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Random walks in one-dimensional random media

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Abstract. The long-time asymptotic behavior of the mean displacement, $\langle x(t)\rangle$, is investigated for continuous- and discrete-time random walks in one-dimensional random media. Exact results are obtained for model systems that contain a finite fraction of 'diodes'. Depending on the model parameters, the behavior of $\langle x(t)\rangle$ changes from $\langle x(t)\rangle \propto t$ to $\langle x(t)\rangle \propto t^{\nu}F(\ln t)$, where $\nu < 1$ and where F is a periodic function. This remarkable phenomenon can be related to temporal self-similarity properties of the stochastic process, and is also observed in models without 'diodes'. Corresponding numerical simulations are presented, and an Effective-Medium approximation is shown to be applicable only if $\langle x(t)\rangle \propto t$.

I. Introduction

Random walks in one-dimensional random media have recently attracted considerable attention, in pure mathematics as well as in the physical sciences. A number of remarkable results have been obtained, both for continuous-time [1–6] and for discrete-time [4, 7–10] models. Some of these results have been used to explain the highly unusual transport properties of certain quasi-one-dimensional materials [11–13].

The models investigated correspond to random walks in the one-dimensional lattice $\mathbb{Z}$ with only nearest-neighbor transition rates $W_n^{\mp}$ (continuous-time models) or transition probabilities $p_n^{\mp}$, $p_n^+ + p_n^- = 1$ (discrete-time models), where $n \in \mathbb{Z}$ labels the lattice sites and where the $\pm$ superscripts refer to transitions to the right ($n \to n+1$) and to the left ($n \to n-1$), respectively. The transition rates, $W_n^{\pm}$, or transition probabilities, p_n^{+}, respectively, are assumed to be random variables with a given probability distribution.

Continuous-time models with symmetric transition rates ($W_n^+ = W_{n+1}^-$), which are identically and independently distributed, have been studied very extensively [1–3]. The asymptotic behaviors of the autocorrelation function, of the mean-square displacement, and of the frequency-dependent conductivity have been shown to exhibit some interesting and unusual features, in particular for transition rate distributions with diverging negative moments. Corresponding models have successfully been used to explain the anomalous transport properties of different quasi-one-dimensional materials [11–13].

Attention has also been directed to models with random, but asymmetric transition rates. For example, asymmetry may be induced by a uniform electric field, in which case one observes interesting crossover effects in the behavior of the field- and frequency-dependent conductivity [6, 14, 15]. Very recently, the present authors have investigated models with intrinsically asymmetric transition rates [4], where the pairs of transition rates associated with the same bond, $\{W_n^+, W_{n+1}^-\}$, are assumed to be independent and identically distributed random variables. These models are thus essentially equivalent to discrete-time models with random transition probabilities p_n^+, for which one has made the remarkable observation that under certain conditions the mean displacement $\langle x(t)\rangle$ increases slower than linearly in time [4, 7–10]. An even more surprising phenomenon, however, has been observed [4] in model systems which contain a finite fraction of 'diodes', $\{W_n^+, W_{n+1}^-\} = \{u, 0\}$. In such systems, the asymptotic behavior of $\langle x(t)\rangle$ can be determined exactly and, in addition to an overall sublinear increase, may exhibit superimposed, nondecaying oscillations as a function of ln t. It has further been conjectured [4] that similar phenomena should also be observed in systems with more general discrete transition rate distributions.

In the present paper, we shall examine the origin and observability of these asymptotic oscillations in some more detail. In Section II, we introduce continuous-time random walks with intrinsically asymmetric transition rates, and derive formal expressions for the mean displacement $\langle x(t)\rangle$ and for its Laplace transform. In Section III, the asymptotic behavior of $\langle x(t)\rangle$ is determined rigorously for a model system that contains a finite fraction of 'diodes', and in Section IV the corresponding results are derived for the discrete-time analogue of the 'diode-model'. In Section V, the observed asymptotic oscillations are interpreted in terms of a selfsimilar clustering of waiting times, and the results of numerical simulations are presented and discussed in Section VI. In Section VII, finally, we introduce and analyze a selfconsistent Effective-Medium approximation. We show that it only leads to reliable results in situations where $\langle x(t)\rangle \propto t$.

II. Continuous-time random walks with random, asymmetric transition rates

We consider a particle whose motion on the one-dimensional lattice $\mathbb{Z}$ is described by a continuous-time random walk with nearest-neighbor transition rates $W_n^\pm$, where $n \in \mathbb{Z}$ labels the lattice sites and where the $\pm$ superscripts refer to transitions to the right and left (from n to $n\pm 1$), respectively. The probabilities $P_n(t)$ of finding the particle at site n at time $t \geq 0$ are then determined by the master equation

$$\frac{dP_n}{dt} = W_{n-1}^+ P_{n-1} + W_{n+1}^- P_{n+1} - (W_n^+ + W_n^-)P_n, \tag{2.1}$$

and we shall always assume that

$$P_n(0) = \delta_{n0} \tag{2.2}$$

i.e. at time $t = 0$ the particle starts at site $n = 0$.

The transition rates $W_n^\pm$ are random variables and in general asymmetric, i.e. the two transition rates associated with the same bond, W_n^+ and W_{n+1}^-, can be different. To be specific, we shall assume that the pairs $\{W_n^+, W_{n+1}^-\}$ are independent random variables, identically distributed according to some given probability density $\rho(w^+, w^-)$.

We shall mainly be interested in the effects of a random asymmetry, W_{n+1}^{-}/W_n^{+}, in particular with respect to the long-time asymptotic behavior of the average mean displacement, $\langle x(t)\rangle$. The latter can be expressed as

$$\langle x(t)\rangle = \sum_{n=-\infty}^{\infty} n\langle P_n(t)\rangle, \tag{2.3}$$

where $\langle\cdot\cdot\cdot\rangle$ refers to the average with respect to the transition rate distribution.

Formally, our master equation (2.1) can be solved by means of a Laplace transformation with respect to t. Taking equation (2.2) into account, the Laplace transform of (2.1) becomes

$$(z + W_n^{+} + W_n^{-})\tilde{P}_n - W_{n-1}^{+}\tilde{P}_{n-1} - W_{n+1}^{-}\tilde{P}_{n+1} = \delta_{n0} \tag{2.4}$$

where

$$\tilde{P}_n(z) = \int_0^{\infty} dt e^{-zt} P_n(t). \tag{2.5}$$

The solution of equation (2.4) can be written in the form

$$\tilde{P}_0(z) = (z + X_0 + Y_0)^{-1}, \tag{2.6a}$$

$$\tilde{P}_n(z) = \tilde{P}_0(z)\prod_{m=1}^{n}\frac{X_{m-1}}{z + X_m}, \qquad n = 1, 2, \ldots, \tag{2.6b}$$

$$\tilde{P}_{-n}(z) = \tilde{P}_0(z)\prod_{m=1}^{n}\frac{Y_{m-1}}{z + Y_m}, \qquad n = 1, 2, \ldots, \tag{2.6c}$$

where the X_n and Y_n are infinite continued fractions, recursively defined by

$$X_n = \frac{W_n^{+}}{1 + \dfrac{W_{n+1}^{-}}{z + X_{n+1}}}, \qquad n = 0, 1, 2, \ldots, \tag{2.7a}$$

$$Y_n = \frac{W_{-n}^{-}}{1 + \dfrac{W_{-n-1}^{+}}{z + Y_{n+1}}}, \qquad n = 0, 1, 2, \ldots. \tag{2.7b}$$

The Laplace transform, $\langle\tilde{x}(z)\rangle$, of the average mean displacement $\langle x(t)\rangle$, can finally be expressed as

$$\langle\tilde{x}(z)\rangle = z^{-1}\langle\tilde{v}(z)\rangle, \tag{2.8}$$

where

$$\begin{aligned}\langle\tilde{v}(z)\rangle &= \sum_{n=0}^{\infty}\langle X_n\tilde{P}_n - Y_n\tilde{P}_{-n}\rangle \\ &\equiv \langle\tilde{v}_{+}(z)\rangle - \langle\tilde{v}_{-}(z)\rangle.\end{aligned} \tag{2.9}$$

In the absence of disorder, i.e. if the transition rates are independent of n, $W_n^{\pm} = W^{\pm}$, one immediately obtains

$$\tilde{v}(z) = z^{-1}(W^{+} - W^{-}), \tag{2.10}$$

so that

$$x(t)=(W^+-W^-)t\equiv vt. \tag{2.11}$$

The mean displacement thus varies linearly with time, and $v=W^+-W^-$ can be identified as a drift velocity.

If the transition rates $W_n^\pm$ are random variables, however, the situation becomes very complex, and rigorous results for the asymptotic behavior of $\langle x(t)\rangle$ have only been obtained for a specific model system [4]. These results, which reveal some remarkable aspects of the general problem, will be derived in some detail in the following section.

III. The 'diode-model'

An interesting and nontrivial model for which the $t\to\infty$ asymptotic behavior of $\langle x(t)\rangle$ can be determined exactly, has been introduced in Ref. [4]. It is characterized by a probability density $\rho(w^+,w^-)$ of the form

$$\rho(w^+,w^-)=(1-p)\,\delta(w^+-u)\,\delta(w^-)+p\,\delta(w^+-\lambda v)\,\delta(w^--v), \tag{3.1}$$

with λ, u and v positive, and with $0<p<1$. Any realization of this so-called 'diode-model' thus contains a fraction $(1-p)$ of 'diodes', $\{W_n^+, W_{n+1}^-\}=\{u,0\}$, and a fraction p of two-way bonds, $\{W_n^+, W_{n+1}^-\}=\{\lambda v, v\}$.

As sketched in Ref. [4], the $t\to\infty$ behavior of $\langle x(t)\rangle$ can be determined via an investigation of the $z\to 0$ behavior of its Laplace transform, $\langle\tilde{x}(z)\rangle$, and in the following we shall give a detailed derivation of the corresponding results. We recall that

$$\tilde{x}(z)=z^{-1}\tilde{v}(z)=z^{-1}[\tilde{v}_+(z)-\tilde{v}_-(z)] \tag{3.2}$$

and, using equation (2.9) and (2.6), we may write

$$\tilde{v}_+(z)=X_0\tilde{P}_0\sum_{n=0}^{\infty}\left(\prod_{m=1}^{n}\frac{X_m}{z+X_m}\right), \tag{3.3a}$$

$$\tilde{v}_-(z)=Y_0\tilde{P}_0\sum_{n=0}^{\infty}\left(\prod_{m=1}^{n}\frac{Y_m}{z+Y_m}\right), \tag{3.3b}$$

where $\tilde{P}_0=(z+X_0+Y_0)^{-1}$ and where an empty product is equal to 1. The quantities X_n and Y_n are defined in equations (2.7a) and (2.7b), respectively, and in general are infinite continued fractions depending on the random variables $W_n^\pm$. For the diode-model, however, their determination is simplified considerably. We have

$$X_n=\begin{cases} u & \text{if} \quad W_{n+1}^-=0\\ \dfrac{\lambda v}{1+\dfrac{v}{z+X_{n-1}}} & \text{otherwise}\end{cases} \tag{3.4a}$$

$$Y_n = \begin{cases} 0 & \text{if} \quad W^-_{-n} = 0 \\ \dfrac{v}{1 + \dfrac{\lambda v}{z + Y_{n+1}}} & \text{otherwise} \end{cases} \tag{3.4b}$$

so that for each segment between two 'diodes' the X_n and Y_n can be calculated separately, and with the same respective recursion.

We now let N denote the smallest integer ≥ 0 such that the bond $(-N-1, -N)$ is a diode, i.e. $W^-_{-N} = 0$. It follows that $Y_N = 0$, and therefore

$$\tilde{v}_-(z) = 0 \quad \text{if} \quad N = 0, \tag{3.5a}$$

and

$$\tilde{v}_-(z) = z\tilde{P}_0 \frac{a_{N-1}}{1 - a_{N-1}} \sum_{n=0}^{N-1} \left(\prod_{m=1}^{n} a_{N-1-m} \right) \quad \text{if} \quad N \geq 1, \tag{3.5b}$$

where

$$a_n = \frac{Y_{N-1-n}}{z + Y_{N-1-n}}, \qquad n = 0, 1, \ldots, N-1. \tag{3.6}$$

The a_n are determined by the recursion

$$a_n = \frac{v}{z + (1+\lambda)v - \lambda v a_{n-1}}, \tag{3.7}$$

with

$$a_0 = \frac{v}{z + (1+\lambda)v}. \tag{3.8}$$

This is solved by

$$a_n = \mu \frac{1 - \kappa^{n+1}}{1 - \kappa^{n+2}}, \qquad n = 0, 1, \ldots, N-1, \tag{3.9}$$

where

$$\mu \equiv \frac{v}{\xi_+}, \qquad \kappa \equiv \frac{\xi_-}{\xi_+}, \tag{3.10}$$

and

$$\xi_\pm = \tfrac{1}{2}\{z + (1+\lambda)v \pm [z^2 + 2(1+\lambda)vz + (1-\lambda)^2 v^2]^{1/2}\}. \tag{3.11}$$

Together with equation (3.5b), this leads to the following expression for $\tilde{v}_-(z)$,

$$\tilde{v}_-(z) = z\tilde{P}_0 \frac{\mu}{1 - \mu + (\mu - \kappa)\kappa^N} \left[\frac{1 - \mu^N}{1 - \mu} - \kappa \frac{\mu^N - \kappa^N}{\mu - \kappa} \right], \tag{3.12}$$

which holds for $N = 0, 1, 2, \ldots$.

Now let N_1 be the smallest integer ≥ 0 such that the bond $(N_1, N_1 + 1)$ is a diode, i.e. $W^-_{N_1+1} = 0$. From equation (3.4a) it follows that $X_{N_1} = u$, and that the quantities

$$b_n = \frac{X_{N_1-1}}{z + X_{N_1-n}}, \qquad n = 0, 1, \ldots, N_1 \tag{3.13}$$

obey the recursion

$$b_n = \frac{\lambda v}{z + (1+\lambda)v - v b_{n-1}}, \tag{3.14}$$

with

$$b_0 = \frac{u}{u+z}. \tag{3.15}$$

The solution of this recursion is given by

$$b_n = \lambda\mu \frac{(u+z)(1-\kappa^n) - u\mu(1-\kappa^{n-1})}{(u+z)(1-\kappa^{n+1}) - u\mu(1-\kappa^n)}, \qquad n = 0, 1, \ldots, N_1, \tag{3.16}$$

with μ and κ as defined in equations (3.10) and (3.11). If we write

$$\tilde{v}_+(z) = X_0 \tilde{P}_0 \sum_{n=0}^{\infty} \left(\prod_{m=1}^{n} \frac{X_m}{z + X_m} \right) = X_0 \tilde{P}_0 \sum_{n=0}^{\infty} q_n, \tag{3.17}$$

we see that equation (3.16) determines

$$q_n = \prod_{m=1}^{n} b_{N_1-m}, \qquad n = 1, 2, \ldots, N_1. \tag{3.18}$$

In particular, we have

$$q_{N_1} = r_{N_1-1}, \tag{3.19}$$

if we define

$$r_n = b_n b_{n-1} \cdots b_0 = \frac{u(1-\kappa)(\lambda\mu)^n}{(u+z)(1-\kappa^{n+1}) - u\mu(1-\kappa^n)}, \qquad n \geq 0, \tag{3.20}$$

and

$$r_{-1} = q_0 = 1. \tag{3.21}$$

The q_n for $n > N_1$ can now be determined in exactly the same way. Let us assume that

$$W_{n+1}^- = 0 \quad \text{if} \quad n = \sum_{i=1}^{k} N_i + k, \qquad N_i \geq 0, \qquad k = 1, 2, \ldots, \tag{3.22}$$

i.e. the kth and $(k+1)$st diode are separated by a segment of N_{k+1} two-way bonds. Inductively, we then obtain

$$q_n = r_{N_1-1} r_{N_2} r_{N_3} \cdots r_{N_k} \prod_{m=0}^{s} b_{N_{k+1}-m}$$
$$\text{for} \quad n = N_1 + \cdots + N_k + k - s, \qquad s = 0, 1, \ldots, N_{k+1}, \tag{3.23}$$

and finally

$$\tilde{v}_+(z) = \frac{z\tilde{P}_0}{1-b_{N_1}} \sum_{k=1}^{\infty} \left(\prod_{j=1}^{k-1} r_{N_j} \right) s_{N_k}, \tag{3.24}$$

where

$$\begin{aligned} r_n &= b_n b_{n-1} \cdots b_0, \qquad n \geq 0, \\ r_{-1} &= 1, \\ s_n &= b_n + b_n b_{n-1} + \cdots + b_n b_{n-1} \cdots b_0, \qquad n \geq 0, \\ s_{-1} &= 0, \end{aligned} \tag{3.25}$$

and where we have observed that

$$X_0 = \frac{z b_{N_1}}{1-b_{N_1}}. \tag{3.26}$$

If we further observe that

$$Y_0 = \frac{z a_{N-1}}{1-a_{N-1}}, \tag{3.27}$$

the prefactor $z\tilde{P}_0 = z/(z + X_0 + Y_0)$ in equations (3.12) and (3.24) can be expressed as

$$z\tilde{P}_0 = \frac{(1-a_{N-1})(1-b_{N_1})}{1-a_{N-1}b_{N_1}}. \tag{3.28}$$

We now have to average $\tilde{v}_+(z)$ and $\tilde{v}_-(z)$ with respect to the transition rate distribution specified by equation (3.1). If we define

$$\langle \cdots \rangle_n \equiv (1-p) \sum_{n=0}^{\infty} p^n \cdots, \tag{3.29}$$

we may write

$$\langle \tilde{v}_-(z) \rangle = \left\langle z\tilde{P}_0 \frac{\mu}{1-\mu+(\mu-\kappa)\kappa^N} \left[\frac{1-\mu^N}{1-\mu} - \kappa \frac{\mu^N - \kappa^N}{\mu - \kappa} \right] \right\rangle_{N,N_1}, \tag{3.30}$$

and

$$\langle \tilde{v}_+(z) \rangle = \left\langle \frac{z\tilde{P}_0}{1-b_{N_1}} s_{N_1} \right\rangle_{N,N_1} + \left\langle \frac{z\tilde{P}_0}{1-b_{N_1}} r_{N_1} \right\rangle_{N,N_1} \frac{\langle s_n \rangle_n}{1-\langle r_n \rangle_n}. \tag{3.31}$$

In Appendix B we show that

$$\left\langle \frac{z\tilde{P}_0}{1-b_{N_1}} s_{N_1} \right\rangle_{N,N_1} \Bigg|_{z=0} = \langle s_n \rangle_n \Big|_{z=0} = (1-p)^{-1}, \tag{3.32}$$

$$\left\langle \frac{z\tilde{P}_0}{1-b_{N_1}} r_{N_1} \right\rangle_{N,N_1} \Bigg|_{z=0} = \langle r_n \rangle_n \Big|_{z=0} = 1, \tag{3.33}$$

$$\langle \tilde{v}_-(z) \rangle \to 0 \quad \text{as} \quad z \to 0, \tag{3.34}$$

so that the $z \to 0$ asymptotic behavior of $\langle \tilde{v}(z) \rangle = \langle \tilde{v}_+(z) \rangle - \langle \tilde{v}_-(z) \rangle$ is determined by

$$\langle\tilde{v}(z)\rangle \sim \frac{(1-p)^{-1}}{1-\langle r_n\rangle_n} \quad \text{as} \quad z \to 0. \tag{3.35}$$

This result implies that in the limit as $z \to 0$ it is sufficient to consider configurations with $N=0$, i.e. for which the bond $(-1,0)$ is a diode $(W_0^- = 0)$. The asymptotic behavior of $1-\langle r_n\rangle_n$ is investigated in Appendix A, and the corresponding results for $\langle\tilde{v}(z)\rangle$ may be summarized as follows:

$$\langle\tilde{v}(z)\rangle \sim \begin{cases} v_\infty/z, & \lambda > p, \\ \alpha(1-p)^{-2}\dfrac{c}{z}\Big/\ln\dfrac{c}{z}, & \lambda = p, \\ (1-p)^{-2}\left(\dfrac{c}{z}\right)^{\nu}\Big/G\left(\beta^{-1}\ln\dfrac{c}{z}\right), & \lambda < p, \end{cases} \tag{3.36}$$

where

$$v_\infty = \frac{(\lambda-p)uv}{pu+(1-p)\lambda v}, \qquad c = \frac{(1-\lambda)^2 uv}{u+(1-\lambda)v}, \tag{3.37}$$

$$\alpha = -\ln p, \qquad \beta = -\ln\lambda, \qquad \nu = \alpha/\beta. \tag{3.38}$$

The function $G(\zeta)$,

$$G(\zeta) = \sum_{k=-\infty}^{\infty} \frac{e^{-\alpha(k-\zeta)}}{1+e^{-\beta(k-\zeta)}}, \tag{3.39}$$

is periodic, $G(\zeta+1) = G(\zeta)$, and can be expressed as

$$G(\zeta) = \sum_{k=-\infty}^{\infty} g_k e^{2\pi i k\zeta}, \tag{3.40}$$

with

$$g_k = \frac{\pi}{\beta}\left(\sin\pi\nu\cosh\frac{2\pi^2 k}{\beta} - i\cos\pi\nu\sinh\frac{2\pi^2 k}{\beta}\right)^{-1}. \tag{3.41}$$

If we observe that $\langle\tilde{x}(z)\rangle = z^{-1}\langle\tilde{v}(z)\rangle$, the long-time asymptotic behavior of $\langle x(t)\rangle$ follows from general theorems about inverse Laplace transforms [16], and we finally obtain

$$\langle x(t)\rangle \sim \begin{cases} v_\infty t, & \lambda > p, \\ \alpha(1-p))^{-2} ct/\ln(ct), & \lambda = p \\ (1-p)^{-2}(ct)^{\nu}F(\beta^{-1}\ln(ct)) & \lambda < p, \end{cases} \tag{3.42}$$

where F is periodic with period 1,

$$F(\tau) = \sum_{n=-\infty}^{\infty} \frac{f_n}{\Gamma(\nu+1+2\pi i n/\beta)} e^{2\pi i n\tau}. \tag{3.43}$$

The coefficients f_n are determined by

$$\sum_{n=-\infty}^{\infty} f_n e^{2\pi i n\tau} = \left(\sum_{k=-\infty}^{\infty} g_k e^{2\pi i k\tau}\right)^{-1}, \tag{3.44}$$

where the g_k are defined in equation (3.41).

For $\lambda < p$, the average mean displacement thus not only increases slower than linearly in time, $\langle x(t)\rangle \sim t^{\nu}$ with $\nu = \ln p/\ln \lambda < 1$, but exhibits superimposed, non-decaying oscillations as a function of $\ln t$. As $\langle x(t)\rangle$ represents an average over all possible configurations of 'diodes' and 'two-way bonds', the persistence of these oscillations is very remarkable and rather surprising. In Section V, we shall analyze the origin of such oscillations from a more intuitive point of view.

We further note, and this will be evident from the results of the following section, that equation (3.42) is also valid for a site-disorder version of our diode-model. In this model, the independent random variables are the pairs, $\{W_n^+, W_n^-\}$, of transition rates associated with the same site, rather than with the same bond as in the model analyzed above.

IV. Discrete-time random walks, and the discrete-time analog of the 'diode-model'

In this section, we turn our attention to discrete-time ($t \in \mathbb{Z}_+$) random walks, and our corresponding one-dimensional models can be characterized as follows. If $x_t = n \in \mathbb{Z}$ denotes the position of the particle at time $t \in \mathbb{Z}_+$, we assume that $x_{t+1} = n+1$ with probability p_n^+, and $x_{t+1} = n-1$ with probability $p_n^- = 1 - p_n^+$. With $P_n(t)$ denoting the probability that $x_t = n$, we thus have

$$P_n(t+1) = p_{n-1}^+ P_{n-1}(t) + p_{n+1}^- P_{n+1}(t), \tag{4.1}$$

and we assume as usual that

$$P_n(0) = \delta_{n0}, \tag{4.2}$$

i.e. at time $t = 0$ the particle starts at the origin. The transition probabilities $p_n^+ (p_n^+ + p_n^- = 1)$ are assumed to be independent and identically distributed random variables.

If we introduce the z-transform [17] $\hat{P}_n(z)$ of $P_n(t)$,

$$\hat{P}_n(z) = \sum_{t=0}^{\infty} P_n(t) z^{-t}, \tag{4.3}$$

it follows from equation (4.1) and (4.2) that

$$z\hat{P}_n - p_{n-1}^+ \hat{P}_{n-1} - p_{n+1}^- \hat{P}_{n+1} = z\delta_{n0}. \tag{4.4}$$

Equation (4.4) is formally solved by

$$\hat{P}_0(z) = \frac{z}{(z-1) + X_0 + Y_0}, \tag{4.5a}$$

$$\hat{P}_n(z) = \frac{X_{n-1}}{(z-1) + X_n} \hat{P}_{n-1}(z), \qquad n \geq 1, \tag{4.5b}$$

$$\hat{P}_{-n}(z) = \frac{Y_{n-1}}{(z-1) + Y_n} \hat{P}_{-(n-1)}(z), \qquad n \geq 1, \tag{4.5c}$$

where X_n and Y_n are infinite continued fractions, recursively determined by

$$X_n = \cfrac{p_n^+}{1 + \cfrac{p_{n+1}^-}{(z-1) + X_{n+1}}}, \qquad n \geq 0, \tag{4.6a}$$

$$Y_n = \cfrac{p_{-n}^-}{1 + \cfrac{p_{-n-1}^+}{(z-1) + Y_{n+1}}}, \qquad n \geq 0, \tag{4.6b}$$

For the z-transform $\hat{x}(z)$ of the mean displacement $x(t)$,

$$x(t) = \sum_{n=-\infty}^{\infty} n P_n(t), \tag{4.7}$$

we finally obtain

$$\begin{aligned}(z-1)\hat{x}(z) &= \sum_{n=0}^{\infty} X_n \hat{P}_n - \sum_{n=0}^{\infty} Y_n \hat{P}_{-n} \\ &= X_0 \hat{P}_0 \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{X_k}{(z-1) + X_k} \right) - Y_0 \tilde{P}_0 \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{Y_k}{(z-1) + Y_k} \right).\end{aligned} \tag{4.8}$$

A function $f(t)$, originally defined for $t \in \mathbb{Z}_+$, can be extended to $t \in \mathbb{R}_+$ by setting

$$f(t) = f(\mathrm{INT}(t)), \qquad t \in \mathbb{R}_+, \tag{4.9}$$

where $\mathrm{INT}(t)$ denotes the largest integer smaller or equal to t. Its Laplace transform,

$$\tilde{f}(s) = \int_0^{\infty} dt f(t) e^{-st}, \tag{4.10}$$

and its z-transform,

$$\hat{f}(z) = \sum_{t=0}^{\infty} f(t) z^{-t}, \tag{4.11}$$

are then related by

$$\tilde{f}(s) = \frac{1 - e^{-s}}{s} \hat{f}(e^s), \tag{4.12}$$

so that

$$\tilde{f}(s) \sim \hat{f}(1+s), \qquad s \to 0. \tag{4.13}$$

The $t \to \infty$ behavior of the average mean displacement, $\langle x(t) \rangle$, can thus be determined from the $s \to 0$ behavior of $\langle \hat{x}(1+s) \rangle$ via the usual theorems about inverse Laplace transforms. We further observe that, as $s \to 0$, $\hat{x}(1+s)$ becomes identical to the $\tilde{x}(s)$ for a site-disordered continuous-time model (see Section III), with $p_n^{\pm}$ playing the role of $W_n^{\pm}$.

Let us now consider the discrete-time version of our 'diode-model', defined by the following probability distribution for the transition probabilities p_n^+,

$$p_n^+ = \begin{cases} 1, & \text{with probability } 1-p, \\ \sigma, & \text{with probability } p, \end{cases} \tag{4.14}$$

where $0<\sigma<1$, and where we recall that $p_n^- = 1-p_n^+$. The $s\to 0$ asymptotic behavior of $\langle \hat{x}(1+s)\rangle$ can then be analyzed with methods similar to those presented in Section III, and we obtain

$$s\langle \hat{x}(1+s)\rangle \sim \frac{(1-p)^{-1}}{1-\langle R\rangle}, \qquad s\to 0, \tag{4.15}$$

where

$$\langle R\rangle = (1-p)\sum_{n=0}^{\infty} p^n R_n, \tag{4.16}$$

$$R_n(1+s) = \frac{\sigma^n(\eta_+ - \eta_-)}{(1+s)(\eta_+^{n+1} - \eta_-^{n+1}) - (1-\sigma)(\eta_+^n - \eta_-^n)}, \tag{4.17}$$

and

$$\eta_\pm = \tfrac{1}{2}[1+s \pm \sqrt{(1+s)^2 - 4\sigma(1-\sigma)}]. \tag{4.18}$$

A comparison with equation (3.20) shows that

$$R_n(1+s) = r_n(s) \tag{4.19}$$

if the following identifications are made,

$$u = 1, \qquad v = 1-\sigma, \qquad \lambda = \frac{\sigma}{1-\sigma}. \tag{4.20}$$

Equation (3.35) then implies that the asymptotic behavior of the discrete-time diode-model is obtained from that of its continuous-time counterpart simply by making the substitutions defined in equation (4.20). In particular, equation (3.42) thus leads to the following longtime behavior of the average mean displacement, $\langle x(t)\rangle$, in the discrete-time diode model:

$$\langle x(t)\rangle \sim \begin{cases} v_\infty t, & p\dfrac{1-\sigma}{\sigma} < 1, \\ \alpha(1-p)^{-2} ct/\ln(ct), & p\dfrac{1-\sigma}{\sigma} = 1, \\ (1-p)^{-2}(ct)^\nu F(\beta^{-1}\ln(ct)), & p\dfrac{1-\sigma}{\sigma} > 1, \end{cases} \tag{4.21}$$

where

$$v_\infty = \frac{\sigma - p(1-\sigma)}{\sigma + p(1-\sigma)}, \qquad c = \frac{(1-2\sigma)^2}{2(1-\sigma)^2}, \tag{4.22}$$

$$\alpha = -\ln p, \qquad \beta = -\ln\frac{\sigma}{1-\sigma}, \qquad \nu = \frac{\alpha}{\beta}, \tag{4.23}$$

and where the periodic function F is defined by equation (3.43), with β and ν as in equation (4.23).

This discrete-time diode-model has first been considered by Solomon [8]. His results about the limiting distribution of x_t, the position of the particle after time t, do not immediately lead to explicit expressions for the average mean displacement $\langle x(t)\rangle$. Our respective results of equations (4.21) to (4.23), however, seem to be in complete agreement with his theorems.

V. Temporal self-similarity aspects

We consider the discrete-time version of the diode-model and shall try to interpret the corresponding motion of a particle in terms of average waiting times at m-tuples of consecutive 'two-way sites'.

We recall that two-way sites are characterized by $p_n^+ = \sigma$, $p_n^- = 1-\sigma$, and that they occur with probability p. Very roughly, the average separation of corresponding m-tuples can thus be estimated as p^{-m}.

The average waiting time, τ_m, at an m-tuple of consecutive two-way sites is defined as the difference between the average and minimum time, respectively, needed to cross such an m-tuple from left to right. Mathematically, this can be expressed as

$$\tau_m(\sigma) = x_m^{(1)}(\sigma) - m, \tag{5.1}$$

where $x_m^{(1)}$ is the 1-component of the m-dimensional vector x_m determined by

$$M_m(\sigma)x_m(\sigma) = a_m(\sigma), \tag{5.2}$$

with $M_m(\sigma)$ denoting the $m \times m$ matrix

$$M_m(\sigma) = \begin{bmatrix} \sigma & -\sigma & & & & 0 \\ \sigma-1 & 1 & -\sigma & & & \\ & \sigma-1 & 1 & -\sigma & & \\ & & \ddots & \ddots & \ddots & \\ 0 & & & \sigma-1 & 1 & -\sigma \\ & & & & \sigma-1 & 1 \end{bmatrix} \tag{5.3}$$

and $a_m(\sigma)$ the m-dimensional vector

$$a_m(\sigma) = \begin{bmatrix} 2-\sigma \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}. \tag{5.4}$$

It follows that

$$\tau_m(\sigma) = 2\left(\frac{1-\sigma}{\sigma}\right)^m \sum_{j=1}^{m} j\left(\frac{\sigma}{1-\sigma}\right)^{j-1}, \tag{5.5}$$

and in particular we have

$$\lim_{m\to\infty} \frac{\tau_m(\sigma)}{\tau_{m-1}(\sigma)} = \frac{1-\sigma}{\sigma}, \qquad 0<\sigma\leq\tfrac{1}{2}. \tag{5.6}$$

Very roughly, we can therefore say that the waiting times are concentrated around values $\tau_m \sim [(1-\sigma)/\sigma]^m$ which occur with respective probabilities $\sim p^m$. On the average, p^{-1} waiting times of $\tau \sim (1-\sigma)/\sigma$ occur before a waiting time of $\tau \sim [(1-\sigma)/\sigma]^2$; p^{-1} clusters of this type occur before a waiting time of $\tau \sim [(1-\sigma)/\sigma]^3$; etc.

If $p > \sigma/(1-\sigma)$, the average waiting time becomes infinite,

$$\bar{\tau} \sim \sum_{m=1}^{\infty} p^m \left(\frac{1-\sigma}{\sigma}\right)^m = \infty, \tag{5.7}$$

and the set of waiting times exhibits self-similar clustering with an average fractal dimension of

$$\nu = \ln p/\ln\left(\frac{\sigma}{1-\sigma}\right) < 1. \tag{5.8}$$

The results of Shlesinger and Hughes [18] about this type of waiting time distributions then imply that

$$\langle x(t)\rangle \sim t^{\nu}\cdot F(\ln t), \qquad t\to\infty, \tag{5.9}$$

where ν is given in equation (5.8), and where F is periodic with period $\beta = \ln[(1-\sigma)/\sigma]$. This is in complete agreement with the exact results of section IV, and we conclude that the oscillating behavior of $\langle x(t)\rangle$ for $p>\sigma/(1-\sigma)$ can be related to the temporal self-similarity properties of the random walk.

The average time, t_k, it takes a particle to go beyond the first k-tuple of consecutive two-way sites it encounters on his path, can now be estimated as follows,

$$t_k \approx p^{-k} + \sum_{m=1}^{k} p^{m-k}\tau_m. \tag{5.10}$$

If $p>\sigma/(1-\sigma)$, then $t_k \approx \tau_k$, i.e. the average waiting time at the first k-tuple of consecutive two-way sites is much larger than the average time it takes the particle to go there. We would therefore expect $\ln[\tau^{-\nu}\langle x(t)\rangle]$ to exhibit minima at $t\approx t_k$, which is quite accurately confirmed by numerical simulations presented in the next section. We further note that

$$\lim_{k\to\infty} (\ln t_{k+1} - \ln t_k) = \ln\frac{1-\sigma}{\sigma}, \tag{5.11}$$

in agreement with our exact result for the asymptotic oscillation period.

Let us now turn our attention to models which have a more general transition probability distribution than our diode-models. The results of Kesten et al. [7] seem to indicate that asymptotic oscillations do not occur if this distribution is non-arithmetic. In the case of arithmetic transition probability distributions, however, much less is known about the precise asymptotic behavior of $\langle x(t)\rangle$. Sinai [9] has restricted his considerations to models with a symmetric distribution of $\log[p_n^+/(1-p_n^+)]$, and Derrida and Pomeau [10] have investigated a special class of binary p_n^+-distributions. Their approach seems to give the correct values for the exponent ν, but is insensitive to possible asymptotic oscillations.

On the other hand, arguments similar to those presented above indicate that such oscillations should not be an exclusive consequence of our diode-models, and that more general arithmetic transition probability distributions could also lead to a self-similar clustering of waiting times. These expectations are actually confirmed by numerical simulations (see the following section) and by the results of a new real-space renormalization approach [19].

VI. Monte Carlo simulations

In order to supplement our asymptotic results for the diode-model and to test our predictions for more general transition probability distributions, we have performed two sets of rather extended Monte Carlo simulations. The first set (Fig. 1) refers to the diode-model, and the second (Fig. 2) to the 'left-right' model of Derrida and Pomeau [10] (see below). In both cases, the numerical results for $\langle x(t)\rangle$ represent averages over at least 10 000 Monte Carlo samples, each corresponding to a different realization of the respective random medium and to a random walk of 200 000 steps. To facilitate a compact and accurate presentation of the results over the entire time range, we have introduced a reflecting boundary at $n=0$ (starting point of the random walks). This conveniently enhances the small t values of $t^{-\nu}\langle x(t)\rangle$, but has no effect on the long time asymptotic behavior.

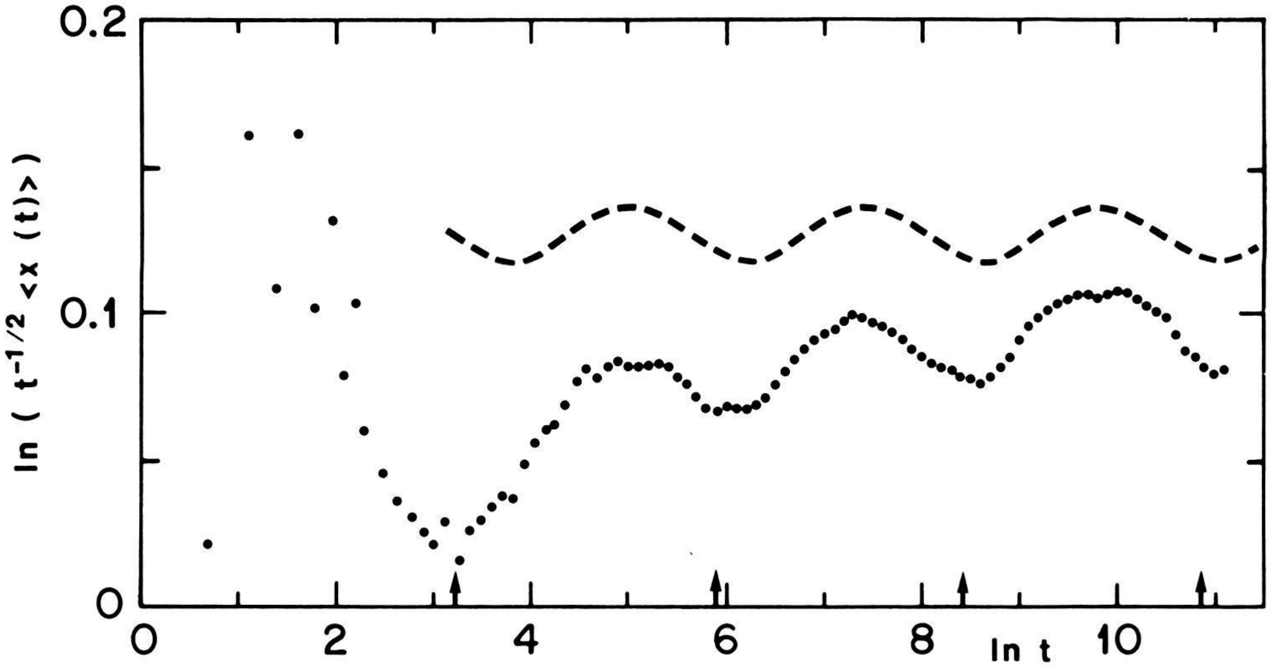

Figure 1
Average mean displacement $\langle x(t)\rangle$ for the discrete-time 'diode-model', equation (6.1), with $p=0.3$ and $\sigma/(1-\sigma)=0.09$. The numerical results represent an average over 10 000 Monte Carlo samples, and the broken curve refers to the exact asymptotic behavior [equation (4.21), case $p(1-\sigma)/\sigma>1$]. The arrows indicate simple estimates for the locations t_k of the first four minima [equations (5.10) and (5.5), $k=1,\ldots,4$].

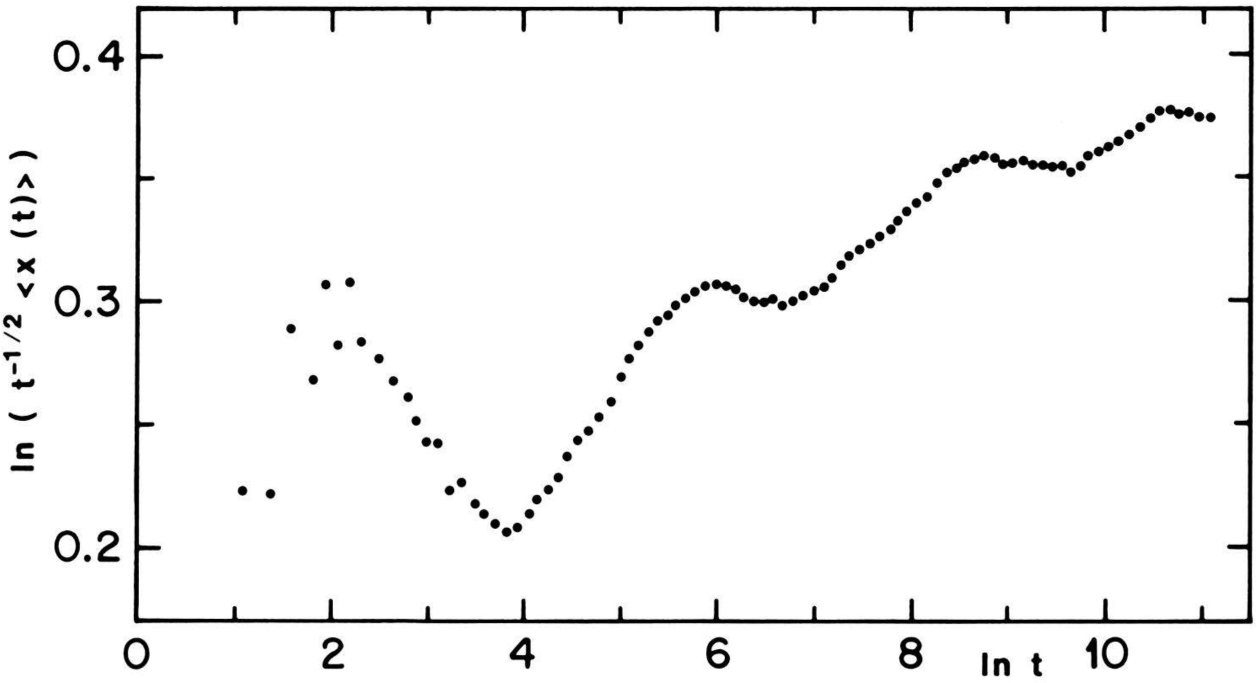

Figure 2
Average mean displacement $\langle x(t)\rangle$ for the 'left-right' model, equation (6.5), with $p=0.2$ and $\sigma/(1-\sigma)=\frac{1}{16}$. The numerical results represent averages over 15 000 Monte Carlo samples.

Without particular program optimization, 10 000 Monte Carlo runs took about 20 hours of CPU-time on a Digital Equipment Corporation VAX-11/780 computer.

The diode-model is characterized by [see equation (4.14)]

$$p_n^+ = \begin{cases} 1, & \text{with probability } 1-p, \\ \sigma, & \text{with probability } p, \end{cases} \tag{6.1}$$

and for our numerical simulations we have chosen $p=0.3$ and $\sigma/(1-\sigma)=0.09$. According to equation (4.21), the asymptotic behavior of $\langle x(t)\rangle$ is thus given by

$$\langle x(t)\rangle \sim (1-p)^{-2}(ct)^{\nu}F(\beta^{-1}\ln(ct)), \tag{6.2}$$

with an exponent ν of

$$\nu = \frac{\ln p}{\ln[\sigma/(1-\sigma)]} = \frac{1}{2} \tag{6.3}$$

and an asymptotic oscillation period of β of

$$\beta = \ln\frac{1-\sigma}{\sigma} \sim 2.41. \tag{6.4}$$

The numerical results for $\ln[t^{-1/2}\langle x(t)\rangle]$ vs. $\ln t$ are plotted in Fig. 1, together with the corresponding analytic predictions for the asymptotic behavior. Even-odd effects are important for small t, but the asymptotic oscillations develop very rapidly. For $4<\ln t<11$ we observe an apparent exponent ν of about 0.503, and the prefactor is only about 4% below its asymptotic value. Period, phase, and amplitude of the oscillations also are already very close to the asymptotic predictions. Our simple estimates of equation (5.10) for t_k, the locations of the minima, are indicated by arrows. They are confirmed very accurately by the numerical simulations.

Our second set of numerical simulations refers to a model that has been investigated by Derrida and Pomeau [10]. It is characterized by a transition probability distribution of the form

$$\rho(p_n^+) = p\,\delta(p_n^+ - \sigma) + (1-p)\,\delta(p_n^+ - (1-\sigma)), \tag{6.5}$$

so that either $p_n^-/p_n^+ = (1-\sigma)/\sigma$ [left bias, probability p] or $p_n^+/p_n^- = (1-\sigma)/\sigma$ [right bias, probability $1-p$]. We therefore call this model the *'left-right' model.*

For $\sigma < p < \frac{1}{2}$ (or $\frac{1}{2} < p < \sigma$), Derrida and Pomeau [10] predict that

$$\langle x(t)\rangle \sim t^{\nu}, \qquad t \to \infty, \tag{6.6}$$

with

$$\nu = \frac{\ln[p/(1-p)]}{\ln[\sigma/(1-\sigma)]}. \tag{6.7}$$

The same result is obtained if the theorems of Kesten et al. [7] are applied to the 'left-right' model, although the former have only been proved for non-arithmetic transition probability distributions. We believe that the expression of equation (6.7) for the exponent ν is exact, but so far no analytic information about the possible occurrence of asymptotic oscillations has been obtained.

In our Monte Carlo simulations of the 'left-right' model we have chosen $p = 0.2$ and $\sigma/(1-\sigma) = \frac{1}{16}$, so that equation (6.7) predicts $\nu = \frac{1}{2}$. Figure 2 shows the corresponding plot of $\ln[t^{-1/2}\langle x(t)\rangle]$ vs. $\ln t$. The approach to the asymptotic behavior appears to be somewhat slower than in the 'diode-model', the apparent exponent ν being about 0.515 for $5 < \ln t < 11$. Asymptotic oscillations (with a period close to $\ln[(1-\sigma)/\sigma] \sim 2.77$), however, have already clearly developed. These numerical simulations thus confirm our expectation that asymptotic oscillations should also be observed in models with arithmetic transition probability distributions that do not contain 'diodes'.

VII. A self-consistent effective-medium approximation

Random walks in media with random, symmetric transition rates ($W_n^+ = W_{n+1}^-$) have frequently, and quite successfully, been analyzed in terms of a self-consistent effective-medium approximation (EMA). The most prominent of these approaches is a straightforward generalization [1, 20] of Kirkpatrick's EMA [21] for random resistor networks, and has repeatedly been rederived with different techniques [22].

This EMA can easily be extended to the case of asymmetric transition rates [4, 23], and in the following we derive and analyze the corresponding self-consistency equations for our one-dimensional models. To be definite, let us consider *continuous-time random walks with bond-disorder* (i.e. the pairs $\{W_n^+, W_{n+1}^-\}$ of transition rates are assumed to be independent random variables). The treatment of continuous-time random walks with site-disorder and of discrete-time random walks, however, is entirely analogous (see below).

We start with equation (2.4), the master equation for the $\tilde{P}_n(z)$,

$$(z + W_n^+ + W_n^-)\tilde{P}_n - W_{n-1}^+\tilde{P}_{n-1} - W_{n+1}^-\tilde{P}_{n+1} = P_n(0) = \delta_{n0}, \tag{7.1}$$

where $\tilde{P}_n(z)$ is the Laplace transform of $P_n(t)$, the probability of finding the particle at site n at time t. In matrix notation this may be written as

$$(z \cdot \mathbb{1} + \underline{\underline{W}})\underline{\tilde{P}} = \underline{P}_0, \tag{7.2}$$

where

$$(\underline{\underline{W}})_{nm} = (W_n^+ + W_n^-)\,\delta_{nm} - W_{n-1}^+\,\delta_{n-1,m} - W_{n+1}^-\,\delta_{n+1,m}, \tag{7.3}$$

and where $\mathbb{1}$ denotes the $(\infty \times \infty)$ unit matrix. For given initial conditions $\underline{P}_0$ we have

$$\langle\underline{\tilde{P}}(z)\rangle = \langle(z \cdot \mathbb{1} + \underline{\underline{W}})^{-1}\rangle\underline{P}_0, \tag{7.4}$$

i.e. the average properties of the random walk are determined by $\langle z \cdot \mathbb{1} + \underline{\underline{W}})^{-1}\rangle$, which thus plays the role of an average resolvent (or Green's function). We now want to represent the random medium by an effective, non-random medium with (z-dependent) transition rates $W_{\text{eff}}^+(z)$ and $W_{\text{eff}}^-(z)$. The corresponding transition rate matrix is denoted by $\underline{\underline{W}}_{\text{eff}}$, and we have

$$(\underline{\underline{W}}_{\text{eff}})_{nm} = (W_{\text{eff}}^+ + W_{\text{eff}}^-)\,\delta_{nm} - W_{\text{eff}}^+\,\delta_{n-1,m} - W_{\text{eff}}^-\,\delta_{n+1,m}. \tag{7.5}$$

We define

$$\underline{\underline{\Delta}} \equiv \underline{\underline{W}}_{\text{eff}} - \underline{\underline{W}}, \tag{7.6}$$

and observe that $\underline{\underline{\Delta}}$ can be written as a sum over single-bond contributions,

$$\underline{\underline{\Delta}} = \sum_i \underline{\underline{\Delta}}_i, \tag{7.7}$$

where $\underline{\underline{\Delta}}_i$ refers to the bond $(i, i+1)$,

$$\underline{\underline{\Delta}}_i = \Delta_i^+\underline{\underline{Q}}_i^+ + \Delta_{i+1}^-\underline{\underline{Q}}_{i+1}^-, \tag{7.8}$$

with

$$\Delta_i^\pm = W_{\text{eff}}^\pm - W_i^\pm \tag{7.9}$$

and

$$(\underline{\underline{Q}}_i^\pm)_{nm} = \delta_{mi}(\delta_{ni} - \delta_{n,i\pm1}). \tag{7.10}$$

We can now define a *single-bond t-matrix*,

$$\underline{\underline{t}}_i = [\mathbb{1} - \underline{\underline{\Delta}}_i(z \cdot \mathbb{1} + \underline{\underline{W}}_{\text{eff}})^{-1}]^{-1}\,\underline{\underline{\Delta}}_i, \tag{7.11}$$

and observe that

$$\underline{\underline{t}}_i = \frac{1}{1 - F_i(z)}\,\underline{\underline{\Delta}}_i. \tag{7.12}$$

The latter equation follows then from the fact that

$$\underline{\underline{\Delta}}_i(z \cdot \mathbb{1} + \underline{\underline{W}}_{\text{eff}})^{-1}\,\underline{\underline{\Delta}}_i = F_i\,\underline{\underline{\Delta}}_i \tag{7.13}$$

where

$$F_i = \Delta_i^+ f_i^+ + \Delta_{i+1}^- f_{i+1}^-, \tag{7.14}$$

$$f_i^\pm = [(z \cdot \mathbb{1} + \underline{\underline{W}}_{\text{eff}})^{-1}]_{ii} - [(z \cdot \mathbb{1} + \underline{\underline{W}}_{\text{eff}})^{-1}]_{i,i\pm 1}, \tag{7.15}$$

and where $\Delta_i^\pm$ is defined in equation (7.9). The $f_i^\pm$ are independent of i, $f_i^\pm = f^\pm$, and can be determined via Fourier transformation:

$$f^\pm = \frac{1}{2\pi} \int_{-\pi}^{\pi} dq \frac{1 - e^{\mp iq}}{(z + W_{\text{eff}}^+ + W_{\text{eff}}^-) - e^{-iq} W_{\text{eff}}^+ - e^{iq} W_{\text{eff}}^-} \tag{7.16}$$

$$= \frac{1}{2\pi i} \oint_{\substack{\text{unit}\\ \text{circle}}} du \frac{u - 1}{W_{\text{eff}}^\pm u^2 - (z + W_{\text{eff}}^+ + W_{\text{eff}}^-) u + W_{\text{eff}}^\mp} \tag{7.17}$$

For Re $z \geq 0$, we thus obtain

$$f^\pm = \frac{1}{2 W_{\text{eff}}^\pm} \{1 + [\pm(W_{\text{eff}}^+ - W_{\text{eff}}^-) - z]/S_{\text{eff}}\}, \tag{7.18}$$

with

$$S_{\text{eff}} = [z^2 + 2(W_{\text{eff}}^+ + W_{\text{eff}}^-) z + (W_{\text{eff}}^+ - W_{\text{eff}}^-)^2]^{1/2}. \tag{7.19}$$

The EMA-requirement that the average single-bond t-matrix should vanish,

$$\langle \underline{\underline{t}}_i \rangle = 0, \tag{7.20}$$

then leads to the following two coupled self-consistency equations for $W_{\text{eff}}^+(z)$ and $W_{\text{eff}}^-(z)$,

$$\left\langle \frac{\Delta_i^+}{1 - \Delta_i^+ f^+ - \Delta_{i+1}^- f^-} \right\rangle = 0 \tag{7.21a}$$

$$\left\langle \frac{\Delta_{i+1}^-}{1 - \Delta_i^+ f^+ - \Delta_{i+1}^- f^-} \right\rangle = 0 \tag{7.21b}$$

where we recall that

$$\Delta_i^\pm = W_{\text{eff}}^\pm - W_i^\pm, \tag{7.22}$$

and where the average is with respect to the probability distribution of the pair $\{W_i^+, W_{i+1}^-\}$ of transition rates.

For *site-disordered systems*, we note that the $\underline{\underline{\Delta}}$ of equation (7.6) can also be written as a sum over single-site contributions,

$$\underline{\underline{\Delta}} = \sum \underline{\underline{\Delta}}_i^{(s)} \tag{7.23}$$

$$\underline{\underline{\Delta}}_i^{(s)} = \Delta_i^+ \underline{\underline{Q}}_i^+ + \Delta_i^- \underline{\underline{Q}}_i^-, \tag{7.24}$$

so that in equations (7.21) we merely have to replace Δ_{i+1}^- by Δ_i^-, and the average then refers to the probability distribution of the pair $\{W_i^+, W_i^-\}$ of transition rates.

We further note that for *discrete-time random walks* the transition rates $W_i^\pm$ are replaced by transition probabilities $p_i^\pm$. As $p_i^- = 1 - p_i^+$, it follows that $p_{\text{eff}}^- = 1 - p_{\text{eff}}^+$, and equations (7.21) thus reduce to a single self-consistency equation for $p_{\text{eff}}^+(z)$.

From the results for non-random media [see equations (2.10) and (2.8)] it follows that the EMA-result for the Laplace transform, $\langle \tilde{x}(z) \rangle$, of the average mean displacement is given by

$$\langle \tilde{x}(z) \rangle = z^{-2} \tilde{v}_{\text{eff}}(z), \tag{7.25}$$

where

$$\tilde{v}_{\text{eff}}(z) = W^{+}_{\text{eff}}(z) - W^{-}_{\text{eff}}(z). \tag{7.26}$$

For $z \to 0$, equation (7.25) also holds for discrete-time random walks (see Section IV], with

$$\tilde{v}_{\text{eff}}(z) = 2p^{+}_{\text{eff}}(z) - 1. \tag{7.27}$$

If $\tilde{v}_{\text{eff}}(0) \neq 0$, the average mean displacement, $\langle x(t) \rangle$, is thus predicted to increase linearly with time,

$$\langle x(t) \rangle \sim \tilde{v}_{\text{eff}}(0) \cdot t, \qquad t \to \infty. \tag{7.28}$$

For continuous-time random walks this turns out to be the case if either $\langle w^{-}/w^{+} \rangle < 1$ or $\langle w^{+}/w^{-} \rangle < 1$, and we obtain

$$\tilde{v}_{\text{eff}}(0) = \begin{cases} \langle 1/w^{+} \rangle^{-1} [1 - \langle w^{-}/w^{+} \rangle], & \text{if} \quad \langle w^{-}/w^{+} \rangle < 1 \\ -\langle 1/w^{-} \rangle^{-1} [1 - \langle w^{+}/w^{-} \rangle], & \text{if} \quad \langle w^{+}/w^{-} \rangle < 1. \end{cases} \tag{7.29}$$

The analogous result for discrete-time random walks is obtained by replacing w^{+} and w^{-} by p^{+} and $1 - p^{+}$, respectively, i.e.

$$\tilde{v}_{\text{eff}}(0) = \begin{cases} \left\langle \dfrac{1}{p^{+}} \right\rangle^{-1} \left[1 - \left\langle \dfrac{1-p^{+}}{p^{+}} \right\rangle \right], & \text{if} \quad \left\langle \dfrac{1-p^{+}}{p^{+}} \right\rangle < 1 \\ -\left\langle \dfrac{1}{1-p^{+}} \right\rangle^{-1} \left[1 - \left\langle \dfrac{p^{+}}{1-p^{+}} \right\rangle \right], & \text{if} \quad \left\langle \dfrac{p^{+}}{1-p^{+}} \right\rangle < 1. \end{cases} \tag{7.30}$$

As a first example let us consider a model with random, but intrinsically symmetric transition rates, $W^{+}_{n} = W^{-}_{n+1} \equiv W_n$. In a uniform electric field E_0, the transition rates become asymmetric,

$$W^{+}_{n}(E_0) = W_n e^{+E_0/kT},$$
$$W^{-}_{n+1}(E_0) = W_n e^{-E_0/kT},$$

where the lattice constant has been set to unity. For this model, equation (7.29) predicts

$$\hat{v}_{\text{eff}}(0) = 2\langle 1/w \rangle^{-1} \sinh(E_0/kT), \tag{7.32}$$

in agreement with the result of Derrida and Orbach [6].

For our continuous-time diode-model, equation (3.1), we have

$$\langle w^{-}/w^{+} \rangle = p/\lambda, \tag{7.33}$$

$$\langle w^{+}/w^{-} \rangle = \infty, \tag{7.34}$$

and the EMA [equations (7.28) and (7.29)] predicts that

$$\langle x(t) \rangle \sim \tilde{v}_{\text{eff}}(0) \cdot t, \quad \text{if} \quad \lambda > p, \tag{7.35}$$

with

$$\tilde{v}_{\text{eff}}(0)=\frac{(\lambda-p)uv}{pu+(1-p)\lambda u}, \tag{7.36}$$

which coincides with the corresponding exact result, equations (3.42) and (3.37), for the case $\lambda > p$. For the discrete-time diode-model, equation (4.14), the EMA [equation (7.28) and (7.30)] similarly leads to

$$\langle x(t)\rangle \sim \frac{\sigma-p(1-\sigma)}{\sigma+p(1-\sigma)}\cdot t, \quad \text{if} \quad p\frac{1-\sigma}{\sigma}<1, \tag{7.37}$$

again in agreement with our exact result [equations (4.21) and (4.22)]. In fact, equations (7.28) and (7.30) coincide with the corresponding general results of Kesten et al. [7] and of Derrida and Pomeau [10]. This suggests that the EMA should always lead to the exact conditions for the existence of an asymptotic drift velocity, and that the latter is then correctly reproduced by $\tilde{v}_{\text{eff}}(0)$.

If both $\langle w^-/w^+\rangle \geq 1$ and $\langle w^+/w^-\rangle \geq 1$ [or $\langle (1-p^+)/p^+\rangle \geq 1$ and $\langle p^+/(1-p^+)\rangle \geq 1$], however, the EMA does not seem to lead to meaningful results for the long-time asymptotic behavior of $\langle x(t)\rangle$. This can, e.g., be demonstrated explicitly for the discrete-time diode-model of equation (4.14). If p and σ are such that $p(1-\sigma)/\sigma > 1$, the EMA leads to

$$\tilde{v}_{\text{eff}}(z) \sim a\cdot z^{1/2}, \qquad z\to 0, \tag{7.38}$$

and therefore predicts that

$$\langle x(t)\rangle \sim \frac{2a}{\pi^{1/2}}\, t^{1/2}, \qquad t\to\infty, \tag{7.39}$$

where $a > 0$ is a constant depending on p and σ. This, however, does not at all reflect the exact asymptotic behavior given by equations (4.21) and (4.22).

VIII. Conclusions

We have presented a detailed analysis of the average mean displacement, $\langle x(t)\rangle$, for continuous- and discrete-time random walks in one-dimensional random media. Exact asymptotic results are derived for model systems that contain a finite fraction of 'diodes', i.e. bonds which can only be traversed in one direction. Depending on the model parameters, the behavior of $\langle x(t)\rangle$ changes from $\langle x(t)\rangle \sim v_\infty t$ to $\langle x(t)\rangle \sim t^\nu F(\ln t)$, where $\nu < 1$ and where F is a periodic function. This remarkable phenomenon of persisting asymptotic oscillations is interpreted in terms of a self-similar clustering of 'waiting times'. It is argued, and demonstrated by numerical simulations, that such oscillations should not only be observed in 'diode-models', but also for other (arithmetic) transition rate distributions. Finally, a self-consistent effective-medium approximation is derived and shown to be applicable only if the transition rate distribution is such that $\langle x(t)\rangle \sim v_\infty t$, in which case, however, v_∞ is reproduced exactly.

Acknowledgement

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Appendix A

In Section III we have shown [equation (3.35)] that for the diode-model

$$\langle \tilde{v}(z)\rangle \sim \frac{(1-p)^{-1}}{1-\langle r_n\rangle_n} \quad \text{as} \quad z\to 0, \tag{A1}$$

and we therefore have to analyze the $z\to 0$ asymptotic behavior of

$$1-\langle r_n\rangle_n = (1-p)\sum_{n=0}^{\infty} p^n(1-r_n). \tag{A2}$$

Using equation (3.20), we may write

$$1-\langle r_n\rangle_n = A(z)-B(z), \tag{A3}$$

where

$$A(z) = (1-p)\sum_{n=0}^{\infty} \frac{p^n}{1+C(z)\kappa^n}, \tag{A4}$$

$$B(z) = (1-p)\sum_{n=0}^{\infty} \frac{p^n B_n(z)}{1+C(z)\kappa^n}, \tag{A5}$$

with

$$C(z) = \frac{u\mu-\kappa(u+z)}{u+z-u\mu}, \tag{A6}$$

and

$$B_n(z) = \frac{(1-\kappa)u}{u+z-u\mu}(\lambda\mu)^n - C(z)\kappa^n. \tag{A7}$$

The z-dependence of μ and κ is given by equations (3.10) and (3.11), and we note in particular that $\mu=1$ and $\kappa=\lambda$ for $z=0$.

If $\underline{\lambda > p}$, $1-\langle r_n\rangle_n$ vanishes linearly in z, and we obtain

$$1-\langle r_n\rangle_n \sim \frac{pu+(1-p)\lambda v}{(1-p)(\lambda-p)uv}\, z, \tag{A8}$$

so that

$$\langle \tilde{v}(z)\rangle \sim \frac{(\lambda-p)uv}{pu+(1-p)\lambda v}\, z^{-1} \equiv v_\infty z^{-1}. \tag{A9}$$

If $\underline{\lambda \le p < 1}$, we have

$$C(z) = \frac{c}{z} + O(1), \qquad c \equiv \frac{(1-\lambda)^2 uv}{u+(1-\lambda)v}, \tag{A10}$$

and

$$B(z) = O(z), \tag{A11}$$

as $z \to 0$. It follows that the leading asymptotic behavior of $1 - \langle r_n \rangle_n$ is determined by $A(z)$, and a careful analysis shows that we may write

$$1 - \langle r_n \rangle_n \sim (1-p) \sum_{n=0}^{\infty} \frac{p^n}{1 + \frac{c}{z} \lambda^n}, \qquad z \to 0. \tag{A12}$$

For $\lambda = p$, this leads to

$$1 - \langle r_n \rangle_n \sim (1-p) \frac{z}{c} \sum_{n=0}^{\infty} \frac{p^n}{\frac{z}{c} + p^n} \tag{A13}$$

$$\sim (1-p) \frac{z}{c} \int_0^{\infty} dx \frac{p^x}{\frac{z}{c} + p^x} \tag{A14}$$

$$\sim \frac{1-p}{\ln p} \frac{z}{c} \ln \frac{z}{c}, \tag{A15}$$

so that

$$\langle \tilde{v}(z) \rangle \sim \frac{-\ln p}{(1-p)^2} \frac{c}{z} \Big/ \ln \frac{c}{z}. \tag{A16}$$

If $\lambda < p$, we define

$$\alpha \equiv -\ln p, \qquad \beta \equiv -\ln \lambda, \qquad \nu = \alpha / \beta, \tag{A17}$$

$$m = m(z) \equiv \mathrm{INT}\left(\beta^{-1} \operatorname{Re} \ln \frac{c}{z}\right), \tag{A18}$$

where $\mathrm{INT}(x)$ denotes the largest integer smaller or equal to x, and

$$\delta = \delta(z) \equiv \beta^{-1} \ln \frac{c}{z} - m(z), \tag{A19}$$

so that equation (A12) becomes

$$1 - \langle r_n \rangle_n \sim (1-p) \left(\frac{z}{c}\right)^{\nu} \sum_{k=-m}^{\infty} \frac{e^{-\alpha(k-\delta)}}{1 + e^{-\beta(k-\delta)}}. \tag{A20}$$

It is straightforward to show, however, that

$$\left(\frac{z}{c}\right)^{\nu} \sum_{k=-\infty}^{-m-1} \frac{e^{-\alpha(k-\delta)}}{1 + e^{-\beta(k-\delta)}} = O(z), \qquad z \to 0, \tag{A21}$$

so that we may write

$$1 - \langle r_n \rangle_n \sim (1-p) \left(\frac{z}{c}\right)^{\nu} G(\delta), \tag{A22}$$

with

$$G(\delta)=\sum_{k=-\infty}^{\infty}\frac{e^{-\alpha(k-\delta)}}{1+e^{-\beta(k-\delta)}}. \tag{A23}$$

As $G(\delta)$ is periodic, $G(\delta+1)=G(\delta)$, it can also be expressed as

$$G(\delta)=\sum_{k=-\infty}^{\infty} g_k e^{2\pi i k\delta}, \tag{A24}$$

and a somewhat lengthy calculation yields

$$g_k=\frac{\pi}{\beta}\left(\sin\pi\nu\cosh\frac{2\pi^2 k}{\beta}-i\cos\pi\nu\sinh\frac{2\pi^2 k}{\beta}\right)^{-1}. \tag{A25}$$

The definition of δ, equation (A19), further implies that

$$G(\delta)=G\left(\beta^{-1}\ln\frac{c}{z}\right), \tag{A26}$$

and we finally obtain

$$\langle\tilde{v}(z)\rangle\sim(1-p)^{-2}\left(\frac{c}{z}\right)^{\nu}\Big/G\left(\beta^{-1}\ln\frac{c}{z}\right). \tag{A27}$$

Appendix B

We have

$$s_n=b_n+b_nb_{n-1}+\cdots+b_nb_{n-1}\cdots b_0, \qquad n\geq 0, \tag{B1}$$

$$s_{-1}=0, \tag{B2}$$

$$r_n=b_nb_{n-1}\cdots b_0, \qquad n\geq 0, \tag{B3}$$

$$r_{-1}=1, \tag{B4}$$

and

$$z\tilde{P}_0=\frac{(1-a_{N-1})(1-b_{N_1})}{1-a_{N-1}b_{N_1}}, \tag{B5}$$

where a_n and b_n are given by equations (3.9) and (3.16), respectively. In particular

$$a_n|_{z=0}=\frac{1-\lambda^{n+1}}{1-\lambda^{n+2}}, \qquad b_n|_{z=0}=1, \tag{B6}$$

so that

$$s_n|_{z=0}=n+1, \qquad r_n|_{z=0}=1, \tag{B7}$$

and

$$\left.\frac{z\tilde{P}_0}{1-b_{N_1}}\right|_{z=0}=1. \tag{B8}$$

We therefore immediately obtain

$$\left\langle \frac{z\tilde{P}_0}{1-b_{N_1}} s_{N_1} \right\rangle_{N,N_1} \Bigg|_{z=0} = (1-p)^2 \sum_{N=0}^{\infty} p^N \sum_{n_1=0}^{\infty} (N_1+1)p^{N_1} = \frac{1}{1-p}, \tag{B9}$$

and

$$\left\langle \frac{z\tilde{P}_0}{1-b_{N_1}} r_{N_1} \right\rangle_{N,N_1} \Bigg|_{z=0} = (1-p)^2 \sum_{n=0}^{\infty} p^N \sum_{n_1=0}^{\infty} p^{N_1} = 1. \tag{B10}$$

A careful analysis of the $z \to 0$ behavior of μ, κ, a_{N-1}, and b_{N_1} further shows that the leading asymptotic behavior of $\langle \tilde{v}_-(z) \rangle$, equation (3.30), is determined by

$$\langle \tilde{v}_-(z) \rangle \sim (1-p)^2 \sum_{N_1=0}^{\infty} p^{N_1} \sum_{N=0}^{\infty} p^N \frac{O(N)}{1+\frac{c}{z}\lambda^{N_1+N}}, \tag{B11}$$

$$\sim (1-p)^2 \sum_{M=1}^{\infty} \frac{p^M}{1+\frac{c}{z}\lambda^M} O(M^2). \tag{B12}$$

Essentially by differentiating the results for $1-\langle r_n \rangle_n$ [see Appendix A], we finally obtain

$$\langle \tilde{v}_-(z) \rangle \sim \text{const.} \cdot z, \qquad \lambda > p, \tag{B13}$$

$$\langle \tilde{v}_-(z) \rangle \sim \text{const.} \left(\frac{z}{c}\right)\left(\ln \frac{c}{z}\right)^3, \qquad \lambda = p, \tag{B14}$$

$$\langle \tilde{v}_-(z) \rangle \sim \text{const.} \left(\frac{z}{c}\right)^{\nu}\left(\ln \frac{c}{z}\right)^2, \qquad \lambda < p, \tag{B15}$$

so that for all parameter values, $\langle \tilde{v}_-(z) \rangle \to 0$ as $z \to 0$.

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