Zeitschrift: Helvetica Physica Acta
Band: 58 (1985)
Heft: 6

Artikel: Gauge invariant fields in nonabelian gauge theories
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DOI: https://doi.org/10.5169/seals-115631

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Helvetica Physica Acta, Vol. 58 (1985) 995–1003

0018–0238/85/060995–09$1.50 + 0.20/0
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Gauge invariant fields in nonabelian gauge theories

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(26. IV. 1985; rev 24. VI. 1985)

Abstract. Fields which carry gauge charges but are invariant under localized gauge transformations are constructed for nonabelian gauge theories without confinement. In a quantized theory these fields can serve to generate the physical states out of the vacuum.

1. Introduction

In a complete quantum field theory there exists a set of basic fields such that the observables are functions of these fields, and a dense set of physical states can be obtained from a 'vacuum state' Ω by application of polynomials in them. In an economical formulation *only* physical states are obtained in this way, and no unobservable degrees of freedom like gauge freedoms are present. In a gauge field theory this requirement amounts to a choice of basic fields which are invariant at least under localized gauge transformations.

A formulation of QED in terms of gauge invariant fields has been given by Mandelstam [1, 2]. His basic fields are the electromagnetic field tensor $F_{\mu\nu}$ and the non-local[1]) string

$$\Psi(x) = \exp\left\{-ie\int_x^\infty d\xi^\mu A_\mu(\xi)\right\}\psi(x), \tag{1.1}$$

where A_μ, ψ, are the electromagnetic potential and the electron field of a conventional gauge-dependent formulation, e.g. a local Gupta–Bleuler gauge. The integral in the exponent runs over a one-dimensional path extending from x to infinity. Ψ depends on the choice of path, but this dependence is physically meaningful. Equations of motion for these fields can be written down and solved perturbatively. Mandelstam's treatment was, however, formal insofar as he did not discuss possible divergences. Indeed, it turns out [4] that the Wightman and Green functions of such a string field develop malignant and very likely incurable infrared divergences already in order e^2, due to the singular behaviour of the string at infinity.[2]) This difficulty can be avoided if (1.1) is replaced by the more

[1]) For the necessity of using non-local fields in physical-gauge formulations of any gauge theory see e.g. [3].

[2]) This behaviour leads to non-integrable singularities of the Feynman integrand in p-space along higher-dimensional manifolds, not only at the origin as in the familiar infrared problem.

general ansatz

$$\Psi(x) = \exp\left\{ie\int d^4y f^\mu(x-y)A_\mu(y)\right\}\psi(x), \tag{1.2}$$

where the f^μ are real functions or distributions with $\partial_\mu f^\mu(x) = \delta^4(x)$. A formulation of QED based on this field and $F_{\mu\nu}$ has been discussed in [4].

In trying to generalize these methods to nonabelian theories we must distinguish between theories with total confinement (the physical states carry no gauge charges) and those without or with only partial confinement. In the first case we demand the basic fields to be invariant under all gauge transformations, not only the localized ones. Fields of this type are the Wilson loops

$$\operatorname{Tr} P \exp\left\{-ie\oint d\xi^\mu A_\mu(\xi)\right\}$$

and the finite strings

$$\bar\psi(x) P \exp\left\{-ie\int_x^y d\xi^\mu A_\mu(\xi)\right\}\psi(y),$$

where A_μ, ψ, are again the gauge potentials and spinor fields of a gauge-dependent formulation, and P denotes path ordering. On the other hand, completely gauge invariant local fields can be introduced if the gauge symmetry is broken by the Higgs mechanism [5]. In the present paper we will be concerned with theories without confinement and without symmetry breaking. In order to generate charged states out of the vacuum we need charged fields, like in QED, i.e. fields which transform covariantly under gauge transformations of the first kind, even though they are invariant under localized transformations.

Let $\mathfrak{G}$ be the gauge group, a Lie group which we consider realized as a matrix group: $\mathfrak{G} \subset \mathfrak{M}_N$, with $\mathfrak{M}_N$ the algebra of (real or complex) $N\times N$-matrices. Let $\mathfrak{g}$ be the corresponding Lie algebra. Let $A_\mu(x)$, with $iA_\mu \in \mathfrak{g}$, be the gauge potentials in a suitable gauge, and

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig[A_\mu(x), A_\nu(x)] \tag{1.3}$$

the corresponding field strengths. Again we have $iF_{\mu\nu} \in \mathfrak{g}$. g is a coupling constant. The components of A_μ, $F_{\mu\nu}$, etc., will be treated classically, i.e. as c-number valued functions, not as operator valued distributions. Under the gauge transformation $U(x) \in \mathfrak{G}$ the fields transform as

$$\left.\begin{aligned} A_\mu(x) &\to U(x)A_\mu(x)U^{-1}(x) - \frac{i}{g}U(x)\,\partial_\mu U^{-1}(x) \\ F_{\mu\nu}(x) &\to U(x)F_{\mu\nu}(x)U^{-1}(x). \end{aligned}\right\} \tag{1.4}$$

$U(x)$ is called localized if $U(x) \Rightarrow 1$ sufficiently fast for $|x| \to \infty$. An x-independent gauge transformation U is called global. We will have no occasion to consider more general $U(x)$.

Let $V(x, A_\mu)$ be a x-dependent functional of A_μ with values in $\mathfrak{M}_N$, such that

$$U(x)V(x, A_\mu) = V\left(x, UA_\mu U^{-1} - \frac{i}{g}U\,\partial_\mu U^{-1}\right) \tag{1.5}$$

for localized $U(x)$ and

$$UVU^{-1} = V(x, UA_\mu U^{-1}) \tag{1.6}$$

for global U. Then

$$\mathfrak{F}_{\mu\nu}(x) = V^{-1}(x, A_\rho)F_{\mu\nu}(x)V(x, A_\sigma) \tag{1.7}$$

is invariant under localized gauge transformations, covariant (i.e. transforming like $F_{\mu\nu}$) under global ones. It is thus a candidate for one of our basic fields. If $\psi(x)$ is a field transforming as $\psi(x) \to U(x)\psi(x)$, then

$$\Psi(x) = V^{-1}(x, A_\rho)\psi(x) \tag{1.8}$$

is again invariant under localized transformations, covariant under global ones, and thus a possible basic field.

An explicit expression for V generalizing the Mandelstam string of QED has been given by Bialynicki-Birula [6]:

$$V(x, A_\mu) = P \exp \left\{ ig \int_x^\infty d\xi^\mu A_\mu(\xi) \right\} \tag{1.9}$$

where again the integral runs over a one-dimensional path and P denotes path ordering. The perturbative quantum theory of the resulting field $\mathfrak{F}$ has been developed by Mandelstam [2], but again in a formal way, without regard to possible divergences. Unfortunately the same severe IR-divergences as in QED are also present here. Therefore a nonabelian analogue of the more general ansatz (1.2) is desirable.

Such an analogue will be derived in the next section, in the form of a formal power series in A. We have no information on the convergence of this series, so that at present our result is only useful in a perturbative context. In Section 3 we will show that the V introduced in Section 2 is unitary, again in the sense of formal power series, if $\mathfrak{G}$ is one of the groups U_N, SU_N, O_N, SO_N, or a subgroup of one of these groups. This unitarity implies that

$$\mathrm{Tr}\,(\mathfrak{F}_{\mu\nu}(x)\mathfrak{F}^{\rho\sigma}(x)) = \mathrm{Tr}\,(F_{\mu\nu}(x)F^{\rho\sigma}(x)),$$

or, if ψ is a Dirac spinor and Γ an operator in spin space,

$$\bar{\Psi}(x)\Gamma\Psi(x) = \bar{\psi}(x)\Gamma\psi(x),$$

etc. Expressions like these are candidates for the observables of the theory, and are local fields by our results, since $F_{\mu\nu}$, ψ, can be chosen as fields in a local gauge.

Finally we will briefly discuss in Section 4 whether it is possible to formulate the dynamics of the model exclusively in terms of our invariant fields, in the sense that equations of motion containing only these fields can be written down, as is the case in QED. The answer seems to be yes for the string case (1.9), no for the more general case of Section 2.

The possible applications of our fields to perturbative quantum field theory have not yet been investigated. In particular, it is not known at present, whether the expected remaining infrared singularities (due to the singularity of h at the origin) will cancel between graphs of the same order, as is the case in QED. This is clearly the direction in which further work will have to proceed.

2. Construction of V

The notion of path ordering is not applicable to an extended object like the exponential in (1.2), so that some more complicated procedure is needed. We start from the observation that path ordering of the exponential (1.9) is usually defined via an expansion into a power series. This leads us to the ansatz

$$V(x, A_\mu) = 1 + \sum_{n=1}^{\infty} (-ig)^n \int \prod_{i=1}^{n} d^4u_i \, {}^nf^{\rho_1\cdots\rho_n}(x-u_1, \ldots, x-u_n) \times A_{\rho_1}(u_1)\cdots A_{\rho_n}(u_n). \tag{2.1}$$

The coefficients ${}^nf^{\cdots}$ are real valued functions or distributions which must be determined such that the condition (1.5) is satisfied. (1.6) obviously holds independently of the choice of ${}^nf$. We assume the behaviour of A_ρ and ${}^nf$ at infinity to be such that the integrals occurring in (2.1) and later on converge. As already noted in the introduction we will, however, not discuss the convergence of the n-sum, but treat (2.1) as a formal power series in A_ρ.

We use the notations $\rho = \{\rho_1, \ldots, \rho_n\}$, $\xi_i = x - u_i$, $\xi = \{\xi_1, \ldots, \xi_n\}$, so that the coefficients in (2.1) are written as ${}^nf^\rho(\xi)$.

Condition (1.5) is satisfied if it is satisfied for infinitesimal transformations: we can restrict ourselves to the case $U(x) = 1 + \varepsilon T(x)$ where $T(x) \in \mathfrak{g}$, $T(x) \to 0$ sufficiently fast for $|x| \to \infty$, and only terms of order ε are retained in our calculations. (1.5) becomes, using the ansatz (2.1):

$$T(x)\left\{1 + \sum_n (ig)^n \int \prod_1^n du_i \, {}^nf^\rho(\xi) \prod A_{\rho_i}(u_i)\right\}$$
$$= \sum_n (-ig)^n \int \prod_1^n du_i \, {}^nf^\rho(\xi) \sum_{\alpha=1}^{n} \left\{\prod_{i=1}^{\alpha-1} A_{\rho_i}(u_i)\left\{[T(u_\alpha), A_{\rho_\alpha}(u_\alpha)] + \frac{i}{g}\partial_\mu T(u_\alpha)\right\} \times \prod_{i=\alpha+1}^{n} A_{\rho_i}(u_i)\right\}. \tag{2.2}$$

The differentiation in $\partial_\mu T(u_\alpha)$ can be shifted to ${}^nf$ by integration by parts, the boundary term vanishing because of $T(\infty) = 0$. After that all the terms in (2.2) are of general form

$$\int \prod_1^\alpha du_i \, dv \prod_1^\beta dw_j c^{\rho\sigma}(x, u_i, v, w_j) \prod_1^\alpha A_{\rho_i}(u_i) T(v) \prod_1^\beta A_{\sigma_j}(w_j)$$

For equation (2.2) to hold it is sufficient that it holds separately for the terms with fixed α, β. This leads to the conditions

$$\partial^1_{\rho_1} \, {}^nf^\rho(\xi) = [\delta^4(\xi_1) - \delta^4(\xi_1 - \xi_2)] \, {}^{n-1}f^{\rho\backslash 1}(\xi\backslash 1), \tag{2.3$'$}$$
$$\partial^\alpha_{\rho_\alpha} \, {}^nf^\rho(\xi) = [\delta^4(\xi_\alpha - \xi_{\alpha-1}) - \delta^4(\xi_\alpha - \xi_{\alpha+1})] \, {}^{n-1}f^{\rho\backslash\alpha}(\xi\backslash\alpha), \qquad 1 < \alpha < n, \tag{2.3$''$}$$
$$\partial^n_{\rho_n} \, {}^nf^\rho(\xi) = \delta^4(\xi_n - \xi_{n-1}) \, {}^{n-1}f^{\rho\backslash n}(\xi\backslash n). \tag{2.3$'''$}$$

Here $\rho\backslash\alpha = \{\rho_1, \ldots, \rho_{\alpha-1}, \rho_{\alpha+1}, \ldots, \rho_n\}$ and analogously for $\xi\backslash\alpha$, and $\partial^\alpha_{\rho_\alpha}$ denotes differentiation with respect to $\xi^{\rho_\alpha}_\alpha$. The equations (2.3′) and (2.3‴) can be subsumed under the form (2.3″) if we agree that $\xi_0 \equiv 0$ and that terms containing ξ_{n+1} must be dropped. (2.3″) with this interpretation will be called equation (2.3). We have defined ${}^0f = 1$.

For $n=1$ (2.3) becomes $\partial_\mu\, {}^1f^\mu(\xi)=\delta^4(\xi)$, the equation that was already encountered in QED. For ${}^1f$ we can therefore choose ${}^1f^\mu=f^\mu$, where f^μ is any of the f's used in QED [4]. The same considerations on the relative merits of various possible choices apply here as well.

Starting from this ${}^1f$ we determine the higher ${}^nf$ recursively by solving the equations (2.3) for $n=2,3,\ldots$. Denote the right-hand side of (2.3) by ${}^nA^{\rho\backslash\alpha}(\xi)$. It is known if ${}^{n-1}f$ is known. An elementary calculation using (2.3) for the $(n-1)$-case shows that the integrability condition

$$\partial^\alpha_{\rho_\alpha}\, {}^nA^{\rho\backslash\beta}(\xi)=\partial^\beta_{\rho_\beta}\, {}^nA^{\rho\backslash\alpha}(\xi) \tag{2.4}$$

is satisfied.

The solution of (2.3) looks simpler in p-space than in x-space. Therefore we introduce the Fourier transforms

$${}^nh^\rho(p)=\int\prod_1^n d\xi_i e^{ip\xi}\, {}^nf^\rho(\xi), \tag{2.5}$$

$$\begin{aligned}{}^nB^{\rho\backslash\alpha}(p)&=\int\prod_1^n d\xi_i e^{ip\xi}\, {}^nA^{\rho\backslash\alpha}(\xi)\\ &={}^{n-1}h^{\rho\backslash\alpha}(\ldots,p_{\alpha-1}+p_\alpha,p_{\alpha+1},\ldots)-{}^{n-1}h^{\rho\backslash\alpha}(\ldots,p_{\alpha-1},p_\alpha+p_{\alpha+1},\ldots),\end{aligned} \tag{2.6}$$

where we have used the notations $p=\{p_1,\ldots,p_n\}$, $p\xi=\sum p_i\xi_i$. The last term in (2.6) is absent for $\alpha=n$. Equation (2.3) becomes

$$p^\alpha_{\rho_\alpha}\, {}^nh^\rho(p)=i\, {}^nB^{\rho\backslash\alpha}(p). \tag{2.7}$$

A solution is given by

$${}^nh^\rho(p)=-i\sum_L i^l\prod_{i\in L}h^{\rho_i}(p_i)\prod_{\substack{i\in L\\ i\neq\gamma_L}}p^i_{\sigma_i}\, {}^nB^{\rho\backslash\gamma_L,\rho_L\to\sigma_L}(p). \tag{2.8}$$

Here $h={}^1h$, the Fourier transform of f. The sum $\sum_L$ extends over all non-empty subsets L of the index set $\{1,\ldots,n\}$, l is the number of elements in L, γ_L is an arbitrarily chosen element of L, and the superscript $\rho\backslash\gamma_L,\rho_L\to\sigma_L$ means that in the index set $\rho\backslash\gamma_L$ the ρ_i with $i\in L$ are replaced by σ_i. Because of (2.4) the value of ${}^nh^\rho$ does not depend on the choice of γ_L.

In order to see that (2.8) solves (2.7) we first apply the derivation $p^\alpha_{\rho_\alpha}$ to a term in (2.8) with $L=L'\cup\{\alpha\}$, $\alpha\notin L'$, $L'\neq\emptyset$. We choose $\gamma_L\in L'$. Using $p^\alpha_{\rho_\alpha}h^{\rho_\alpha}(p_\alpha)=i$ we obtain

$$i^l\prod_{i\in L'}h^{\rho_i}(p_i)\prod_{\substack{i\in L\\ i\neq\gamma_L}}p^i_{\sigma_i}\, {}^nB^{\rho\backslash\gamma_L,\rho_L\to\sigma_L}.$$

This cancels the L'-term in $p^\alpha_{\rho_\alpha}\, {}^nh^\rho$, as one sees by the renaming of a dummy index $\rho_\alpha\to\sigma_\alpha$. The only remaining term in $p^\alpha_{\rho_\alpha}\, {}^nh^\rho$ is the term with $L=\{\alpha\}$, and this is $i\, {}^nB^{\rho\backslash\alpha}$, as desired.

The expression (2.8) is not the only solution of the system (2.6). We could add to it any solution of the homogeneous equation

$$p^\alpha_{\rho_\alpha}\, {}^nh^\rho_{\text{hom}}(p)=0. \tag{2.9}$$

The most general solution of (2.9) is

$$ {}^n h^{\rho}_{\text{hom}}(p) = \prod_{i=1}^{n} \left(\delta^{\rho_i \sigma_i} - \frac{p_i^{\rho_i} p_i^{\sigma_i}}{p_i^2} \right) H_{\sigma}(p) \tag{2.10} $$

with an arbitrary H_σ. But (2.8) seems to be the simplest solution (in a not easily definable way), and we will not consider the more general solutions any further.

3. Unitarity of V

In this section we assume that all elements of $\mathfrak{g}$ are antihermitian, so that $A^*_\mu(x) = A_\mu(x)$. The unitarity equations

$$ V(x, A_\mu) V^*(x, A_\mu) = V^*(x, A_\mu) V(x, A_\mu) = 1 \tag{3.1} $$

are satisfied in the sense of formal power series if

$$ \left. \begin{aligned} &\sum_{\alpha=0}^{n} (-1)^{\alpha}\, {}^{\alpha}h^{\rho'}(p_1, \ldots, p_\alpha)\, {}^{n-\alpha}h^{\rho''}(p_n, \ldots, p_{\alpha+1}) = 0 \\ &\sum_{\alpha=0}^{n} (-1)^{n-\alpha}\, {}^{\alpha}h^{\rho'}(p_\alpha, \ldots, p_1)\, {}^{n-\alpha}h^{\rho''}(p_{\alpha+1}, \ldots, p_n) = 0 \end{aligned} \right\} \tag{3.2} $$

where $\rho' = \{\rho_1, \ldots, \rho_\alpha\}$ or $\rho' = \{\rho_\alpha, \ldots, \rho_1\}$ respectively and similarly for ρ'', and ${}^0h = 1$.

We prove only the first of the equations (3.2), the proof of the second one is analogous. The proof proceeds by induction with respect to n. (3.2) is obviously satisfied for $n = 1$.

Firstly we prove that the product ${}^{\alpha}h^{\rho'}(p')\, {}^{n-\alpha}h^{\rho''}(p'')$, with $p' = \{p_1, \ldots, p_\alpha\}$, $p'' = \{p_n, \ldots, p_{\alpha+1}\}$, is given by the expression (2.8) with ${}^nB^{\cdots}$ replaced by

$$ -ip^{\gamma}_{\rho_\gamma}\, {}^{\alpha}h^{\rho'}(p')\, {}^{n-\alpha}h^{\rho''}(p''). \tag{3.3} $$

For convenience we have dropped the subscript L in γ_L. If $\alpha = 0, n$, the statement follows immediately from the results of Section 2. Therefore, assume $0 < \alpha < n$. Insert (3.3) into (2.8). Consider first the contributions with $L \subset \{1, \ldots, \alpha\}$. All of them contain ${}^{n-\alpha}h^{\rho''}(p'')$ as a factor, and the rest adds to ${}^{\alpha}h^{\rho'}(p')$ by the definition of ${}^{\alpha}h$. Hence these terms contribute ${}^{\alpha}h^{\rho'}(p')\, {}^{n-\alpha}h^{\rho''}(p'')$. The same is true by the same argument for the terms with $L \subset \{n, \ldots, \alpha+1\}$. There remain the terms with $L \cap \{1, \ldots, \alpha\} = L' \neq \emptyset$, $L \cap \{n, \ldots, \alpha+1\} = L'' \neq \emptyset$. In all of them we choose $\gamma \in L'$. We obtain

$$ \begin{aligned} -&\left\{ i \sum_{L'} i^{l'} \prod_{i \in L'} h^{\rho_i}(p_i) \prod_{\substack{i \in L' \\ i \neq \gamma}} p^{i}_{\sigma_i}\, {}^{\alpha}B^{\rho' \backslash \gamma, \rho_L \to \sigma_L}(p') \right\} \\ &\times \left\{ \sum_{L''} i^{l''} \prod_{i \in L''} h^{\rho_i}(p_i) \prod_{i \in L''} p^{i}_{\sigma_i}\, {}^{n-\alpha}h^{\rho'', \rho_L \to \sigma_L}(p'') \right\}. \end{aligned} \tag{3.4} $$

The factor ${}^{\alpha}B^{\cdots}(p')$ can be replaced by $-i\, {}^{\alpha}h^{\cdots}(p')$, whereupon the first curly bracket gets the same structure as the second one. But by inserting (2.7) into (2.8) we find

$$ {}^n h^{\rho}(p) = -\sum_{l} i^{l} \prod_{i \in L} \{h^{\rho_i}(p_i) p^{i}_{\sigma}\}\, {}^n h^{\rho, \rho_L \to \sigma_L}(p). \tag{3.5} $$

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Hence the expression (3.4) is equal to $-{}^{\alpha}h^{\rho'}(p')\,{}^{n-\alpha}h^{\rho''}(p'')$, which, together with the previously executed sums, gives the desired result ${}^{\alpha}h\,{}^{n-\alpha}h$.

Since (2.8) is linear in B we find as a corollary of this result that the first sum in (3.2) is also expressible in the form (2.8), with ${}^{n}B^{\cdots}$ replaced by

$$\mathfrak{B} = -ip^{\gamma}_{\rho_\gamma} \sum_{\alpha=1}^{n} (-1)^{\alpha}\,{}^{\alpha}h^{\rho'}(p')\,{}^{n-\alpha}h^{\rho''}(p''). \tag{3.6}$$

But $\mathfrak{B}=0$ as is seen as follows: we use (2.6–7) and find

$$\begin{aligned}\mathfrak{B} = \sum_{\alpha=1}^{\gamma-2} &(-1)^{\alpha}\,{}^{\alpha}h^{\rho'}(p')[{}^{n-\alpha-1}h^{\rho''\setminus\gamma}(\ldots, p_\gamma+p_{\gamma+1}, p_{\gamma-1}, \ldots, p_{\alpha+1})\\ &-{}^{n-\alpha-1}h^{\rho''\setminus\gamma}(\ldots, p_{\gamma+1}, p_\gamma+p_{\gamma-1}, \ldots, p_{\alpha+1})]\\ &+(-1)^{\gamma-1}\,{}^{\gamma-1}h^{\rho'}(p')\,{}^{n-\gamma}h^{\rho''\setminus\gamma}(\ldots, p_{\gamma+2}, p_\gamma+p_{\gamma+1})\\ &+(-1)^{\gamma}\,{}^{\gamma-1}h^{\rho'\setminus\gamma}(\ldots, p_{\gamma-1}+p_\gamma)\,{}^{n-\gamma}h^{\rho''}(p'')\\ &+\sum_{\alpha=\gamma+1}^{n}(-1)^{\alpha}[{}^{\alpha-1}h^{\rho'\setminus\gamma}(\ldots, p_{\gamma-1}+p_\gamma, \ldots)\\ &-{}^{\alpha-1}h^{\rho'\setminus\gamma}(\ldots, p_\gamma+p_{\gamma+1}, \ldots)]\,{}^{n-\alpha}h^{\rho''}(p'').\end{aligned}$$

Collecting the terms containing the variable $p_\gamma+p_{\gamma+1}$ we find that they are a sum of the form (3.2) with $n-1$ variables, hence vanish by the inductive assumption. The same is true for the terms containing the variable $p_\gamma+p_{\gamma-1}$.

The same method of proof can be used to solve another problem. If $\mathfrak{G}$ is abelian then the ordering of the fields A_{ρ_i} in (2.1) is immaterial, hence only the totally symmetric part

$${}^{n}S^{\rho}(p) = (n!)^{-1}\sum_{P}{}^{n}h^{P\rho}(Pp) \tag{3.7}$$

of ${}^{n}f$ contributes to V. Here we have summed over all permutations P of the indices $\{1, \ldots, n\}$. It can be shown that

$${}^{n}S^{\rho}(p) = (n!)^{-1}\prod_{i=1}^{n} h^{\rho_i}(p_i), \tag{3.8}$$

so that in the abelian case our V is the familiar QED expression occurring in (1.2). In order to prove (3.8) one considers

$$A = {}^{n}S^{\rho}(p) - (n!)^{-1}\prod h^{\rho_i}(p_i),$$

shows first, as in the proof of unitarity, that A is given by (2.8) with ${}^{n}B^{\cdots}$ replaced by $-ip^{\gamma}_{\rho_\gamma}A$, and then he shows, using equations (2.6–7), that $p^{\gamma}_{\rho_\gamma}A=0$.

4. Equations of motion

In this section we state some results on whether it is possible in a pure Yang–Mills theory to find equations of motion involving only the gauge invariant fields $\mathfrak{F}_{\mu\nu}$. Since the answer is essentially negative we refrain from giving proofs, which involve lengthy calculations.

We start from the equations of motion of the gauge-dependent fields A_μ, $F_{\mu\nu}$:

$$D_\nu F^{\mu\nu}(x)=0, \qquad D_\nu {}^*F^{\mu\nu}(x)=0, \tag{4.1}$$

where $D_\nu=\partial_\nu+ig[A_\nu(x),.]$ is the covariant derivative and ${}^*F^{\mu\nu}=\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$.

First we consider the string case, i.e. an $\mathfrak{F}^{\mu\nu}$ constructed with the V of (1.9). We write this V in the form

$$V(x,A_\mu)=P\exp\left\{ig\int_0^\infty d\xi^\mu A_\mu(\xi-x)\right\} \tag{4.2}$$

in order to make clear that the form of the path is considered to be fixed once and for all, i.e. is not treated as a variable. In this we differ from Refs. [1, 6]. The equations of motion

$$\begin{aligned}\partial_\nu\mathfrak{F}^{\mu\nu}(x)&=ig\int_0^\infty d\xi^\rho[\mathfrak{F}^{\mu\nu}(x),\mathfrak{F}_{\rho\nu}(\xi-x)],\\ \partial_\nu {}^*\mathfrak{F}^{\mu\nu}(x)&=ig\int_0^\infty d\xi^\rho[{}^*\mathfrak{F}^{\mu\nu}(x),\mathfrak{F}_{\rho\nu}(\xi-x)]\end{aligned} \tag{4.3}$$

can then be derived essentially as a corollary to equation (4.2) of Ref. [2]. Here $\partial_\nu=\partial/\partial x^\nu$ is the ordinary derivative, not the covariant one. However, we have already remarked in Section 1 that these string fields are apt to lead to intractable divergence problems in an attempt to quantize the theory. Therefore the equations (4.3) are at best of limited value.

Starting from the more general ansatz (2.1, 8) we can derive the following equation up to order g^2:

$$\begin{aligned}\partial_\nu\mathfrak{F}^{\mu\nu}(x)=&-ig\int du f^\rho(x-u)[\mathfrak{F}^{\mu\nu}(x),\mathfrak{F}_{\nu\rho}(u)]\\ &-g^2\int du\,dv\,dw f^\rho(x-u-w)f^\sigma(x-v-w)f^\tau(w)\\ &\times[\mathfrak{F}^{\mu\nu}(x),[\mathfrak{F}_{\nu\rho}(u),\mathfrak{F}_{\tau\sigma}(v)]]+O(g^3),\end{aligned} \tag{4.4}$$

and an analogous equation in which $\mathfrak{F}^{\mu\nu}(x)$ is everywhere replaced by ${}^*\mathfrak{F}^{\mu\nu}(x)$. Unfortunately there are strong indications, though no proof, that similar equations do not hold to order g^3 or higher. This means that the corresponding quantum theory cannot be formulated exclusively in terms of $\mathfrak{F}^{\mu\nu}$, as was done in [4] for QED. We see no other way of calculating the Wightman or Green functions of $\mathfrak{F}$ than by using the definition of $\mathfrak{F}$ in terms of A and starting from the known Green functions of these auxiliary fields. But even so the fields $\mathfrak{F}^{\mu\nu}$ may be of value, since they generate the physical states and contain therefore information on the nature of these states.

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