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Note on relative entropy and thermodynamical limit

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Abstract. Let φ, $\psi \in \mathcal{M}_{*+}$ be faithful normal positive linear functionals on a von Neumann algebra $\mathcal{M}$. We call φ 'thermodynamically convertible into ψ' iff $R(\varphi/\psi) = -\log(\Delta_{\Phi,\Psi}) \cdot \chi_{[\varepsilon,\infty]}(\Delta_{\Phi,\Psi}) + \log(\Delta_\Psi) \geqslant 0$, a Clausius-type version of the second principle of thermodynamics. The relation of this definition with the KMS-condition is established. Let $\mathcal{M} \neq \mathbb{C}$ be a factor. $\mathcal{M}$ is of type III_1 if for any faithful φ, $\psi \in \mathcal{M}^1_{*+}$, $\inf \psi(S(\varphi(u \cdot u^*)/\psi)) = 0$ where the infimum is taken over the unitaries of $\mathcal{M}$, and $S(\varphi_1/\varphi_2) = \varphi_2(R(\varphi_1/\varphi_2))$ denotes Araki's relative entropy.

§1. Introduction

The thermodynamical limit idealization of a physical system by an infinite volume and an infinite particle number for finite mean particle density is usually advocated by the following arguments: 1. Finite systems exclude phase transitions. 2. Closed finite systems show almost periodic behaviour in time which excludes ergodicity. 3. Perfectly extensive quantities A cannot exist but in infinite systems (because of the boundary effects in the finite case), and 4. the relative fluctuations $\Delta A(N)/N$ vanish with $N^{-1/2}$ (N being the particle number), such that the corresponding intensive quantities $a := \lim_{N\to\infty}(A(N)/N)$ assume sharp values as in phenomenological equilibrium thermodynamics. 5. The limit can be regarded as a coupling of a finite realistic system to a reservoir of the same matter, or 6. as a tool to pass from a 'microscopic' to a 'macroscopic' level.

In this paper we extract a Clausius-type formulation [8] of the second principle of thermodynamics from the KMS-condition. This leads to a positivity condition of an operator which we construct using the Tomita–Takesaki theory of von Neumann algebras. There exists a relation between this R-operator and the type III_1-property. Hence we are provided with a link between the second principle and the thermodynamical limit, at least in those cases where the representations of a quasilocal C^*-algebra, arising in the limit, are factors (resp. direct integrals of factors) of type III_1. This includes thermodynamical equilibrium representations of prominent physical examples. So the above given limit arguments could be supplemented, or replaced.

In §2 we expose some material of the modular theory. In §3 we construct the R-operator from the KMS-condition, and prove its self-adjointness if it is

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positive. To translate the KMS-condition into a Clausius-type statement we employ Araki's theory of perturbation of states [1]. The relation between the R-operator and the type-III_1 property is shown in §4. There we make use of a theorem of Connes and Størmer on the homogeneity of a type III_1-factor state space [11].

§2. Ingredients of the modular theory

Assume that a concrete physical system can be described by a σ-finite von Neumann algebra in standard form [3, 15, 10] with Hilbert space $\mathcal{H}$, and natural positive cone $\mathcal{P} \subset \mathcal{H}$. The states of the system shall be the set $\mathcal{M}_{*+}$ of normal positive linear functionals on $\mathcal{M}$. There exists a homeomorphic bijection

$$\psi \in \mathcal{M}_{*+} \mapsto \xi(\psi) =: \Psi \in \mathcal{P} \tag{1a}$$

uniquely representing the states $\mathcal{M}_{*+}$ by the vectors in $\mathcal{P}$ such that

$$\psi(x) = \langle \Psi, x\Psi \rangle, \qquad x \in \mathcal{M} \tag{1b}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{H}$. For two cyclic (and therefore separating) vectors $\Phi, \Psi \in \mathcal{P}$ the 'relative modular operator' $\Delta_{\Phi,\Psi}$ is defined by the polar decomposition

$$S_{\Phi,\Psi} =: J(\Delta_{\Phi,\Psi})^{1/2} \tag{2a}$$

of the closure $S_{\Phi,\Psi}$ of the operator

$$x\Psi \mapsto x^*\Phi, \qquad x \in \mathcal{M}. \tag{2b}$$

J is an antiunitary involution, and $\Delta_{\Phi,\Psi}$ a positive selfadjoint operator. For $\Phi = \Psi$ we write $\Delta_{\Psi,\Psi} =: \Delta_\Psi$; the one-parameter 'modular automorphism group' is defined as

$$\sigma_t^\psi(x) := (\Delta_\Psi)^{it} x (\Delta_\Psi)^{-it}, \qquad x \in \mathcal{M}, \qquad t \in \mathbb{R}. \tag{3}$$

For arbitrary faithful pairs $\varphi, \psi \in \mathcal{M}_{*+}$ (with cyclic and separating representing vectors $\Phi, \Psi \in \mathcal{P}$) there exists a unitary cocycle [9]

$$(D\varphi : D\psi)_t = (\Delta_{\Phi,\Psi})^{it}(\Delta_\Psi)^{-it} \in \mathcal{M}, \qquad t \in \mathbb{R} \tag{4a}$$

strongly continuous in t, such that

$$(D\varphi : D\psi)_t \sigma_t^\psi(x)(D\varphi : D\psi)_t^* = \sigma_t^\varphi(x), \qquad x \in \mathcal{M}, t \in \mathbb{R}. \tag{4b}$$

All information in comparing two (faithful) states $\varphi, \psi \in \mathcal{M}_{*+}$ is contained in the densely defined operator $(\Delta_{\Phi,\Psi})^{1/2}(\Delta_\Psi)^{-1/2}$,

$$\begin{aligned}
(\Delta_{\Phi,\Psi})^{1/2}(\Delta_\Psi)^{-1/2} j(x)\Psi &= (\Delta_{\Phi,\Psi})^{1/2} JJ(\Delta_\Psi)^{-1/2} JxJ\Psi \\
&= (\Delta_{\Phi,\Psi})^{1/2} J(\Delta_\Psi)^{1/2} x\Psi \\
&= (\Delta_{\Phi,\Psi})^{1/2} S_\Psi x\Psi \\
&= (\Delta_{\Phi,\Psi})^{1/2} x^*\Psi \\
&= JS_{\Phi,\Psi} x^*\Psi \\
&= Jx\Phi \\
&= j(x)\Phi, \qquad x \in \mathcal{M}
\end{aligned} \tag{5}$$

where

$$x \in \mathcal{M} \mapsto j(x) := JxJ \in \mathcal{M}' \tag{6}$$

is an anti-linear isomorphism, with $\mathcal{M}'$ the commutant, and

$$J\Omega = \Omega, \qquad \Omega \in \mathcal{P}. \tag{7}$$

$(\Delta_{\Phi,\Psi})^{1/2}(\Delta_\Psi)^{-1/2}$ is affiliated with $\mathcal{M}$:

$$\begin{aligned} & j(u^*)(\Delta_{\Phi,\Psi})^{1/2}(\Delta_\Psi)^{-1/2} j(u) j(x)\Psi \\ &\quad = j(u^*) j(u) j(x)\Phi \\ &\quad = j(x)\Phi \\ &\quad = (\Delta_{\Phi,\Psi})^{1/2}(\Delta_\Psi)^{-1/2} j(x)\Psi, \qquad u^* = u^{-1} \in \mathcal{M}, \qquad x \in \mathcal{M}. \end{aligned} \tag{8}$$

A comprehensive treatment of the Tomita–Takesaki theory can be found e.g. in [7, 18, 6].

§3. KMS-condition and second principle of thermodynamics

A Clausius-like version of the second principle of thermodynamics states whether a process $\varphi \to \psi$, for states φ, ψ goes by itself, i.e. without transmitting energy to the system. If this is the case the entropy ('Verwandlungsinhalt') characterizing the conversions $\varphi \to \psi$ increases [8]. Evidently the work of Clausius refers to a classical context. But a priori there are no reasons to restrict his fundamental principle to classical (i.e. commutative) systems. Therefore we admit states of non-commutative systems in the second law and consider processes $\varphi \in \mathcal{M}_{*+} \mapsto \psi \in \mathcal{M}_{*+}$.

From a mathematical point of view the operator $(\Delta_{\Phi,\Psi})^{1/2}(\Delta_\Psi)^{-1/2}$ characterizes the transition $\varphi \in \mathcal{M}_{*+} \mapsto \psi \in \mathcal{M}_{*+}$. To construct a thermodynamically reasonable expression from this operator, we refer to the KMS-condition which we believe to characterize thermodynamical equilibrium states [13, 7]. The following Proposition 2 decomposes the KMS-property into two independent parts. One expresses stationarity. Therefore the other one should cover the second principle.

Definition 1. a) Let M be a W^*-algebra and $\{\alpha_t\}_{t\in\mathbb{R}}$ a weak*-continuous one parameter group of automorphisms on M. Define

$$\begin{aligned} \mathcal{D}_\beta &:= \{z \in \mathbb{C} \mid 0 < \operatorname{Im} z < \beta\}, \qquad \beta \geq 0, \\ \mathcal{D}_\beta &:= \{z \in \mathbb{C} \mid \beta < \operatorname{Im} z < 0\}, \qquad \beta \leq 0, \end{aligned} \tag{9}$$

and let $\bar{\mathcal{D}}_\beta$ be the closure of $\mathcal{D}_\beta$ if $\beta \neq 0$, and $\bar{\mathcal{D}}_\beta = \mathbb{R}$ if $\beta = 0$. A normal positive linear functional $\omega \in \mathcal{M}_{*+}$ is said to fulfill the (α, β)-KMS-condition for $\beta \in \mathbb{R}$ if for any a, $b \in M$, there exists a complex function $F_{a,b}$ which is analytic on $\mathcal{D}_\beta$, and continuous on $\bar{\mathcal{D}}_\beta$, such that

$$\begin{aligned} F_{\alpha,b}(t) &= \omega(a\alpha_t[b]), \\ F_{a,b}(t + i\beta) &= \omega(\alpha_t[b]a), \qquad t \in \mathbb{R}. \end{aligned} \tag{10}$$

b) Let $\mathcal{M}$ be a von Neumann algebra in standard form with Hilbert space $\mathcal{H}$, H a self-adjoint operator on $\mathcal{H}$, and $\alpha_t(x) := e^{iHt}\mathcal{H}e^{-iHt}$, $x \in \mathcal{M}$, $t \in \mathbb{R}$. A normal positive linear functional $\omega \in \mathcal{M}_{*+}$ is said to fulfill the (H, β)-KMS-condition for $\beta \in \mathbb{R}$ if

$$\omega \text{ is } (\alpha, \beta)\text{-KMS, and} \tag{11(i)}$$

$$H\Omega = 0. \tag{11(ii)}$$

For $\beta \neq 0$ the (α, β)-KMS-condition for ω implies the stationarity of ω. To ensure the stationarity of the respresenting vector Ω condition (ii) has to be added explicitly. Clearly (ii) is not independent of (i), since (ii) by itself implies the stationarity of ω.

The following proposition contains an operator theoretical version of the Roepstorff–Araki–Sewell correlation inequality which has been proven to be equivalent to the KMS-condition [7].

Proposition 2. *Let $\mathcal{M}$ be a von Neumann algebra in standard form with Hilbert space $\mathcal{H}$, and $\psi \in \mathcal{M}_{*+}$ a faithful state. Let $\alpha_t(x) = e^{iHt}xe^{-iHt}$, $x \in \mathcal{M}$, $t \in \mathbb{R}$, be the time evolution such that the common domain $\mathcal{D}(H) \cap \mathcal{D}(\log \Delta_\Psi)$ of the self-adjoint operators H and $\log \Delta_\Psi$ contain a core $\mathcal{M}_0\Psi$ of H, $\mathcal{M}_0\Psi \subset \mathcal{M}\Psi$, $\mathcal{M}_0^* = \mathcal{M}_0$. Then φ is (H, β)-KMS iff*

$$H\Psi = 0, \tag{12}$$

$$\beta H \geqslant -\log \Delta_\Psi \tag{13}$$

Proof. If ψ is (H, β)-KMS, $\Delta_\Psi = e^{-\beta H}$. Therefore the above inequality assumes equality. Now suppose (13). For all $a \in \mathcal{M}$ with $a\Psi \in \mathcal{D}(H) \cap \mathcal{D}(\log \Delta_\Psi)$ the convexity of $-\log$ implies

$$\begin{aligned} \beta \frac{\langle a\Psi, Ha\Psi\rangle}{\langle a\Psi, a\Psi\rangle} &\geqslant -\log \frac{\langle a\Psi, \Delta_\Psi a\Psi\rangle}{\langle a\Psi, a\Psi\rangle} \\ &= \log \frac{\langle a\Psi, \alpha\Psi\rangle}{\langle a^*\Psi, a^*\Psi\rangle}. \end{aligned}$$

By (12) the Roepstorff–Araki–Sewell inequality follows which is equivalent to the (α, β)-KMS-condition by Theorem 5.3.15 in [7]. qed

Condition (13) relates the 'energy-operator' H to a state ψ. In the case

$$\beta H = -\log \Delta_\Psi - h \tag{14}$$

for some $h = h^* \in \mathcal{M}$, we can interpret (13) as a relation between two states using Araki's theory of perturbed states [4]:

$$\log \Delta_\Psi + h = \log (\Delta_{\xi(\psi^h), \Psi}), \tag{15}$$

where

$$\begin{aligned} &\xi(\psi^h) = \exp\{\tfrac{1}{2}(\log \Delta_\Psi + h)\}\Psi, \\ &\psi^h(x) = \langle \xi(\psi^h), x\xi(\psi^h)\rangle, \qquad x \in \mathcal{M}. \end{aligned} \tag{16}$$

Therefore inequality (13) can be written

$$R(\psi^h \mid \psi) \geqslant 0, \tag{17a}$$

$$R(\psi^h \mid \psi) := -\log(\Delta_{\xi(\psi^h),\Psi}) + \log \Delta_\Psi$$
$$= -h. \tag{17b}$$

Note that if ψ is faithful so ψ^h too (Corollary 4.4 in [1]).

Now the R-operator can be defined for arbitrary faithful states, such that the thermodynamical positivity relation can be extended. This positivity relation will imply a unique self-adjoint extension of the R-operator.

Definition 3. Given two faithful states φ, $\psi \in \mathcal{M}_{*+}$ of a von Neumann algebra $\mathcal{M}$. Let $\chi_{[a,b]}$ denote the characteristic function of the real interval $[a, b]$, and put $\chi_n := \chi_{[-n,\infty]}(\log \Delta_{\Phi,\Psi})$. We define

$$R(\varphi \mid \psi) := s - \lim_{n\to\infty} \{-\log(\Delta_{\Phi,\Psi})\chi_n + \log(\Delta_\Psi)\} \tag{18}$$

on those vectors for which the strong limit exists. φ is called 'thermodynamically convertible into ψ' if

$$R(\varphi \mid \psi) \geqslant 0. \tag{19}$$

Definition (18) generalizes (17b). For $h = h^* \in \mathcal{M}$, the domains of $\log \Delta_\Psi$ and $\log \Delta_{\xi(\psi^h),\Psi}$ are identical such that the limit in (18) exists for all vectors of this domain, and equals $-h$.

Theorem 4. If $R(\varphi/\psi) \geqslant 0$ there exists a unique self-adjoint extension affiliated with $\mathcal{M}$.

Proof. Let $\mathcal{M}_0 \subseteq \mathcal{M}$ be the set of analytic elements

$$x_f := \int_{-\infty}^{\infty} \sigma_t^\psi(x) f(t)\, dt, \qquad x \in \mathcal{M}, \qquad f \in C_0^\infty(\mathbb{R}) \tag{20}$$

where $C_0^\infty(\mathbb{R})$ means that C^∞-functions with compact support. $\mathcal{M}_0$ is a weak*-dense subspace of $\mathcal{M}$ [7], such that $\mathcal{M}_0\Psi$ is dense in $\mathcal{H}$. Moreover

$$\lim_{t\to 0} \frac{1}{it}((\Delta_\Psi)^{it} - \mathbb{1})x_f\Psi = x_{-f'}\Psi.$$

So $\mathcal{M}_0\Psi$ is a core for $\log(\Delta_\Psi)$. On the other hand

$$n \geqslant \frac{\langle x\Psi, -\log(\Delta_{\Phi,\Psi})\chi_n x\Psi\rangle}{\langle x\Psi, x\Psi\rangle} = \frac{\langle x\Psi, -\log e^{\log(\Delta_{\Phi,\Psi})\chi_n} x\Psi\rangle}{\langle x\Psi, x\Psi\rangle}$$
$$\geqslant -\log \frac{\langle x\Psi, e^{\log(\Delta_{\Phi,\Psi})\chi_n} x\Psi\rangle}{\langle x\Psi, x\Psi\rangle}$$
$$= -\log \frac{\langle x\Psi, \Delta_{\Phi,\Psi}\chi_{[e^{-n},\infty)}(\Delta_{\Phi,\Psi})x\Psi\rangle}{\langle x\Psi, x\Psi\rangle}$$
$$= -\log \frac{\langle (\Delta_{\Phi,\Psi})^{1/2}x\Psi, \chi_{[e^{-n},\infty)}(\Delta_{\Phi,\Psi})(\Delta_{\Phi,\Psi})^{1/2}x\Psi\rangle}{\langle x\Psi, x\Psi\rangle}$$
$$\geqslant -\log \frac{\langle (\Delta_{\Phi,\Psi})^{1/2}x\Psi, (\Delta_{\Phi,\Psi})^{1/2}x\Psi\rangle}{\langle x\Psi, x\Psi\rangle} = -\log \frac{\langle x^*\Phi, x^*\Phi\rangle}{\langle x\Psi, x\Psi\rangle}$$

since log is concave, and monotone increasing. Therefore the symmetric quadratic form

$$t_n(x\Psi, x\Psi) := \langle x\Psi, \{-\log(\Delta_{\Phi,\Psi})\chi_n + \log(\Delta_\Psi)\}x\Psi\rangle, \qquad x \in \mathcal{M}_0 \tag{21}$$

is densely defined. In addition it is closable. Now by assumption the quadratic form

$$t_\infty(x\Psi, x\Psi) := \lim_{n\to\infty} t_m(x\Psi, x\Psi), \qquad x \in \mathcal{M}_0 \tag{22}$$

is positive,

$$\begin{aligned} 0 &\leq t_\infty(x\Psi, x\Psi) \\ &\leq \langle x\Psi, \log(\Delta_\Psi)x\Psi\rangle - \log\left(\frac{\varphi(xx^*)}{\psi(x^*x)}\right)\psi(x^*x), \qquad x \in \mathcal{M}_0. \end{aligned}$$

It is densely defined, and by Theorem S.14 in [17] it is closable. Thus the closure t_R of t_∞ uniquely defines a self-adjoint operator [17] which we again denote $R(\varphi \mid \psi)$.

At the same time

$$\lim_{n\to\infty} \langle x\Psi, \log(\Delta_{\Phi,\Psi})x_n x\Psi\rangle = \langle x\Psi, \log(\Delta_{\Phi,\Psi})x\Psi\rangle, \qquad x \in \mathcal{M}_0. \tag{23}$$

Therefore

$$\begin{aligned} &\langle x\Psi, R(\varphi \mid \psi)x\Psi\rangle \\ &\quad = \langle x\Psi, \{-\log \Delta_{\Phi,\Psi} + \log \Delta_\Psi\}x\Psi\rangle \\ &\quad = \left\langle x\Psi, \lim_{t\to 0} \frac{i}{t}\{(\Delta_{\Phi,\Psi})^{it}(\Delta_\Psi)^{-it} - \mathbb{1}\}x\Psi\right\rangle \\ &\quad = \left\langle x\Psi, j(u^*)j(u) \lim_{t\to 0} \frac{i}{t}\{(\Delta_{\Phi,\Psi})^{it}(\Delta_\Psi)^{-it} - \mathbb{1}\}x\Psi\right\rangle \\ &\quad = \left\langle x\Psi, j(u^*) \lim_{t\to 0} \frac{i}{t}\{(\Delta_{\Phi,\Psi})^{it}(\Delta_\Psi)^{-it} - \mathbb{1}\}j(u)x\Psi\right\rangle \\ &\quad = \langle x\Psi, j(u^*)R(\varphi \mid \psi)j(u)x\Psi\rangle \end{aligned} \tag{24a}$$

for $x \in \mathcal{M}_0$, and all unitaries $u \in \mathcal{M}$, where we have used (4a). This shows that $R(\varphi \mid \psi)$ is affiliated with $\mathcal{M}$. qed

Remark. The conclusion of the theorem remains correct if we only assume that $R(\varphi \mid \psi)$ is semi-bounded instead of positive. We will shortly discuss the physical aspects of this fact later on.

Corollary 5. *$R(\varphi \mid \psi)$ coincides with the strong derivation of $(D\varphi : D\psi)_t$ at $t = 0$:*

$$R(\varphi \mid \psi) = s - \lim_{t\to 0} \frac{i}{t}\{(\Delta_{\Phi,\Psi})^{it}(\Delta_\Psi)^{-it} - \mathbb{1}\}. \tag{24}$$

To provide a link between the relations (17a) and (19) we need the following lemmas. The first one can easily be read off from [1]:

Lemma 6. *Given $\psi \in \mathcal{M}_{*+}$. Then for any $\varphi \in \mathcal{M}_{*+}$ there exists a sequence $\{h_n^\varphi\}$, $h_n^\varphi = (h_n^\varphi)^* \in \mathcal{M}$ such that $\varphi = \lim_{n\to\infty} \psi^{h_n^\varphi}$.*

Proof. Let J be the modular involution associated with the standard form of $\mathcal{M}$. According to Proposition 5.4 in [1] and the remark following it,

$$\mathcal{M}^{\mathcal{P}} := \{e^b \mid b \in \mathcal{M},\ Jb\Psi = b\Psi,\ \sigma_t^\psi(b) \text{ has an analytic continuation } \sigma_z^\psi(b) \text{ for } \operatorname{Im}(z) \in [-1/2, 0]\}$$

is dense in $\mathcal{P}$, and for all $a \in \mathcal{M}^{\mathcal{P}}$ there exists $a\ h = h^* = \mathcal{M}$ such that $a\Psi = \xi(\psi^h)$. Now the proposition follows from the inequalities [3, 7]

$$\|\Phi_1 - \Phi_2\|^2 \leqslant \|\varphi_1 - \varphi_2\| \leqslant \|\Phi_1 + \Phi_2\|\, \|\Phi_1 - \Phi_2\|, \qquad \varphi_i \in \mathcal{M}_{*+}, \qquad i = 1, 2. \quad \text{qed} \tag{25}$$

Lemma 7. *Given faithful φ, $\psi \in \mathcal{M}_{*+}$. For a sequence $\{\varphi_k\}$ of faithful $\varphi_k \in \mathcal{M}_{*+}$ with $\lim_{k\to\infty} \|\varphi_k - \varphi\| = 0$, we have*
(a) $\lim_{k\to\infty} (\Delta_{\Phi_k,\Psi})^{1/2} = (\Delta_{\Phi,\Psi})^{1/2}$,
(b) $\lim_{k\to\infty} (\Delta_{\Phi_k,\Psi})^{it} = (\Delta_{\Phi,\Psi})^{it}$,

both limits taken in the strong topology, the last one being uniformly in t for t varying in a compact interval.

Proof. (a)

$$\begin{aligned}
&\|(\Delta_{\Phi_k,\Psi})^{1/2} x\psi - (\Delta_{\Phi,\Psi})^{1/2} x\Psi\| \\
&\quad = \|J(\Delta_{\Phi_k,\Psi})^{1/2} x\Psi - J(\Delta_{\Phi,\Psi})^{1/2} x\Psi\| \\
&\quad = \|S_{\Phi_k,\Psi} x\Psi - S_{\Phi,\Psi} x\Psi\| \\
&\quad = \|x^*\Phi_k - x^*\Phi\| \\
&\quad \leqslant \|x^*\|\, \|\Phi_k - \Phi\| \\
&\quad = \|x\|\, \|\Phi_k - \Phi\| \\
&\quad \leqslant \|x\|\, \|\varphi_k - \varphi\|^{1/2}.
\end{aligned}$$

The last inequality follows from (25).
(b) The implication (a) $\Rightarrow$ (b) is well known (see e.g. [19]). qed

An immediate consequence of the preceding lemmas combined with Theorem VIII.21 in [17] is the following corollary.

Corollary 8. *For any pair of faithful states φ, $\psi \in \mathcal{M}_{*+}$ with $R(\varphi \mid \psi)$ bounded from below, there exists a sequence $\{h_n^\varphi\}$, $h_n^\varphi = (h_n^\varphi)^* \in \mathcal{M}$ with $\varphi = \lim_{n\to\infty} \psi^{h_n^\varphi}$ such that*

$$\lim_{n\to\infty} R(\psi^{h_n^\varphi} \mid \psi) = \lim_{n\to\infty} (-h_n^\varphi) = R(\varphi/\psi)$$

in the strong resolvent limit.

Corollary 8 insures the consistency of the positivity relation $R(\varphi/\psi) \geqslant 0$ with its extension from pairs of states, one being a 'perturbation' of the other, to general pairs of $\mathcal{M}_{*+}$ with equal support.

Supposed $-h/\beta$ in (14) corresponds to an energy transfer being positive if the work is absorbed by the system in the state ψ, an idea supported by the stability properties of the KMS-states [14, 16, 7]. Then Definition 3 appeals to be taken as a Clausius-type version of the second principle of thermodynamics: The process $\varphi \in \mathcal{M}_{*+} \mapsto \psi \in \mathcal{M}_{*+}$ is possible without energy supply from outside the system.

Definition 3 is an abstract statement. It says nothing about the concrete realization of irreversible processes.

§4. Type III_1-criterions

The expectation value of the operator $R(\varphi/\psi)$ in the state ψ coincides with Araki's definition of a relative entropy [5]:

$$\begin{aligned} S(\varphi/\psi) &:= \lim_{n\to\infty} \langle \Psi, -\log(\Delta_{\Phi,\Psi})\chi_n \Psi\rangle \\ &= \langle \Psi, R(\varphi/\psi)\Psi\rangle. \end{aligned} \tag{26}$$

This expression may be finite or $+\infty$; for normalized φ, ψ it is non-negative [5]. In special cases $S(\varphi/\psi)$ coincides with familiar thermodynamical expressions usually interpreted as a relative entropy (see §5). Now since $R(\varphi/\psi)$ is thermodynamically relevant by its relation to the KMS-condition, this relevance carries over to $S(\varphi/\psi)$, and vice versa. The following theorem gives a sufficient characterization of the type-III_1-property in terms of $S(\varphi/\psi)$.

Theorem 9. *Let $\mathcal{M}$ be a factor von Neumann algebra with separable predual, not coinciding with the complex numbers. $\mathcal{M}$ is of type III_1 if for arbitrary normalized faithful states φ, $\psi \in \mathcal{M}_{*+}$*

$$\inf_{u\in U} S(\varphi_u/\psi) = 0, \tag{27}$$

where $U := \{u \in \mathcal{M} \mid u^* = u^{-1}\}$, $\varphi_a(x) := \varphi(a^*xa)$, $\quad x \in \mathcal{M}$.

To prove the theorem we expose a result of Connes and Størmer [11].

Lemma 10. *Let $\mathcal{M}$ be a factor von Neumann algebra with separable predual, not coinciding with the complex numbers. Denote the normalized states by $\mathcal{M}^1_{*+}$, and the normalized faithful states by $\mathcal{M}^{f\,1}_{*+}$. The following statements are equivalent:*

(i) $\mathcal{M}$ is of type III_1;
(ii) $\forall(\varphi, \psi \in \mathcal{M}^1_{*+})\,\forall(\varepsilon > 0)\,\exists(u \in U): \|\varphi_u - \psi\| \leq \varepsilon$;
(iii) $\forall(\varphi, \psi \in \mathcal{M}^{f\,1}_{*+})\,\forall(\varepsilon > 0)\,\exists(u \in U): \|\varphi_u - \psi\| \leq \varepsilon$.

Proof.
(i) $\Leftrightarrow$ (ii): [11].
(ii) $\Rightarrow$ (iii): The statement (iii) is covered by (ii).
(iii) $\Rightarrow$ (ii): Given an arbitrary $\omega \in \mathcal{M}^{f\,1}_{*+}$, and $0 < \delta < 1$. Then $\tilde{\varphi} := (1-\delta)\varphi + \delta\omega$, $\tilde{\psi} := (1-\delta)\psi + \delta\omega$ are normalized and faithful for arbitrary φ, $\psi \in \mathcal{M}^1_{*+}$. Now

$$\|\varphi_u - \psi\| = \|\tilde{\varphi}_u - \tilde{\psi} + \delta(\varphi_u - \omega_u - \psi + \omega)\| \leq \|\tilde{\varphi}_u - \tilde{\psi}\| + 4\delta,$$

and $\|\varphi_u - \psi\| \leq \varepsilon$, if $\|\tilde{\varphi}_u - \tilde{\psi}\| \leq \varepsilon/2$ and $\delta = \varepsilon/8$, for any preassigned $\varepsilon > 0$ and an appropriate $u \in U$. qed

Proof of Theorem 9

$$\begin{aligned}
\langle \Psi, R(\varphi/\psi)\Psi\rangle^{1/2} &= \langle \Psi, (-\log \Delta_{\Phi,\Psi})\Psi\rangle^{1/2} \\
&= 2^{1/2}\langle \Psi, (-\log (\Delta_{\Phi,\Psi})^{1/2})\Psi\rangle^{1/2} \\
&\geq 2^{1/2}\langle \Psi, (-(\Delta_{\Phi,\Psi})^{1/2} + \mathbb{1})\Psi\rangle^{1/2} \\
&= \langle \Psi, [(\Delta_{\Phi,\Psi})^{1/2} - \mathbb{1}]^2\Psi\rangle^{1/2} \\
&= \|[(\Delta_{\Phi,\Psi})^{1/2} - \mathbb{1}]\Psi\| \\
&= \|\Phi - \Psi\| \\
&\geq \|\Phi + \Psi\|^{-1}\, \|\varphi - \psi\| \\
&\geq \tfrac{1}{2}\|\varphi - \psi\|, \qquad \varphi, \psi \in \mathcal{M}^{f\,1}_{*+}.
\end{aligned}$$

The first inequality uses $-\log(x) \geq -x + 1$, the second (25). Now the conclusion follows from Lemma 10. qed

The author was told by Prof. Araki that there exists another proof of the inequality $\|\varphi - \psi\| \leq 2\,(2S(\varphi/\psi))^{1/2}$ [20]. Lemma 7 and Theorem VIII.21 in [17] imply the following necessary condition for the type III_1-property:

Proposition 11. *Let $\mathcal{M}$ be a type III_1 factor von Neumann algebra with separable predual. Then for normalized faithful states φ, $\psi \in \mathcal{M}^{f\,1}_{*+}$ there exists a sequence of unitaries $u_k \in \mathcal{M}$ such that in the strong resolvent sense*

$$\lim_{k\to\infty} R(\varphi_{u_k} \mid \varphi) = 0. \tag{28}$$

§5. Discussion

1. In the case of a matrix algebra, if ρ_ω denotes the density matrix of a state ω we have [4],

$$R(\varphi/\psi) = -\log \rho_\varphi + \log \rho_\psi$$
$$S(\varphi/\psi) = \mathrm{tr}\,[\rho_\psi(\log \rho_\psi - \log \rho_\varphi)].$$

(ρ_φ, ρ_ψ are supposed to be strictly positive.) In the case of a commutative von Neumann algebra $L_\infty(\Omega, \mu)$ and strictly positive probability distributions f, $g \in L_1(\Omega, \mu)$, we have

$$R(f/g) = -\log f + \log g$$
$$S(f/g) = \int_\Omega \mu(d\omega) g(\omega) \log \frac{g(\omega)}{f(\omega)}$$

2. If $R(\varphi/\psi) > 0$ then φ and ψ cannot have the same norm. This is seen in the following way. Assume $\varphi = \psi^h$ and $\psi^h(\mathbb{1}) = 1$ for $h = h^* \in \mathcal{M}$. Then Araki's generalization of the Peierls–Bogoliubov inequality [2],

$$\psi^h(\mathbb{1}) \geq \exp \psi(h) \tag{29}$$

implies $\psi(h) \leqslant 0$, i.e. a negative part in the spectrum of h. But his generalization of the Golden–Thompson inequality [2]

$$\psi(e^h) \geqslant \psi^h(\mathbb{1}) \tag{30}$$

implies $\psi(e^h) \geqslant 1$, i.e. a positive part in the spectrum of h. This contradicts $R(\psi^h/\psi) = -h > 0$.

We have focussed our discussion on dissipativity. However it should be kept in mind that in physical processes dissipation and fluctuation jointly occur, the latter being not considered here. Therefore the problem of norm conservation in the dynamics of states, related to the possibility of a probabilistic interpretation, cannot be discussed definitely in the framework of the present note.

3. While a purely positive R-operator indicates stability, bounded negative parts in the spectrum of R indicate local instability. This could possibly be applied for the discussion of chaotic behaviour in the sense of sensitive dependence on initial conditions when the dynamics of states is non-linear.

4. Thermodynamical equilibrium representations of physical systems in the infinite volume limit idealization with non-zero and non-infinite temperatures lead to type III-factors (resp. direct integrals of type III-factors). In particular the type III_1 arises for important physical examples. The case of type III_λ, $\lambda \neq 1$, may come from an oversimplification. For example [12] an infinite chain of harmonic oscillators all having exactly the same frequency exhibits the type III_λ-property, $\lambda \neq 1$, but this arrangement cannot be controlled experimentally.

Theorem 9 relates the type III_1-property to a thermodynamically relevant quantity. Therefore it should be possible to replace the infinite volume limit arguments. To be specific, we could characterize a thermodynamical reservoir as a system where for each process $\varphi \in \mathcal{M}_{*+} \mapsto \psi \in \mathcal{M}_{*+}$ possible entropic restricitions can be circumvented by reversible (unitarily implemented) inner excitations. Then Theorem 9 implies the type III_1-property.

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