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Purification of KMS states*

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Abstract. We present some basic facts about KMS states in a novel way suggested both by the problem of purification and also by the Bisognano–Wichmann situation (and the related application to quantum field theory on black holes.) We also use the resulting framework to continue a discussion of linear Bose fields begun in a companion paper. The results here will be utilized in a subsequent construction of such fields on black holes.

§1. Introduction

In the quantum mechanics of a finite system, there is a simple procedure by which a statistical mixture may be regarded as the restriction to the system of a pure state on a system twice as big: Take for definiteness an (assumed trace-class) Gibbs density operator $\rho = Z^{-1}e^{-\beta H}$ on some Hilbert space $\mathcal{H}$. (We choose a basis ψ_i with $H\psi_i = E_i\psi_i$ so $Z = \sum_i e^{-\beta E_i}$.) Then, defining the vector state $\Omega = Z^{-1/2}\sum_i e^{-\beta E_i/2}\psi_i \otimes \psi_i$ on $\mathcal{H}\otimes\mathcal{H}$, we have, for any observable A on $\mathcal{H}$:

$$(\mathrm{tr}\,\rho A)_{\mathcal{H}} = \langle \Omega \mid (\mathbb{1}\otimes A)\Omega\rangle_{\mathcal{H}\otimes\mathcal{H}}$$

The adjoining of a duplicate system in this way may be regarded in general as just an artificial trick. However, it has recently[1]) acquired new interest in the description of thermal states on black holes, since there the duplicate system has an interesting interpretation (as the system on the other side of the Schwarzschild throat. Cf. also the Bisognano–Wichmann situation [2] which as Sewell [3] has pointed out is a flat-space-time analogy.)

In this note, we explain how, for thermal equilibrium states, this doubling procedure (sometimes called purification) extends to infinite systems (possibly, but not necessarily black holes!) As is well known, on infinite systems, Gibbs states typically cannot be described by density matrices in the vacuum sector, and it is

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[1]) Since the fundamental work of Hawking [1].

more appropriate to adopt the algebraic approach to quantum mechanics[2]) [4, 5] in which Gibbs states are characterized by the KMS condition [6]. Once we adopt this language, we require little more than a re-organization of some known results from a rather unusual point of view.

This 'little more' will be explained in §2. In §3, we illustrate our general discussion by reconsidering and further developing earlier results on linear Bose fields which were obtained in [7].

Our main reason for recording these things here is to lay suitable foundations for the rigorous construction of linear quantum fields on black hole backgrounds to be discussed separately ([8], see also [9]). It is also hoped that the language developed here will be generally of use in discussing pure-state/mixed-state aspects of the Hawking effect.

Finally, the discussion here is intended to be complementary to the existing literature on purification ([10–13], see also [14]. See footnote 6 and Note (5) in §2.)

§2. The general case

We assume we are given some quantum dynamical system in the form of a *algebra[3]) $\mathfrak{A}$ together with a one parameter group $\alpha(t)$ of automorphisms of $\mathfrak{A}$ describing time-evolution. (In the quantum mechanical example of §1, one could take for $\mathfrak{A}$ the set of all bounded operators $\mathscr{B}(\mathscr{H})$ on $\mathscr{H}$ and $\alpha(t)A = e^{iHt}Ae^{-iHt}$ for $A \in \mathfrak{A}$.)

Our first step is to construct a quantum dynamical system which is twice as big as $(\mathfrak{A}, \alpha(t))$ and which contains $(\mathfrak{A}, \alpha(t))$ as a subsystem. Specifically, we seek a *double (quantum dynamical) system* $(\tilde{\mathfrak{A}}, \tilde{\alpha}(t), \iota)$ where $\tilde{\mathfrak{A}}$ consists of the tensor product[4]) $\mathfrak{A}^L \otimes \mathfrak{A}^R$ of two commuting subalgebras $\mathfrak{A}^L$ and $\mathfrak{A}^R$, $\tilde{\alpha}(t)$ is an automorphism of $\tilde{\mathfrak{A}}$ which maps $\mathfrak{A}^L \to \mathfrak{A}^L$ and $\mathfrak{A}^R \to \mathfrak{A}^R$, and ι is an involutary antiautomorphism[5]) ($\iota^2 = 1$) on $\tilde{\mathfrak{A}}$ which commutes eith $\tilde{\alpha}(t)$ ($\tilde{\alpha}(t) \circ \iota = \iota \circ \tilde{\alpha}(t)$) and which maps $\mathfrak{A}^L \to \mathfrak{A}^R$ and $\mathfrak{A}^R \to \mathfrak{A}^L$. We say that such a double system $(\tilde{\mathfrak{A}}, \tilde{\alpha}(t), \iota)$ extends $(\mathfrak{A}, \alpha(t))$ if we can (and do!) identity $(\mathfrak{A}, \alpha(t))$ with $(\mathfrak{A}^R, \tilde{\alpha}(t) \upharpoonright_{\mathfrak{A}^R})$.

We may always construct such a double system in the following way: Let ℓ be any anti-linear involution on $\mathfrak{A}$, and define[4]) $\tilde{\mathfrak{A}} = \mathfrak{A} \otimes \mathfrak{A}$, so that $\mathfrak{A}^L = \mathfrak{A} \otimes \mathbb{1}$ and

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[2]) We assume a general familiarity with the algebraic approach to quantum theory [4, 5] (more in these footnotes, less in the main text) especially with KMS states (see footnote 7 here and for more information, see e.g. the article by Hugenholtz in [4]) and with the GNS construction and the concepts of pure and mixed states (see footnotes 8 and 9 here and for more information, e.g. the article by Simon in [4]).

[3]) We assume $\mathfrak{A}$ contains an identity $\mathbb{1}$. Typically, we have in mind a C^* algebra but with suitable interpretations (for such constructions as tensor products, limits and commutants $\rho(\mathfrak{A})'$ of representations ρ of $\mathfrak{A}$) everything we say would also apply to more general *algebras. Recall that a state on $\mathfrak{A}$ means a positive linear functional on $\mathfrak{A}$. We shall adopt the convention throughout this paper that in addition, a state is required to be such that its GNS Hilbert space is separable (see footnote 8).

[4]) In the C^* case, take the C^* tensor product as defined in Vol. I of [5].

[5]) By antiautomorphism, we mean $\iota(A+B) = \iota(A) + \iota(B)$, $\iota(AB) = \iota(A)\iota(B)$, $\iota(A^*) = [\iota(A)]^*$, $\iota(\alpha A) = \bar{\alpha}\iota(A)$, $\iota(\mathbb{1}) = \mathbb{1}$.

$\mathfrak{A}^R = \mathbb{1}\otimes\mathfrak{A}$ (with the obvious identification $\mathbb{1}\otimes\mathfrak{A}\leftrightarrow\mathfrak{A}$). Then define $\tilde{\alpha}(t)$ and ι by their action on elements of form $A\otimes B$ by $\tilde{\alpha}(t)(A\otimes B) = \ell\alpha(t)\ell A\otimes\alpha(t)B$ and $\iota(A\otimes B) = \ell B\otimes\ell A$. It is easy to see that all double systems extending $(\mathfrak{A}, \alpha(t))$ are equivalent in a suitable sense[6]) so from now on, we shall talk about *the* double system extending $(\mathfrak{A}, \alpha(t))$. In practice, it would usually be convenient to work with a particular concrete realization as above, choosing for ℓ some preferred symmetry operation which reverses the sense of time (such as PCT) so that $\ell\alpha(t) = \alpha(-t)\ell$ and we then have $\tilde{\alpha}(t) = \alpha(-t)\otimes\alpha(t)$. In our 'finite system' example, we may choose $\tilde{\mathfrak{A}} = \mathscr{B}(\mathscr{H})\otimes\mathscr{B}(\mathscr{H})$, $(\approx\mathscr{B}(\mathscr{H}\otimes\mathscr{H}))$ $\tilde{\alpha}(t)(A\otimes B) = (e^{iHt}\otimes e^{-iHt})^{-1}(A\otimes B)(e^{iHt}\otimes e^{-iHt})$ and $\iota(A\otimes B) = CBC\otimes CAC$ where C is the antiunitarity involution $C\sum\alpha_i\psi_i = \sum\bar{\alpha}_i\psi_i$ corresponding to out choice of basis ψ_i.

We can now state the key facts[7]) about KMS states in the following way: Any KMS state ω_β on $(\mathfrak{A}, \alpha(t))$ (with inverse temperature β) extends to an $\alpha(t)$- and ι-invariant state $\tilde{\omega}_\beta$ on $\tilde{\mathfrak{A}}$ such that the corresponding GNS[8]) representation $\tilde{\rho}_\beta$ on a (separable[3])) Hilbert space $\tilde{\mathscr{H}}$ with cyclic vector Ω satisfying $\langle\Omega\mid\tilde{\rho}_\beta(A)\Omega\rangle = \tilde{\omega}_\beta(A)$ satisfies

(A) Ω *is cyclic for* $\tilde{\rho}_\beta(\mathfrak{A}^R)$ alone (and hence also for $\tilde{\rho}_\beta(\mathfrak{A}^L)$ alone).

(B) There is a 'Hamiltonian' $\tilde{H}$ which satisfies[9])

$$\tilde{\rho}_\beta(\tilde{\alpha}(t)A) = e^{i\tilde{H}t}\tilde{\rho}_\beta(A)e^{-i\tilde{H}t}$$

and $e^{-i\tilde{H}t}\Omega = \Omega$ (i.e. $\tilde{H}\Omega = 0$) and also satisfies $\tilde{\rho}_\beta(\mathfrak{A}^R)\subset\mathscr{D}(e^{-\beta\tilde{H}/2})$ and

$$e^{-\beta\tilde{H}/2}\tilde{\rho}_\beta(A)\Omega = \tilde{\rho}_\beta(\iota A^*)\Omega\ \ \forall A\in\mathfrak{A}^R$$

(and in consequence the same statement with $R\to L$ and $e^{-\beta\tilde{H}/2}\to e^{\beta\tilde{H}/2}$).

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6) Given two candidates $(\tilde{\mathfrak{A}}_1, \tilde{\alpha}_1(t), \iota_1)$, $(\tilde{\mathfrak{A}}_2, \tilde{\alpha}_2(t), \iota_2)$ extending $(\mathfrak{A}, \alpha(t))$, there is an isomorphism $\beta_{21}: \tilde{\mathfrak{A}}_1\to\tilde{\mathfrak{A}}_2$ s.t. $\beta_{21}\restriction_{\mathfrak{A}^R} = id$, $\beta_{21}: \mathfrak{A}_1^L\to\mathfrak{A}_2^L$, $\beta_{21}\tilde{\alpha}_1(t) = \tilde{\alpha}_2(t)\beta_{21}$, $\beta_{21}\iota_1 = \iota_2\beta_{21}$. (For the proof, use the identification $\mathfrak{A}_1^R = \mathfrak{A}_2^R$ and define β_{21} on elements of form $A_1^L\otimes A_1^R\in\tilde{\mathfrak{A}}_1$ to be $(\iota_2\iota_1A_1^L)\otimes A_1^R$, extending by linearity, multiplicativity and continuity.) In the case where $\tilde{\mathfrak{A}}_1$ is constructed as in the text from some ℓ_1, and $\tilde{\mathfrak{A}}_2$ from some ℓ_2, β_{21} is simply $\ell_2\ell_1\otimes id$. Finally, to make the link with [11, 12, 13] note that another realization is given by choosing $\tilde{\mathfrak{A}} = \mathfrak{A}^0\otimes\mathfrak{A}$ where $\mathfrak{A}^0$ is the so-called opposite algebra, setting $\iota(A^0\otimes B) = (B^{0*}\otimes A^*)$ and $\tilde{\alpha}(t)(A^0\otimes B) = (\alpha(t)A)^0\otimes\alpha(t)B$ (where A^0 etc. denotes A regarded as an element of $\mathfrak{A}^0$).

7) A convenient statement of the KMS condition for our purposes is that ω_β should be an $\alpha(t)$-invariant state on $\mathfrak{A}$ whose GNS structure (cf. footnotes 8 and 9) $(\rho_\beta, \mathscr{H}, \tilde{U}(t), \Omega)$ has $\mathscr{H}$ separable and $\tilde{U}(t)$ strongly continuous (so we can write $\tilde{U}(t) = e^{-i\tilde{H}t}$) and satisfies the property: $\exists$ an antiunitary involution J on $\mathscr{H}$ s.t. $[J, e^{-it\tilde{H}}] = 0$ and $\rho_\beta(\mathfrak{A})\Omega\subset\mathscr{D}(e^{-\beta\tilde{H}/2})$ with $e^{-\beta\tilde{H}/2}\rho_\beta(A)\Omega = J\rho_\beta(A^*)\Omega$. Note that with this structure, it is a theorem that $J\rho_\beta(\mathfrak{A})J\subseteq\rho_\beta(\mathfrak{A})'$ [6] (actually $J\rho_\beta(\mathfrak{A})''J = \rho_\beta(\mathfrak{A})'$. We shall prove this here as part (i) of Theorem 1 in §2. See also footnote 11).

8) Recall that for any state (such as $\tilde{\omega}$) on an algebra (such as $\tilde{\mathfrak{A}}$) there is always such a GNS triple $(\tilde{\rho}_\beta, \tilde{\mathscr{H}}, \Omega)$ characterized up to equivalence by
(i) Ω is cyclic for $\rho_\beta(\tilde{\mathfrak{A}})$
(ii) $\tilde{\omega}(A) = \langle\Omega\mid\rho_\beta(A)\Omega\rangle\ \forall A\in\tilde{\mathfrak{A}}$
Later we shall also need that a state is pure if and only if its GNS representation is irreducible [4, 5].

9) The $\alpha(t)$-invariance of $\tilde{\omega}_\beta$ guarantees there is a unique unitary group implementing $\tilde{\alpha}(t)$ in the $\tilde{\rho}_\beta$ representation which maps $\Omega\mapsto\Omega$. The assumption here is that it is also strongly continuous. Note that the ι-invariance of $\tilde{\omega}_\beta$ implies similarly a unique antiunitary J with $J\rho(A)J = \rho(\iota A)$ and $J\Omega = \Omega$. Moreover, since $\tilde{\alpha}(t)\circ\iota = \iota\circ\tilde{\alpha}(t)$, we have $[J, e^{-i\tilde{H}t}] = 0$.

In fact these properties are equivalent to the KMS condition. That is

Proposition 1. *ω_β is KMS for $(\mathfrak{A}, \alpha(t))$ if and only if it extends to an $\tilde{\alpha}(t)$- and ι-invariant state $\tilde{\omega}$ on $\tilde{\mathfrak{A}}$ whose GNS triple $(\tilde{\rho}, \tilde{H}, \Omega)$ satisfies (A) and (B) above.*[10]

We shall call such an $\tilde{\omega}$ a double KMS state.

Notes

(1) In our 'finite system' example, ω on $\mathcal{B}(\mathcal{H})$ is $\operatorname{tr}(\rho\,\cdot)$, $\tilde{\omega}(A\otimes B)$ is $\langle\Omega \mid A\otimes B\Omega\rangle$ where Ω is defined in §1. The GNS triple may be taken to be $(\tilde{\rho}, \mathcal{H}\otimes\mathcal{H}, \Omega)$ where $\tilde{\rho}$ is the identity representation. $\tilde{H} = \mathbb{1}\otimes H - H\otimes\mathbb{1}$. J (see footnote 9) is given by $J(\psi_i\otimes\psi_j) = \psi_j\otimes\psi_i$.

(2) The essential innovation in the above discussion is that for any KMS state on $(\mathfrak{A}, \alpha(t))$ the 'modular involution' J (see footnote 9) always arises as the implementor of the same involution ι which we regard as a fixed attribute of our underlying dynamical system.

(3) In the Bisognano–Wichmann situation [2], we may identify $\tilde{\mathfrak{A}}$ with the spacelike double wedge algebra of a quantum field on Minkowski space-time (with $\mathfrak{A}^L$ the left wedge subalgebra, and $\mathfrak{A}^R$ the right wedge subalgebra), $\tilde{\alpha}(t)$ with the action of the double-wedge-preserving Lorentz boosts and ι with the $P'CT$ operation (P' = wedge reflection). The Minkowski vacuum – restricted to the double wedge – is then a double KMS state on $\tilde{\mathfrak{A}}$ with $\beta = 2\pi$, and the properties (A) and (B) above correspond respectively to the Reeh–Schlieder property for the right wedge, and to the Bisognano–Wichmann Theorem (see Theorem 1 in the first article of [2] or Theorem 3 in [15]). In a sense, what we are doing here is to give an exposition of some basic facts about KMS states in which the Bisognano–Wichmann situation may be seen as typical of a generic KMS state.

Finally, for any double KMS state $\tilde{\omega}$ on any double dynamical system $(\tilde{\mathfrak{A}}, \tilde{\alpha}(t), \iota)$ (whether or not we view it as the extension of an ω on an $(\mathfrak{A}, \alpha(t))$ we have the following

Theorem 1.[11] (i) *The von-Neumann algebras generated by $\mathfrak{A}^R$ and $\mathfrak{A}^L$ in the $\tilde{\rho}_\beta$-representation are commutants – i.e. $\tilde{\rho}_\beta(\mathfrak{A}^L)'' = \tilde{\rho}_\beta(\mathfrak{A}^R)'$. If, in addition Ω is the*

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[10] That (A) and (B) imply the KMS condition is immediate by footnote 7. To prove the converse, start with the formulation of footnote 7 and *define* for $A^L \in \mathfrak{A}^L$, $A^R \in \mathfrak{A}^R$

$$\tilde{\rho}_\beta(A^L\otimes A^R) = J\rho_\beta(\iota A^L)J\rho_\beta(A^R)$$

($= \rho_\beta(A^R)J\rho_\beta(\iota A^L)J$ by the theorem mentioned in footnote 7). (Notice that $J\rho_\beta(\iota A^L)J$ and $\rho_\beta(A^R)$ may possibly have non-zero overlap.)

[11] This theorem holds in the C^* algebra case (as proved here) or – with a suitable interpretation for $\tilde{\rho}_\beta(\mathfrak{A}^R)'$, $\tilde{\rho}_\beta(\mathfrak{A}^R)''$ etc. for more general algebras. The proof of (i) given here is adapted from Rigotti [15] (see especially Rigotti's Theorems 4–7) which proved a special case of (i) (duality of wedge algebras in the Bisognano–Wichmann situation – see Note 7 in text) for certain algebras of unbounded operators. For other proofs of (i) alone which do not use Tomita Takesaki theory, see [6] in the C^* algebra case and e.g. the proof of Theorem 2 in the first article cited in [2] for more general algebras. Rigotti's method adapts nicely to proving (i), (ii), (iii) together.

only eigenvector of $\tilde{H}$ with eigenvalue zero – i.e. the condition

$$(\alpha)\ \forall\Psi\in\mathscr{D}(\tilde{H}),\qquad \tilde{H}\Psi=0\Rightarrow\Psi=\lambda\Omega$$

holds, then we also have

(ii) $\tilde{\rho}_\beta(\mathfrak{A}^R)''$, $\tilde{\rho}_\beta(\mathfrak{A}^L)''$ *are factors – i.e.* $\tilde{\rho}_\beta(\mathfrak{A}^R)'\cap\tilde{\rho}_\beta(\mathfrak{A}^R)''=\{\lambda\mathbb{1}\}$ *(and similarly for* $R\to L$*). and*

(iii) $\tilde{\rho}_\beta(\tilde{\mathfrak{A}})$ *is irreducible – i.e.* $\tilde{\omega}$ *is a pure state.*

Further notes

(4) In the presence of (i), (ii) and (iii) are of course equivalent.

(5) We can now clarify the link between our discussion and the existing literature on purification [10–13]. Woronowicz [12] defines a purification map $\omega\mapsto\tilde{\omega}$ for arbitrary factor states on C^* algebras. Our doubling map $\omega\mapsto\tilde{\omega}$ for KMS states reduces to Woronowicz' purification for KMS factor states. (The "exactness" of $\tilde{\omega}_\beta$ corresponds to (i) of Theorem 1. For 'j-positivity' see footnote 6 and calculate

$$\begin{aligned}\tilde{\omega}_\beta(A^{0*}\otimes A)&=\tilde{\omega}_\beta([\iota(\mathbb{1}\otimes A)][\mathbb{1}\otimes A])\\&=\langle\rho_\beta(\mathbb{1}\otimes A)\Omega\mid e^{-\beta\tilde{H}/2}\rho_\beta(\mathbb{1}\otimes A)\Omega\rangle\geqslant 0)\end{aligned}$$

From this point of view, what we have done above is

(a) to explain how the KMS property for a state ω_β may be expressed in terms of properties of its doubling $\tilde{\omega}_\beta$.

(b) to give a simple sufficient condition (Condition (α) in Theorem 1) for when ω_β is a factor state, in which case the doubling map constitutes a purification (Theorem 1 parts (ii) and (iii).)

(6) For the quasi-free Bose states to be discussed in §3, we shall see that parts (i), (ii), (iii) of Theorem 1 hold without the need for Condition (α).

(7) In the Bisognano–Wichmann situation (see note (3) above) (i), (ii), (iii) correspond respectively to the duality of the left and right wedge algebras, the factor property for single wedge algebras, and the irreducibility of the double wedge algebra.

Proof of Theorem 1.[11]) It is straightforward to extend $\tilde{\alpha}(t)$ to $e^{iHt}\cdot e^{-iHt}$ which maps $\tilde{\rho}_\beta(\mathfrak{A})''\to\tilde{\rho}_\beta(\mathfrak{A})''$ with $\tilde{\rho}_\beta(\mathfrak{A}^R)''\to\tilde{\rho}_\beta(\mathfrak{A}^R)''$ and $(R\to L)$. Similarly, ι is extended by $J\cdot J$ (see footnote 9) with $\tilde{\rho}_\beta(\mathfrak{A}^R)''\to\tilde{\rho}_\beta(\mathfrak{A}^L)''$ and $R\leftrightarrow L$). Since $[\mathfrak{A}^R,\mathfrak{A}^L]=0$, we easily have $\rho_\beta(\mathfrak{A}^R)''\subset\rho_\beta(\mathfrak{A}^L)'$ and so since Ω is cyclic for $\rho_\beta(\mathfrak{A}^L)$ (or for $\rho_\beta(\mathfrak{A}^L)''$) it is also separating for $\rho_\beta(\mathfrak{A}^R)''$ and hence by $L\leftrightarrow R$ it is cyclic and separating for each of $\rho_\beta(\mathfrak{A}^R)''$, $\rho_\beta(\mathfrak{A}^L)''$. The next step is to show (β) that property (B) extends to $\rho(\mathfrak{A}^R)''$, $\rho(\mathfrak{A}^L)''$ – i.e. that $\rho(\mathfrak{A}^R)''\subset\mathscr{D}(e^{-\beta\tilde{H}/2})$ and

$$e^{-\beta\tilde{H}/2}X\Omega=JX^*\Omega\ \forall X\in\rho(\mathfrak{A}^R)''\qquad(\beta)$$

and (γ) that moreover $\rho(\mathfrak{A}^R)''\Omega$ is a core for $e^{-\beta\tilde{H}/2}$ (and similarly for $R\to L$ and $e^{-\beta\tilde{H}/2}\to e^{\beta\tilde{H}/2}$).

First note that by Rigotti's lemma (§A2), $\rho(\mathfrak{A}^R)\Omega$ is a core for $e^{-\beta\tilde{H}/2}$. (β) then follows by a simple calculation showing $\langle X\Omega \mid e^{-\beta\tilde{H}/2}\Psi\rangle = \langle JX^*\Omega \mid \Psi\rangle\ \forall\Psi \in \rho(\mathfrak{A}^R)\Omega$, (use X commutes eith JY^*J for $y \in \rho(\mathfrak{A}^R)$.) and ($\gamma$) is then immediate. We may now conclude by (β) that $Je^{-\beta\tilde{H}/2} \restriction_{\rho(\mathfrak{A}^R)''\Omega}$ may be identified with the operator S associated by Tomita's theorem (§A1) to the pair $(\rho(\mathfrak{A}^R)'', \Omega)$, and by ($\gamma$) that $Je^{-\beta\tilde{H}/2} = \bar{S}$. Therefore, by the uniqueness of the polar decomposition, J may be identified with the J of Tomita's theorem, and $e^{-\beta\tilde{H}/2}$ with $\Delta^{1/2}$. By part (iii) of Tomita's theorem, we then have $J\rho(\mathfrak{A}^R)''J = \rho(\mathfrak{A}^R)'$ which gives part (i) of our theorem since, as we have already observed, $J\rho(\mathfrak{A}^R)''J = \rho(\mathfrak{A}^L)''$.

Now, assume Condition (α) and let $X \in \tilde{\rho}(\mathfrak{A}^R)' \cap \tilde{\rho}(\mathfrak{A}^R)''$. Then, by (i) $X \in \tilde{\rho}(\mathfrak{A}^L)'' \cap \tilde{\rho}(\mathfrak{A}^R)''$ so that using (β) above, $X\Omega \in \mathscr{D}(e^{-\beta\tilde{H}/2}) \cap \mathscr{D}(e^{\beta\tilde{H}/2})$ and $e^{-\beta\tilde{H}/2}X\Omega = e^{\beta\tilde{H}/2}X\Omega$ whereupon $X\Omega \in \mathscr{D}(e^{\beta\tilde{H}})$ and $e^{\beta\tilde{H}}X\Omega = X\Omega$ – i.e. $X\Omega$ is an eigenvector of $\tilde{H}$ with eigenvalue 0 and hence by Condition (α) $X\Omega = \lambda\Omega$, whereupon, since Ω is separating (say for $\rho(\mathfrak{A}^R)''$) $X = \lambda$. This proves (ii). Finally, for (iii), let $X \in \rho(\tilde{\mathfrak{A}})''$. Then, $X \in \rho(\mathfrak{A}^R)' \cap \rho(\mathfrak{A}^L)' = \rho(\mathfrak{A}^R)' \cap \rho(\mathfrak{A}^R)'' = \{\lambda\mathbb{1}\}$.

§3. The quasi-free Bose case

In this section, we explain how the ideas of §2 work out in detail for a class of KMS states on linear Bose systems. We assume familiarity with [7] (which we shall refer to from now on as *I*) and §A3, §A4 which review some further properties of second quantization not mentioned in *I*. Note that we shall reserve the symbol $\hbar$ here to denote one particle Hilbert spaces (denoted by $\mathscr{H}$ in *I*).

In *I*, we discussed a class of quantum dynamical systems $(\mathfrak{A}, \alpha(t))$ where $\mathfrak{A}$ arose as the Weyl algebra (equation 2.2 of *I*) $\mathscr{W}(D, \sigma)$ over some linear symplectic space (D, σ) and $\alpha(t)$ was defined by $\alpha(t)W(\Phi) = W(\mathscr{T}(t)\Phi)$ where $\mathscr{T}(t)$ was a given one parameter group of linear symplectic transformations on (D, σ). In the remainder of this chapter, we shall assume we are given an $(\mathfrak{A}, \alpha(t))$ of this form. In this case, a double system $(\tilde{\mathfrak{A}}, \tilde{\alpha}(t), \iota)$ extending $(\mathfrak{A}, \alpha(t))$ may be obtained from a double (classical) linear dynamical system $(\tilde{D}, \tilde{\sigma}, \tilde{\mathscr{T}}(t), \mathscr{I})$ (see Definition 2 of §5.1 in *I*) extending $(D, \sigma, \mathscr{T}(t))$ by setting $\tilde{\mathfrak{A}} = \mathscr{W}(\tilde{D}, \tilde{\sigma})$, $\tilde{\alpha}(t)W(\Phi) = W(\tilde{\mathscr{T}}(t)\tilde{\Phi})$, $\iota W(\tilde{\Phi}) = W(\mathscr{I}\Phi)$. A standard candidate for such an extension – depending on a choice of preferred antisymplectic (i.e. $(T\Phi, T\Psi) = -\sigma(\Phi, \Psi)$) involution T reversing the sense of time (i.e. $T\mathscr{T}(t) = \mathscr{T}(-t)T$) on (D, σ) – may be inferred from the construction of §5.1 of *I*: It amounts to setting (cf §2) $\tilde{\mathfrak{A}} = \mathfrak{A} \otimes \mathfrak{A}$, $\tilde{\alpha}(t) = \alpha(-t) \otimes \alpha(t)$ and $\iota(A \otimes B) = \ell B \otimes \ell A$ where $\ell W(\Phi) = W(T\Phi)$.

Now fix β and assume the existence of a double KMS one-particle structure $(\tilde{K}^\beta, \tilde{\hbar}, e^{-it\tilde{h}}, j)$ defined (up to equivalence – see Theorem 2 of *I*) over $(\tilde{D}, \tilde{\sigma}, \tilde{\mathscr{T}}(t), \mathscr{I})$ by Definition 3 of *I*. Then $\tilde{\omega}_\beta$ – defined by $\tilde{\omega}_\beta(W(\Phi)) = \exp(-\frac{1}{2}\|\tilde{K}^\beta\Phi\|^2_{\tilde{\hbar}})$ is a double KMS state over $(\tilde{\mathfrak{A}}, \tilde{\alpha}(t), \iota)$ in the sense of §2. We shall sometimes call this $\tilde{\omega}_\beta$ the *standard double KMS state over* $(\tilde{\mathfrak{A}}, \tilde{\alpha}(t), \iota)$. For this state, we may identify $\rho_\beta(W(\tilde{\Phi}))$ on $\tilde{\mathscr{H}}$ with (see §2.3 of *I* and §A2, §A3) $W^{\mathscr{F}}(\tilde{K}^\beta\Phi)$ on $\mathscr{F}(\tilde{\hbar})$, $\tilde{H}$ with $d\Gamma(\tilde{h})$, and Ω with $\Omega^{\mathscr{F}}$. In particular, conditions (1), (3), (5) of Definition 3 of *I* imply $\tilde{\omega}_\beta$ is an $\tilde{\alpha}(t)$- and ι-invariant state, condition (2) implies property (A), and condition (6) property (B).

In the case we begin with an $(\mathfrak{A}, \alpha(t))$ arising from a $(D, \sigma, \mathscr{T}(t))$ which admits a ground one particle structure (see §2 of *I*) $(K, \hbar, e^{-iht})$ satisfying in addition the 'regularity' condition $KD \subset \mathscr{D}(h^{-1/2})$, then we showed in I that we can always construct a double KMS one particle structure $(\tilde{K}, \tilde{\mathscr{k}}, e^{-i\tilde{h}t}, j)$ over any $(\tilde{D}, \tilde{\sigma}, \tilde{\mathscr{T}}(t), \mathscr{I})$ extending $(D, \sigma, \mathscr{T}(t))$ and hence as described above an $\tilde{\omega}_\beta$ on any $(\tilde{\mathfrak{A}}, \tilde{\alpha}(t), \iota)$ extending $(\mathfrak{A}, \alpha(t))$. Moreover, the details of that construction imply a simple relationship between $\tilde{\omega}_\beta$ and the ground state ω_0 on $(\mathfrak{A}, \alpha(t))$ (defined by $\omega_0(W(\Phi)) = \exp(-\frac{1}{2}\|K_0\Phi\|^2_{\mathscr{k}})$) which we now explain: Take for simplicity[12]) a standard candidate for $(\tilde{\mathfrak{A}}, \tilde{\alpha}(t), \iota)$ as mentioned earlier arising from a T which is implemented at the one-particle level by a complex conjugation C (i.e. $KT = CK$) so that $[C, h] = 0$. Define $\tilde{\omega}_0$ on $\tilde{\mathfrak{A}} = \mathfrak{A} \otimes \mathfrak{A}$ by $\tilde{\omega}_0(A \otimes B) = \omega_0(A)\omega_0(B)$. (The corresponding GNS representation is $\tilde{\rho}_0(W(\Phi) \otimes W(\Phi)) = W^{\mathscr{F}}(K\Phi) \otimes W^{\mathscr{F}}(K\Psi)$ on $\mathscr{F}(\mathscr{k}) \otimes \mathscr{F}(\mathscr{k})$, with cyclic vector $\Omega = \Omega^{\mathscr{F}} \otimes \Omega^{\mathscr{F}}$; $\tilde{\alpha}(t)$ is implemented by $\exp(itd\Gamma(h)) \otimes \exp(-itd\Gamma(h))$ and J which implements ι is given by $J(x \otimes y) = \Gamma(-C)y \otimes \Gamma(-C)x$.) Extend $\tilde{\omega}_0$ by $\omega_0'' = \langle \Omega | . \Omega \rangle$ to the von-Neumann algebra $\rho_0(\tilde{\mathfrak{A}})''$ $(= \mathscr{B}(\mathscr{F}(\mathscr{k}) \otimes \mathscr{F}(\mathscr{k})))$ generated by $\tilde{\rho}_0(\tilde{\mathfrak{A}})$. Also extend $\tilde{\alpha}(t)$ to $\tilde{\alpha}(t)'' = \exp(-itd\Gamma(h)) \otimes \exp(itd\Gamma(h)) \cdot \exp(itd\Gamma(h)) \otimes \exp(-itd\Gamma(h))$ and ι to $\iota'' = J \cdot J$. Then, the discussion in §5 of I tells us that the state $\tilde{\omega}_\beta$ arises as $\tilde{\omega}_0'' \circ \tau_\beta$ on $\tilde{\mathfrak{A}}$ where τ_β is the isomorphism $\tilde{\mathfrak{A}} \to \rho_0(\tilde{\mathfrak{A}})''$ defined by (recalling the isomorphism $\mathscr{W}^{\mathscr{F}}(h) \otimes \mathscr{W}^{\mathscr{F}}(h) \cong \mathscr{W}(h \oplus h)$) $\tau_\beta(W(\Phi) \otimes W(\Psi)) = W^{\mathscr{F}}(\mathscr{T}_\beta(-K\Phi \oplus K\Psi))$ where $\mathscr{T}_\beta$ is the possibly unbounded Bogolubov transformation from $KD \oplus KD$ to $\mathscr{k} \oplus \mathscr{k}$ (writing $-K\Phi \oplus K\Psi$ as a column vector)

$$\mathscr{T}_\beta = \begin{pmatrix} chZ^\beta & CshZ^\beta \\ CshZ^\beta & chZ^\beta \end{pmatrix}$$

where $\tan hZ^\beta = \exp(-\beta h/2)$
(The condition $KD \subset \mathscr{D}(h^{-1/2})$ guarantees $KD \oplus KD \subset \mathscr{D}(\mathscr{T}_\beta)$ – see §A2 of I.) Note that $\tau_\beta \circ \iota = \iota'' \circ \tau_\beta$ and $\tau_\beta \circ \alpha(t) = \alpha(t)'' \circ \tau_\beta$ so that for *any* $0 < \beta < \infty$, the implementors of $\tilde{\alpha}(t)$ and ι in $\tilde{\rho}_\beta$ are the same as those quoted above for (the $\beta = \infty$ case) $\tilde{\rho}_0$. That is, we always have $\tilde{H} = -d\Gamma(h) \otimes \mathbb{1} \oplus \mathbb{1} \otimes d\Gamma(h)$ and $J(x \otimes y) = \Gamma(-C)y \otimes \Gamma(-C)x$.

Much of the above discussion can be pieced together in one way or another from the literature on quasi-free states of the late '60's and early '70's. However, the present discussion contains a number of improvements. One is the weakness of our condition $KD \subset \mathscr{D}(h^{-1/2})$ (which is weaker than demanding that h has a mass gap). Another is the need (in the case $\mathscr{T}_\beta : KD \oplus KD \mapsto KD \oplus KD$ to view τ_β as an isomorphism $\tilde{\mathfrak{A}} \to \rho_0(\tilde{\mathfrak{A}})''$ (see §A4) and not as an automorphism $\tilde{\mathfrak{A}} \to \tilde{\mathfrak{A}}$ (cf [11]). Another improvement is that we do not require KD to be complex linear. See I for further discussion. These technical improvements are essential for the constructions in [8].

Finally, we explain how for the class of states introduced here, Theorem 1 may be strengthened by proving an analogous result at the one-particle level and

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[12]) It is straightforward to generalize this discussion to any $(\tilde{\mathfrak{A}}, \tilde{\alpha}(t), \iota)$ which arises from some $(\tilde{D}, \tilde{\sigma}, \tilde{\mathscr{T}}(t), \mathscr{I})$ s.t. $(D^R, \sigma, \mathscr{T}(t))$ (with $\sigma = \tilde{\sigma}\restriction_{D^R}$, $\mathscr{T}(t) = \mathscr{T}(t)\restriction_{D^R}$) admits a ground one-particle structure $(K, \mathscr{k}, e^{-iht})$ satisfying $KD \subset \mathscr{D}(h^{-1/2})$. Simply choose for C any complex conjugation s.t. $[C, h] = 0$, modify the definition of $\tilde{\omega}_0$ to read $\tilde{\omega}_0(A^L \otimes A^R) = \omega_0(\mathscr{I}A^L)\omega_0(A^R)$, $A^L \in \mathfrak{A}^L$, $A^R \in \mathfrak{A}^R$ and modify the definition of τ_β to read

$$\tau_\beta(W(\tilde{\Phi})) = W^{\mathscr{F}}(\mathscr{T}^\beta(-CK\mathscr{I}\Phi^L \oplus K\Phi^R))$$

where $\tilde{\Phi} = \Phi^L \oplus \Phi^R$, $\Phi^L \in D^L$, $\Phi^R \in D^R$, etc.

then second quantizing. Again assume we are given a double KMS one-particle structure $(\tilde{K}^\beta, \tilde{\mathscr{k}}, e^{-it\tilde{h}}, j)$ over a double classical linear dynamical system $(\tilde{D}, \tilde{\sigma}, \tilde{\mathscr{T}}(t), \mathscr{I})$ as in *I*. Then, defining the (not necessarily closed) real linear subspaces $R = \tilde{K}^\beta D^R$, $L = \tilde{K}^\beta D^L$ of $\tilde{\mathscr{k}}$, and denoting, for any real linear subspace M of $\tilde{\mathscr{k}}$ the symplectic complement by M' (see §A4) we have

Theorem 2
(i) $\bar{L} = R'$, $\bar{R} = L'$
(ii) $\bar{R} \cap R' = \{0\}$, $\bar{L} \cap L' = \{0\}$
(iii) $R + L$ $(= \tilde{K}^\beta \tilde{D})$ is dense in $\tilde{\mathscr{k}}$.

Notes

(1) In the presence of (i), (ii) and (iii) are, of course, equivalent
(2) We immediately obtain the Corollary to Theorem 2: The resulting standard double KMS state $\tilde{\omega}_\beta$ always possesses properties (i), (ii), (iii) of Theorem 1. (For the proof, use (iv), (vi) and (v) of Theorem in §A4 respectively.) Note that this is a stronger result for these quasi-free Bose states than could have been obtained by application of Theorem 1, since Condition (α) of Theorem 1 is *not* implied by the condition ((4) of Definition 3 of I) that $\tilde{h}$ has no zero eigenvalues.

Proof. We mimic the proof given above for Theorem 1, substituting the Tomita theorem of §A1 with the pre-Tomita theorem of §A5. First note some immediate consequences of Definition 3 of *I*:

(a) $e^{-it\tilde{h}}$: $R \to R$, $L \to L$
(b) $[e^{-it\tilde{h}}, j] = 0$ on $\tilde{K}^\beta \tilde{D} = R + L$
(c) $jR = L$, $jL = R$
(d) $\bar{R} + i\bar{R}$ is dense in $\tilde{\mathscr{k}}$, $\bar{L} + i\bar{L}$ is dense in $\tilde{\mathscr{k}}$ (pre-cyclicity, cf. §A4)
(e) $\bar{L} \subset R'$, $\bar{R} \subset L'$
(f) $\bar{R} \cap i\bar{R} = \{0\}$, $(\bar{L} \cap i\bar{L}) = \{0\}$
(pre-separating property, cf. §A4)

((f) follows from (d) and (e) on taking symplectic complements in (d).)

The next step is to show that Property (6) of Definition 3 of *I* holds with R replaced by $\bar{R}$ and L replaced by $\bar{L}$. This follows on using Rigotti's Lemma (§A2) to conclude that $R + iR$ is a core for $e^{-\beta\tilde{h}/2}$ and $L + iL$ is a core for $e^{\beta\tilde{h}/2}$. Now, take $\bar{R}$ ($\bar{L}$ is similar). From (d) and (f) above, we know that $\bar{R}$ satisfies the hypotheses of the pre-Tomita theorem (§A5). Moreover, from our strengthened Property (6) we see that $je^{-\beta\tilde{h}/2} = \mathscr{s}$ where $\mathscr{s}$ is as in the pre-Tomita Theorem (§A5). By the uniqueness of the polar decomposition, we may therefore identity j with the j of pre-Tomita, and $e^{-\beta\tilde{h}}$ with δ. By part (iii) of the pre-Tomita theorem, we may then obtain $\bar{L} = j\bar{R} = R'$ (and similarly, or by taking symplectic complements $\bar{R} = L'$) which is part (i) of our theorem. Now, let $x \in \bar{R} \cap R'$, then by (i), $x \in \bar{R} \cap \bar{L}$ so again by our strengthened Property (6), we must have $x \in \mathscr{D}(e^{\beta\tilde{h}})$ and $e^{\beta\tilde{h}}x = x$ whereupon x is an eigenvector with eigenvalue zero of $\tilde{h}$ and hence by Property (4), $x = 0$. This gives (ii) of our theorem. (iii) now follows from (i) and (ii) on taking symplectic complements.

Appendix

We recall some background material needed to make the proofs of Theorems 1 and 2 self-contained. §§A3, A4 will also be generally helpful for §3.

§A1. *Tomita's theorem*

Let $\mathcal{A}$ be a von-Neumann algebra and Ω a cyclic and separating vector for $\mathcal{A}$. Define the real-linear operator

$$S : A\Omega \to A^*\Omega$$

Then S is closable and in the polar decomposition $\bar{S} = J\Delta^{1/2}$ of $\bar{S}$, we have

(i) *J is an antiunitary involution*
(ii) *$\Delta^{1/2}$ is complex-linear (positive)*
(iii) *$J\mathcal{A}J = \mathcal{A}'$*
(iv) *$\Delta^{it} \cdot \Delta^{-it} : \mathcal{A} \to \mathcal{A}, \mathcal{A}' \to \mathcal{A}'$*

§A2. *Rigotti's Lemma* [15]

Let K be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Let f be a real Borel function bounded on the compacts. If $\Delta \subset D(f(K))$ is a dense subspace invariant for the group e^{-itK}, then Δ is a core for $f(K)$.

More About Second Quantization [16, 17, 18]

The next two sections continue the discussion begun in §§2–3 of *I* (ref. [7]) concerning the Fock representation triple $(W \to W^{\mathcal{F}}, \mathcal{F}(\hbar), \Omega^{\mathcal{F}})$ of Weyl operators over a complex ('one-particle') Hilbert space $\hbar$ (characterized by $\langle \Omega^{\mathcal{F}} \mid W^{\mathcal{F}}(x)\Omega^{\mathcal{F}} \rangle_{\mathcal{F}(\hbar)} = \exp(-\frac{1}{2}\|x\|_{\hbar}^2)$ and the second quantization map Γ from operators on $\hbar$ to operators on $\mathcal{F}(\hbar)$ which satisfies, for unitaries and antiunitaries, $W^{\mathcal{F}}(Ux) = \Gamma(U)W^{\mathcal{F}}(x)\Gamma(U)^{-1}$ and $d\Gamma(A) \geq 0$ for $A > 0$ where $\Gamma(e^{iA}) = \exp(\mathrm{i}d\,\Gamma(A))$.

§A3

Given two one-particle Hilbert spaces $\hbar_1$ and $\hbar_2$, there is a natural isomorphism $\mathcal{F}(\hbar_1 \oplus \hbar_2) \approx \mathcal{F}(\hbar_1) \otimes \mathcal{F}(\hbar_2)$ in such a way that $W(x_1 \oplus x_2)$ corresponds to $W(x_1) \otimes W(x_2)$. In the special case where $\hbar_1 = \hbar_2$, we can write operators on $\hbar \oplus \hbar$ as 2×2 matrices of operators on h. Then under this isomorphism, we have $\Gamma\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \approx \Gamma(A) \otimes \Gamma(B)$ and $\Gamma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Xi$ where $\Xi(\alpha \otimes \beta) = \beta \otimes \alpha$ for $\alpha, \beta \in \mathcal{F}(\hbar)$.

§A4

We quote some results from the very nice discussion of Leyland, Roberts, Testard [18] on the von-Neumann algebra $\mathcal{W}^{\mathcal{F}}(\hbar)''$ generated by the $W^{\mathcal{F}}(x)$ for

x's in an arbitrary one-particle Hilbert space $\mathscr{h}$. These results concern subalgebras $\mathcal{W}^{\mathcal{F}}(M)''$ of $\mathcal{W}^{\mathcal{F}}(\mathscr{h})''$ generated by real subspaces M of $\mathscr{h}$. We use the notation (as usual) $\mathscr{A}'$ to denote the commutant when $\mathscr{A}$ is an algebra of operators, and (without risk of confusion) M' to denote the symplectic complement when M is a real subspace of $\mathscr{h}$. So $x \in M' \Leftrightarrow \operatorname{Im}\langle x \mid y\rangle = 0\ \forall y \in M$. Considering $\mathscr{h}$ as a real Hilbert space with inner product $\operatorname{Re}\langle\cdot \mid \cdot\rangle$, M' is of course just $iM^{\perp}$. Note that $M'' = \bar{M}$ and $(M+iM)' = M' \cap iM'$. More generally, we have $(M+N)' = M' \cap N'$ for arbitrary real subspaces M, N.

Theorem (= Theorem I.3.2 of [18]). *Let $\mathscr{h}$ be a complex Hilbert space, M, N real subspaces of $\mathscr{H}$ and $\bar{M}$ the closure of M etc. Then*

(i) $\mathcal{W}^{\mathcal{F}}(M)'' = \mathcal{W}^{\mathcal{F}}(\bar{M})$

(ii) *Ω is cyclic for $\mathcal{W}^{\mathcal{F}}(M)''$ if and only if $M+iM$ is dense in $\mathscr{h}$.*

(iii) *Ω is separating for $\mathcal{W}^{\mathcal{F}}(M)''$ if and only if $\bar{M} \cap i\bar{M} = \{0\}$.*

(iv) $\mathcal{W}^{\mathcal{F}}(M)'' = \mathcal{W}^{\mathcal{F}}(M')'$

(v) $(\mathcal{W}^{\mathcal{F}}(M)'' \cup \mathcal{W}^{\mathcal{F}}(N)'')'' = \mathcal{W}^{\mathcal{F}}(M+N)''$

(vi) *$\mathcal{W}^{\mathcal{F}}(M)'' \cap \mathcal{W}^{\mathcal{F}}(N)'' = \mathcal{W}^{\mathcal{F}}(\bar{M} \cap \bar{N})''$ and consequently $\mathcal{W}^{\mathcal{F}}(M)''$ is a factor if and only if $\bar{M} \cap M' = \{0\}$.*

Note that (iv) is the so-called abstract duality property originally isolated and proved by Araki [19]. (See end of §A5.)

§A5. *Pre-Tomita theorem*

Let M be a closed real subspace of a complex Hilbert space $\mathscr{h}$ s.t. $M+iM$ is dense and $M \cap iM = \{0\}$. Define

$$\mathfrak{s} : M+iM \to M+iM$$

$$\alpha + i\beta \to -\alpha + i\beta$$

Then $\mathfrak{s}$ is closable and in the polar decomposition $\bar{\mathfrak{s}} = j\delta^{1/2}$ of $\bar{\mathfrak{s}}$ (on ($\mathscr{h}$, $\operatorname{Re}\langle\cdot \mid \cdot\rangle$)) we have

(i) *j is an antiunitary involution*

(ii) *$\delta^{1/2}$ is complex linear, (positive)*

(iii) $jM = M'$

(iv) $\delta^{it} : M \to M,\ M' \to M'$.

For the proof see [18] or [20]. Note that for the Fock representation of the Weyl algebra over a complex Hilbert space $\mathscr{h}$, the results in §A4 link together the pre-Tomita theorem with the full Tomita theorem. (Thus consider $(\mathcal{W}^{\mathcal{F}}(M)'', \Omega^{\mathcal{F}})$ for an M as in Pre-Tomita. Then by §A4, $\Omega^{\mathcal{F}}$ is cyclic and separating and we may identify: $S = \Gamma(\mathfrak{s})$, $J = \Gamma(j)$ $\Delta^{it} = \Gamma(\delta^{it})$.) (In fact, in [18] one uses the Tomita and pre-Tomita theorems to prove the theorem in §A4.)

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