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Resonances defined by modified dilations

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Abstract. We construct a method generalizing the usual complex scaling method of Aguilar, Balslev and Combes to describe resonances of Schrödinger operators. The method is based on a modified dilation in the momentum space and allows to treat multicenter potentials and potentials with compact support. We give also some assertions on the locations of resonances.

1. Introduction

The study of resonances in Schrödinger operator theory has been an object of growing interest in the last 10 years. There are several mathematical concepts but there is no general theory which can describe all physical phenomena considered as resonances.

One very powerful method is the complex scaling which was introduced by Aguilar, Balslev and Combes [1, 4]. It was extended in many directions (see [7] for a survey).

This method, while very fruitful, works only for the restricted class of dilation analytic potentials which are essentially functions having as radial part the restriction of an analytic function to the positive real axis.

This class clearly excludes potentials with compact support and potentials with multicenter singularities (multicenter Coulomb potentials for example) for which one expects resonances to exist. There are several methods trying to overcome these restrictions (see [2, 3, 15] for example). We mention especially a recent paper of Sigal [14].

We propose here a method which has some similarities with that of Sigal but which is simpler and has therefore, we believe, a wider range of applications. It is a further development of ideas discussed in [9] which go back to Mourre [12], see also [8]. As the usual complex scaling, our method is based essentially on a transformation (a modified dilation) ϕ_θ of the underlying (momentum) space $\mathbb{R}^n$ depending on a real parameter Θ. This induces a family of unitary maps U_θ in $L^2(\mathbb{R}^n)$ which give rise to family $H(\theta)$ of operators unitary equivalent to the considered Schrödinger operator $H = H_0 + V$.

In contrast to usual complex scaling and also to Sigal's method we do not need a group structure of $\Theta \mapsto U_\theta$ which gives a considerable larger freedom for the choice of ϕ_θ. If chosen suitably $\Theta \to \phi_\theta$ can be continued (component-wise) analytically into a complex domain. The complex discrete eigenvalues of the associated 'deformed' Hamiltonian $H(\Theta)$ (also suitable continued) are then called resonances of H.

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We discuss in this paper mainly an explicit choice of ϕ_θ which has, if continued, bounded imaginary part. This is the technical reason why we can consider potentials with compact support or translated singularities.

Moreover this explicit choice of ϕ_θ has the advantage that one can calculate the essential spectrum of the deformed Hamiltonian explicitely. In the case we discuss here it is a (slightly deformed) lower branch of a parabola. The explicit knowledge of $H(\theta)$ allows also to make some assertions as to the location of the resonances. We note, however, that the method is not restricted to this special choice of ϕ_θ. Any local distortion of the essential spectrum can be obtained by a suitable choice of ϕ_θ. The paper is organized as follows.

In Section 2 we give a general theorem which allows to define resonances by general deformations of the Hamiltonian. We then turn to a concrete deformation and discuss some of its properties. Furthermore we introduce a class of potentials which can be treated by this method, and give some examples.

In Section 3 we prove some estimates on the resolvent of the deformed Hamiltonian which imply that resonances do not occur in a certain region near the reals axis. These results are in accordance to one-dimensional results of Nussenzveig [13].

We use throughout the paper the notations $\mathscr{H} := L^2(\mathbb{R}^n)$ and $\mathscr{H}_2 := \{\phi \in \mathscr{H} \mid \Sigma D^\alpha \phi \in \mathscr{H}, |\alpha| \leq 2\}$, for the Sobolev space.

2. The definition of resonances

We begin with a theorem which describes the general situation of complex scaling. This is an extension of an idea of Combes [5].

Theorem 1. *Let $\{U_\theta\}$ $\theta \in I$ a family of unitary mappings, I being an open interval in $\mathbb{R}$. Let H be a self-adjoint operator in the Hilbert space $\mathscr{H}$ and denote $H(\theta) := U_\theta H U_\theta^{-1}$.*

Assume that there is a complex neighbourhood $\mathcal{O}$ of I and a dense set $A \subseteq \mathscr{H}$ such that $\theta \to U_\theta \psi$, $U_\theta^{-1}\psi$, for $\psi \in A$ has analytic continuations and that $H(\theta)$ has an analytic continuation of type (A) (which we also denote by $H(\theta)$) into $\mathcal{O}$.

Assume furthermore that for $\theta \in \mathcal{O} \setminus I$ $\sigma_{ess}(H(\theta))$ is a one-dimensional manifold in $\mathbb{C}^{+-} := \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0, \operatorname{Im} z \leq 0\}$. Denote by S_θ the union of connected components of $\mathbb{C}^{+-} \setminus \sigma_{ess}(H(\theta))$ having an open intersection with $\mathbb{R}^+$. Then

(i) *for any $\theta \in \mathcal{O} \setminus I$; $\phi, \psi \in A$ $z \to (\phi, (H - z)^{-1}\psi)$ has a meromorphic continuation from $\mathbb{C}^{++} := \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0, \operatorname{Im} z > 0\}$ through the positive real axis into S_θ, which is given by*

$$f_\theta(z) := (U_{\bar\theta}\phi, (H(\theta) - z)^{-1} U_\theta \psi), \qquad z \in S_\theta.$$

(ii) *this continuation is unique in the following sense. Let $\theta_1, \theta_2 \in \mathcal{O} \setminus I$, $\theta_1 \neq \theta_2$ then $f_{\theta_1}(z) = f_{\theta_2}(z)$ for all $z \in S_{\theta_1} \cap S_{\theta_2}$.*

Remark. 1(ii) implies that the poles of $f_\theta(z)$ (which are of finite order) are independent of θ as long as z stays away from $\sigma_{ess}(H(\theta))$.

2. Since $(H(\theta) - z)^{-1}$ is analytic in S_θ except of discrete eigenvalues of $H(\theta)$ (see [10, p. 176]), finite order poles of f_θ can only occur at discrete eigenvalues of

$H(\theta)$. If in addition $\{U_\theta\psi/\psi \in A\}$ is dense in $\mathcal{H}$, discrete eigenvalues give rise to arbitrary values of $|f_\theta(z)|$ if ψ, ϕ are suitably chosen.

This motivates the definition

Definition. We call the discrete non-real eigenvalues of $H(\theta)$ in S_θ the *resonances* of H.

Proof (of Theorem 1). Let $\theta \in \mathcal{O}$; $\phi, \psi \in A$ and consider $f_\theta(z)$ as defined above. Assume first that $z \in \mathbb{C}^{++}$. As long as $\theta \in I$ U_θ is unitary and $f(z) := (\phi, (H-z)^{-1}\psi)$ coincides with $f_\theta(z)$. By analyticity in θ this is still true for $\theta \in \mathcal{O} \setminus I$. But since $z \to (H(\theta)-z)^{-1}$ is meromorphic in $z \in S_\theta$, $f_\theta(\cdot)$ is obviously a meromorphic continuation of $f(\cdot)$ into S_θ. The uniqueness of $f_\theta(\cdot)$ is an obvious consequence of the uniqueness theorem for analytic functions on connected sets in $\mathbb{C}$, since it is unique for $z \in \mathbb{C}^{++}$. $\square$

The simplest example for Theorem 1 is the usual complex scaling (see [1] or [4]) for Schrödinger operators. In this case we have $H = H_0 + V$ in $\mathcal{H} := L^2(\mathbb{R}^n)$ where V is a dilation analytic potential (see [18] for definition). A is the set of analytic vectors in $\mathcal{H}$, i.e. the vectors of which $U_\theta\phi(x) := e^{(n/2)\theta}(e^\theta x)$, $\theta \in \mathbb{R}$ has an analytic continuation in θ into a strip $\mathcal{O}$ in $\mathbb{C}$ around the real axis. Then

$$\sigma_{ess}(H(\theta)) = e^{-2\theta}\mathbb{R}^+$$

and $S_\theta = \text{conv hull } \{\mathbb{R}^+, \sigma_{ess}(H(\theta))\} \setminus \sigma_{ess}(H(\theta))$.

Theorem 1 covers also all other cases where resonances are defined by any kind of complex scaling (see [7, Chapter 8]). But in contrast to these constructions no group property of U_θ for $\theta \in \mathbb{R}$ is needed here.

We will now introduce a special family of 'deformations' leading to resonance theory for a large class of potentials. It has some similarities with Sigals [14] construction but we need considerably less structure here.

Let $\theta \in \mathbb{R}$. Consider the map (in momentum space) $\phi_\theta : \mathbb{R}^n \to \mathbb{R}^n$

$$\xi \to \phi_\theta(\xi) := \xi - \theta h(\xi)$$

where

$$h(\xi) := \xi \frac{g(|\xi|)}{|\xi|}$$

and

$$g(|\xi|) := c_0 \int_0^{|\xi|} \chi(s)\, ds$$

with $\chi \in C^\infty$, $\chi \geq 0$, $\chi(0) = 1$, $\chi(s) = 0$ for $s \geq s_0$, $s_0 > 0$ suitably and

$$c_0 := \left[\int_0^{s_0} \chi(s)\, ds\right]^{-1}.$$

This ϕ_θ induces a unitary map in $L^2(\mathbb{R}^n)$.

$$U_\theta\psi(\xi) := [I_\theta(\xi)]^{1/2}\psi(\phi_\theta(\xi)), \qquad \xi \in \mathbb{R}^n, \qquad \psi \in L^2(\mathbb{R}^n)$$

where $I_\theta(\xi) := \det[D\phi_\theta(\xi)]$ is the Jacobian of ϕ_θ. Note that

$$I_\theta(\xi) = \det\left[(1-\theta\tilde{g}(\xi))1 - \theta\frac{\tilde{g}'(\xi)}{|\xi|}(\xi_i\xi_j)\right]$$

for $\tilde{g}(\xi) = |\xi|^{-1} g(|\xi|)$ and where $(\xi_i \ \xi_j)$ denotes the matrix

$$\begin{pmatrix} \xi_1\cdot\xi_1 & \xi_1\cdot\xi_2 & \cdots \\ \xi_2\cdot\xi_1 & \xi_2\cdot\xi_2 & \cdots \\ \vdots & & \end{pmatrix}$$

Note also that $\tilde{g}(\xi)$ and $\tilde{g}'(\xi)\cdot|\xi|$ are bounded uniformly in $\xi\in\mathbb{R}^n$. Thus

$$I_\theta(\xi) = 1 + \theta P_\xi(\theta)$$

where $P_\xi(\theta)$ is a polynomial of $n-1$-th order in θ with coefficients bounded uniformly in $\xi\in\mathbb{R}^n$. This implies that

$$I_\theta(\xi) > 0 \quad \text{for any } \xi\in\mathbb{R}^n$$

and therefore that U_θ is a well-defined unitary operator in $L^3(\mathbb{R}^n)$ as long as $\theta\in I := (-\theta_0, \theta_0)$, $\theta_0 > 0$ suitably.

Remark. Note that ϕ_θ is no flow in $\mathbb{R}^n$ (there is no group property in θ). Thus the family of $\{U_\theta\}$, $\theta\in I$ has not group structure.

Now consider the dense set

$$A := \{\psi\in L^2(\mathbb{R}^n) \mid \check{\psi}\in C_0^\infty(\mathbb{R}^n)\}$$

where $\check{}$ denotes the inverse Fouriertransformation. Note that $\psi\in A$ can be analytically continued into $\mathbb{C}^n$ by the Paley–Wiener theorem. Therefore for $\psi\in A$ $U_\theta\psi$ and $U_\theta^{-1}\psi$ can easily extended into $\theta\in\mathcal{O}_{\theta_0}$ where

$$\mathcal{O}_{\theta_0} := \{z\in\mathbb{C} \mid |z|\le\theta_0\}.$$

Before we prove the analyticity of these extensions we give some useful properties of the map ϕ_θ. We denote by $|\cdot|$ the usual norm in $\mathbb{R}^n$ and $\mathbb{C}^n$ respectively, i.e.

$$|\xi| = \left(\sum_{i=1}^n \xi_i^2\right)^{1/2} \quad \text{for} \quad \xi\in\mathbb{R}^n$$

and $|z| = (|\text{Re } z|^2 + |\text{Im } z|^2)^{1/2}$, for $z\in\mathbb{C}^n$.

Lemma 2. *Let $\theta\in\mathcal{O}_{\theta_0}$ and ϕ_θ as above. Then there exist constants $c_1, c_2 > 0$ such that*

(i) $c_1|\xi| \le |\phi_\theta(\xi)| \le c_2|\xi|$, $\xi\in\mathbb{R}^n$

(ii) $c_1|\xi_7 - \xi_2| \leqq |\phi_\theta(\xi_1) - \phi_\theta(\xi_2)| \leqq c_2|\xi_1 - \xi_2|$ $\xi_1, \xi_2\in\mathbb{R}^n$.

(iii) *Image* $\phi_\theta(\xi_1)$ − *Image* $\phi_\theta(\xi_2) \subseteq \Lambda_\theta$, $\xi_1, \xi_2\in\mathbb{R}^n$ *where*

$$\Lambda_\theta := \{z\in\mathbb{C}^n \mid |\text{Im } z| \leqq \min(\tilde{c}\,|\text{Re } z|, c\,|\text{Im }\theta|)\}$$

for suitable $\tilde{c}, c > 0$.

Proof. (i) follows by inspection, i.e. since

$$|\phi_\theta(\xi)| = |\xi(1-\theta\tilde{g}(\xi))|$$

we have

$$|\phi_\theta(\xi)| \leq |\xi|\,(1+|\theta|\,|\tilde{g}(\xi)|)$$

and

$$|\phi_\theta(\xi)| \geq |\xi|\,(1-|\theta|\,|\tilde{g}(\xi)|)$$

which gives (i) for $|\theta|$ small enough since $\tilde{g}(\xi)$ is uniformly bounded.

(ii) Note that

$$|\phi_\theta(\xi_1)-\phi_\theta(\xi_2)-(\xi_1-\xi_2)| = |\theta|\,|h(\xi_1)-h(\xi_2)|.$$

Since $|h(\xi)|$ is uniformly bounded in $\xi \in \mathbb{R}^n$ and has a finite derivative, there is a $c>0$ such that

$$|\phi_\theta(\xi_1)-\phi_\theta(\xi_2)-(\xi_1-\xi_2)| \leqq |\theta|\,c\,|\xi_1-\xi_2|$$

(iii) Since $h(\cdot)$ has a bounded derivative we have for a suitable c_1

$$|h(\xi_1)-h(\xi_2)| \leqq c_1\,|\xi_1-\xi_2| + o\,|\xi_1-\xi_2| \quad \text{for} \quad |\xi_1-\xi_2| \to 0.$$

Furthermore $|h(\xi)| \leqq c_2$.

Thus for $\theta \in \mathcal{O}_{\theta_0}$

$$\phi_\theta(\xi_1)-\phi_\theta(\xi_2) = \xi_1-\xi_2+(\operatorname{Re}\theta)(h(\xi_1)-h(\xi_2)) - i(\operatorname{Im}\theta)(h(\xi_1)-h(\xi_2))$$

and for any $z \in \text{Image}\,\phi_\theta - \text{Image}\,\phi_\theta$ we have

$$|\operatorname{Im} z| \leqq \tilde{c}\,|\operatorname{Re} z| \quad \text{for a suitable } \tilde{c}>0$$

and

$$|\operatorname{Im} z| \leqq c_2\,|\operatorname{Im}\theta|.$$

This implies (iii). □

We remark that it is obvious that for $\xi \in \mathbb{R}^n$ the map from $\mathcal{O}_{\theta_0}$ into $\mathbb{C}^n$

$$\theta \mapsto \phi_\theta(\xi) \text{ is analytic}$$

(in the sense that each component is analytic). This has a consequence that the unitary operators U_θ and U_θ^{-1} for $\theta \in I := (-\theta_0, \theta_0)$ have analytic continuations $\mathcal{O}_{\theta_0}$ on A as we show in the following

Lemma 3. *Let $\psi \in A$ and U_θ, U_θ^{-1} defined as above for $\theta \in I$. Then there is a $\theta_1 > 0$ such that*

(i) $U_\theta\psi$ and $U_\theta^{-1}\psi$ have analytic continuations into $\mathcal{O}_{\theta_1} := \{z \in \mathbb{C} \mid |z| \leqq \theta_1\}$

(ii) $U_\theta(A)$ is dense in $\mathscr{H}$ for $\theta \in \mathcal{O}_{\theta_1}$

Proof. (i) We prove (i) only for U_θ.

For U_θ^{-1} the proof is similar.

Step 1. We first show that $U_\theta\psi$ is locally (in θ) bounded. Then for $\xi\in\mathbb{R}^n$, using integration by parts and that $|\mathrm{Im}\ \phi_\varepsilon(\phi)|\leqq|\theta|\cdot c$ for suitable $c>0$ we get

$$|U_\theta\psi(\xi)|\leq|I_\theta^{1/2}(\xi)|\int|e^{ix\phi_\theta(\xi)}\psi(x)|\,dx$$

$$\leq C_N\frac{e^{Rc|\theta|}}{(1+|\xi|)^N},\quad\text{for any } N\in\mathbb{N}$$

and suitable $C_N>0$; where $R:=\mathrm{diam}\,(\mathrm{supp}\,\psi)$. Thus $\|U_\theta\psi\|$ is bounded for θ in a compact set.

Step 2. We show that $\theta\mapsto(\phi, U_\theta\psi)$ is analytic for all $\phi\in A$ and θ contained in bounded subsets of $\mathcal{O}_{\theta_0}$. Note that the map $\langle\xi,\theta\rangle\mapsto\bar{\phi}(\xi)U_\theta\psi(\xi)$ is continuous in $\xi\in\mathbb{R}^n$ and analytic in $\theta\in\mathcal{O}_{\theta_0}$. Now consider

$$F_k(\theta):=\int_{|\xi|\leq k}d\xi\bar{\phi}(\xi)U_\theta\psi(\xi),\quad\text{for}\quad k\in\mathbb{N}.$$

then $F_k(\theta)$ is analytic in θ by Fubini and Cauchy's integral formula. Since $F_k(\theta)$ converges uniformly on compact subsets of $\mathcal{O}_{\theta_0}$ to $F(\theta):=(\phi, U_\theta\psi)$, as $k\to\infty$, $F(\theta)$ is locally analytic. Step 1 and Step 2 now imply that $U_\theta\psi$ is analytic in $\theta\in\mathcal{O}_{\theta_0}$ (see [11, p. 365]).

(ii) Consider the bounded operator J in $\mathcal{H}$, defined by:

$$(J\psi)(\xi):=e^{-\xi^2}*\psi(\xi),\qquad\xi\in\mathbb{R}^n,\qquad\psi\in\mathcal{H},\qquad D(J):=\mathcal{H},$$

where $*$ denotes the convolution, and define for $\theta\in\mathcal{O}_{\theta_0}$ the operator $V_\theta:=U_\theta\circ J$. We show that V_θ is bounded and analytic in $\mathcal{H}$. Let $\psi\in\mathcal{H}$ then

$$V_\theta\psi(\xi)=\int dx e^{-ix\xi}p(\xi,x)\check{\psi}(x)$$

where

$$p(x,\xi):=e^{i\theta h(\xi)\cdot x}I_\theta^{1/2}(\xi)e^{-x^2}$$

and $\check{}$ is the inverse Fouriertransformation. Thus V_θ is a pseudodifferential operator which is bounded by the theorem of Calderon and Vaillancourt (see Taylor [17, p. 347]), since the symbol p has bounded derivations in x and ξ. Additionally one can show similar as in (i) above that V_θ is bounded analytic in $\mathcal{O}_{\theta_0}$.

Now consider the weighted Hilbert space

$$\mathcal{H}_1:=\{\psi\in L^2\mid e^{\xi^2}*\psi(\xi)\in L^2\}$$

and denote the restriction of V_θ

$$\tilde{V}_\theta:=V_\theta\upharpoonright\mathcal{H}_1.$$

This is obviously also an analytic family of bounded operators from $\mathcal{H}_1$ into $\mathcal{H}$. Thus $\tilde{V}_\theta$ and also $\tilde{V}_\theta^*$ is continuous in θ in the generalized sense (see [11, p. 366 and p. 206]), furthermore $\tilde{V}_\theta^*$ converges to $\tilde{V}_0^*$ (for $\theta\to0$) in the generalized sense. Note that $\tilde{V}_0^*=J$. Since $J:\mathcal{H}\to\mathcal{H}_1$ has a bounded inverse this is also true for $\tilde{V}_\theta^*$ and θ in a suitable neighbourhood $\mathcal{O}_{\tilde{\theta}_0}$, for $\tilde{\theta}_0>0$ suitably [11, Theorem

2.23, p. 206]. Thus $\tilde{V}^*_\theta$, $\theta \in \mathcal{O}_{\tilde{\theta}_0}$ is injective and therefore we know that $\tilde{V}_\theta$ has a dense range in $\mathcal{H}$. Since $J(A) = A \subseteq \mathcal{H}_1$ and since $\tilde{V}_\theta : \mathcal{H}_1 \to \mathcal{H}$ is bounded, we have

$$\tilde{V}_\theta(A) = U_\theta \circ J(A) = U_\theta(A)$$

is dense in for $\theta \in \mathcal{O}_{\tilde{\theta}_0}$. Now choose $\theta_1 := \min\{\theta_0, \tilde{\theta}_0\}$. □

The next lemma shows why the above choice of ϕ_θ is useful for Schrödinger operators. We first consider the 'free' Schrödinger operator in momentum space and calculate its spectrum.

Lemma 4. *Let $H_0 = \xi^2$ be the representation of the Laplacian in momentum space and $\theta \in I$. Then $H_0(\theta) := U_\theta H_0 U_\theta^{-1}$ has a continuation which is analytic of type (A) into $\theta \in \mathcal{O}_{\theta_1}$ (which we denote also by $H_0(\theta)$). This continuation has in momentum space the representation*

$$H_0(\theta) = \xi^2 + \theta^2 g^2(|\xi|) - 2\theta\,|\xi|\,g(|\xi|), \qquad \xi \in \mathbb{R}^n, \qquad \theta \in \mathcal{O}_{\theta_1}, \tag{1}$$

and

$$\sigma(H_0(\theta)) = \sigma_{ess}(H_0(\theta)) = \{z \in \mathbb{C} \mid z = \xi^2 + \theta^2 g^2(|\xi|) - 2\theta\,|\xi|\,g(|\xi|),\ \xi \in \mathbb{R}^n\}$$

Proof. Let $\theta \in I$ and $\psi \in A$. Then for $\xi \in \mathbb{R}^n$.

$$\begin{aligned} H_0(\theta)\psi(\xi) &= U_\theta H_0 U_\theta^{-1}\psi(\xi) \\ &= [\det(D\phi_\theta(\xi))]^{1/2}(\phi_\theta(\xi))^2[\det(D\phi_\theta^{-1}(\phi_\theta(\xi)))]^{1/2}\psi(\phi_\theta^{-1}(\phi_\theta(\xi))) \\ &= (\phi_\theta(\xi))^2 \\ &= \xi^2 + \theta^2 g^2(|\xi|) - 2\theta\,|\xi|\,g(|\xi|). \end{aligned}$$

This expression has an obvious analytic continuation of type (A) in $\theta \in \mathcal{O}_{\theta_1}$. Since this is a multiplication operator in $L^2(\mathbb{R}^n)$ the assertion on $\sigma(H_0(\theta))$ follows by the spectral mapping theorem. □

Remark. If $\theta = i\beta$, $\beta > 0$ then $\sigma(H_0(\theta))$ is a line $\mathbb{C}^{+-}$ which starts linearly at 0 with some angle (since $g(s) \simeq s$ for small s) and ends up to be the lower part of a parabola (since $g(s) = 1$ for $s \geq s_0$).

We introduce now a class of perturbations which leave the essential spectrum invariant.

Definition. We call a symmetric operator V ϕ-analytic if
(i) V is H_0-compact
(ii) The operator family $V(\theta) := U_\theta V U_\theta^{-1}$, $\theta \in I := (-\theta_1, \theta_1)$ has an analytic type (A) continuation into a neighbourhood $\mathcal{O}_V$ of I in $\mathbb{C}$ as a family of bounded operators from $\mathcal{H}_{+2}$ to $\mathcal{H}$.

Theorem 5. *Assume that V is ϕ-analytic and consider the Schrödinger operator $H := H_0 + V$. Then*
(i) $H(\theta) := U_\theta H U_\theta^{-1}$, $\theta \in I := (-\theta_1, \theta_1)$ has an analytic continuation as a family of type (A) into $\mathcal{O} := \mathcal{O}_V \cap \mathcal{O}_{\theta_1}$ (which we also denote by $H(\theta)$).
(ii) Let $\theta \in \mathcal{O}$ then $\sigma_{ess}(H(\theta)) = \sigma(H_0(\theta))$.

Proof. Since both, $H_0(\theta)$ and $V(\theta)$ are analytic families of type (A) with $D(H_0(\theta)) = \mathcal{H}_{+2} \subseteq D(V(\theta))$ for $\theta \in \mathcal{O}$, (i) is obvious. Since $f(\theta) := V(\theta)(H_0(\theta)+1)^{-1}$ is compact for $\theta \in I$ and an analytic bounded operator-valued function in $\theta \in \mathcal{O}$, it is compact for all $\theta \in \mathcal{O}$ (see [18, p. 126]). Thus $V(\theta)$ is $H_0(\theta)$-compact for $\theta \in \mathcal{O}$ and (ii) follows from a theorem of Sigal [15]. □

Corollary 6. *Let V be ϕ-analytic, $\theta = i\beta$, $\beta > 0$ and $H(\theta)$ defined as above. Then $\sigma_{ess}(H(\theta))$ is a concave (parabola-shaped) line in $\mathbb{C}^{+-}$ and the discrete eigenvalues of $H(\theta)$ in $S_\theta = \text{conv hull}\,\{\mathbb{R}^+ \cup \sigma_{ess}(H(\theta))\} \setminus \sigma_{ess}(H(\theta))$ (which we call resonances of H) are independent of θ as long as they stay away from $\sigma_{ess}(H(\theta))$. They coincide with the finite order poles of the continuation of $(\phi, (H-z)^{-1}\psi)$ for suitable $\phi, \psi \in A$.*

Proof. The Statement on $\sigma_{ess}(H(\theta))$ follows from Lemma 4 and Theorem 5. Clearly U_θ and H fulfill the conditions of Theorem 1. Thus the singularities of $(\phi, (H-z)^{-1}\psi)$ do not depend on θ. Furthermore, since $\{U_\theta\psi \mid \psi \in A\}$, $\theta \in \mathcal{O}$ is dense by Lemma 2(ii). Thus poles of $(\phi, (H-z)^{-1}\psi)$ occur at and only at discrete eigenvalues of $H(\theta)$ in S_θ. □

We give now a class of examples of ϕ-analytic potentials.

Lemma 7 (Sigal's class [14]). *Let V be a real-valued H_0-compact multiplication operator in $L^2(\mathbb{R}^n)$, such that is Fouriertransformation $\hat{V}$ exists and has analytic continuation into*

$$\Lambda := \{z \in \mathbb{C}^n / |\text{Im } z| \leqq c_1 |\text{Re } z| (1 + c_2 |\text{Re } z|)^{-\alpha}\}$$

for $c_1, c_2 > 0$ suitably and $0 < \alpha \leq 1$, such that

$$|\hat{V}(z)| \leq |W(\text{Re } z)|, \qquad z \in \Lambda$$

where W is a suitable real-valued function with $W \in L^1(\mathbb{R}^n) + L^r(\mathbb{R}^n)$ for $r < (n/n-2)$ if $n > 2$ and $r \leq \infty$ for $n \leq 2$. Then V is ϕ-analytic.

Proof. We have only to show that $U_\theta V U_\theta^{-1}$ has bounded analytic continuation as operator from $\mathcal{H}_{+2}$ to $\mathcal{H} := L^2(\mathbb{R}^n)$. We work in momentum space. Denote $V(\theta) := U_\theta \mathcal{F} V \mathcal{F}^{-1} U_\theta^{-1}$. $\mathcal{F}$:= Fourier transform. Then for $\tilde{\psi} \in A$ and $\psi := (1/1+\xi^2)\tilde{\psi}$, $\theta \in I$, $\xi \in \mathbb{R}^n$

$$V(\theta)\psi(\xi) = U_\theta \hat{V} * (U_\theta^{-1}\psi)(\xi)$$

$$= U_\theta \int \hat{V}(\xi - \phi_\theta(\tilde{\xi})) I_\theta^{1/2}(\tilde{\xi}) \psi(\tilde{\xi})\, d\tilde{\xi}$$

$$= \int I_\theta^{1/2}(\xi) \hat{V}(\phi_\theta(\xi) - \phi_\theta(\tilde{\xi})) I_\theta^{1/2}(\tilde{\xi}) \psi(\tilde{\xi})\, d\tilde{\xi}$$

By Lemma 2(iii) and the assumptions $V(\theta)\psi(\xi)$ is analytic in θ in a suitable neighbourhood of θ pointwise for $\xi \in \mathbb{R}^n$ and by similar arguments as in Lemma 3 we can conclude that $(\phi, V(\theta)\psi)$ is locally analytic for all $\phi \in A$. So we are left to show that $V(\theta)\psi$ is locally bounded (see [11, p. 365]). Denote $h(\xi) := (1 + |\xi|^2)^{-1}$.

Since

$$|V(\theta)h \cdot \psi(\xi)| \leq \int d\tilde{\xi}\, |I_\theta^{1/2}(\xi)|\, |\hat{V}(\phi_\theta(\xi) - \phi_\theta(\tilde{\xi}))|\, I_\theta^2\, |\tilde{\xi}|\, |h(\tilde{\xi})|\, \psi(\tilde{\xi})$$

$$\leq c \int d\tilde{\xi}\, |W(\xi - \tilde{\xi})|\, |h(\tilde{\xi})|\, \psi(\tilde{\xi})$$

$$= c\, |W| * |h| \cdot |\psi|\, (\xi), \qquad \theta \in \mathcal{O}_{\theta_0}$$

thus by Young's and Hölder's inequalities

$$\|V(\theta)(H_0 + 1)^{-1}\psi\|_2 \leq \|W\|_r\, \|h\|_s\, \|\psi\|_2$$

for $s > n/2$ and $r < n/n - 2$ if $n > 2$ or $r \leq \infty$ if $n \leq 2$. If $W \in L^1$ we can set $s = \infty$ and $r = 1$. Thus $V(\theta)(H_0 + 1)^{-1}$ is bounded for all $\theta \in \mathcal{O}_{\theta_1}$. This together with the local weak analyticity above implies that $\theta \to V_\theta(H_0 + 1)^{-1}$ is bounded analytic operator-valued. □

Remark. If $V \in C_0^N(\mathbb{R}^n)$, $N > n$ i.e. V is N-times continuous differentiable with compact support then $V(\theta)$ is a family of operators bounded uniformly in $\theta \in \mathcal{O}$, a fact which eases the arguments in Lemma 7 above considerably.

To see this we use integrations by parts

$$|\hat{V}(\phi_\theta(\xi) - \phi_\theta(\tilde{\xi}))| = \left| \int e^{-ix(\phi_\theta(\xi) - \phi_\theta(\tilde{\xi}))} V(x)\, dx \right|$$

$$\leq \tilde{C}_N(|\phi_\theta(\xi) - \phi_\theta(\tilde{\xi})|)^{-N} e^{Rc_0} R^n\, \|D^N V\|_\infty \quad \text{for} \quad \xi, \tilde{\xi} \in \mathbb{R}^n$$

where $R := \operatorname{diam}(\operatorname{supp} V)$, $\tilde{C}_N > 0$ suitably

$$c_0 := \sup |\operatorname{Im}(\phi_\theta(\xi) - \phi_\theta(\tilde{\xi}))|$$

and

$$D^N V := \sum_{|\alpha| \leq N} D^\alpha V.$$

Since by Lemma 2(ii)

$$|\phi_\theta(\xi) - \phi_\theta(\tilde{\xi})|^{-1} \leq \tilde{c}\, |\xi - \tilde{\xi}|^{-1}$$

we have

$$|\hat{V}(\phi_\theta(\xi) - \phi_\theta(\tilde{\xi}))| \leq c_N(|\xi - \tilde{\xi}|)^{-N} =: W(\xi - \tilde{\xi})$$

where $W \in L^1(\mathbb{R}^n)$ for $N > n$. Thus by Young's inequality for any $\psi \in L^2(\mathbb{R}^n)$

$$\|V(\theta)\psi\| \leq \|W * \psi\|_2 \leq \|W\|_1\, \|\psi\|_2, \text{ i.e.}$$

$V(\theta)$ is a bounded operator with $\|V(\theta)\| \leq \|W\|_1$.

Example 1 (Multi-center Coulomb potentials). Let $n \neq 1$, $R_j \in \mathbb{R}^n$, $j = 1, \ldots, k$ and

$$V(x) := \sum_{j=1}^{k} |x - R_j|^{-1}, \quad x \in \mathbb{R}^n.$$

Then we have

$$\hat{V}(\xi)=\sum_{j=1}^{k} e^{-i\xi\cdot R_j}\hat{V}_c(\xi)$$

for $V_c(x):=1/|x|$. Thus (see [10, p. 187] for example)

$$\hat{V}_c(\xi)=c_0\,|\xi|^{1-n},\quad c_0>0 \text{ suitably},\quad \xi\in\mathbb{R}^n.$$

This has an obvious analytic continuation into $\mathbb{C}^n\setminus\{0\}$ given by

$$\hat{V}_c(z):=\langle z_1,\ldots,z_n\rangle\to\left(\sum_{l=1}^{n} z_l^2\right)^{1-n/2}.$$

Thus

$$|\hat{V}(z)|\le c_0\left(\sum_{j=1}^{k} e^{|\mathrm{Im}\,z||R_j|}\right)\left(\sum_{l=1}^{n}|\mathrm{Re}\,z_l|^2\right)^{(1-n)/2}$$
$$\le \tilde{c}\,|\xi|^{1-n}\quad \text{for}\quad z\in\Lambda,\ \tilde{c}>0 \text{ suitably}$$

where

$$\Lambda:=\{z\in\mathbb{C}^n\mid |\mathrm{Im}\,z|\le\min\{c_1\,|\mathrm{Re}\,z|,c_2\}\};\ c_1,c_2>0 \text{ suitably}.$$

The last estimate above holds since $|\mathrm{Im}\,z|$ is bounded and $\mathrm{Re}\,z=\xi$. Thus if we set

$$W(\xi):=\tilde{c}\,|\xi|^{1-n}.$$

$W\in L^1(\mathbb{R}^n)+L^2(\mathbb{R}^n)$, for $1<r<n/n-2$ and fulfills the conditions of Lemma 7. (See also [14, Lemma 2.2].)

Example 2. Let $m\le 2$, $V\in L^\infty(\mathbb{R}^n)$ with compact support. Then $\hat{V}(\cdot)$ can be continued to a function which is analytic everywhere in $\mathbb{C}^n$ and

$$|\hat{V}(z)|\le e^{R|\mathrm{Im}\,z|}R^n\,\|V\|_\infty=:W$$

for $z\in\Lambda$, where

$$\Lambda:=\{z\in\mathbb{C}^n\mid |\mathrm{Im}\,z|\le\min\{c_1\,|\mathrm{Re}\,z|,c_2\}\}$$

for $c_1,c_2>0$ suitably and $R:=\mathrm{diam}(\mathrm{supp}\,V)$. This means that V is in the class discussed in Lemma 7.

Remark. The above example contains in particular 'double barrier' potentials which is the typical case for so-called shape resonances to occur (see [6] for example).

3. On the location of resonances

We will prove now some estimates on the 'deformed' resolvent of the Schrödinger operator. They will imply that resonances do not occur in a certain region near the real axis. Note that the imaginary part of the resonance can be physically interpreted as the inverse of the life time of the resonance state of the quantum mechanical system, described by the Schrödinger operator in question. First we give an estimate for the spectrum of the free Hamiltonian.

Lemma 8. *Let $\theta = i\beta$, $0 < \beta < \theta_0$, $H_0(\theta) = H_0(i\beta)$ defined as above and let $z = \lambda - i\eta$, $\lambda \geq 0$, $\eta \geq 0$ and denote by*

$$S_{i\beta} := \text{conv. hull}\, \{\mathbb{R}^+, \sigma(H_0(i\beta))\}.$$

Then there exist $\beta_0 \in (0, \beta)$ and $\lambda_0 > 0$ such that

$$\|(H_0(i\beta) - z)^{-1}\| \leq c_1 \lambda^{-1/2} \tag{2}$$

for $z = \lambda - i\eta \in S_{i\beta_0} \cap \{z \in \mathbb{C} \mid \lambda > \lambda_0\}$
($c_1 > 0$ suitably) where $S_{i\beta_0}$ is defined analogously.

Proof. Note that $g(s) = 1$ for $s \geq s_0$. Now choose $\lambda > 2s_0$. By the representation (1) given in Lemma 4 we have for $z \in S_{i\beta_0}$, $\beta_0 \in (0, (1/\sqrt{2})\beta)$ and suitable $\tilde{c} > 0$

$$\begin{aligned}
\|(H_0(i\beta) - z)^{-1}\| &= \sup_{\xi \in \mathbb{R}^n} |(\xi^2 - \beta^2 g^2(|\xi|) - i2\beta\, |\xi|\, g(|\xi|) - z)^{-1}| \\
&\leq \tilde{c} \sup_{\substack{\xi \in \mathbb{R}^n \\ z \in S_{i\beta_0}}} |(\xi^2 - \beta^2 - 2i\beta\, |\xi| - \lambda - i\eta)^{-1}| \\
&= \tilde{c} \sup_{\substack{\xi \in \mathbb{R}^n \\ z \in S_{i\beta_0}}} [(\xi^2 - \beta^2 - \lambda)^2 + (2\beta\, |\xi| - \eta)^2]^{-1/2} \\
&\leq \tilde{c} \sup_{\xi \in \mathbb{R}^n} [(\xi^2 - \beta^2 - \lambda)^2 + (2\beta\, |\xi| - 2\beta_0 \sqrt{\lambda + \beta_0^2})^2]^{-1/2}.
\end{aligned}$$

To estimate the term

$$F(\xi, \lambda) := [(\xi^2 - \beta^2 - \lambda)^2 + (2\beta\, |\xi| + 2\beta_0 \sqrt{\lambda + \beta_0^2})^2]^{-1/2}$$

we consider two cases.

1. Case. Let $\xi^2 - \beta^2 - \lambda \geq -\frac{1}{2}(\lambda + \beta^2)$ then $|\xi| \geq (1/\sqrt{2})\sqrt{\lambda + \beta^2}$ and we can estimate

$$\begin{aligned}
F(\xi, \lambda) &\leq (\sqrt{2}\beta \sqrt{\lambda + \beta^2} - 2\beta_0 \sqrt{\lambda + \beta_0^2})^{-1} \\
&\leq \left[2\left(\frac{1}{\sqrt{2}}\beta - \beta_0\right)\sqrt{\lambda + \beta_0^2}\right]^{-1} \\
&\leq c_1 \lambda^{-1/2}
\end{aligned}$$

for $c_1 := \frac{1}{2}((1/\sqrt{2})\beta - \beta_0)^{-1} > 0$.

2. Case. Let $\xi^2 - \beta^2 - \lambda \leq -\frac{1}{2}(\lambda + \beta^2) < 0$. Then $|\xi^2 - \beta^2 - \lambda| \geq \frac{1}{2}(\lambda + \beta^2)$ and $F(\xi, \lambda) \leq 2(\lambda + \beta^2)^{-1} \leq 2\lambda^{-1}$. Therefore we have

$$\begin{aligned}
F(\xi, \lambda) &\leq \min\{c_1 \lambda^{-1/2}, 2\lambda^{-1}\} \\
&\leq c_1 \lambda^{-1/2}
\end{aligned}$$

for λ large enough which proves (2). □

Now we use this estimate to show an assertion on the location of resonances for a special class of ϕ-analytic potentials.

Theorem 9. *Let $\theta = i\beta$, $\beta \in (0, \theta_0)$. Let V $H_0^{1/2}$-compact and assume that $V(\theta)$*

has a $H_0^{1/2}$-bounded analytic continuation from $(-\theta_0, \theta_0)$ into $\mathcal{O}$. Let

$$H(i\beta) := H_0(i\beta) + V(i\beta) \quad \text{and} \quad S_{i\beta}$$

defined as above. Then for any $\beta_0 \in (0, 1/\sqrt{2}\beta)$ there is a $\lambda_0 > 0$ such that H has no resonances in the set

$$\tilde{S} := S_{i\beta_0} \cap \{z \in \mathbb{C} \mid \operatorname{Re} z \geq \lambda_0\}.$$

Remark. Note that V's fulfilling the conditions of Theorem 9 are ϕ-analytic since $H_0^{1/2}$-compact are also H_0-compact.

Proof (of Theorem 9). Let β, β_0 as above and $z = \lambda - i\eta \in S_{i\beta}$. Then we have by the resolvent equation

$$(H(i\beta) - z)^{-1} = (H_0(i\beta) - z)^{-1}[1 - V(i\beta)(H_0(i\beta) - z)^{-1}]^{-1}. \tag{3}$$

The first factor on the RHS above is clearly analytic in $z \in S_{i\beta}$ by the definition of $S_{i\beta}$. We will show that the second factor has a bounded inverse for z in the desired region. By a similar estimate as in Lemma 8 we can see that

$$(H_0 + 1)^{1/2}(H_0(i\beta) - z)^{-1} \tag{4}$$

is a family of operators uniformly bounded in $z \in S_{i\beta_0}$ if $\operatorname{Re} z = \lambda > \lambda_1$ for suitable $\lambda_1 > 0$.

Considering the representation of (4) in momentum space we see that

$$(\xi^2 - 1)^{1/2}(\xi^2 - \beta^2 g^2(|\xi|) - 2i\beta\, |\xi|\, g(|\xi|) - z)^{-1} \to 0$$

as $\operatorname{Re} z = \lambda \to \infty$, $z \in S_{i\beta_0}$, pointwise for every $\xi \in \mathbb{R}^n$. Since the operator (4) is bounded it follows by Lebesgue's dominated convergence theorem that

$$(H_0 + 1)^{1/2}(H_0(i\beta) - z)^{-1} \to 0 \quad \text{as} \quad \operatorname{Re} z = \lambda \to \infty$$

in the strong sense uniformly for $z \in S_{i\beta_0}$. Furthermore, since $V(i\beta)$ is $(H_0 + 1)^{1/2}$-compact we have that

$$\|V(i\beta)(H_0(i\beta) - z)^{-1}\|$$
$$= \|V(i\beta)(H_0 + 1)^{-1/2}(H_0 + 1)^{1/2}(H_0(i\beta) - z)^{-1}\| \to 0 \quad \text{as} \quad \operatorname{Re} z \to \infty$$

as long as $z \in S_{i\beta_0}$.

Therefore, for any $\beta_0 \in (0, (1/\sqrt{2})\beta)$ we can choose a $\lambda_0 > \lambda_1 \geq s_0 > 0$ such that

$$\|V(i\beta)(H_0(i\beta) - z)^{-1}\| \leq \delta_0 < 1, \quad (\delta_0 \text{ suitably})$$

for $z \in S_{i\beta_0} \cap \{z \in \mathbb{C} \mid \operatorname{Re} z \geq \lambda_0\} =: \tilde{S}$. This implies that both factors in the RHS of (3) have no singularities in $\tilde{S}$ and this means that there are no resonances in this region. □

Remark. (1) Note that the 'borderline' for the distance of the resonances to the real axis (if $\operatorname{Re} z$ is large enough) is the lower branch of the parabola $\sigma_{ess}(H(i\beta_0))$. This picture is in accordance with a one-dimensional result of Nussenzveig [13, p. 219, ff].

(2) If we assume that $V \in C_0^N(\mathbb{R}^n)$ as in the remark after Lemma 7 we get

some more detailed estimates since

$$\|V(i\beta)\| \le c_1 e^{Rc_0} R^n \|D^N V\|_\infty$$

and

$$\|(H_0(i\beta)-z)^{-1}\| \le c_2 \lambda^{-1/2}$$

we get

$$\|V(i\beta)(H_0(i\beta)-z)^{-1}\| < 1$$

if and only if

$$(c_3 e^{Rc_0} R^n \|D^N V\|_\infty)^2 < \lambda = \operatorname{Re} z. \tag{5}$$

Thus the region where the resonances can occur grows at most like the L.H.S. of (5) above.

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