{Tr}\,(r_i \cdot s) = \mathrm{Tr}\,(r' \cdot s')$ for $i = 1, 2$ (resp. $\mathrm{Tr}\,(r \cdot p) = \mathrm{Tr}\,(r' \cdot p')$ and $\mathrm{Tr}\,(r \cdot s) = \mathrm{Tr}\,(r' \cdot s')$). Hence by applying once more Lemma 8, we conclude that either $r' = S(r_1)$ or $r' = S(r_2)$ (resp. that $r' = S(r)$). Q.E.D.

The above lemma allows us to consider only surjective isometries by restricting the image space $\mathcal{K}$ to $\mathcal{K}_0$ defined by $\mathbb{P}(\mathcal{K}_0) = S[\mathbb{P}(\mathcal{H})]$.

The essential result in our proof of Wigner's theorem is the following lemma.

Lemma 11. *Let $S:\mathbb{P}(\mathcal{H})\to\mathbb{P}(\mathcal{H})$ be a bijective isometry. Then there are only two possibilities (depending only on S):*

Either: $\mathrm{Tr}\,(\mathrm{S(p)}\cdot\mathrm{S(q)}\cdot\mathrm{S(r)})=\mathrm{Tr}\,(p\cdot q\cdot r),\ \forall p, q, r\in\mathbb{P}(\mathcal{H})$.

Or: $\mathrm{Tr}\,(S(p)\cdot S(q)\cdot S(r))=\overline{Tr\,(p\cdot q\cdot r)},\ \forall p, q, r\in\mathbb{P}(\mathcal{H})$.

Remark. The main step in the proof of Lemma 11 is the fact that the domain of a certain continuous function is connected. In order to prove this fact we shall repeatedly use the following result (which is a special case of Proposition 3.7 of Chapter 1, §11 of [3]; see the Appendix for a short proof).

Proposition. *Let X, Y be topological spaces, $f:X\to Y$ a continuous map. Assume that Y is path-connected and:*

(1) *f has a continuous section; i.e. there is a continuous function σ: $Y\to X$, with $f\circ\sigma(y)=y$ for all $y\in Y$.*
(2) *$f^{-1}(y)$ is path-connected for each $y\in Y$.*

Then X is path-connected.

Proof of Lemma 11. We shall suppose $p\neq q\neq r\neq p$, otherwise the lemma is trivial.

(1) For any $\varphi, \psi, \chi\in\mathcal{S}(\mathcal{H})$, one has: $\mathrm{Tr}\,(\hat{\varphi}\cdot\hat{\psi}\cdot\hat{\chi})=(\varphi,\psi)(\psi,\chi)(\chi,\varphi)$ so that for any $p, q, r\in\mathbb{P}(\mathcal{H})$, one has:

$$|\mathrm{Tr}\,(p\cdot q\cdot r)|^2=\mathrm{Tr}\,(p\cdot q)\cdot\mathrm{Tr}\,(q\cdot r)\cdot\mathrm{Tr}\,(r\cdot p)=|\mathrm{Tr}\,(S(p)\cdot S(q)\cdot S(r))|^2.$$

Now let us write:

$$\begin{aligned}\mathrm{Tr}\,(p\cdot q\cdot r)&=\mathrm{Re}\,[\mathrm{Tr}\,(p\cdot q\cdot r)]+i\,\mathrm{Im}\,[\mathrm{Tr}\,(p\cdot q\cdot r)]\\&=\mathrm{Tr}\left(\frac{p\cdot q+q\cdot p}{2}r\right)+\mathrm{Tr}\left(\frac{p\cdot q-q\cdot p}{2}r\right)\end{aligned}$$

(2) First we shall study the real part of $\mathrm{Tr}\,(p\cdot q\cdot r)$. By taking the square and the trace and using the relation $p\cdot q\cdot p=p\cdot\mathrm{Tr}\,(p\cdot q)$, one verifies immediately that:

$$p\vee q=\frac{p+q-p\cdot q-q\cdot p}{1-\mathrm{Tr}\,(p\cdot q)}.$$

Thus:

$$\mathrm{Re}\,[\mathrm{Tr}\,(p\cdot q\cdot r)]=\mathrm{Tr}\,(p\cdot r)+\mathrm{Tr}\,(q\cdot r)-(1-\mathrm{Tr}\,(p\cdot q))\,\mathrm{Tr}\,((p\vee q)\cdot r).$$

Choosing p_0, $q_0\in\mathbb{P}(\mathcal{H})$ such that: $p_0\perp q_0$ and p_0, $q_0\leqq p\vee q$. One has: $p\vee q=p_0\vee q_0=p_0+q_0$. Then:

$$\begin{aligned}\mathrm{Tr}\,((p\vee q)\cdot r)&=\mathrm{Tr}\,(p_0\cdot r)+\mathrm{Tr}\,(q_0\cdot r)=\mathrm{Tr}\,(S(p_0)\cdot S(r))+\mathrm{Tr}\,(S(q_0)\cdot S(r))\\&=\mathrm{Tr}\,((S(p_0)\vee S(q_0))\cdot S(r))=\mathrm{Tr}\,((S(p)\vee S(q))\cdot S(r)).\end{aligned}$$

In the last equality we have used the fact that $S(p_0) \perp S(q_0)$ and

$$\mathrm{Tr}\,(S(p)\cdot S(p_0)) + \mathrm{Tr}\,(S(p)\cdot S(q_0)) = 1,$$
$$\mathrm{Tr}\,(S(q)\cdot S(p_0)) + \mathrm{Tr}\,(S(q)\cdot S(q_0)) = 1,$$

so that $S(p),\ S(q) \in L(S(p_0), S(q_0))$ and thus: $S(p_0) \vee S(q_0) = S(p) \vee S(q)$. In conclusion:

$$\mathrm{Re}\,[\mathrm{Tr}\,(p\cdot q\cdot r)] = \mathrm{Re}\,[\mathrm{Tr}\,(S(p)\cdot S(q)\cdot S(r)].$$

(3) Let us now study the imaginary part of $\mathrm{Tr}\,(p\cdot q\cdot r)$. Taking into account the above results, we have only two possibilities:

$$\mathrm{Im}\,[\mathrm{Tr}\,(p\cdot q\cdot r)] = \pm \mathrm{Im}\,[\mathrm{Tr}\,(S(p)\cdot S(q)\cdot S(r))]. \tag{6}$$

We must still prove that the sign does not depend on the triplet $(p, q, r) \in [\mathbb{P}(\mathscr{H})]^3$ but only on S. For that we define $F\colon \tilde{\mathscr{C}} \to \mathbb{R}$ by:

$$\tilde{\mathscr{C}} \overset{d}{=} \{(p, q, r) \in [\mathbb{P}(\mathscr{H})]^3 \mid \mathrm{Im}\,[\mathrm{Tr}\,(p\cdot q\cdot r)] \neq 0\},$$
$$F((p, q, r)) = \frac{\mathrm{Im}\,[\mathrm{Tr}\,(p\cdot q\cdot r)]}{\mathrm{Im}\,[\mathrm{Tr}\,(S(p)\cdot S(q)\cdot S(r))]}.$$

First we remark that F is symmetric in any two arguments, both the numerator and denominator changing their sign at any transposition of the variables. So it is enough to study F on

$$\mathscr{C} = \{(p, q, r) \in [\mathbb{P}(\mathscr{H})]^3 \mid \mathrm{Im}\,[\mathrm{Tr}\,(p\cdot q\cdot r)] > 0\},$$

because if: $\mathrm{Im}\,[\mathrm{Tr}\,(p\cdot q\cdot r)] < 0$ then $\mathrm{Im}\,[\mathrm{Tr}\,(q\cdot p\cdot r)] > 0$, so that $(q, p, r) \in \mathscr{C}$ but $F(p, q, r) = F(q, p, r)$. Then we remark that because of (6) we have $F(p, q, r) = \pm 1$ and F is evidently continuous on $\mathscr{C}$. So if we prove that $\mathscr{C}$ is connected, the conclusion of Lemma 11 will follow.

In order to prove that $\mathscr{C}$ is connected we shall first define a function $u\colon \mathscr{A} \to \mathbb{P}(\mathscr{H})$ by

$$\mathscr{A} = \{(p, q) \in [\mathbb{P}(\mathscr{H})]^2 \mid 0 < \mathrm{Tr}\,(p\cdot q) < 1\},$$
$$u(p, q) = \frac{p\cdot q - q\cdot p}{2i[\mathrm{Tr}\,(p\cdot q)(1 - \mathrm{Tr}\,(p\cdot q))]^{1/2}} + \tfrac{1}{2} p \vee q$$

As one can easily verify (take the square and the trace), $u(p, q)$ is a rank one projection, i.e. belongs to $\mathbb{P}(\mathscr{H})$, and depends continuously on $(p, q) \in \mathscr{A}$. Evidently $\mathrm{Tr}\,(u\cdot p) = \mathrm{Tr}\,(u\cdot q) = \frac{1}{2}$. Then $u(p, q) \in L(p, q)$ because u is orthogonal to each projection which is orthogonal to p and q. We remark that $\mathrm{Im}\,[\mathrm{Tr}\,(p\cdot q\cdot r)] > 0$ is equivalent to: $0 < \mathrm{Tr}\,(p\cdot q) < 1$ and $\frac{1}{2}\mathrm{Tr}\,((p \vee q)\cdot r) < \mathrm{Tr}\,(u(p, q)\cdot r)$. In fact: $\mathrm{Tr}\,(p\cdot q) = 0 \Rightarrow \mathrm{Tr}\,(p\cdot q\cdot r) = 0$, and $\mathrm{Tr}\,(p\cdot q) = 1 \Rightarrow \mathrm{Tr}\,(p\cdot q\cdot r) \in \mathbb{R}$. Thus:

$$\mathscr{C} = \{(p, q, r) \in [\mathbb{P}(\mathscr{H})]^3 \mid 0 < \mathrm{Tr}\,(p\cdot q) < 1$$
$$\text{and } \tfrac{1}{2}\mathrm{Tr}\,((p \vee q)\cdot r) < \mathrm{Tr}\,(u(p, q)\cdot r)\}.$$

In order to prove that $\mathscr{C}$ is connected, we shall use the proposition stated after the statement of Lemma 11, by taking $X = \mathscr{C}$, $f = \pi|_{\mathscr{C}}$ and $y = \mathscr{A}$, where $\pi : [\mathbb{P}(\mathscr{H})]^3 \to [\mathbb{P}(\mathscr{H})]^2$ is defined by: $\pi(p, q, r) = (p, q)$. We show now that the assumptions of the proposition are fulfilled.

(a) *$\mathscr{A}$ is path-connected*

In order to prove this fact we shall once more use the proposition, by taking: $X = \mathscr{A}$, $f(p, q) = \operatorname{Tr}(p \cdot q)$; $Y = (0, 1)$. We still have to prove two facts:

(i) The existence of a continuous section: Let us choose $p_0 \in \mathbb{P}(\mathscr{H})$ and $q_0 \in \mathbb{P}(\mathscr{H})$ such that $p_0 \perp q_0$, and φ, $\psi \in \mathscr{S}(\mathscr{H})$ such that $\hat{\varphi} = p_0$, $\hat{\psi} = q_0$. Let $\xi_t = t\varphi + (1-t)\psi$ with $0 < t < 1$, then: $\sigma(t) = (p_0, \hat{\xi}_t)$ gives us a continuous section σ: $(0, 1) \to \mathscr{A}$ for the application $(p, q) \to \operatorname{Tr}(p \cdot q)$.

(ii) The inverse image of each point is connected: Let us fix an arbitrary $t \in (0, 1)$, and two arbitrary points (p, q) and (p', q') in $\mathscr{A}_t \overset{d}{=} \{(r, s) \in [\mathbb{P}(\mathscr{H})]^2 \mid \operatorname{Tr}(r \cdot s) = t\}$. Let us choose $\varphi, \varphi', \psi, \psi' \in \mathscr{S}(\mathscr{H})$ such that $\hat{\varphi} = p$, $\hat{\varphi}' = p'$, $\hat{\psi} = q$, $\hat{\psi}' = q'$ and $(\varphi, \psi) = |(\varphi, \psi)| = \sqrt{\operatorname{Tr}(p \cdot q)} = \sqrt{t} = \sqrt{\operatorname{Tr}(p' \cdot q')} = |(\varphi', \psi')| = (\varphi', \psi')$. There always exists a strongly continuous one-parameter family $\{U_s\}_{s \in R}$ of unitary operators in $\mathscr{H}$ such that: $U_1\varphi = \varphi'$. Thus:

$$[0, 1] \ni s \to (\widehat{U_s\varphi}, \widehat{U_s\psi}) \in \mathscr{A}_t$$

is a continuous path in $\mathscr{A}_t$ from (p, q) to (p', q_1), where $q_1 = \widehat{U_1\psi}$. Then we consider the three vectors: φ', $U_1\psi \overset{d}{=} \psi_1$, ψ'. Evidently: $(\varphi', \psi_1) = (U_1\varphi, U_1\psi) = (\varphi, \psi) = (\varphi', \psi') \neq 0$. Thus there is a new one-parameter strongly continuous family of unitary operators in $\mathscr{H}$ such that: $U_s\varphi' = \varphi'$ for any $s \in \mathbb{R}$, and $U_1\psi_1 = \psi'$, (because $\psi_1 - p'\psi_1$ and $\psi' - p'\psi'$ have equal norms and are orthogonal to φ'). Now:

$$[0, 1] \ni s \to (p', \widehat{U_s\psi_1}) \in \mathscr{A}_t$$

is a continuous path, contained in $\mathscr{A}_t$ (because $\operatorname{Tr}(p' \cdot \widehat{U_s\psi_1}) = (\varphi', U_s\psi_1)^2 = (U_s\varphi', U_s\psi_1)^2 = (\varphi', \psi_1)^2 = t$) and joining (p', q_1) to (p', q').

(b) *$f : \mathscr{C} \to \mathscr{A}$ has a continuous section*

Indeed, we have: $\operatorname{Tr}(u(p, q) \cdot u(p, q)) = 1 > \frac{1}{2} = \frac{1}{2}\operatorname{Tr}((p \vee q) \cdot u(p, q))$, so that $(p, q) \to (p, q, u(p, q))$ is a continuous section for f.

(c) *The inverse image of each point from $\mathscr{A}$ is connected.*

For any $(p, q) \in \mathscr{A}$ let us define:

$$\mathscr{B}(p, q) = \{r \in \mathbb{P}(\mathscr{H}) \mid \operatorname{Tr}(u(p, q) \cdot r) > \tfrac{1}{2}\operatorname{Tr}(p \vee q) \cdot r)\},$$

so that: $f^{-1}(p, q) = \{(p, q, r) \in [\mathbb{P}(\mathscr{H})]^3 \mid r \in \mathscr{B}(p, q)\} = (p, q) \times \mathscr{B}(p, q)$, and it is enough to prove that $\mathscr{B}(p, q)$ is connected for each $(p, q) \in \mathscr{A}$. Now $(p, q) \in \mathscr{A}$ implies $p \neq q$. Let us define: $v(p, q) \overset{d}{=} (p \vee q) - u(p, q)$. From now on we shall simply write u and v, p and q being fixed. With these notations:

$$\mathscr{B}(p, q) = \{r \in \mathbb{P}(\mathscr{H}) \mid \mathrm{Tr}\,(u \cdot r) > \mathrm{Tr}\,(v \cdot r)\},$$

where $u \perp v$ and $u, v < p \vee q$.

Let us consider two arbitrary points r_0 and r_1 in $\mathscr{B}(p, q)$. In order to construct a continuous path in $\mathscr{B}(p, q)$ joining them we shall choose: $\varphi, \xi_0, \xi_1, \psi \in \mathscr{S}(\mathscr{H})$ such that:

$$\begin{aligned}
&\hat{\varphi} = u \\
&\hat{\xi}_0 = r_0 \quad \text{and} \quad (\varphi, \xi_0) = |(\varphi, \xi_0)| = \sqrt{\mathrm{Tr}\,(u \cdot r_0)}, \\
&\hat{\xi}_1 = r_1 \quad \text{and} \quad (\varphi, \xi_1) = |(\varphi, \xi_1)| = \sqrt{\mathrm{Tr}\,(u \cdot r_1)}, \\
&\hat{\psi} = v \quad \text{and} \quad (\psi, \xi_0) = |(\psi, \xi_0)| = \sqrt{\mathrm{Tr}\,(v \cdot r_0)},
\end{aligned}$$

and we shall consider the path:

$$[0, 1] \ni s \to r_s \stackrel{d}{=} \frac{\widehat{\xi_s}}{\|\xi_s\|} \in \mathbb{P}(\mathscr{H}),$$

where: $\xi_s = s\xi_1 + (1-s)\xi_0$. Then this is a continuous path joining r_0 to r_1. We shall verify now that $r_s \in \mathscr{B}(p, q)$ for any $s \in [0, 1]$. Indeed

$$\begin{aligned}
\mathrm{Tr}\,(u \cdot r_s) &= |(\varphi, s\xi_0 + (1-s)\xi_1)|^2 = s^2\,\mathrm{Tr}\,(u \cdot r_0) + (1-s)^2\,\mathrm{Tr}\,(u \cdot r_1) \\
&\quad + 2s(1-s)\sqrt{\mathrm{Tr}\,(u \cdot r_0)\,\mathrm{Tr}\,(u \cdot r_1)}, \\
\mathrm{Tr}\,(v \cdot r_s) &= |(\psi, s\xi_0 + (1-s)\xi_1)|^2 = s^2\,\mathrm{Tr}\,(v \cdot r_0) + (1-s)^2\,\mathrm{Tr}\,(v \cdot r_1) \\
&\quad + 2s(1-s)\sqrt{\mathrm{Tr}\,(v \cdot r_0)}\,\mathrm{Re}\,(\psi, \xi_1).
\end{aligned}$$

But:

$$\begin{aligned}
&\mathrm{Tr}\,(u \cdot r_0) > \mathrm{Tr}\,(v \cdot r_0) \geqq 0, \\
&\mathrm{Tr}\,(u \cdot r_1) > \mathrm{Tr}\,(v \cdot r_1) \geqq 0; \\
&\sqrt{\mathrm{Tr}\,(u \cdot r_1)} > \sqrt{\mathrm{Tr}\,(v \cdot r_1)} = |(\psi, \xi_1)| \geqq \mathrm{Re}\,(\psi, \xi_1),
\end{aligned}$$

so that: $\mathrm{Tr}\,(u \cdot r_s) > \mathrm{Tr}\,(v \cdot r_s)$ for any $s \in [0, 1]$. Q.E.D.

Lemma 12. *Let S: $\mathbb{P}(\mathscr{H}) \to \mathbb{P}(\mathscr{K})$ be an isometry. Then the induced linear isometry (cf. Lemma 5) $\tilde{S}$: $\mathscr{F}_2(\mathscr{H}) \to \mathscr{F}_2(\mathscr{K})$ is either multiplicative or antimultiplicative.*

Proof. As stated after Lemma 10, we may consider only surjective (and hence bijective) isometries S: $\mathbb{P}(\mathscr{H}) \to \mathbb{P}(\mathscr{K})$. We have also seen that for any $p, q, r \in \mathbb{P}(\mathscr{H})$ we have: $\overline{\mathrm{Tr}\,(p \cdot q \cdot r)} = \mathrm{Tr}\,(q \cdot p \cdot r)$. Now, as proved in Lemma 11, we have only two alternatives:

(1) either: $\mathrm{Tr}\,(S(p) \cdot S(q) \cdot S(r)) = \mathrm{Tr}\,(p \cdot q \cdot r)$ for all p, q, r so that:

$$\begin{aligned}
\langle S(p), \tilde{S}(q \cdot r)\rangle_2 &= \langle p, q \cdot r\rangle_2 = \mathrm{Tr}\,(p \cdot q \cdot r) \\
&= \mathrm{Tr}\,(S(p) \cdot S(q) \cdot S(r)) = \langle S(p), S(q) \cdot S(r)\rangle_2,
\end{aligned}$$

(2) or: $\mathrm{Tr}\,(S(p)\cdot S(q)\cdot S(r))=\overline{Tr\,(p\cdot q\cdot r)}$ for all p, q, r so that:

$$\langle S(p), \tilde{S}(q\cdot r)\rangle_2=\langle p, q\cdot r\rangle_2=\mathrm{Tr}\,(p\cdot q\cdot r)$$
$$=\mathrm{Tr}\,(S(p)\cdot S(r)\cdot S(q))=\langle S(p), S(r)\cdot S(q)\rangle_2.$$

Taking now any $p'\in\mathbb{P}(\mathcal{K})$ and any $q, r\in\mathbb{P}(\mathcal{H})$, because of the surjectivity of S we can find a unique $p\in\mathbb{P}(\mathcal{H})$ such that $p'=S(p)$, and with the above alternative we have one of the two situations:

either: $\langle p', \tilde{S}(q\cdot r)\rangle_2=\langle p', S(q)\cdot S(r)\rangle_2$ for all p', q, r,

or: $\langle p', \tilde{S}(q\cdot r)\rangle_2=\langle p', S(r)\cdot S(q)\rangle_2$ for all p', q, r.

Because $\mathcal{F}_2(\mathcal{H})$ is linearly generated by $\mathbb{P}(\mathcal{H})$ (and similarly for $\mathcal{K}$) we have:
either:

$$\langle X, \tilde{S}(A\cdot B)\rangle_2=\langle X, S(A)\cdot S(B)\rangle_2 \quad \text{for all } X\in\mathcal{F}_2(\mathcal{K}) \text{ and } A, B\in\mathcal{F}_2(\mathcal{H}),$$

or:

$$\langle X, \tilde{S}(A\cdot B)\rangle_2=\langle X, S(B)\cdot S(A)\rangle_2 \quad \text{for all } X\in\mathcal{F}_2(\mathcal{K}) \text{ and } A, B\in\mathcal{F}_2(\mathcal{H}),$$

so that $\tilde{S}$ is either multiplicative or antimultiplicative. Q.E.D.

4. The Jauch–Misra–Gibson theorem

By applying the results of Section 2 we shall now give a simple proof of the Jauch–Misra–Gibson theorem as stated in the Introduction.

The implication (2)$\Rightarrow$(1) is obvious, so that we shall discuss the implication (1)$\Rightarrow$(2). Thus suppose we are given a family of isometries W_t: $\mathcal{H}\to\mathcal{K}$ for $t\geqq 0$, we denote by $\hat{W}_t$: $\mathbb{P}(\mathcal{H})\to\mathbb{P}(\mathcal{K})$ the induced isometries, i.e. $\hat{W}_t\hat{\varphi}=\widehat{W_t\varphi}=W_t\hat{\varphi}W_t^*$ and we suppose that for each $p\in\mathbb{P}(\mathcal{H})$, the limit: $\lim_{t\to\infty}\hat{W}_tp\overset{d}{=}\omega(p)\in\mathbb{P}(\mathcal{K})$ exists. We shall denote by $\tilde{\omega}$: $\mathcal{F}_2(\mathcal{H})\to\mathcal{F}_2(\mathcal{K})$ the unique linear isometric extension of ω given by Lemma 5. Then $\tilde{\omega}$ has the following properties for all A, $B\in\mathcal{F}_2(\mathcal{H})$:

(i) $\tilde{\omega}(A)=\lim_{t\to\infty}W_tAW_t^*$.
(ii) $\tilde{\omega}(A^*)=\tilde{\omega}(A)^*$.
(iii) $\tilde{\omega}(A\cdot B)=\tilde{\omega}(A)\cdot\tilde{\omega}(B)$.

Proof. Any $A\in\mathcal{F}_2(\mathcal{H})$ may be written as: $A=\sum_{i=1}^r\lambda_ip_i$, where $\lambda_i\in\mathbb{C}$, $p_i\in\mathbb{P}(\mathcal{H})$ for each $i\leqq r$. If A is self-adjoint we can choose $\lambda_i\in\mathbb{R}$. In particular (ii) will clearly be true. Then:

$$\tilde{\omega}(A)=\sum_{i=1}^r\lambda_i\omega(p_i)=\sum_{i=1}^r\lambda_i\lim_{t\to\infty}W_tp_iW_t^*=\lim_{t\to\infty}W_tAW_t^*$$

which proves (i). Finally (iii) follows from $W_t ABW_t^* = W_t AW_t^* W_t BW_t^*$ and the continuity of the multiplication in $\mathcal{F}_2(\mathcal{H})$. Q.E.D.

By applying now Lemma 7 to the morphism $\tilde{\omega}$: $\mathcal{F}_2(\mathcal{H}) \to \mathcal{F}_2(\mathcal{H})$ we get the existence of a unique (up to a multiplicative complex factor of modulus one) linear isometry $W: \mathcal{H} \to \mathcal{H}$ such that: $\omega(\hat{\varphi}) = W\hat{\varphi}W^* = \widehat{W\varphi}$. More precisely, in the proof of Lemma 7 we have constructed W in a quasi-explicit way: if $\varphi \in \mathcal{H}$

$$W\varphi = \tilde{\omega}(|\varphi\rangle\langle e|)e',$$

where $e \in \mathcal{S}(\mathcal{H})$ is an arbitrary fixed vector, and $e' \in \mathcal{S}(\mathcal{H})$ is fixed, such that $\hat{e}' = \omega(\hat{e})$. Thus: $\hat{e}' = \omega(\hat{e}) = \lim_{t\to\infty} W_t\hat{e}W_t^* = \lim_{t\to\infty} \widehat{W_t e}$ and applying Lemma 3, we get the existence of a sequence $\{\xi_t\}_{t\geqq 0}$ of complex numbers of modulus one such that: $e' = s - \lim_{t\to\infty} \xi_t W_t e$. Then, if $\varphi \in \mathcal{H}$, we have:

$$W_t\xi_t\varphi = |W_t\xi_2\varphi\rangle\langle W_t\xi_t e|\, W_t\xi_t e.$$

But:

$$|W_t\xi_t\varphi\rangle\langle W_t\xi_t e| = W_t\,|\varphi\rangle\langle e|\, W_t^* \to \tilde{\omega}(|\varphi\rangle\langle e|)$$

In $\mathcal{F}_2(\mathcal{H})$ as we have remarked before. Clearly then;

$$\lim_{t\to\infty} W_t\xi_t\varphi = \tilde{\omega}(|\varphi\rangle\langle e|)e' = \widehat{W\varphi},$$

which finishes the proof of the theorem (the uniqueness was proved in Lemma 7).

Remarks. (1) In the above proof we used only Lemmas 5, 7 and 3 of Section 2. We think this provides a very simple proof for the results of [7]. Remark that Theorem 3 of [7] is a particular case of our Lemma 7.

(2) Let us explain how the theorem of Jauch–Misra–Gibson is used in scattering theory. Assume that in $\mathcal{H}$ are given two strongly continuous one-parameter groups of unitary operators $\{V_t\}_{t\in\mathbb{R}}$ and $\{U_t\}_{t\in\mathbb{R}}$. One says that $\{U_t\}_{t\in\mathbb{R}}$ is a free evolution associated to $\{V_t\}_{t\in\mathbb{R}}$ at $t\to+\infty$ if for each (pure) state $\hat{\varphi} \in \mathbb{P}(\mathcal{H})$ there is a state $\hat{\psi} \in \mathbb{P}(\mathcal{H})$ such that:

$$\lim_{t\to\infty} d(\widehat{V_t\psi}, \widehat{U_t\varphi}) = 0.$$

This is physically natural in view of the discussion at the beginning of the introduction (see (1)). Using (2) for example, one sees that the above condition is equivalent to condition (1) in the statement of the theorem of Jauch–Misra–Gibson, where $W_t = V_t^* U_t$.

Appendix

We give here a proof of the proposition stated after Lemma 11. Let $x_0, x_1 \in X$ and set $y_0 = f(x_0)$, $y_1 = f(x_1)$, $x_0' = \sigma(y_0)$ and $x_1' = \sigma(y_1)$. Since Y is path-connected, there exists a continuous path $\tilde{\gamma}: [0, 1] \to Y$ such that $\tilde{\gamma}(0) = y_0$

and $\tilde{\gamma}(1)=y_1$. Now

$$f(x_0)=y_0=(f\circ\sigma)(y_0)=f(x_0'),$$

so that $x_0, x_0'\in s^{-1}(y_0)$. Similarly we have $x_1, x_1'\in f^{-1}(y_1)$. By assumption (2) of the proposition there are two continuous paths

$$\gamma^{(1)}: [0, 1]\rightarrow f^{-1}(y_0)\subset X \quad \text{such that} \quad \gamma^{(1)}(0)=x_0,\ \gamma^{(1)}(1)=x_0',$$

$$\gamma^{(3)}: [0, 1]\rightarrow f^{-1}(y_1)\subset X \quad \text{such that} \quad \gamma^{(3)}(0)=x_1',\ \gamma^{(3)}(1)=x_1.$$

Let $\gamma^{(2)}=\sigma\circ\tilde{\gamma}$. Then $\gamma^{(2)}$ is a continuous path in X satisfying $\gamma^{(2)}(0)=x_0'$, $\gamma^{(2)}(1)=x_1'$. The composition of the three paths $\gamma^{(1)}$, $\gamma^{(2)}$, $\gamma^{(3)}$ gives a continuous path in X joining x_0 to x_1. Q.E.D.

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