Zeitschrift: Helvetica Physica Acta

Band: 59 (1986)

Heft: 3

Artikel: On the decomposition of relativistic quantum field theories into pure phases

Autor: Driessler, Wulf / Summers, Stephen J. / [s.n.]

DOI: https://doi.org/10.5169/seals-115699

Helvetica Physica Acta, Vol. 59 (1986) 331–348

0018–0238/86/030331–18$1.50 + 0.20/0

On the decomposition of relativistic quantum field theories into pure phases

By Wulf Driessler[1]) and Stephen J. Summers[2])

Fachbereich Physik, Universität Osnabrück, D-4500 Osnabrück, Federal Republic of Germany

(17. V 1985)

Abstract. We give two new, independent, sufficient conditions that individually insure that the extremal decomposition of a Wightman state on the polynomial algebra (equivalently, the Borchers algebra) of a relativistic quantum field is actually a decomposition into pure phases, i.e. the clustering property is satisfied in each extremal state occurring in the decomposition. Moreover, the corresponding representation also decomposes into a direct integral of irreducible representations with unique vacua.

I. Introduction

For more than twenty years [1] integral decompositions of representations and states on the polynomial algebra $\mathscr{P}(\mathbb{R}^d)$ of a relativistic quantum field $\varphi(x)$ satisfying the Wightman axioms have been studied. The primary aim from the point of view of physics has been to decompose a given quantum theory into its component pure phases. It is known [2] that if ω is a vacuum state on $\mathscr{P}(\mathbb{R}^d)$, then there exists a decomposition $\omega = \int \omega_\zeta \, d\nu(\zeta)$ of ω into extremal, vacuum states ω_ζ on $\mathscr{P}(\mathbb{R}^d)$.

Here extremal means that both the strong and weak commutants of $\mathscr{P}(\mathbb{R}^d)$ in the corresponding representation consist of multiples of the identity (for details, see Section 2). The representation of $\mathscr{P}(\mathbb{R}^d)$ associated with ω also decomposes into the direct integral of the representations associated with the ω_ζ occurring in the decomposition of ω. However, it has been shown [2, 3] that, as opposed to the situation in the bounded algebraic formulation of relativistic quantum theory [4, 5], the extremal states ω_ζ are not necessarily pure phases, i.e. the cluster decomposition property [6, 7] may not hold in each ω_ζ appearing in the decomposition of ω.

Abstract necessary and sufficient conditions have been given [8] under which the elements ω_ζ of the decomposition of ω do indeed fulfill the cluster decomposition property. However, these conditions do not seem to be readily

---

[1]) Present address: Sternstraße 5, D-5800 Hagen 1, Federal Republic of Germany.
[2]) Address after July 1, 1985: Dept. of Math., University of Rochester, Rochester, NY 14627, USA.

verifiable in concrete cases. Until now, there appear to be two known sets of usable sufficient conditions under which the extremal decomposition must yield only pure phases. The first one [1, 3] requires, stated for a scalar, Hermitian field, that each symmetric field operator $\varphi(f)$ be essentially self-adjoint on the standard domain D_1 and the spectra resolutions of the self-adjoint closures $\overline{\varphi(f) \upharpoonright D_1}$ and $\overline{\varphi(g) \upharpoonright D_1}$ commute if the supports of f and g are spacelike to one another. The second one [9, 10] demands that the Schwinger functions obtained by analytically continuing the Wightman functions of $\{\omega, \varphi(x)\}$ to the purely imaginary time points are the moments of a Euclidean-invariant probability measure that satisfies a certain positivity property (this approach excludes Fermi fields). For both of these sets of sufficient conditions there are examples of nontrivial quantum field models in which they hold.

In this paper we present two new, independent, sufficient conditions, one of which clearly weakens the assumptions made by Borchers [1, 3]; the second condition holds in many known quantum field models and is expected to obtain in all models with a positive mass gap. The first condition requires that there exists at least one 'intrinsically local' field operator $\varphi(f_s)$ [11], i.e. a field operator whose closure $\overline{\varphi(f_s) \upharpoonright D_1}$ (which need not be selfadjoint) is 'local' with respect to its own Poincaré transforms – see Section 2 for an exact definition. The second condition assumes that all operators $\varphi(f)$, with $f \in \mathscr{S}(\mathbb{R}^d)$ having a Fourier transform with compact support, satisfy a generalized H-bound, i.e. for each such test function f there exists a strictly positive function F_f on $[0, \infty)$ such that $\|\varphi(f)F_f(H)\| < \infty$, where H is the generator of the time translations for the field $\varphi(x)$. In both cases we shall show that a von Neumann algebra $\mathscr{A}$ can be constructed to which some closed extension of every field operator $\varphi(f)$ is affiliated ([12] – see Section 2), and that the central decomposition of $\mathscr{A}$ yields a direct integral decomposition of the representation of $\mathscr{P}(\mathbb{R}^d)$ associated to any vacuum state ω into irreducible representations with unique vacua and satisfying the rest of the Wightman axioms. Further results that may have independent interest are proven along the way.

For the sake of simplicity of presentation, we shall consider only scalar, Hermitian quantum fields, but all the results can be straightforwardly extended for any relativistic quantum field with finitely many components satisfying the axioms of [7]. We shall also assume the fields to be tempered distributions, but here again the results are valid for larger classes of generalized functions, e.g. for those considered by Jaffe [13]. All results holds for space-time dimension $d = 2$, 3 or 4.

In Section 2 further background and the main results are given and briefly discussed. Section 3 presents the proofs in the case of fields satisfying generalized H-bounds, and Section 4 gives the necessary details for fields with a suitable intrinsically local operator $\varphi(f_s)$.

II. Main results

In this section we shall introduce notation and state the main results of the paper, postponing most proofs to the following sections. $\varphi(x)$ will denote a scalar, Hermitian relativistic quantum field satisfying the usual axioms [7] with the possible exception of the uniqueness of the vacuum. Thus there exists a separable Hilbert space $\mathscr{H}$, in which a strongly continuous, unitary representation $U(\mathscr{P}_+^\uparrow)$ of (the universal covering group of) the Poincaré group $\mathscr{P}_+^\uparrow$ satisfying the spectrum condition acts. If P_0 is the projection in $\mathscr{H}$ onto the subspace of all vectors invariant under the translation subgroup of $U(\mathscr{P}_+^\uparrow)$, it is assumed there exists a vector $\Omega \in P_0\mathscr{H}$ such that $D_0 \equiv \mathscr{P}_0(\mathbb{R}^d)\Omega$ is dense in $\mathscr{H}$, where $\mathscr{P}_0(\mathbb{R}^d)$ is the $*$-algebra of all polynomials of operators $\{\varphi(f) \mid f \in \mathscr{S}(\mathbb{R}^d)\}$ including the identity I on $\mathscr{H}$. It follows easily from the axioms of [7] that there exists, for any $n \in \mathbb{N}$ and any $f \in \mathscr{S}(\mathbb{R}^{dn})$, an operator $\varphi\{f\}$ symbolically defined by

$$\varphi\{f\} = \int d^dx_1 \cdots d^dx_n f(x_1, \ldots, x_n)\varphi(x_1) \cdots \varphi(x_n) \quad (\text{on } D_0).$$

The polynomial algebra of such operators is well-defined and denoted by $\mathscr{P}(\mathbb{R}^d)$, and $D_1 \equiv \mathscr{P}(\mathbb{R}^d)\Omega$ is a dense, invariant domain of definition for all elements of $\mathscr{P}(\mathbb{R}^d)$. Other subalgebras of $\mathscr{P}(\mathbb{R}^d)$ we shall have need of are the following: $\mathscr{P}_{\text{loc}}(\mathbb{R}^d)$ (resp. $\mathscr{P}_c(\mathbb{R}^d)$) is the polynomial algebra of operators $\{\varphi(f) \mid f \in \mathscr{D}(\mathbb{R}^d)\}$ (resp. $\{\varphi(f) \mid f \in \mathscr{K}(\mathbb{R}^d)\}$, where $\mathscr{K}(\mathbb{R}^d) \equiv \{f \mid \tilde{f} \in \mathscr{D}(\mathbb{R}^d)\}$ with $\tilde{f}$ the Fourier transform of f).

It will be assumed that the decomposition theory [14] of von Neumann algebras is familiar to the reader. A von Neumann algebra $\mathscr{A}$ in a separable Hilbert space $\mathscr{H}$ is decomposed with respect to a maximally Abelian algebra $\mathscr{Z}$ in $\mathscr{A}'$, the commutant of $\mathscr{A}$, to yield a standard Borel measure space (S, ν) and measurable families $(\zeta \to \mathscr{H}(\zeta))$ of Hilbert spaces and $(\zeta \to \mathscr{A}(\zeta))$ of von Neumann algebras such that

$$\mathscr{H} = \int_S^\oplus \mathscr{H}(\zeta)\, d\nu(\zeta) \quad \text{and} \quad \mathscr{A} = \int_S^\oplus \mathscr{A}(\zeta)\, d\nu(\zeta). \tag{2.1}$$

For ν-almost all ζ, $\mathscr{A}(\zeta)$ is irreducible in $\mathscr{H}(\zeta)$. It is also possible to use Choquet theory to decompose any state ω on $\mathscr{A}$ into pure states $\{\omega_\lambda\}$ on $\mathscr{A}$, so that the associated GNS representations $\{\pi_\lambda\}$ of $\mathscr{A}$ are irreducible. However, unless ω is a normal state on $\mathscr{A}$, (2.1) will not, in general, hold for this decomposition.

Similarly, given $\mathscr{P}(\mathbb{R}^d)$ (in $\mathscr{H}$) and the state ω on $\mathscr{P}(\mathbb{R}^d)$ determined by $\langle \Omega, \cdot\, \Omega \rangle$, it is possible to consider the decomposition of either the state or the representation of $\mathscr{P}(\mathbb{R}^d)$ in $\mathscr{H}$. The former has been carried out for a general quantum field theory in [15] using Choquet theory and the nuclear spectral theorem. The decomposition of the representation, which includes the former as a special case and carries more information, was studied in [2], where, due to the fact that $\mathscr{P}(\mathbb{R}^d)$ is an algebra of unbounded operators, it was found necessary in general to extend the representation in a certain sense and to decompose this extension. The extension was necessitated by the fact that the weak commutant $(\mathscr{P}(\mathbb{R}^d))'_w$ (see (2.2)) of $\mathscr{P}(\mathbb{R}^d)$, which is the analog to $\mathscr{A}'$, is in general not an algebra, and a decomposition with respect to a maximally Abelian algebra in $(\mathscr{P}(\mathbb{R}^d))'_w$ will not necessarily result in a decomposition into irreducible representations, i.e. representations in which the weak commutant is trivial.

However, whether one decomposes the state or the representation, it can happen [2] that the extremal states in the decomposition do not satisfy the cluster decomposition property, i.e. in the associated irreducible representations the subspace of Poincaré-invariant vectors has more than one dimension. As previously mentioned, necessary and sufficient conditions that assure that the resulting extremal states *do* cluster have been given in [8]. These conditions are difficult to verify directly in concrete models. Two sets of sufficient conditions have been isolated [1, 3, 9, 10] that *are* useful in certain models.

We shall be occupied with the consequences for the decomposition theory of a relativistic quantum field that follow from two new, independent conditions on the field. Under either of these conditions the extension of the representation of $\mathscr{P}(\mathbb{R}^d)$ performed in [2] will not be necessary, and the resulting extremal states in the decomposition will satisfy clustering, i.e. the extremal decomposition of the representation results in a decomposition into pure phases of the theory. First we define what it will mean to say that the field $\varphi(x)$ satisfies a generalized H-bound (H is the generator of the time translation subgroup of $U(\mathscr{P}_+^\uparrow)$).

Definition 2.1. A quantum field $\varphi(x)$ will be said to satisfy a generalized H-bound if for each $f \in \mathscr{K}(\mathbb{R}^d)$ there exists a monotone decreasing function F_f on $[0, \infty)$, with $F_f(x) > 0$ for all $x \in [0, \infty)$, such that $F_f(H)\mathscr{H} \subseteq D(\varphi(f)^-)$ (where $\varphi(f)^-$ is the closure of $\varphi(f)$ taken on D_1) and $\|\varphi(f)^- F_f(H)\| < \infty$.

Any quantum field with a positive mass gap is expected to satisfy a generalized H-bound, and this expectation has been verified in a large number of concrete field models [16, 17, 18].

The second condition is a special case of a concept that was introduced in [11]. If X is a closed operator and $X = U\,|X|$ is its polar decomposition, the von Neumann algebra $a(X)$ generated by X is that generated by U and the spectral projections of $|X|$. If $\mathscr{A}$ is a von Neumann algebra and X is a closed operator, X is affiliated with $\mathscr{A}$ (in symbols, $X\eta\mathscr{A}$) in the sense of [12] if for every unitary $U \in \mathscr{A}'$, the commutant of $\mathscr{A}$, $UXU^{-1} = X$. $a(X)$ is, in fact, the smallest von Neumann algebra to which X is affiliated.

Definition 2.2. If $\varphi(x)$ is a quantum field, the operator $\varphi(f_s)(f_s \in \mathscr{S}(\mathbb{R}^d))$ will be said to be intrinsically local and locally associated with the bounded set $\mathcal{O}_s \subset \mathbb{R}^d$ if $D_s \equiv \mathscr{P}_{0s}(\mathbb{R}^d)\Omega$ is dense in $\mathscr{H}$, where $\mathscr{P}_{0s}(\mathbb{R}^d)$ is the polynomial algebra of the operators $\{U(\lambda)\varphi(f_s)U(\lambda)^{-1} \mid \lambda \in \mathscr{P}_+^\uparrow\}$, and if the von Neumann algebra $\overline{a(\varphi(f_s) \restriction D_s)} \equiv a_s$ generated by the closure of $\varphi(f_s) \restriction D_s$ satisfies: $U(\lambda)a_s U(\lambda)^{-1} \subset a_s'$ wherever $\mathcal{O}_{s,\lambda}$ is spacelike relative to $\mathcal{O}_s$, $\lambda \in \mathscr{P}_+^\uparrow$.

Of course, if the field satisfies the assumptions made in [1] (see Introduction), then *every* field operator $\varphi(f)$ with $f \in \mathscr{S}(\mathbb{R}^d)$ having e.g. strictly nonvanishing Fourier transform is intrinsically local. A simple example of a Bose field that is not known to conform to the hypotheses of [1] but for which all $\varphi(f)$ are intrinsically local, $f \in \mathscr{S}(\mathbb{R}^d)$ as above, is $\varphi(f) =: \varphi_0^n : (f)$ for any odd $n \geq 3$, where $\varphi_0(x)$ is the free scalar field (for even $n \geq 4$ one must simply restrict $\mathcal{H}$ to the even particle subspace of Fock space; then such operators are intrinsically local in this subspace).

Before stating the main result, we must introduce further notation. If $\mathcal{P}$ is a $*$-subalgebra of $\mathcal{P}(\mathbb{R}^d)$, the weak commutant $\mathcal{P}'_w$ of $\mathcal{P}$ is defined to be:

$$\mathcal{P}'_w \equiv \{A \in \mathcal{B}(\mathcal{H}) \mid \langle A\Phi, X\Psi\rangle = \langle X^*\Phi, A^*\Psi\rangle, \forall X \in \mathcal{P}, \Phi, \Psi \in D_1\}. \tag{2.2}$$

$\mathcal{B}(\mathcal{H})$ is the set of all bounded, linear operators on $\mathcal{H}$. The strong commutant $\mathcal{P}'_s$ of $\mathcal{P}$ is given by

$$\mathcal{P}'_s \equiv \{A \in \mathcal{B}(\mathcal{H}) \mid AD_1 \subseteq (\bar{X}) \quad \text{and} \quad [A, \bar{X}]\Phi = 0, \forall X \in \mathcal{P}, \Phi \in D_1\}.$$

$\bar{X}$ (or X^-) will always signify the closure of the operator $X \restriction D_1$. And X^* will always mean $(X \restriction D_1)^*$. Note that if $A \in \mathcal{P}'_s$, then $AD(\bar{X}) \subseteq D(\bar{X})$ and $[A, \bar{X}] = 0$ on $D(\bar{X})$ for any $X \in \mathcal{P}$. Of course, $\mathcal{P}'_s \subseteq \mathcal{P}'_w$. For further, general properties of such sets of operators we refer the reader to [19, 2].

In the following, it will be important to know that if $\mathcal{A}$ is a von Neumann algebra contained in $\mathcal{P}'_w$, then for each $X \in \mathcal{P}$ there exists a closed extension $\tilde{X}$ of $X \restriction D_1$ (not necessarily equal to $\bar{X}$) such that $\tilde{X}\eta\mathcal{A}'$. Indeed, $\tilde{X}$ is the closure of $\mathcal{A}D_1$ of the operator defined by $\tilde{X}A\Phi \equiv AX\Phi$, $\Phi \in D_1$, $A \in \mathcal{A}$. Finally, we remark that, in the following, sets of vectors with overbars signify the norm closures of the given sets.

Theorem 2.3. *Let $\varphi(x)$ be a scalar, Hermitian quantum field satisfying the axioms of* [7], *possibly excepting the uniqueness of the vacuum. Then if either* (1) *$\varphi(x)$ satisfies a generalized H-bound or* (2) *there exists an $f_s \in \mathcal{D}(\mathbb{R}^d)$ with $\tilde{f}_s(p) \neq 0$, $\forall p \in \mathbb{R}^d$, such that $\varphi(f_s)$ is intrinsically local (and locally associated with $\mathcal{O}_s \supset \operatorname{supp}(f_s)$), then there exists a von Neumann algebra $\mathcal{A}$ such that*

(i) $\mathcal{A}' = \mathcal{Z}(\mathcal{A}) \equiv \mathcal{A} \cap \mathcal{A}'$;
(ii) $\mathcal{A}' = \mathcal{P}(\mathbb{R}^d)'_w$;
(iii) $\overline{\mathcal{A}'\Omega} = P_0\mathcal{H}$ *and* $U(\mathcal{P}^\uparrow_+) \subset \mathcal{A}$;
(iv) *under the central decomposition of $\mathcal{A}$* [14] *the Wightman functions of $\varphi(x)$ decompose into a direct integral of Wightman functions, i.e. there exists a standard Borel measure space (S, ν) and measurable families of Hilbert spaces $\zeta \to \mathcal{H}(\zeta)$ and von Neumann algebras $\zeta \to \mathcal{A}(\zeta)$ such that for any $n \in \mathbb{N}$, $\{f_i\}_{i=1}^n \subset \mathcal{S}(\mathbb{R}^d)$,*

$$\left\langle \Omega, \left(\prod_{i=1}^n \varphi(f_i)\right)\Omega \right\rangle = \int_S W_\zeta(f_1, \ldots, f_n)\, d\nu(\zeta),$$

and $\{W_\zeta(x_1, \ldots, x_n)\}_{n\in\mathbb{N}}$ satisfies all Wightman axioms [7] *including clustering for all $\zeta \in N \subseteq S$ with $\nu(S\backslash N) = 0$; moreover*

$$\mathcal{H} = \int_S^\oplus \mathcal{H}(\zeta)\, d\nu(\zeta), \qquad \mathcal{A} = \int_S^\oplus \mathcal{A}(\zeta)\, d\nu(\zeta),$$

$\mathcal{A}(\zeta) = \mathcal{B}(\mathcal{H}(\zeta))$ ν-almost everywhere, and for all $\zeta \in N$, $\mathcal{H}(\zeta)$ is the

Hilbert space obtained by the reconstruction theorem of [7] *from* $\{W_\zeta(x_1, \ldots, x_n)\}_{n\in\mathbb{N}}$, *with* $\Omega(\zeta)$ *the (up to a factor) unique vacuum vector in* $\mathscr{H}(\zeta)$ *and* $\Omega = \int_S^\oplus \Omega(\zeta)\, d\nu(\zeta)$;

(v) *all conclusions of Theorem* 3.3 *of* [2] *hold with* $(\mathscr{P}(\mathbb{R}^d), D_1)$ *(our notation)* $= (\mathscr{A}, \mathscr{D})$ *(their notation)*;

(vi) *for any* $f \in \mathscr{S}(\mathbb{R}^d)$ *for which* $\|\varphi(f)^- g(H)\| < \infty$, *the estimate* $\|\varphi(f)^-(\zeta) g(H(\zeta))\| < \infty$ *holds for* ν*-almost all* ζ, *where* $H = \int_S^\oplus H(\zeta)\, d\nu(\zeta)$.

Remarks. 1) What is actually shown in the following proof is that if there exists a von Neumann algebra $\mathscr{A}$ satisfying (i)–(iii), then (iv)–(vi) hold. It is proven in the subsequent sections that (1) or (2) implies the existence of $\mathscr{A}$ satisfying (i)–(iii).

2) Clearly, condition (2) weakens the assumptions made in [1]. On the other hand, the hypotheses of [9, 10] would seem to exclude application to Fermi fields, while, we reemphasize, the methods of this paper are applicable to any quantum field with finitely many components.

3) For reasons of space we do not list the conclusions of Theorem 3.3 of [2], which concern continuity properties of the decomposition of the representation and can be, in any case, read off the following proof.

4) The assertion (vi) is an extension of a result in [20].

5) There is an abundance of test functions $f_s \in \mathscr{D}(\mathbb{R}^d)$ satisfying $\tilde{f}_s(p) \neq 0$ for all $p \in \mathbb{R}^d$, as is easily verified [11].

Proof. Under the assumption (1), the existence of $\mathscr{A}$ satisfying (i)–(iii) follows from Prop. 3.6 and 3.10. Under the assumption (2) these results are given by Prop. 4.1 and 4.2.

Turning to (iv), let (S, ν) be the standard Borel space and $\zeta \to \mathscr{H}(\zeta)$ and $\zeta \to \mathscr{A}(\zeta)$ be the measurable families of Hilbert spaces and von Neumann algebras arising from the central decomposition of $\mathscr{A}$ [14]. Then

$$\mathscr{H} = \int_S^\oplus \mathscr{H}(\zeta)\, d\nu(\zeta), \qquad \mathscr{A} = \int_S^\oplus \mathscr{A}(\zeta)\, d\nu(\zeta),$$

and by (i) and (iii), $\mathscr{A}(\zeta) = \mathscr{B}(\mathscr{H}(\zeta))$ and $\dim((P_0\mathscr{H})(\zeta)) = 1$, for ν-almost all ζ. Since $U(\mathscr{P}_+^\uparrow) \subset \mathscr{A}$, one can use [21] to conclude that there exists a measurable family $\zeta \to U(\mathscr{P}_+^\uparrow)(\zeta)$ of strongly continuous, unitary representations of the (covering group of the) Poincaré group satisfying the spectral condition such that $U(\mathscr{P}_+^\uparrow) = \int_S^\oplus U(\mathscr{P}_+^\uparrow)(\zeta)\, d\nu(\zeta)$ and such that $(P_0\mathscr{H})(\zeta) = P_0(\zeta)\mathscr{H}(\zeta)$ for ν-almost all ζ. Thus the vacuum in ν- almost all phases $\mathscr{H}(\zeta)$ is unique (up to a factor). Of course, with $\Omega = \int_S^\oplus \Omega(\zeta)\, d\nu(\zeta)$ one has $\Omega(\zeta) \in P_0(\zeta)\mathscr{H}(\zeta)$ and $U(\lambda)(\zeta)\Omega(\zeta) = \Omega(\zeta)$, $\forall \lambda \in \mathscr{P}_+^\uparrow$, for ν-almost all ζ (independent of λ).

By (ii), for any $X \in \mathscr{P}(\mathbb{R}^d)$ there exists a closed extension $\tilde{X}$ that is affiliated to $\mathscr{A}$. From [22, 23] it is known that this extension also decomposes under the central decomposition of $\mathscr{A}$ into densely defined, closed operators $\tilde{X} = \int_S^\oplus \tilde{X}(\zeta)\, d\nu(\zeta)$ for which $D(\tilde{X}) \ni \Phi = \int_S^\oplus \Phi(\zeta)\, d\nu(\zeta)$ is equivalent to $\Phi(\zeta) \in D(\tilde{X}(\zeta))$ ν-almost everywhere and $\int_S \|\tilde{X}(\zeta)\Phi(\zeta)\|^2\, d\nu(\zeta) < \infty$. These assertions hold, in particular, if $X = \varphi(f)$, for any $f \in S(\mathbb{R}^d)$. By Theorem 6.3 in [24], if $\{\varphi(f_i)^{\sim}\}\prod_{i=1}^{N}$ is a finite set of such decomposable closed operators, then the closure of $\prod_{i=1}^{N} \varphi(f_i)^{\sim}$ on its natural domain of definition (which contains D_1) is also decomposable, and for ν-almost all ζ, $(\prod_{i=1}^{N} \varphi(f_i)^{\sim})^{-}(\zeta) = (\prod_{i=1}^{N} \varphi(f_i)^{\sim}(\zeta))^{-}$, where the bar signifies the closure of the operator on the natural domain. Hence,

$$\left\langle \Omega, \left(\prod_{i=1}^{N} \varphi(f_i)\right)\Omega \right\rangle = \left\langle \Omega, \left(\prod_{i=1}^{N} \varphi(f_i)^{\sim}\right)^{-} \Omega \right\rangle$$

$$= \int_S \left\langle \Omega(\zeta), \left(\prod_{i=1}^{N} \varphi(f_i)^{\sim}\right)^{-} (\zeta)\Omega(\zeta) \right\rangle d\nu(\zeta)$$

$$= \int_S \left\langle \Omega(\zeta), \left(\prod_{i=1}^{N} \varphi(f_i)^{\sim}(\zeta)\right)^{-} \Omega(\zeta) \right\rangle d\nu(\zeta).$$

Since $\Phi \in D(\varphi(f)^{\sim})$ implies $\Phi(\zeta) \in D(\varphi(f)^{\sim}(\zeta))$ for ν-almost all ζ, this equals

$$\int_S \left\langle \Omega(\zeta), \left(\prod_{i=1}^{N} \varphi(f_i)^{\sim}(\zeta)\right)\Omega(\zeta) \right\rangle d\nu(\zeta).$$

$\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{P}_+^{\uparrow}$ are separable, and it is possible to pick a countable, dense, linear subset $\mathscr{S} \subset \mathscr{S}(\mathbb{R}^d)$ and a countable, dense subgroup $\mathscr{P} \subset \mathscr{P}_+^{\uparrow}$ so that the induced action of $\mathscr{P}$ on $\mathscr{S}(\mathbb{R}^d)$ maps $\mathscr{S}$ into itself. Then there exists a set $N \subseteq S$ with $\nu(S \backslash N) = 0$ such that for each $\zeta \in N$, $(\prod_{i=1}^{N} \varphi(f_i)^{\sim}(\zeta))\Omega(\zeta)$ is multilinear on $\mathscr{S}$, $\forall N \in \mathbb{N}$, and such that

$$U(\lambda)(\zeta)\left(\prod_{i=1}^{N} \varphi(f_i)^{\sim}(\zeta)\right)\Omega(\zeta) = \left(\prod_{i=1}^{N} \varphi(f_{i,\lambda})^{\sim}(\zeta)\right)\Omega(\zeta), \qquad \forall \lambda \in \mathscr{P},\ N \in \mathbb{N}. \tag{2.3}$$

(f_λ is the image of f under the induced action of $\lambda \in \mathscr{P}_+^{\uparrow}$ on $s(\mathbb{R}^d)$.) From [25] or [26, Section I.4.4] one may conclude that there exists a set $N_1 \subseteq N$ with $\nu(S \backslash N_1) = 0$ such that for all $\zeta \in N_1$ the mapping

$$T_\zeta : \left(\prod_{i=1}^{n} \varphi(f_i)\right)\Omega \to \left(\prod_{i=1}^{n} \varphi(f_i)^{\sim}(\zeta)\right)\Omega(\zeta), \qquad n \in \mathbb{N}, \tag{2.4}$$

is nuclear as a map from D_1, viewed as a nuclear vector space with the standard topology (see [1]), into $\mathscr{H}(\zeta)$, and there exists for each $n \in \mathbb{N}$ a norm $\|\cdot\|_{\mathscr{S},n}$ continuous in the topology on $\mathscr{S}(\mathbb{R}^{dn})$ such that

$$\left|\left\langle \Omega(\zeta), \left(\prod_{i=1}^{n} \varphi(f_i)^{\sim}(\zeta)\right)\Omega(\zeta) \right\rangle\right| \le C(\zeta)\, \|f_1 \otimes \cdots \otimes f_n\|_{S,n},\ C(\zeta) < \infty, \tag{2.5}$$

for all $\{f_i\}_{i=1}^{n} \subset \mathscr{S}$. Hence, by (2.3) and (2.5), the strong continuity of $U(\mathscr{P}_+^{\uparrow})(\zeta)$ and the continuity of the induced action of $\mathscr{P}_+^{\uparrow}$ on $\mathscr{S}(\mathbb{R}^d)$,

$$W_\zeta(f_1, \ldots, f_n) \equiv \left\langle \Omega(\zeta), \left(\prod_{i=1}^{n} \varphi(f_i)^{\sim}(\zeta)\right)\Omega(\zeta) \right\rangle, \qquad \zeta \in N_1,$$

can be extended uniquely to a Poincaré-invariant, tempered distribution on $\mathscr{S}(\mathbb{R}^{dn})$, for any $n \in \mathbb{N}$. The remaining axioms of hermiticity, local commutativity and positive definiteness of $\{W_\zeta(x_1, \ldots, x_n)\}_{n\in\mathbb{N}}$ can be shown similarly, at the possible cost of throwing out a set of ν-measure zero; this is left to the reader. The clustering property of the Wightman functions $W_\zeta(x_1, \ldots, x_n)$, $n \in \mathbb{N}$, for ν-almost all ζ is implied by the uniqueness of the vacuum in $\mathscr{H}(\zeta)$ and the other Wightman axioms [6, 7].

In light of (2.4) and (2.5) the assertions made in Theorem 3.3 of [2] hold with $(\mathscr{P}(\mathbb{R}^d), D_1, T_\zeta)$ (our notation) $= (\mathscr{A}, \mathscr{D}, E_\lambda)$ (their notation), as is easily verified.

Finally, since the group $U(\mathscr{P}_+^\uparrow)$ is contained in $\mathscr{A}$, H must be affiliated to $\mathscr{A}$ and is, therefore, decomposable into positive, selfadjoint operators $H(\zeta)$ [23] that are the generators of the time translation subgroup of $U(\mathscr{P}_+^\uparrow)(\zeta)$ [21]. For any $f \in \mathscr{S}(\mathbb{R}^d)$ such that $\|\varphi(f)^- g(H)\| < \infty$, one has, calling once again upon Theorem 6.3 in [24],

$$\varphi(f)^- g(H)\Phi = \varphi(f)^{\sim} g(H)\Phi = \int_S^\oplus (\varphi(f)^{\sim} g(H)\Phi)(\zeta)\, d\nu(\zeta)$$

$$= \int_S^\oplus \varphi(f)^{\sim}(\zeta) g(H)(\zeta)\Phi(\zeta)\, d\nu(\zeta), \qquad \forall \Phi \in \mathscr{H}.$$

But $g(H)(\zeta) = g(H(\zeta))$ for ν-almost all ζ [21]. Therefore, $\|\varphi(f)^{\sim}(\zeta) g(H(\zeta))\|$ must be finite for ν-almost all ζ, since $\|\varphi(f)^- g(H)\|$ is equal to the $L^\infty(S, d\nu)$-norm of $\|(\varphi(f)^{\sim} g(H))(\zeta)\|$. □

III. Fields satisfying generalized H-bounds

Throughout this section we shall assume that $\varphi(x)$ is a scalar, Hermitian quantum field satisfying the axioms of [7], with the possible exception of the uniqueness of the vacuum, and a generalized H-bound in the sense of Def. 2.1. We first remark that if $\varphi(x)$ satisfies such a bound, then for *any* $G \in \mathscr{D}(\mathbb{R})$,

$$\|\varphi(f)^- G(H)\| < \infty, \qquad \forall f \in \mathscr{K}(\mathbb{R}^d). \tag{3.1}$$

This is clear since with $f \in \mathscr{K}(\mathbb{R}^d)$ and F_f as in Def. 2.1, $\|F_f(H)^{-1} G(H)\| < \infty$ by the spectral calculus theorem. Moreover, it is known that (3.1) implies that there exists a Schwartz space norm $\|\cdot\|_{\mathscr{S},G}$ such that

$$\|\varphi(f)^- G(H)\| \le \|f\|_{\mathscr{S},G} \qquad \forall f \in \mathscr{S}(\mathbb{R}^d). \tag{3.2}$$

For a proof of this, see the Appendix. A trivial consequence of the H-bound (3.2) is that $P_0\mathscr{H} \subset D(\varphi(f)^-)$ and $\varphi(f)^- \upharpoonright P_0\mathscr{H}$ is bounded for any $f \in \mathscr{S}(\mathbb{R}^d)$. In fact, P_0 can be replaced in these assertions by any spectral projection $E_{(-\infty, a]}$ of H with $a < \infty$. Henceforth, a vector $\Phi \in \mathscr{H}$ such that $E_{(-\infty, a]}\Phi = \Phi$ for some $a < \infty$ will be said to have bounded energy support.

Lemma 3.1. *If $\varphi(x)$ satisfies a generalized H-bound, then for any $X \in$*

$\mathscr{P}_c(\mathbb{R}^d)$, $D(X^-) \supset P_0(\mathscr{P}_0(\mathbb{R}^d)\Omega)$ and $X^- \upharpoonright P_0(\mathscr{P}_0(\mathbb{R}^d)\Omega)$ is bounded. Thus, $P_0\mathscr{H} \subset D(X^-)$.

Proof. Let $X \in \mathscr{P}_c(\mathbb{R}^d)$, resp. $Y \in \mathscr{P}_0(\mathbb{R}^d)$, be a monomial of degree k, resp. m, in the field. The general case follows easily from this one. To begin, one notes that $P_0 Y\Omega = s\text{-}\lim_{n\to\infty} G_n(H)Y\Omega$, with $G_n(0) = 1$, $G_n \in \mathscr{D}(\mathbb{R})$, $G_n \geq 0$, and $\operatorname{supp}(G_n) \subset [0, n^{-1}]$, for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, $G_n(H)Y\Omega \in \{\varphi\{g\}\Omega \mid g \in \mathscr{S}(\mathbb{R}^{dm})\} \subset D_1$, so that $G_n(H)Y\Omega \in D(X)$. Since $X = \prod_{i=1}^k \varphi(f_i)$ with $\{f_i\}_{i=1}^k \subset \mathscr{K}(\mathbb{R}^d)$ and the energy support of $G_n(H)Y\Omega$ is contained in $[0, n^{-1}]$, for any fixed $N \in \mathbb{N}$ there exist functions $\{F_i\}_{i=1}^k \subset \mathscr{D}(\mathbb{R})$ with $F_i \geq 0$, such that for any $j \in \{1, \ldots, k\}$,

$$F_j(H)\left(\prod_{i=1}^{j} \varphi(f_i)F_i(H)\right)G_N(H)Y\Omega = \left(\prod_{i=1}^{j} \varphi(f_i)\right)G_N(H)Y\Omega.$$

In fact, one chooses $F_1(x) = 1$ for $x \in [0, N^{-1}]$, $F_2(x) = 1$ for $x \in [-\eta_1, N^{-1} + \eta_1]$ where $[-\eta_1, \eta_1]$ is the smallest symmetric interval containing $\operatorname{supp}(\tilde{f}_1)$, and so forth. One has then $XG_n(H)Y\Omega = A_X G_n(H)Y\Omega$, $\forall n \geq N$, where $A_X = \prod_{i=1}^k (\varphi(f_i)^- F_i(H)) \in \mathscr{B}(\mathscr{H})$. This entails at once that $P_0 Y\Omega \in D(X^-)$ and $X^- P_0 Y\Omega = A_X P_0 Y\Omega$. □

Given a field $\varphi(x)$ that satisfies a generalized H-bound, we define the $*$-algebra $\mathscr{A}_0$ to be that generated by the bounded operators

$$\{\varphi(f)^- F(H), \qquad G(H) \mid f \in \mathscr{K}(\mathbb{R}^d), \qquad f, G \in \mathscr{D}(\mathbb{R})\}. \tag{3.3}$$

We shall say that a monomial $A \in \mathscr{A}_0$ in the generators (3.3) is of order n if it contains n field operators, irrespective of the other factors in the product. $\mathscr{A}_0$ has been defined so that $\mathscr{A} \equiv \mathscr{A}_0''$, the von Neumann algebra generated by $\mathscr{A}_0$, contains all bounded functions of H; therefore $P_0 \in \mathscr{A}$. Moreover, the energy support of $A\Phi$ is bounded for any $A \in \mathscr{A}_0$, $\Phi \in \mathscr{H}$.

Lemma 3.2. *If $\varphi(x)$ satisfies a generalized H-bound, then $\overline{\mathscr{A}_0\Omega} = \mathscr{H}$ and $\mathscr{A}_0 D_1 \subseteq D_1$.*

Proof. For every monomial $X = \prod_{i=1}^n \varphi(f_i)$, $\{f_i\}_{i=1}^n \subset \mathscr{K}(\mathbb{R}^d)$, one has $X\Omega = A_X\Omega$, with $A_X = \prod_{i=1}^n (\varphi(f_i)F_i(H))$ and $\{F_i\}_{i=1}^n \subset \mathscr{D}(\mathbb{R})$ chosen as in the proof of Lemma 3.1. But as $\mathscr{P}_c(\mathbb{R}^d)\Omega$ is dense in $\mathscr{H}$, the first assertion of the lemma follows at once. The second claim is clear since $F(H)D_1 \subseteq D_1$ for any $F \in \mathscr{D}(\mathbb{R})$. □

In the next lemma we show that we can replace the test functions $f \in \mathscr{K}(\mathbb{R}^d)$ in (3.3) by test functions $f \in \mathscr{D}(\mathbb{R}^d)$ and still obtain the same von Neumann algebra.

Lemma 3.3. *If $\varphi(x)$ satisfies a generalized H-bound, then*

$$\mathscr{A} = \mathscr{B} \equiv \{\varphi(f)^- F(H), G(H) \mid f \in \mathscr{D}(\mathbb{R}^d), F, G \in \mathscr{D}(\mathbb{R})\}''.$$

Proof. Since $\mathscr{K}(\mathbb{R}^d)$ is dense in $\mathscr{S}(\mathbb{R}^d)$, given any $f \in \mathscr{D}(\mathbb{R}^d)$ there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathscr{K}(\mathbb{R}^d)$ such that $\{f_n\}_{n\in\mathbb{N}}$ converges to f in the topology of $\mathscr{S}(\mathbb{R}^d)$. By (3.2), for any $F \in \mathscr{D}(\mathbb{R})$ and $\Phi \in D_1$,

$$\|(\varphi(f_n) - \varphi(f))F(H)\Phi\| = \|\varphi(f_n - f)F(H)\Phi\| \leq \|f_n - f\|_{\mathscr{S},F}\|\Phi\|,$$

so that $\varphi(f_n)^- F(H)$ converges in norm to $\varphi(f)^- F(H)$. It follows that any generator of $\mathscr{B}$ can be approximated in the norm operator topology by operators from (3.3), entailing $\mathscr{B} \subseteq \mathscr{A}$. $\mathscr{A} \subseteq \mathscr{B}$ can be shown by the same argument using the fact that $\mathscr{D}(\mathbb{R}^d)$ is dense in $\mathscr{S}(\mathbb{R}^d)$. □

Let $\bar{D} \equiv \bar{\mathscr{P}}_c(\mathbb{R}^d)P_0\mathscr{H}$, where $\bar{\mathscr{P}}_c(\mathbb{R}^d) \equiv \{X^- \mid X \in \mathscr{P}_c(\mathbb{R}^d)\}$. By Lemma 3.1 this set of vectors is well-defined. Since every vector $\Phi \in \bar{D}$ has bounded energy support, (3.2) entails that $\Phi \in D(\varphi(f)^-)$ for every $f \in \mathscr{S}(\mathbb{R}^d)$. Moreover, arguing as in Lemmas 3.1 and 3.3, it is easy to see that if $\{f_n\}_{n\in\mathbb{N}} \subset \mathscr{S}(\mathbb{R}^d)$ converges to $f \in \mathscr{S}(\mathbb{R}^d)$ in the topology of $\mathscr{S}(\mathbb{R}^d)$, then $\varphi(f_n)^-$ converges strongly to $\varphi(f)^-$ on $\bar{D}$. From [3, Theorem 1] we now know that for any $X \in \mathscr{P}_{\text{loc}}(\mathbb{R}^d)$, $P_0X\Omega \in D(Y^-)$ for all $Y \in \mathscr{P}_{\text{loc}}(\mathbb{R}^d)$ (indeed, it is easy to see that $(ZY)^-P_0X\Omega = Z^-Y^-P_0X\Omega$, $\forall X, Y, Z \in \mathscr{P}_{\text{loc}}(\mathbb{R}^d)$). Thus, for every such Y the operator $P_0Y^-P_0$ can be defined in $P_0\mathscr{H}$. Theorem 2 in [3] informs us that the map $\mathscr{P}_{\text{loc}}(\mathbb{R}^d) \ni Y \to P_0Y^-P_0$ maps $\mathscr{P}_{\text{loc}}(\mathbb{R}^d)$ into an Abelian algebra in $P_0\mathscr{H}$. We show that if the quantum field satisfies a generalized H-bound, then also $P_0\bar{\mathscr{P}}_c(\mathbb{R}^d)P_0$ is an Abelian algebra.

Lemma 3.4. *If $\varphi(x)$ satisfies a generalized H-bound, then $(P_0\bar{\mathscr{P}}_c(\mathbb{R}^d)P_0)''$ is a maximally Abelian algebra on $P_0\mathscr{H}$.*

Proof. Since $P_0\bar{\mathscr{P}}_c(\mathbb{R}^d)P_0 \subset \mathscr{B}(P_0\mathscr{H})$ and $P_0\bar{\mathscr{P}}_c(\mathbb{R}^d)\Omega$ is dense in $P_0\mathscr{H}$, it remains only to show that $P_0\bar{\mathscr{P}}_c(\mathbb{R}^d)P_0$ is Abelian. It suffices to consider two monomials $X = \prod_{i=1}^n \varphi(g_i)$ and $Y = \prod_{i=1}^m \varphi(f_i)$ in $\mathscr{P}_c(\mathbb{R}^d)$. For each g_i and f_j pick a sequence $\{g_i^{(k)}\}_{k\in\mathbb{N}}$ and $\{f_j^{(l)}\}_{l\in\mathbb{N}}$ in $\mathscr{D}(\mathbb{R}^d)$ converging to g_i and f_j in the topology of $\mathscr{S}(\mathbb{R}^d)$. If $(k_1, \ldots, k_n) \in \mathbb{N}^n$, $(l_1, \ldots, l_m) \in \mathbb{N}^m$ and $\Phi, \Psi \in P_0\mathscr{P}_{\text{loc}}(\mathbb{R}^d)\Omega$, then

$$\left\langle P_0\left(\prod_{i=1}^n \varphi(g_i^{(k_i)})\right)^- \Phi, \left(\prod_{j=1}^m \varphi(f_j^{(l_j)})\right)^- \Psi\right\rangle$$

$$= \left\langle \left(\prod_{j=1}^m \varphi(f_j^{(l_j)})\right)^* \Phi, P_0\left(\prod_{i=1}^n \varphi(g_i^{(k_i)})\right)^* \Psi\right\rangle,$$

by [3, Theorem 2]. By using the remarks made in the paragraph preceding this lemma, one can take the limit $k_1 \to \infty$ and obtain

$$\left\langle \left(\prod_{i=2}^n \varphi(g_i^{(k_i)})^-\right)\Phi, \varphi(g_1^*)^- P_0\left(\prod_{j=1}^m \varphi(f_j^{(l_j)})\right)^- \Psi\right\rangle$$

$$= \left\langle \left(\prod_{i=2}^n \varphi(g_i^{(k_i)})\right)^- P_0\left(\prod_{j=1}^m \varphi(f_j^{(l_j)})\right)^* \Phi, \varphi(g_1^*)^- \Psi\right\rangle,$$

employing the simple fact that $\varphi(g)^* \restriction \bar{D} = \varphi(g^*)^- \restriction \bar{D}$, $\forall g \in \mathcal{S}(\mathbb{R}^d)$, where g^* is the complex conjugate of g. This process can be continued, yielding

$$\left\langle P_0\left(\prod_{i=1}^{n} \varphi(g_i)^-\right)\Phi, \left(\prod_{j=1}^{m} \varphi(f_j)^-\right)\Psi\right\rangle = \left\langle\left(\prod_{j=1}^{m} \varphi(f_j)\right)^* \Phi, P_0\left(\prod_{i=1}^{n} \varphi(g_i)\right)^* \Psi\right\rangle.$$

Since $P_0X^-P_0$ is bounded for every $X \in \mathcal{P}_c(\mathbb{R}^d)$ and $P_0\mathcal{P}_{\text{loc}}(\mathbb{R}^d)\Omega$ is dense in $P_0\mathcal{H}$, the proof of the lemma is completed. □

We note that if $\varphi(x)$ satisfies a generalized H-bound, then $f \to \varphi(f)^-$ defines a linear map from $\mathcal{S}(\mathbb{R}^d)$ into the set of linear operators defined on $\bar{D}$ such that $f \to \langle \Psi, \varphi(f)^-\Phi\rangle$ is a tempered distribution for all $\Psi \in \mathcal{H}$, $\Phi \in \bar{D}$ (use 3.2)). Thus, $\langle \Psi, (\prod_{i=1}^{n} \varphi(f_i)^-)\Phi\rangle$, $\Psi \in \mathcal{H}$, $\Phi \in \bar{D}$, $\{f_i\}_{i=1}^{n} \subset \mathcal{K}(\mathbb{R}^d)$, is continuous in the topology induced on $\mathcal{K}(\mathbb{R}^d)$ by $\mathcal{S}(\mathbb{R}^d)$ in each variable $f_i \in \mathcal{K}(\mathbb{R}^d)$ singly. In particular, it is separately continuous in each variable in the usual topology on $\mathcal{K}(\mathbb{R}^d)$ (which is simply the analog to that on $\mathcal{D}(\mathbb{R}^d)$). Since $\mathcal{K}(\mathbb{R}^d)$ is nuclear for all d and $\mathcal{K}(\mathbb{R}^d) \otimes \mathcal{K}(\mathbb{R}^1) = \mathcal{K}(\mathbb{R}^{d+1})$, the nuclear theorem entails that $\langle \Psi, (\prod_{i=1}^{n} \varphi(f_i)^-)\Phi\rangle$ defines a unique functional $T_{\Psi,\Phi} \in \mathcal{K}'(\mathbb{R}^{dn})$. It is easy to see that, in fact,

$$T_{\Psi,\Phi}\{g\} = \langle \Psi, \varphi\{g\}^-\Phi\rangle, \forall g \in \mathcal{K}(\mathbb{R}^{dn}), \qquad \Psi \in \mathcal{H}, \Phi \in \bar{D}. \tag{3.4}$$

We shall need the following simple lemma.

Lemma 3.5. *If $\varphi(x)$ satisfies a generalized H-bound, then for any $F \in \mathcal{D}(\mathbb{R})$ and any $\Phi \in P_0\mathcal{H}$, $F(H)\varphi\{g\}^-\Phi = \varphi\{h\}^-\Phi$ for any $g \in \mathcal{K}(\mathbb{R}^{dn})$, where*

$$\tilde{h}(p_1, \ldots, p_n) = F\left(\sum_{i=1}^{n} p_i^0\right)\tilde{g}(p_1, \ldots, p_n).$$

Proof. Given (3.4) and the obvious fact that $U(a)\varphi\{g\}^-\Phi = \varphi\{g_a\}^-\Phi$, where $g_a(x_1, \ldots, x_n) = g(x_1 - a, \ldots, x_n - a)$, the proof follows that of Lemma 49 on page 224 of [27]. □

Proposition 3.6. *If $\varphi(x)$ satisfies a generalized H-bound, then $\mathcal{A}'$ is Abelian and $\overline{\mathcal{A}'\Omega} = P_0\mathcal{H}$.*

Proof. First it is asserted that for any $X \in \mathcal{P}_c(\mathbb{R}^d)$ and $A \in \mathcal{A}$,

$$\langle P_0X^-P_0\Psi, P_0AP_0\Phi\rangle = \langle P_0A^*P_0\Psi, P_0X^*P_0\Phi\rangle, \forall \Phi, \Psi \in \mathcal{H}.$$

It suffices to consider a monomial A of order n: $A = G(H)(\prod_{i=1}^{n} \varphi(f_i)^-F_i(H)) \in \mathcal{A}$, $\{f_i\}_{i=1}^{n} \subset \mathcal{K}(\mathbb{R}^d)$. By repeated use of Lemma 3.5, there exists a $g \in \mathcal{K}(\mathbb{R}^{dn})$ such that $A\chi = \varphi\{g\}^-\chi$ for any $\chi \in P_0\mathcal{H}$. Thus, $P_0AP_0\Phi = P_0\varphi\{g\}^-P_0\Phi$. Since the linear hull of test functions of the form $\prod_{i=1}^{n} f_i(x_i)$, $\{f_i\}_{i=1}^{n} \subset \mathcal{K}(\mathbb{R}^d)$, is dense in $\mathcal{K}(\mathbb{R}^{dn})$ in the usual topology, one may use (3.4) and Lemma 3.4 to conclude

$$\langle P_0X^-P_0\Psi, P_0\varphi\{g\}^-P_0\Phi\rangle = \langle P_0\varphi\{g\}^*P_0\Psi, P_0X^*P_0\Phi\rangle.$$

The assertion then follows easily since $\varphi\{g\}^* = \varphi\{g^*\}^-$ on $P_0\mathcal{H}$, $\forall g \in \mathcal{K}(\mathbb{R}^{dn})$, where $g^*(x_1, \ldots, x_n)$ is the complex conjugate of $g(x_n, \ldots, x_1)$.

Therefore, $(P_0\bar{\mathcal{P}}_c(\mathbb{R}^d)P_0)'' \subseteq (P_0\mathcal{A}P_0)'$, implying $P_0\mathcal{A}P_0 \subseteq (P_0\bar{\mathcal{P}}_c(\mathbb{R}^d)P_0)' = (P_0\bar{\mathcal{P}}_c(\mathbb{R}^d)P_0)''$, since $(P_0\bar{\mathcal{P}}_c(\mathbb{R}^d)P_0)''$ is maximally Abelian on $P_0\mathcal{H}$. Hence $P_0\mathcal{A}P_0$ is maximally Abelian in $P_0\mathcal{H}$, by Lemma 3.2. Since $P_0\mathcal{A}'P_0 = (P_0\mathcal{A}P_0)'$ and the central support of P_0 in $\mathcal{A}$ is the identity (this is a trivial consequence of Lemma 3.2 and [14, Corollaire I.1.1]), $\mathcal{A}'$ is isomorphic to $P_0\mathcal{A}'P_0$ [14, Prop. I.2.2], and hence Abelian. Moreover, $\mathcal{A}'\Omega = P_0\mathcal{A}\Omega$ is dense in $P_0\mathcal{H}$ (Lemma 3.2). □

We recall a well-known result about the weak commutant of $\mathcal{P}(\mathbb{R}^d)$, stated in a form suited to our purposes.

Lemma 3.7. *For any $\mathcal{P} \subset \mathcal{P}(\mathbb{R}^d)$ such that $U(a)\mathcal{P}U(a)^{-1} \subseteq \mathcal{P}$ for all $a \in \mathbb{R}^d$ and such that $\{X^-\Phi \mid X \in \mathcal{P}, \Phi \in P_0\mathcal{H}\}$ is dense in $\mathcal{H}$, one has $\mathcal{P}'_w \subset U(\mathbb{R}^4)'$, i.e. the operators in $\mathcal{P}'_w$ are translation-invariant.*

Proof. A simple proof using only the spectrum condition can be inferred from [4]. □

We may now prove the assertions made in Section 2 about the connection between the von Neumann algebra $\mathcal{A}$ and the polynomial algebras of the field $\varphi(x)$.

Proposition 3.8. *If $\varphi(x)$ satisfies a generalized H-bound, then $(\mathcal{P}_c(\mathbb{R}^d))'_s$, $(\mathcal{P}_{\text{loc}}(\mathbb{R}^d))'_s$ and $(\mathcal{P}(\mathbb{R}^d))'_s$ are contained in $\mathcal{A}'$.*

Proof. By the previous lemma, all three algebras commute with all translations, therefore they commute with all bounded functions of H. It follows easily that $(\mathcal{P}_c\mathbb{R}^d))'_s \supseteq \mathcal{A}'$. Similarly, using Lemma 3.3, $(\mathcal{P}_{\text{loc}}(\mathbb{R}^d))'_s \subseteq \mathcal{A}'$. Since $(\mathcal{P}(\mathbb{R}^d))'_s \subseteq (\mathcal{P}_c(\mathbb{R}^d))'_s$, the proposition is proven. □

Proposition 3.9. *If $\varphi(x)$ satisfies a generalized H-bound, then $\mathcal{A}' \subseteq (\mathcal{P}_c(\mathbb{R}^d)'_s$.*

Proof. Let $X, Y \in \mathcal{P}_c(\mathbb{R}^d)$ be monomials and $A' \in \mathcal{A}'$. Then $A'XY\Omega = A'A_{XY}\Omega$, with $A_{XY} \in \mathcal{A}$ constructed as in the proof of Lemma 3.1. Thus, $A'XY\Omega = A_{XY}A'\Omega$. But since $P_0 \in \mathcal{A}$, one has $A'\Omega \in P_0\mathcal{H}$. Therefore, it is easy to see that $A_{XY}A'\Omega = \overline{XY}A'\Omega$ by using results established in the proof of Lemma 3.1. Of course, by the same argument, $A'\bar{Y}\Omega = \bar{Y}A'\Omega$, so that $A'Y\Omega \in D(X^-)$ and $[A', X^-] = 0$ on $\mathcal{P}_c(\mathbb{R}^d)\Omega$. However, since $\mathcal{P}_c(\mathbb{R}^d)\Omega$ is a core for X^-, if $\Phi \in D(X^-)$ there exists a sequence $\{\Phi_n\}_{n\in\mathbb{N}} \subset \mathcal{P}_c(\mathbb{R}^d)\Omega$ such that $\Phi_n \xrightarrow{s.} \Phi$ and $X^-\Phi_n \xrightarrow{s.} X^-\Phi$. Then $A'\Phi_n \xrightarrow{s.} A'\Phi$, and since $X^-A'\Phi_n = A'X\Phi_n$ for any $n \in \mathbb{N}$, one must have $A'\Phi \in D(X^-)$ and $X^-A'\Phi = A'X^-\Phi$. □

Thus, if $\varphi(x)$ satisfies a generalized H-bound, $(\mathcal{P}_c(\mathbb{R}^d))'_s$ is a von Neumann algebra and is equal to $\mathcal{A}'$.

Proposition 3.10. *If $\varphi(x)$ satisfies a generalized H-bound, then $\mathcal{A}' = (\mathcal{P}(\mathbb{R}^d))'_w$. Furthermore, $U(\mathcal{P}_+^\uparrow) \subset \mathcal{A}$.*

Proof. Let $B' \in (\mathcal{P}(\mathbb{R}^d))'_w$. Then B' is translation-invariant, and since the generators (3.3) of $\mathcal{A}$ leave D_1 invariant, it follows that for any $A \in \mathcal{A}$, $\langle B'\Phi, A\Psi\rangle = \langle A^*\Phi, B'^*\Psi\rangle$ for all $\Phi, \Psi \in D_1$. Since B' and A are bounded, one concludes $(\mathcal{P}(\mathbb{R}^d))'_w \subseteq \mathcal{A}'$.

On the other hand, $A' \in \mathcal{A}'$ commutes strongly and thus weakly with $\mathcal{P}_c(\mathbb{R}^d)$ (Prop. 3.9). But any operator in $\mathcal{P}(\mathbb{R}^d)$ (on D_1) is a strong limit of operators in $\mathcal{P}_c(\mathbb{R}^d)$, using the now-obvious argument, so that $\mathcal{A}' \subseteq (\mathcal{P}(\mathbb{R}^d))'_w$.

Finally, by Theorem 3 in [1] all vectors in $P_0\mathcal{H}$ are $U(\mathcal{P}_+^\uparrow)$-invariant. Since $U(\lambda)(\mathcal{P}(\mathbb{R}^d))'_w\, U(\lambda)^{-1} = (\mathcal{P}(\mathbb{R}^d))'_w$ for all $\lambda \in \mathcal{P}_+^\uparrow$, and since Ω is separating for $\mathcal{A}'$ (use Lemma 3.2), Prop. 3.6 entails that $U(\lambda)A'U(\lambda)^{-1} = A'$ for any $A' \in \mathcal{A}'$ and $\lambda \in \mathcal{P}^\uparrow$. □

IV. Fields with intrinsically local operators

Let $f_s \in \mathcal{D}(\mathbb{R}^d)$ have a Fourier transform that vanishes nowhere. We shall assume that $\varphi(f_s)$ is an intrinsically local operator in the sense of Def. 2.2. If $W_R = \{x \in \mathbb{R}^d \mid |x^0| < x^1\}$, where x^0 is the time coordinate, define

$$\mathcal{A}(W_R) \equiv \{a(\overline{\varphi(f_{s,\lambda}) \upharpoonright D_s}) \mid \operatorname{supp}(f_{s,\lambda}) \subset W_R,\ \lambda \in \mathcal{P}_+^\uparrow\}''$$

and

$$\mathcal{A} \equiv \{a(\overline{\varphi(f_{s,\lambda}) \upharpoonright D_s}) \mid \lambda \in \mathcal{P}_+^\uparrow\}'',$$

using notation established in Section 2. Furthermore, if $\mathcal{W} \equiv \{W_{R,\lambda} \mid \lambda \in \mathcal{P}_+^\uparrow\}$ (for $d = 2$, the transformation $x \to -x$ is included in defining $\mathcal{W}$) is the set of wedges and $\mathcal{K}$ is the set of double cones (interiors of intersections of some forward light cone and some backward light cone), we define

$$\begin{aligned} \mathcal{A}(W_{R,\lambda}) &\equiv U(\lambda)\mathcal{A}(W_R)U(\lambda)^{-1}, & \lambda &\in \mathcal{P}_+^\uparrow,\\ \mathcal{A}(\mathcal{O}) &\equiv \cap\{\mathcal{A}(W) \mid \mathcal{O} \subset W \in \mathcal{W}\}, & \mathcal{O} &\in \mathcal{K},\\ \mathcal{A}(\mathcal{O}') &\equiv \{\mathcal{A}(W) \mid \mathcal{W} \ni W \subset \mathcal{O}'\}'', & \mathcal{O} &\in \mathcal{K}; \end{aligned}$$

then $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \in \mathcal{W} \cup \mathcal{K} \cup \mathcal{K}'}$ defines a Poincaré-covariant net of local von Neumann algebras that satisfies the special condition of duality. That is to say, $\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O}')'$ for any $\mathcal{O} \in \mathcal{W} \cup \mathcal{K} \cup \mathcal{K}'$ and if $\{V(t)\}_{t\in\mathbb{R}}$ is the Abelian subgroup of $U(\mathcal{P}_+^\uparrow)$ representing the Lorentz velocity transformations in the 0–1 direction, then the dense sets $\mathcal{A}(W_R)\Omega \subset D(V(i\pi))$ and $\mathcal{A}(W_R')\Omega \subset D(V(-i\pi))$ satisfy

$$JV(i\pi)A\Omega = A^*\Omega, \qquad \forall A \in \mathcal{A}(W_R), \tag{4.1}$$

$$JV(-i\pi)B\Omega = B^*\Omega, \qquad \forall B \in \mathcal{A}(W_R'), \tag{4.2}$$

with the antiunitary involution $J = U(\pi_1, 0)\,\Theta_0$, with Θ_0 the TCP-operator for the field $\varphi(x)$ and $U(\pi_1, 0)$ the rotation by angle π around the 1-axis. Furthermore,

$$\mathcal{A}(W_R)' = \mathcal{A}(W_R') = J\mathcal{A}(W_R)J. \tag{4.3}$$

See [11] for further details and results. In addition, it is known [28, 29] that

$$JV(i\pi)X\Omega = X^*\Omega, \forall X \in \mathcal{P}_0(W_R), \tag{4.4}$$

$$JV(-i\pi)Y\Omega = Y^*\Omega, \forall Y \in \mathcal{P}_0(W_R'), \tag{4.5}$$

where for $\mathcal{O} \subset \mathbb{R}^d$, $\mathcal{P}_0(\mathcal{O})$ is the polynomial algebra of operators $\{\varphi(f) \mid f \in \mathcal{S}(\mathbb{R}^d), \operatorname{supp}(f) \subset \mathcal{O}\}$. In the following, $\mathcal{P}_{0s}(\mathcal{O})$ will denote the polynomial algebra of operators $\{\varphi(f_{s,\lambda}) \mid \lambda \in \mathcal{P}_+^\uparrow, \operatorname{supp}(f_{s,\lambda}) \subset \mathcal{O}\}$.

The following proposition collects some further results that have been proven in some of our other papers.

Proposition 4.1. *Given the assumptions and definitions stated above, the following are true*:

(i) Ω *is cyclic for* $\mathcal{A}$ *in* $\mathcal{H}$;
(ii) $\mathcal{A}' = Z(\mathcal{A}) \equiv \mathcal{A} \cap \mathcal{A}' = Z(\mathcal{A}(W)) \equiv \mathcal{A}(W) \cap \mathcal{A}(W)', \forall W \in \mathcal{W}$;
(iii) $\overline{\mathcal{A}'\Omega} = P_0\mathcal{H}$;
(iv) $U(\mathcal{P}_+^\uparrow) \subset \mathcal{A}$.

Proof. (i) is shown in [11] and is a straightforward consequence of the requirement that $\tilde{f}_s(p) \neq 0$ for any $p \in \mathbb{R}^d$. (ii), (iii) and $U(\mathbb{R}^d) \subset \mathcal{A}$ follow from Prop. 3.1 of [5]. $U(L_+^\uparrow) \subset \mathcal{A}$ is entailed by the special condition of duality and Prop. 5.3 of [5]. $\square$

Finally, we must show the following to be true.

Proposition 4.2. *Given the assumptions and definitions stated above,* $\mathcal{A}' = (\mathcal{P}(\mathbb{R}^d))_w'$.

Proof. It will first be shown that $\mathcal{A}' \subseteq (\mathcal{P}(\mathbb{R}^d))_w'$. By definition of $\mathcal{A}$, $\varphi(f_{s,\lambda})^- \eta \mathcal{A}$, for all $\lambda \in \mathcal{P}_+^\uparrow$. Therefore, for any $\lambda \in \mathcal{P}_+^\uparrow$

$$\langle \varphi(f_{s,\lambda})\Phi, A'\Psi\rangle = \langle A'^*\Phi, \varphi(f_{s,\lambda}^*)\Psi\rangle, \qquad \forall A' \in \mathcal{A}', \Phi, \Psi \in D_1.$$

But since the linear hull of $\{f_{s,a} \mid a \in \mathbb{R}^d\}$ is dense in $\mathcal{S}(\mathbb{R}^d)$ [11], it follows easily that for any $f \in \mathcal{S}(\mathbb{R}^d)$

$$\langle \varphi(f)\Phi, A'\Psi\rangle = \langle A'^*\Phi, \varphi(f)^*\Psi\rangle, \qquad \forall A' \in \mathcal{A}', \Phi, \Psi \in D_1. \tag{4.6}$$

Since D_1 is invariant under $\mathcal{P}_0(\mathbb{R}^d)$, iteration of equation (4.6) yields $A' \in (\mathcal{P}_0(\mathbb{R}^d))_w'$. Of course, $(\mathcal{P}_0(\mathbb{R}^d))_w' = (\mathcal{P}(\mathbb{R}^d))_w'$, completing the proof of the required containment.

It remains to show that $(\mathcal{P}(\mathbb{R}^d))_w' \subseteq \mathcal{A}'$. Some preparation is necessary. By definition and [12],

$$\mathcal{A}(W)' \subseteq (\{\varphi(f_{s,\lambda}) \mid \lambda \in \mathcal{P}_+^\uparrow, \operatorname{supp}(f_{s,\lambda}) \subset W\})_s', \qquad \forall W \in \mathcal{W}$$

(note that $\overline{\varphi(f) \restriction D_1} = \overline{\varphi(f) \restriction D_s}$, $\forall f \in \mathcal{S}(\mathbb{R}^d)$ [11]).

Thus, for any $A \in \mathcal{A}(W_R) = \mathcal{A}(W_R')'$, $B \in (\mathcal{P}_{0s}(W_R'))_w'$, and $\prod_{i=1}^n \varphi(f_i) \in$

$\mathscr{P}_{0s}(W_R')$,

$$\left\langle AB\Phi, \left(\prod_{i=1}^{n} \varphi(f_i)\right)\Psi\right\rangle = \left\langle B\Phi, \varphi(f_1)^{-}A^*\left(\prod_{i=2}^{n} \varphi(f_i)\right)\Psi\right\rangle$$
$$= \left\langle \varphi(f_1)^*B\Phi, A^*\left(\prod_{i=2}^{n} \varphi(f_i)\right)\Psi\right\rangle = \left\langle B\varphi(f_1)^*\Phi, A^*\left(\prod_{i=2}^{n} \varphi(f_i)\right)\Psi\right\rangle,$$
$$\forall \Phi, \Psi \in D_1$$

(because $B \in (\mathscr{P}_{0s}(W_R'))_w'$ implies that $BX^* = X^*B$ on D_1, for any $X \in \mathscr{P}_{0s}(W_R')$). This process may be continued to yield $AB \in (\mathscr{P}_{0s}(W_R'))_w'$. Similarly, $BA \in (\mathscr{P}_{0s}(W_R'))_w'$. Moreover, since $\mathscr{P}_{0s}(W_R')\Omega$ is dense in $\mathscr{H}$ and $V(t)\mathscr{P}_{0s}(W_R')V(t)^{-1} = \mathscr{P}_{0s}(W_R')$, $\forall t \in \mathbb{R}$, Lemma 13(c) of [28] implies that $(\mathscr{P}_{0s}(W_R'))_w'\Omega \subseteq D(V(i\pi))$ and $V(i\pi)B\Omega = JB^*\Omega$ for any $B \in (\mathscr{P}_{0s}(W_R'))_w'$, where J and $V(i\pi)$ are as in (4.1)-(4.5). Since $AV(t)B^*V(t)^{-1} \in (\mathscr{P}_{0s}(W_R'))_w'$ for all $t \in \mathbb{R}$, $A \in \mathscr{A}(W_R)$ and $B \in (\mathscr{P}_{0s}(W_R'))_w'$, one may use Lemma 14 of [28] to conclude that

$$[A, JBJ]\Omega = 0, \qquad \forall A \in \mathscr{A}(W_R), \qquad B \in (\mathscr{P}_{0s}(W_R'))_w'. \tag{4.7}$$

Let $A = A_1A_2 \in \mathscr{A}(W_R)$. Then iteration of (4.7) yields $[A_1, JBJ]A_2\Omega = 0$. Since Ω is cyclic for $\mathscr{A}(W_R)$, this implies that $J(\mathscr{P}_{0s}(W_R'))_w'J \subseteq \mathscr{A}(W_R)'$. Using (4.3), one must conclude that $(\mathscr{P}_{0s}(W_R'))_w' \subseteq \mathscr{A}(W_R)$. In fact, since $\mathscr{A}(W_R) \subseteq (\mathscr{P}_{0s}(W_R'))_w'$, they are equal. The same argument yields $(\mathscr{P}_{0s}(W_R))_w' = A(W_R')$.

But $(\mathscr{P}(\mathbb{R}^d))_w' \subseteq (\mathscr{P}_{0s}(W))_w'$ for any $W \in \mathscr{W}$. Thus, $(\mathscr{P}(\mathbb{R}^d))_w' \subseteq \mathscr{A}(W_R) \cap \mathscr{A}(W_R') = \mathscr{A}(W_R) \cap \mathscr{A}(W_R)' = \mathscr{A}'$, by Proposition 4.1 (ii). □

Remark. Although Bisognano and Wichmann assume the uniqueness of the vacuum in [28, 29], all of their results excepting the factorialty of the wedge algebras still hold when one drops this assumption.

Acknowledgements

The authors wish to express their gratitude to Frau Trentmann for the patient typing of a difficult text.

Appendix

In this appendix we present a useful result on bounded-operator-valued generalized functions, along with a consequence for quantum fields satisfying a generalized H-bound that finds application in the main text. The authors learned of a special case of the former from E. H. Wichmann [30] and wish to thank Prof. Wichmann for the permission to publish a version of it here.

Theorem A.1 (Wichmann). *Let $f \to A(f)$ be a linear mapping of a countably normed, linear topological space $\mathscr{F}$ into $\mathscr{B}(\mathscr{H})$ such that the mapping $f \to \langle \Phi, A(f)\Psi \rangle$ is in $\mathscr{F}'$ for all $\Phi \in \mathscr{M}$, $\Psi \in \mathscr{N}$, where $\mathscr{M}$ and $\mathscr{N}$ are dense subsets* *of $\mathcal{H}$. Then the mapping $f \to A(f)$ is continuous relative to the norm topology on $\mathcal{B}(\mathcal{H})$. Furthermore, there exists a norm $|\cdot|$ continuous in the topology of $\mathcal{F}$ such that $\|A(f)\| \le |f|$, $\forall f \in \mathcal{F}$.*

Proof. Since $A(f)$ is bounded and $\mathcal{F}'$ is weakly sequentially complete [31, Section 7], i.e. if $\{F_n(f)\}_{n\in\mathbb{N}}$ is Cauchy for each $f \in \mathcal{F}$ and $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{F}'$ then there exists a unique $f \in \mathcal{F}'$ such that $\lim_{n\to\infty} F_n(f) = F(f) \forall f \in \mathcal{F}$, one may conclude that $f \to \langle \Phi, A(f)\Psi\rangle$ is in $\mathcal{F}'$ for any $\Phi, \Psi \in \mathcal{H}$. Hence, $\langle A(f)\Phi, A(g)\Psi\rangle$ is continuous in each variable f, g, separately, and thus jointly [31, Section 7]. This entails that if $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ converges to zero, then $A(f_n)\Phi$ converges strongly to zero, $\forall \Phi \in \mathcal{H}$.

From this it is possible to conclude that for any $\Phi \in \mathcal{H}$ there exist a constant $C(\Phi)$ and a norm $|\cdot|_\Phi$ continuous in the topology on $\mathcal{F}$ such that $\|A(f)\Phi\| \le C(\Phi)|f|_\Phi \forall f \in \mathcal{F}$. In fact, assume the contrary. Then there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ such that $\|A(f_n)\Phi\| > n|f_n|_n$, where $\{|\cdot|_n\}_{n\in\mathbb{N}}$ is a set of seminorms determining the topology on f indexed in nondecreasing order. Let $g_n \equiv f_n(n|f_n|)^{-1}$. Then $\{g_n\}_{n\in\mathbb{N}}$ converges to zero in $\mathcal{F}$, but $\|A(g_n)\Phi\| > 1 \forall n \in \mathbb{N}$. This contradicts what was shown above. One may, henceforth, take $|\cdot|_\Phi = |\cdot|_{n(\Phi)}$ for some $n(\Phi) \in \mathbb{N}$. Since the sequence $\{|\cdot|_n\}_{n\in\mathbb{N}}$ is nondecreasing, there exists a $k(\Phi) \in$ such that $\|A(f)\Phi\| \le k(\Phi)\, |f|_{k(\Phi)}$, $\forall f \in \mathcal{F}$.

For any $n \in \mathbb{N}$ and $f \in \mathcal{F}$, let $V_n(f) \equiv \{\Phi \in \mathcal{H} |\ \|A(f)\Phi\| \le n\, |f|_n\}$. Since $A(f) \in \mathcal{B}(\mathcal{H})$, the mapping $\Phi \to A(f)\Phi$ is norm-continuous; so $V_n(f)$ is norm-closed. Let $V_n \equiv \cap \{V_n(f) \,|\, f \in \mathcal{F}\}$. V_n is also norm-closed. But $\mathcal{H} = \bigcup_{n\in\mathbb{N}} V_n$; hence by the Baire Category Theorem at least one V_n, say V_N, has a nonempty interior. Therefore, there exist a $\Phi_0 \in \mathcal{H}$ and a $\rho > 0$ such that $\|A(f)\Phi\| \le N\, |f|_N$ for all $f \in \mathcal{F}$ and all $\Phi \in \mathcal{H}$ such that $\|\Phi - \Phi_0\| < \rho$. Thus, for all $\Psi \in \mathcal{H}$ with $\|\Psi\| < \rho$, $\|A(f)\Psi\| \le 2N|f|_N \forall f \in \mathcal{F}$. It follows easily that $\|A(f)\Phi\| \ge 2N\rho^{-1}\, |f|_N \|\Phi\| \forall f \in \mathcal{F}$. □

Corollary A.2. *Let $\varphi(x)$ be an operator-valued tempered distribution satisfying the domain and continuity assumptions of* [7] *and let $F \in \mathcal{S}(\mathbb{R})$ be such that $\|\varphi(f)^- F(H)\| < \infty$ for all $f \in \mathcal{V}$, where $\mathcal{V}$ is a dense, linear subset of $\mathcal{S}(\mathbb{R}^d)$. Then there exists a norm $|\cdot|$ continuous in the topology on $\mathcal{S}(\mathbb{R}^d)$ such that $\|\varphi(f)^- F(H)\| \le |f|$, $\forall f \in \mathcal{S}(\mathbb{R}^d)$.*

Proof. Since $F(H)D_1 \subset D_1$, $f \to \langle \Phi, \varphi(f)^- F(H)\Psi\rangle = \langle \Phi, \varphi(f)F(H)\Psi\rangle$ is in $\mathcal{S}'(\mathbb{R}^d)$ for all $\Phi, \Psi \in D_1$. Apply Theorem A.1 with $\mathcal{F} = \mathcal{V}^0$, where $\mathcal{V}^0$ is $\mathcal{V}$ with the topology induced by $\mathcal{S}(\mathbb{R}^d)$. Since the same countable set of seminorms determines the topologies on $\mathcal{V}^0$ and $\mathcal{S}(\mathbb{R}^d)$, there exists a norm $|\cdot|$ continuous in the topology on $\mathcal{S}(\mathbb{R}^d)$ such that $\|\varphi(f)^- F(H)\| \le |f|$, $\forall f \in \mathcal{V}$. Let $f \in \mathcal{S}(\mathbb{R}^d)$ and $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{V}$ converge to f in the topology of $\mathcal{S}(\mathbb{R}^d)$. Then the sequence $\{\varphi(f_n)^- F(H)\}_{n\in\mathbb{N}}$ is norm-Cauchy and converges to an $A \in \mathcal{B}(\mathcal{H})$. However, $\{\varphi(f_n)^- F(H)\Phi\}_{n\in\mathbb{N}} = \{\varphi(f_n)F(H)\Phi\}_{n\in\mathbb{N}}$ converges strongly to $\varphi(f)F(H)\Phi$, $\forall \Phi \in D_1$. Thus, A must coincide with $\varphi(f)F(H)$ on D_1. If $\Phi \in \mathcal{H}$ and $\{\Phi_n\}_{n\in\mathbb{N}} \subset D_1$ converges strongly to Φ, then $\|\varphi(f)F(H)(\Phi_n - \Phi_m)\| = \|A(\Phi_n - \Phi_m)\| \le \|A\|$

$\|\Phi_n - \Phi_m\|$. This implies that $F(H)\Phi \in D(\varphi(f)^-)$ and $\|\varphi(f)^- F(H)\| \leq \|f\|$, $\forall f \in \mathscr{S}(\mathbb{R}^d)$. $\square$

REFERENCES

[1a] H. J. BORCHERS, On the structure of the algebra of field operators, *Nuovo Cim., 24,* 214–236 (1962).

[1b] K. MAURIN, Mathematical structure of Wightman formulation of quantum field theory, *Bull. Acad. Polon. Sci., Sér. sci. math., astr., et phys., 11,* 115–119 (1963).

[1c] K. MAURIN, On some theorems of H. J. Borchers, *Bull. Acad. Polon. Sci., Sér. sci. math., astr., et phys., 11,* 121–123 (1963).

[2] H. J. BORCHERS and J. YNGVASON, On the algebra of field operators: the weak commutant and integral decompositions of states, *Commun. Math. Phys., 42,* 231–252 (1975).

[3] H. J. BORCHERS, On the structure of the algebra of field operators, II, *Commun. Math. Phys., 1,* 49–56 (1965).

[4] H. ARAKI, On the algebra of all local observables, *Prog. Theor. Phys., 32,* 844–854 (1964).

[5] W. DRIESSLER and S. J. SUMMERS, Central decomposition of Poincaré-invariant nets of local field algebras and absence of spontaneous symmetry breaking of the Lorentz group, to appear in *Ann. Inst. Henri Poincaré*.

[6] K. HEPP, R. JOST, D. RUELLE, and O. STEINMANN, Necessary condition on Wightman functions, *Helv. Phys. Acta, 34,* 542–544 (1961).

[7] R. F. STREATER and A. S. WIGHTMAN, *PCT, Spin and Statistics, and All That,* Benjamin Pub. Co., Reading, Massachusetts, 1964.

[8] H. J. BORCHERS and J. YNGVASON, Integral representations for Schwinger functionals and the moment problem over nuclear spaces, *Commun. Math. Phys., 43,* 255–271 (1975).

[9] J. FRÖHLICH, The pure phases, the irreducible quantum fields, and dynamical symmetry breaking in Symanzik-Nelson positive quantum field theories, *Ann. Phys., 97,* 1–54 (1976).

[10] J. YNGVASON, On the decomposition of Wightman functionals in the Euclidean framework, *Rep. Math. Phys., 13,* 101–115 (1978).

[11] W. DRIESSLER, S. J. SUMMERS, and E. H. WICHMANN, On the affiliation of quantum field operators to local algebras of observables, submitted to Commun. Math. Phys.

[12] F. J. MURRAY and J. VON NEUMANN, On rings of operators, *Ann. Math., 37,* 116–229 (1936).

[13] A. JAFFE, High-energy behavior in quantum field theory, *Phys. Rev., 158,* 1454–1461 (1967).

[14] J. DIXMIER, *Les Algèbres d'Opérateurs dans l'Espace Hilbertien,* 2me édition, Gauthier-Villars, Paris, 1969.

[15] G. C. HEGERFELDT, Extremal decomposition of Wightman functions and of states on nuclear $*$-algebras by Choquet theory, *Commun. Math. Phys., 45,* 133–135 (1975).

[16] J. GLIMM and A. JAFFE, *Quantum Physics,* Springer-Verlag, New York and Berlin, 1981.

[17] W. DRIESSLER and J. FRÖHLICH, The Reconstruction of local observable algebras from the Euclidean Green's functions of relativistic quantum field theory, *Ann. Inst. Henri Poincaré, A27,* 221–236 (1977).

[18] J. HERTEL, Lokale Quantentheorie und Felder am Punkt, DESY, preprint T-81/01, 1981.

[19] R. POWERS, Self-adjoint algebras of unbounded operators, *Commun. Math. Phys., 21,* 85–124 (1971).

[20] O. BRATTELI, Conservation of estimates in quantum field theory, *Comm. Pure Appl. Math., 25,* 759–779 (1972).

[21] J.-P. JURZAK, Decomposable operators: application to K.M.S. weights in a decomposable von Neumann algebra, *Rep. Math. Phys., 8,* 203–228 (1975).

[22] R. PALLU DE LA BARRIÈRE, Décomposition des opérateurs non bornés dans les sommes continues d'espaces de Hilbert, *C. R. Acad. Sci. Paris, 232A,* 2071–2073 (1951).

[23] A. E. NUSSBAUM, Reduction theory for unbounded closed operators in Hilbert space, *Duke Math. J., 31,* 33–44 (1964).

[24] M. J. J. LENNON, On sums and products of unbounded operators in Hilbert spaces, *Trans. Am. Math. Soc., 198,* 273–285 (1974).

[25] K. MAURIN, Allgemeine Eigenfunktionsentwicklungen, *Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., 7,* 471–479 (1959).

[26] I. M. GEL'FAND and N. YA. VILENKIN, *Generalized Functions,* Vol. 4, Academic Press, New York, 1964.

[27] H. BAUMGÄRTEL and M. WOLLENBERG, *Mathematical Scattering Theory,* Birkhäuser Verlag, Basel, Boston and Stuttgart, 1983.
[28] J. J. BISOGNANO and E. H. WICHMANN, On the duality condition for a Hermitian scalar field, *J. Math. Phys., 16,* 985–1007 (1975).
[29] J. J. BISOGNANO and E. H. WICHMANN, On the duality condition for quantum fields, *J. Math. Phys., 17,* 303–321 (1976).
[30] E. H. WICHMANN, Private communication.
[31] B. SIMON, Distributions and their Hermite expansions, *J. Math. Phys., 12,* 140–148 (1971).