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Momentum operators for large systems

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Abstract. For infinite systems a momentum operator can always be constructed in a suitable representation. It then commutes with all quasilocal observables. The associated W^*-algebra factorizes into a commutative W^*-algebra generated by the momentum operator and a non-commutative residual part. Extremal translation-invariant states factorize accordingly and are eigenstates of momentum.

I. Motivations

It seems to be a characteristic of infinite systems that proper Galilei-transformations ("boosts") cannot be implemented unitarily in the GNS-Hilbert spaces associated with factor representations of the underlying quasilocal C^*-algebra of observables ([1]–[3]). Covariant representations of boost-transformations, however, can be constructed in the Hilbert spaces belonging to direct integrals of suitable factor representations. As, roughly speaking, the σ-algebra of the corresponding central measures transforms under boost transformations like a system of imprimitivities of a momentum operator, one could conjecture that a momentum operator of an infinite system, if it exists, might be an element of the center of the corresponding W^*-algebra of observables and hence a classical observable. It is the objective of this article to show that this, in fact, is true under quite general conditions. Furthermore it turns out that the W^*-algebra thus obtained factorizes into a nonatomic commutative algebra generated by the momentum operator and a non-abelian W^*-algebra representing the remaining of the infinite medium.

There are various rationales for the special behavior of infinite systems under boost transformations. On the heuristic level it is often argued that "to change the velocity of an infinite system (one) would have to impart on it an infinite amount of energy and momentum. No unitary operator upon a Hilbert space can do this" [4]. On a more formal level some insight can be gained from the study of simple model systems, the most simple being quasifree representations of the CAR-algebra of a Galilei-relativistic Fermi field. Let $\mathscr{H}$ be a separable Hilbert space with a basis of one particle state functions. The CAR-algebra $\mathscr{A}$ is generated by an antilinear map $a: \mathscr{H} \to \mathscr{A}$ such that

$$a(f)a^*(g) + a^*(g)a(f) = \langle f \mid g \rangle$$
$$a(f)a(g) + a(g)a(f) = 0, \qquad f, g \in \mathscr{H},$$

where $\langle \cdot | \cdot \rangle$ denotes the scalar product in $\mathscr{H}$. A quasifree gauge-invariant state $\omega_A : \mathscr{A} \to \mathbb{C}$ is defined by

$$\omega_A\{a^*(f_n) \cdots a^*(f_1)a(g_1) \cdots a(g_m)\} = \delta_{nm} \det \{\langle f_i \mid \hat{A} g_k \rangle\}$$

where $\hat{A}$ is an operator on $\mathscr{H}$ with $0 \le \hat{A} \le 1$. Unitary transformations in $\mathscr{H}$ induce quasifree automorphisms of $\mathscr{A}$

$$\tau_g\{a(f)\} = a(U_g f)$$

where $g \to U_g$, $g \in G$, is a projective representation of some group G in $\mathscr{H}$. To simplify the discussion we restrict the action of the Galilei group to space translations $r \to U_r$ and boosts $v \to U_v$ in one dimension. A quasifree state ω_A is translation-invariant, if $U_r^* \hat{A} U_r = \hat{A}$ for all $r \in \mathbb{R}$ or, equivalently, if

$$A \in \{U_r\}''$$

where $\{U_r \mid r \in \mathbb{R}\}''$ is the von Neumann algebra generated by the generators of space translations, i.e. by the momentum operator $\hat{P}$ on $\mathscr{H}$. The operator $\hat{A}$ is a measurable function of $\hat{P}$ accordingly

$$\hat{A} = A(\hat{P})$$

transforming under boost transformations as

$$\hat{A} \to \hat{A}_v = U_v^* \hat{A} U_v = A(\hat{P} - mv\hat{I})$$

where m is the one-particle-mass. It follows that $\hat{A}^{1/2} - \hat{A}_v^{1/2}$ and $(\hat{I} - \hat{A}^{1/2}) - (\hat{I} - \hat{A}_v^{1/2})$ have absolute continuous spectrum and hence do not belong to the class of Hilbert–Schmidt operators on $\mathscr{H}$. This implies by a well-known theorem [5] that ω_A and ω_{A_v} are *disjoint factor states.* (There are two exceptions, the case $\hat{A} = 0$ representing the vacuum state and the case $\hat{A} = \hat{I}$ representing the "plenum" state.)

Not all the states on a quasilocal C^*-algebra are physically meaningful. It is only the GNS-representation with the class of physically relevant states which yields a meaningful W^*-description of a system. But what are the requirements for a state on a quasilocal C^*-algebra to be physically meaningful? The first condition is local normality since it guarantees a W^*-description with separable predual. A second condition may be the existence of the time evolution (which in general is not an automorphism of the C^*-algebra). But there is a third condition: one must be sure that in a given representation physically important observables such as momentum do exist.

This contribution is devoted to representations of Galilean quantum fields carrying a momentum operator and it is shown that this operator is necessarily a classical observable under quite general assumptions. These are stated in Section II. Section III contains some immediate consequences and the main results.

II. Assumptions

We consider a quasilocal C^*-algebra $\mathscr{A}$ (obtained as an inductive limit of an increasing sequence of local C^*-algebras $\mathscr{A}_n$, $n = 1, 2, \ldots$) together with a norm-continuous automorphic representation of space translations

$$q \to \bar{\alpha}_q \in \operatorname{Aut} \mathscr{A}, \qquad q \in \mathbb{R},$$

and momentum translations ("boost")

$$p \to \bar{\beta}_p \in \operatorname{Aut} \mathscr{A}, \qquad p \in \mathbb{R},$$

which are assumed to commute, i.e.

$$\bar{\alpha}_q \circ \bar{\beta}_p = \bar{\beta}_p \circ \bar{\alpha}_q.$$

With no loss of generality we thus restrict the action of the full Galilei group to a subgroup of space and momentum translations in one dimension. Every local C^*-algebra $\mathscr{A}_n$, $n = 1, 2, \ldots,$ is supposed to be represented on a separable Hilbert space $\mathscr{H}_n$, $\mathscr{A}_n \subseteq \mathscr{B}(\mathscr{H}_n)$.

Assumption I. The action of space translations is required to be norm-asymptotically abelian, i.e.

$$\lim_{|q| \to \infty} \|[\bar{\alpha}_q(x), y]\| = 0 \qquad \text{for all } x, y \in \mathscr{A}.$$

Asymptotic abelianess with respect to space translations of a quasilocal algebra of observables has a clear physical interpretation: "It states that observations performed at large distances in space do not influence each other mutually" ([6]: p. 61).

As we intend to characterize the relevant W^*-algebras by GNS-representations of $\mathscr{A}$ with physically relevant states we restrict our considerations to states $\omega \in (\mathscr{A}^*)_1^+$ fulfilling

Assumption II. ω is a locally normal, translation-invariant state on $\mathscr{A}$, i.e. $\omega \circ \bar{\alpha}_q = \omega$ for all $q \in \mathbb{R}$.

A state ω on a quasilocal algebra $\mathscr{A}$ is said to be locally normal if the restriction of ω to $\mathscr{A}_n$ extends to a normal state on the W^*-algebra $\{\mathscr{A}_n\}''$.[1])

Assumption III. ω is quasiinvariant under boost transformations, i.e. $\omega \circ \bar{\beta}_p$ and ω are quasiequivalent.

Assumptions II and III imply that space translations $\bar{\alpha}_q$ and boost transformations $\bar{\beta}_p$ can be extended to pointwise σ-weakly continuous actions α_p and β_p

[1]) Generalizing (Haag, Kadison, Kastler [7]: Prop. 8) it can be shown that the GNS-Hilbert space $\mathscr{H}_\omega$ of a locally normal state ω is separable under the above conditions.

on the weak closure

$$\mathcal{M} \stackrel{\text{def}}{=} \pi_\omega(\mathcal{A})''$$

of $\pi_\omega(\mathcal{A})$ (Observation 1). Assumption II reflects the fact that we are dealing with infinite systems. It excludes systems with finitely many degrees of freedom because finite systems other than the vacuum do not admit (normal) translation invariant states commonly associated with pure phases of infinite systems.

Assumption IV. There exists a system of imprimitivities for the covariant representation $\pi_\omega \circ \beta_p$ of the boost transformation in the GNS representation $\pi_\omega(\mathcal{A})$ of $\mathcal{A}$.

Assumption IV is the traditional way of establishing the existence of a momentum operator [8, 9]. This method, however, is taylored to commuting observables associated with abelian groups (as in our case). The question of non-commuting observables with respect to non-commutative groups has been studied by one of the authors [10, 11] and has led to a reformulation of the whole problem. In this set-up an observable with respect to a group G is given in terms of embeddings of $\mathcal{L}_\infty(\mathcal{G})$ in $\mathcal{M}$, the algebra of observables. It is this formalism which will be used in the sequel. It has to be stressed that both methods are equivalent for abelian groups and commuting observables. The new formulation is preferred mainly for convenience.

Let us denote by $\mathcal{L}_\infty(\mathbb{R})$ the W^*-algebra of essentially bounded functions on $\mathbb{R}$ (modulo null sets) and let us denote by $\{\mathrm{Ad}\,\lambda(p) \mid p \in \mathbb{R}\}$ the action of boosts on $\mathcal{L}_\infty(\mathbb{R})$ defined by

$$\{\mathrm{Ad}\,\lambda(p)f\}(p') \stackrel{\text{def}}{=} f(p'-p); \qquad p, p' \in \mathbb{R}; \qquad f \in \mathcal{L}_\infty(\mathbb{R}).$$

This action corresponds to the *left regular representation* of $\mathbb{R}$ on $\mathcal{L}^2(\mathbb{R})$. We are now in position to give the following

Definition. A *momentum operator* is a normal *-isomorphism $\tau: \mathcal{L}_\infty(\mathbb{R}) \to \mathcal{M}$ of $\mathcal{L}_\infty(\mathbb{R})$ into $\mathcal{M}$ with the transformation properties

(i) $\beta_p \circ \tau = \tau \circ \mathrm{Ad}\,\lambda(p), \qquad p \in \mathbb{R},$

(ii) $\alpha_q \circ \tau = \tau, \qquad q \in \mathbb{R}.$

It is in this form that the existence of a momentum operator affiliated to $\mathcal{M}$ (assumption IV) will be used in what follows.

Assumption IV may be regarded as fairly strong, both from a mathematical and from a physical point of view, and one could ask for its derivation from weaker requirements. To this end we consider a separable C^*-algebra $\mathcal{A}$, a norm-asymptotically abelian action $\bar{\alpha}$ of $\mathbb{R}$ on $\mathcal{A}$ commuting with an action $\bar{\beta}$ of $\mathbb{R}$ on $\mathcal{A}$ together with an extremal $\bar{\alpha}$-invariant state ϕ on $\mathcal{A}$ with the property $\phi \circ \beta_p \neq \phi \ \forall p \in \mathbb{R}$.

Lemma. *Under the above conditions there exists a state ω on $\mathscr{A}$, invariant under $\bar{\alpha}$ and quasiinvariant under $\bar{\beta}$, such that the associated W^*-system $(\pi_\omega(\mathscr{A})'', \mathbb{R}, \beta)$ contains a momentum operator (Assumption IV).*

The proof of this lemma can be found in the appendix.

III. Observations and results

Observation 1. The extensions α_q and β_p of the actions of space translations $\{\bar{\alpha}_q \mid q \in \mathbb{R}\}$ and boosts $\{\bar{\beta}_p \mid p \in \mathbb{R}\}$ from $\pi_\omega(\mathscr{A})$ to the weak closure $\pi_\omega(\mathscr{A})'' = \mathscr{M}$ are pointwise σ-weakly continuous.

Proof. Since $\mathscr{M}$ has a separable predual, the σ-weak topology on $\mathscr{M}_1 = \{x \in \mathscr{M} \mid \|x\| \leq 1\}$ has a countable basis. Using Kaplansky's theorem we infer that every operator $x \in \mathscr{M}$ is the σ-weak limit of a *sequence* $(\pi_\omega(x_n))_{n\in\mathbb{N}}$, $x_n \in \mathscr{A}$. The functions

$$\mathbb{R} \ni q \to \bar{\alpha}_q(x_n) \in \mathscr{A}, \qquad n \in \mathbb{N},$$

are norm-continuous. Therefore the functions

$$\mathbb{R} \ni q \to \phi(\pi_\omega(\bar{\alpha}_q(x_m))), \qquad n \in \mathbb{N},\ \phi \in \mathscr{M}_*$$

are continuous and in particular Borel. Thus

$$\mathbb{R} \ni q \to \phi(\alpha_q(x))$$

is pointwise approximated by a sequence of Borel functions, therefore itself Borel for every $\phi \in \mathscr{M}_*$ (Cohn [12]: 2.1.4) and continuous (Moffat [13]: Coroll. 1). The same holds for $\{\beta_p \mid p \in \mathbb{R}\}$.

Observation 2. The fixed-point algebra $\mathscr{M}^\alpha \stackrel{\text{def}}{=} \{x \in \mathscr{M} \mid \alpha_q(x) = x,\ q \in \mathbb{R}\}$ is part of the center $\mathscr{Z}(\mathscr{M})$ of $\mathscr{M}$.

The proof follows immediately from (Guichardet [14]: Proposition V.3).

Observation 2 is the algebraic aspect of the well known fact that the extremal α_q-invariant components of ω are mutually disjoint. (This is a special feature of asymptotically abelian systems [15]: Th. 2.)

Theorem 1. *Under the assumptions of section II the momentum operator $\tau\{\mathscr{L}_\infty(\mathbb{R})\}$ is part of the center $\mathscr{Z}(\mathscr{M})$ of $\mathscr{M}$. Furthermore there exists a $*$-isomorphism j of $\mathscr{M}$ onto $\mathscr{M}^\beta \bar{\otimes} \mathscr{L}_\infty(\mathbb{R})$ such that*

(i) $j\{\tau(f)\} = 1 \otimes f, \qquad f \in \mathscr{L}_\infty(\mathbb{R})$

(ii) $j\{x\} = x \otimes 1, \qquad x \in \mathscr{M}^\beta$

(iii) $j \circ \beta_p \circ j^{-1} = \mathrm{Id} \otimes \mathrm{Ad}\, \lambda(p), \qquad p \in \mathbb{R}$

where $\mathcal{M}^\beta$ denotes the fixed-point algebra of the boost-transformations β_p and Id *stands for the identity map on $\mathcal{M}^\beta$.*

Proof

$$\tau(\mathcal{L}_\infty(\mathbb{R})) \subseteq \mathcal{M}^\alpha \subseteq \mathcal{Z}(\mathcal{M}).$$

In the following the duality theory of W^*-systems is used (cf. Nakagami, Takesaki [16]; Stratila, Voiculescu, Zsido [17]). $\mathcal{L}^2(\mathbb{R})$ denotes the Hilbert space of square-integrable functions on $\mathbb{R}$ with respect to Lebesgue-measure; $\mathcal{R}(\mathbb{R})$ denotes the right-regular representation algebra on $\mathcal{L}^2(\mathbb{R})$; $M: f \to M_f$, $f \in \mathcal{L}_\infty(\mathbb{R})$, stands for the multiplication representation of $\mathcal{L}_\infty(\mathbb{R})$ on $\mathcal{L}^2(\mathbb{R})$.

$\tau: \mathcal{L}_\infty(\mathbb{R}) \to \mathcal{M}$ implements a coaction $\delta: \mathcal{M}^\beta \to \mathcal{M}^\beta \bar{\otimes} \mathcal{R}(\mathbb{R})$ of $\mathbb{R}$ on $\mathcal{M}^\beta$. There exists an isomorphism j of $\mathcal{M}$ onto the W^*-algebra generated by $\delta(\mathcal{M}^\beta)$ and $\{1 \otimes M_f \mid f \in \mathcal{L}_\infty(\mathbb{R})\}$, such that

(a) $j(\tau(f)) = 1 \otimes M_f, \qquad f \in \mathcal{L}_\infty(\mathbb{R}),$

(b) $j(x) = \delta(x), \qquad x \in \mathcal{M}^\beta,$

(c) $j \circ \beta_p = \hat{\delta}_p \circ j, \qquad p \in \mathbb{R},$

hold (s. Nakagami, Takesaki [16]: p. 25–27). $\{\hat{\delta}_p \mid p \in \mathbb{R}\}$ is the dual representation of the corepresentation δ.

Since $\tau(\mathcal{L}_\infty(\mathbb{R}) \subseteq \mathcal{Z}(\mathcal{M})$, the implemented coaction δ is trivial, i.e.

$$\delta(x) = x \otimes 1, \qquad \forall x \in \mathcal{M}^\beta.$$

Thus $j: \mathcal{M} \to \mathcal{M}^\beta \bar{\otimes} \mathcal{L}_\infty(\mathbb{R}) \subseteq \mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{L}^2(\mathbb{R}))$ is the desired *-isomorphism. (i), (ii), and (iii) follow from (a), (b), and (c), respectively.

Theorem 1 describes the factorization of the algebra of observables into an abelian algebra, generated by the momentum operator where the boosts act faithfully and the space translations act trivially, and into a non-commutative W^*-algebra $\mathcal{M}^\beta$ where the boosts act trivially and the space translations act faithfully. This factorization carries over to the state ω if $\mathcal{M}^\beta$ is a factor (Theorem 2 below).

Theorem 2. *If $\mathcal{Z}(\mathcal{M})^\alpha = \tau(\mathcal{L}_\infty(\mathbb{R}))$, i.p. if $\mathcal{M}^\beta$ is a factor, there exists a faithful normal state Φ_0 on $\mathcal{M}^\beta$ and a faithful normal state Φ_1 on $\mathcal{L}_\infty(\mathbb{R})$ such that ω coincides with the product state $\{\Phi_0 \otimes \Phi_1\} \circ j$.*

Proof. ω is a normal state on $\mathcal{M}$, invariant under $\{\alpha_q \mid q \in \mathbb{R}\}$. The support of ω is an element of $\mathcal{M}^\alpha \subseteq \mathcal{Z}(\mathcal{M})$.

If ω is faithful on $\mathcal{Z}(\mathcal{M})$, it is thus faithful on $\mathcal{M}$. Assume to the contrary that there is an $x > 0$, $x \in \mathcal{Z}(\mathcal{M})$, such that $\omega(x) = \langle \Omega \mid x\Omega \rangle = 0$

$$\Rightarrow \quad \langle x^{1/2}\Omega \mid x^{1/2}\Omega \rangle = \|x^{1/2}\Omega\|^2 = 0 \Rightarrow x^{1/2}\Omega = 0$$

$$\Rightarrow \quad x^{1/2}(x^{1/2}\Omega) = x\Omega = 0$$

$$\Rightarrow \quad x\pi_\omega(a)\Omega = \pi_\omega(a)x\Omega = 0 \qquad \forall a \in \mathcal{A}$$

$$\Rightarrow \quad x = 0 \text{ (since } \Omega \text{ is cyclic for } \pi_\omega(\mathcal{A})).$$

Thus ω is faithful.

Due to a theorem of Kovács and Szücs (s. Guichardet [14]: Th. II.1) there exists a normal α-invariant conditional expectation $E:\mathcal{M}\to\mathcal{M}^{\alpha}=\tau(\mathcal{L}_\infty(\mathbb{R}))$, such that $\phi\cdot E=\phi$ for every normal α-invariant state ϕ on $\mathcal{M}$. E is faithful and surjective, since ω is faithful.
We define $\phi_0:=\omega|_{\mathcal{M}^\beta}$, $\phi_1:=\omega\circ\tau$. Every element $E(x)$, $x\in\mathcal{M}$, is of the form $\tau(f)$ for a suitable $f\in\mathcal{L}_\infty(\mathbb{R})\Rightarrow j(E(x))=j(\tau(f))=1\otimes f$.

$$\Rightarrow \omega(x)\overset{\#}{=}\omega(E(x))=\omega(\tau(f))=\phi_1(f)=(\phi_0\otimes\phi_1)(1\otimes f)$$

$$=(\phi_0\otimes\phi_1)\circ j(E(x))\overset{\#}{=}(\phi_0\otimes\phi_1)\circ j(x),\qquad x\in\mathcal{M}.$$

(At (#) the α-invariance of ω and $(\phi_0\otimes\phi_1)\circ j$ was used.)

As pure phases are usually connected with extremal translation-invariant states it is of physical interest to consider the properties of these states in the present situation. It turns out that they are characterized by sharp, i.e. dispersion-free values for the momentum, that they are finitely additive ("singular") and of product form on $\mathcal{M}^\beta\bar{\otimes}\mathcal{L}_\infty(\mathbb{R})$. This holds equally for extremal α_q-invariant states on $\mathcal{M}$ (Theorem 3) and on $\mathcal{A}$ (Theorem 4).

Theorem 3. *Every extremal α-invariant state ϕ on $\mathcal{M}$ (necessarily singular) is dispersion-free on $\tau(\mathcal{L}_\infty(\mathbb{R}))$. Furthermore $\phi\circ j^{-1}$ is a 'product state', i.e.*

$$(\phi\circ j^{-1})(x\otimes f)=\{(\phi\circ j^{-1})(x\otimes 1)\}\{(\phi\circ j^{-1})(1\otimes f)\},\quad x\in\mathcal{M}^\beta,\quad f\in\mathcal{L}_\infty(\mathbb{R}).$$

Proof. Consider the GNS-construction $(\mathcal{H}_\phi,\pi_\phi)$ for ϕ. By standard reasoning (cf. Bratteli, Robinson [18]: 4.3.17) it is shown that $\pi_\phi(\mathcal{Z}(\mathcal{M})^\alpha)=\mathbb{C}\circ 1\Rightarrow \pi_\phi\tau(\mathcal{L}_\infty(\mathbb{R}))=\mathbb{C}\circ 1$. From this the assertions are an immediate consequence.

Theorem 4. *Assume $\mathcal{A}$ to be simple and let Ψ be an extremal α-invariant state on $\mathcal{A}$. Then there exists an extremal α-invariant state $\tilde{\Psi}$ on $\mathcal{M}$ such that $\tilde{\Psi}\circ\pi_\omega=\Psi$. In particular, the extension $\tilde{\Psi}$ has a fixed dispersion-free momentum value.*

Proof. By the Hahn–Banach theorem the state Ψ admits an extension Ψ_0 to $\pi_\omega(\mathcal{A})''=:\mathcal{M}$ (Pedersen [19]: 3.1.6). Consider the group $\mathbb{R}$ with the *discrete* topology D. $(\mathbb{R},D)$ is amenable as an abelian group (Berberian [20]: 29.5). Let η denote a mean on $\mathcal{B}(\mathbb{R})$, (the set of bounded, complex-valued functions on $\mathbb{R}$). Define $\Psi_1(x):=\eta\{q\to\Psi_0(\alpha_q(x))\}$, $x\in\mathcal{M}$. Then Ψ_1 is an α-invariant state on $\mathcal{M}$ such that $\Psi_1\circ\pi_\omega=\Psi$. Denote $S:=\{\phi\in(\mathcal{M}^*)_1^+\mid\phi\circ\alpha_q=\phi,\ \phi\circ\pi_\omega=\Psi\}$. S is convex and weak* compact and has therefore (Krein–Mil'man) an extremal point $\tilde{\Psi}$. Assume $\tilde{\Psi}$ can be decomposed

$$\tilde{\Psi}=\lambda\phi_1+(1-\lambda)\phi_2$$

where ϕ_1 and ϕ_2 are α-invariant states on $\mathcal{M}$.

$$\Rightarrow\quad \Psi=\lambda\phi_1|_{\pi_\omega(\mathcal{A})}+(1-\lambda)\phi_2|_{\pi_\omega(\mathcal{A})}.$$

Since Ψ is extremal α-invariant, we infer $\phi_1, \phi_2 \in S$; since $\tilde{\Psi}$ is extremal in S, we have $\tilde{\Psi} = \phi_1 = \phi_2$.

Remark. The dispersion-free momentum value of an extremal α-invariant state ϕ on $\mathcal{M}$ might be infinite. It is infinite if and only if $\phi|_{\tau(C_{00}(\mathbb{R}))} \equiv 0$, where $C_{00}(\mathbb{R})$ denotes the continuous functions on $\mathbb{R}$ with compact support.

IV. Concluding remarks

Momentum operators are constructed in Lemma 2 by integrating disjoint representations $\{\pi_\phi \circ \bar{\beta}_p \mid p \in \mathbb{R}\}$. We stress that for this the asymptotic abelianess of the action $\bar{\alpha}$ commuting with $\bar{\beta}$ is important: If it is omitted, the construction of Lemma 2 may fail (cf. Dixmier [21]: Coroll. 4; Bratteli, Kishimoto [22]: Example 2.3). Nevertheless, the existence of an asymptotic abelian action commuting with $\bar{\beta}$ is not a necessary condition (cf. Guichardet [23]: chapter I, §3, lemme 3; Baker [24]: Theorem 6.15). The delicate problems arising in this context seem not to be entirely clarified.

A further point concerns the detailed physical interpretation of the operator thus obtained. It is intimately connected with the interpretation of the Weyl group used for the characterization of the operators. There are several possible interpretations: The operator may describe

- the velocity of an infinite system with respect to some rest frame
- the momentum of finite mass quasi-particles in an infinite medium
- an average momentum per mass unit or per particle number in the sense of Hepp's macroscopic operators [25].

Appendix: Proof of the lemma of section II[2])

Let $E(\mathcal{A})$ denote the state space of the C^*-algebra $\mathcal{A}$, and $E(\mathcal{A})^{\bar{\alpha}}$ it's $\bar{\alpha}$-invariant part. $E(\mathcal{A})$ is a Polish space (Pedersen [19]: 3.7.2). The mapping $\Phi: \mathbb{R} \ni p \rightarrow \phi \circ \beta_p \in E(\mathcal{A})^{\bar{\alpha}} \subseteq E(\mathcal{A})$ is injective and (w^*)-continuous. Therefore (Cohn [12]: 8.3.5, 8.3.7) $\Phi(\mathbb{R})$ is a Borel subset of $E(\mathcal{A})$ and $\Phi: \mathbb{R} \rightarrow \Phi(\mathbb{R})$ is a Borel isomorphism.

Consider a probability measure μ on $\mathbb{R}$, quasiequivalent to the Haar measure. Define a measure ν on $E(\mathcal{A})$ by $\nu(B) \overset{\text{def}}{=} \mu(\Phi^{-1}(B))$ where B is an arbitrary Borel set on $E(\mathcal{A})$. Since $E(\mathcal{A})$ is Polish, μ is a *regular* Borel measure on $E(\mathcal{A})$ (s. Cohn [12]: 8.1.10).

The state $\omega(\cdot) \overset{\text{def}}{=} \int_{E(\mathcal{A})} \Psi(\cdot)\, d\nu(\Psi) = \int_{\mathbb{R}} (\phi \circ \bar{\beta}_p)(\cdot)\, d\mu(p)$ is invariant under $\bar{\alpha}$. Due to (Bratteli, Robinson [18]: p. 374) $(\mathcal{A}, \omega)$ is $\mathbb{R}$-central with respect to the action $\bar{\alpha}$. Since all the states $\phi \circ \bar{\beta}_p$, $p \in \mathbb{R}$, are extremally $\bar{\alpha}$-invariant, the

[2]) The first part of this proof is due to J. Pöttinger.

measure μ is a maximal measure on $E(\mathscr{A})^{\bar{\alpha}}$ ([18]: 4.1.10). From ([18]: 4.3.14) it is referred that ν is subcentral, i.e., the associated mapping κ_ν ([18]: 4.1.21) is a *-isomorphism of $\mathscr{L}_\infty(E(\mathscr{A}), \nu)$ into the center $\mathscr{Z}(\pi_\omega(\mathscr{A})'')$ of $\pi_\omega(\mathscr{A})''$ (where $(\pi_\omega, \mathscr{H}_\omega)$ is the GNS-representation of $\mathscr{A}$ associated to ω). Define

$$\mathscr{H} \overset{\text{def}}{=} \int_{E(\mathscr{A})}^{\otimes} \mathscr{H}_\Psi \, d\nu(\Psi) = \int_{\mathbb{R}}^{\otimes} \mathscr{H}_{\phi\circ\bar{\beta}_p} d\mu(p)$$

$$\pi \overset{\text{def}}{=} \int_{E(\mathscr{A})}^{\otimes} \pi_\Psi \, d\nu(\Psi) = \int_{\mathbb{R}}^{\otimes} \pi_{\phi\circ\bar{\beta}_p} \, d\mu(p)$$

$$D \overset{\text{def}}{=} \int_{E(\mathscr{A})}^{\otimes} \{\mathbb{C}1_{\mathscr{H}_\Psi}\} \, d\nu(\Psi) \cong \mathscr{L}_\infty(E(\mathscr{A}), \nu) \cong \mathscr{L}_\infty(\mathbb{R}, \mu).$$

Due to ([18]: 4.4.9) there exists a unitary operator $U: \mathscr{H}_\omega \to \mathscr{H}$ such that

(i) $U\pi_\omega(x)U^* = \pi(x), \quad \forall x \in \mathscr{A}$

(ii) $U\kappa_\nu(\mathscr{L}_\infty(\nu))U^* = D.$

Therefore D is part of the center of the W^*-algebra $\pi(\mathscr{A})''$.

The direct integral operators $\int_{\mathbb{R}}^{\otimes} \pi_{\phi\circ\bar{\beta}_p}(x) \, d\mu(p)$, $x \in \mathscr{A}$, can be regarded as continuous and i.p. measurable mappings from $\mathbb{R}$ into $\mathscr{F} \overset{\text{def}}{=} \pi_\phi(\mathscr{A})''$ (cf. Takesaki [26]: chapter IV.7). Recall that $\pi_{\phi\circ\bar{\beta}_p}(x) = \pi_\phi(\bar{\beta}_p(x))$, $x \in \mathscr{A}$, $p \in \mathbb{R}$, holds in a natural identification of the GNS-Hilbert spaces $\mathscr{H}_{\phi\circ\bar{\beta}_p}$ and $\mathscr{H}_\phi$. The automorphisms $\{\bar{\alpha}_q \mid q \in \mathbb{R}\}$ are implemented by the constant functions $\{p \to U_\phi(q), p \in \mathbb{R}\}$, where $U_\phi(q)$ is the unitary operator implementing $\bar{\alpha}_q$ on $\mathscr{H}_\phi$ (cf. Bratteli, Robinson [18]: 2.3.17). Therefore $\kappa_\nu(\mathscr{L}_\infty(\nu)) = U^*DU$ is pointwise invariant under $\{\alpha_q \mid q \in \mathbb{R}\}$.

The algebra of all measurable mappings from $\mathbb{R}$ to $\mathscr{F}$ is naturally isomorphic to $\mathscr{L}_\infty(\mathbb{R})\bar{\otimes}\mathscr{F}$, where D corresponds to $\mathscr{L}_\infty(\mathbb{R})\bar{\otimes}\mathbb{C}$. Since $\tilde{\beta}_{p_0} \overset{\text{def}}{=} \text{Ad}\,\lambda(-p_0) \otimes \text{Id}$, $p_0 \in \mathbb{R}$, (Id is the identity mapping on $\mathscr{F}$) transforms the functions $\{p \to \pi_{\phi\circ\bar{\beta}_p}(x)\}$, $x \in \mathscr{A}$, into $\{p \to \pi_{\phi\circ\bar{\beta}_{p+p_0}}(x)\} = \{p \to \pi_{\phi\circ\bar{\beta}_p}(\bar{\beta}_{p_0}(x))\}$, it follows by the above spatial isomorphism that ω is $\bar{\beta}$-quasiinvariant. Furthermore the associated W^*-system $(\pi_\omega(\mathscr{A})'', \mathbb{R}, \beta)$ contains a momentum operator $(\kappa_\nu(\mathscr{L}_\infty(\nu)), \mathbb{R}, \beta) \cong (\mathscr{L}_\infty(\mathbb{R})\bar{\otimes}\mathbb{C}, \mathbb{R}, \tilde{\beta}|_{\mathscr{L}_\infty(\mathbb{R})\bar{\otimes}\mathbb{C}}) \cong (\mathscr{L}_\infty(\mathbb{R}), \mathbb{R}, \text{Ad}\,\lambda)$.

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