Zeitschrift: Helvetica Physica Acta
Band: 58 (1985)
Heft: 4

Artikel: Some results on magnetic monopoles and vacuum decay

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DOI: https://doi.org/10.5169/seals-115619

Helvetica Physica Acta, Vol. 58 (1985) 531–596 0018–0238/85/040531–66$1.50 + 0.20/0

Some results on magnetic monopoles and vacuum decay

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(30. VIII. 1984)

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I. Introduction

Theoretical attempts beyond the Standard Model of the electroweak and strong interactions often imply only few testable predictions at accessible energies. In comparison with the grand unification scale, accelerator energies will always remain modest. For this reason there is a growing interest of particle physicists in cosmology. Indeed, the universe is a supplement to the accelerator as a 'laboratory' in which new theories concerning very high energy phenomena may be tested. Since it is likely, that energies comparable to the Planck energy were present in the very early universe, specific aspects of grand unified gauge theories can be probed. Unfortunately, this marvelous 'laboratory' shut down over ten billion years ago, so that one must search for fossils which remain from the earliest epochs.

An interesting example for the profound interrelation between cosmology and particle physics is the role which *magnetic monopoles* play in cosmological scenarios. Magnetic monopoles were originally introduced by Dirac in order to explain the quantization of the electric charge. Dirac monopoles are singular objects and their mass is undetermined. After a period of intensive search for Dirac poles from 1951 to about 1975, all with negative results, interest in these particles declined.

The situation changed radically with the discovery of magnetic monopole solutions in non-Abelian gauge theories by 't Hooft and Polyakov in 1974. These authors independently constructed an extended regular monopole solution (a soliton) of a spontaneously broken SU(2) gauge theory, which behaves asymptotically, i.e. for distances large compared to the Compton wave length of the massive gauge bosons, like a Dirac pole. It soon became clear, that spontaneously broken gauge theories for *simple gauge groups* have always monopole solutions, if the unbroken symmetry contains at least an $U(1)$ factor. These conditions are fulfilled by grand unified gauge theories (GUTs), because the unbroken gauge group contains the electromagnetic group $U(1)$. The magnetic flux of these monopoles is quantized for topological reasons, and hence they are stable.

If matter can be described, somewhat below the Planck energy, by a GUT, then one would expect that superheavy magnetic monopoles have been produced during the *phase transition* in the very early universe, when the GUT symmetry got broken down to the $SU(3)\times SU(2)\times U(1)$ gauge group of the Standard Model. A substantial fraction of these monopoles should still be around in the present universe, because only a small fraction of monopoles and antimonopoles will annihilate. Many cosmological scenarios predict by far too many monopoles, or else they have other serious difficulties.

It was first shown by Khlopov and Zeldovich, and later by Prescill, that the abundance of magnetic monopoles, which would be produced in a second-order phase transition, would be at least 10 orders of magnitude larger than is allowed by observational limits.

In order to reduce the monopole-entropy ratio to an acceptable level, it was then suggested, that the phase transition should be delayed by at least a factor 10^{-2}. A. Guth and E. Weinberg studied the possibility that the supercooling ends by a quantum-mechanical tunneling of the vacuum, from the higher metastable symmetric state to the lower-energy stable, but asymmetric vacuum. A severe problem with this attempt of solving the *cosmic monopole puzzle* is that the tunneling process is random. The bubbles, which appear in an uncorrelated fashion, expand so fast that they can never coalesce and thermalize the energy in the walls. Thus one would expect that the universe remains grossly inhomogeneous.

The lack of a smooth ending of the original inflationary scenario of Guth prompted Linde, and independently Albrecht and Steinhardt, to propose the *new inflationary scenario* which preserves the desirable features of the original scenario while overcoming the troublesome features. In this version it is assumed that the symmetry breaking is radiatively induced, in a manner first discussed by E. Weinberg and Coleman. The barrier, that maintains the metastability of the symmetric phase becomes unimportant after substantial supercooling and the effective potential is then very flat for small values of the order parameter. Hence the motion of this parameter to the stable phase is very slow and thus the energy density remains large, implying an exponential expansion for many e-folding times. The result is that a single fluctuation can grow so much that it encompasses the present observable universe.

The new inflationary scenario explains the cosmological homogeneity, isotropy, flatness, and monopole puzzles. If its major aspects survive in one form or another, it provides a beautiful example of the fruitful interrelation between the physics of the smallest and the physics of the largest objects.

The scenario is, however, not without difficulties. The mechanism of exponential inflation requires much fine-tuning, which is unstable against various sources of perturbations (e.g. gravitationally induced). In addition, it turns out that the resulting matter inhomogeneities are much too big. Only superdense objects and black holes, and no galaxies, would be formed.

In all computations of the termination of the phase transition the generalized WKB-approximation of Langer and Coleman has been used. However, the *determinant* in front of the Gamov factor $\exp[-S]$ was never really computed. That this one-loop contribution may have an important effect under some circumstances is one result of this thesis. Also, *anisotropies* in the universe could have dramatic effects on the tunnel probability for the transition from the false to the true vacuum.

This thesis deals with some, mainly mathematical, problems concerning magnetic monopoles and the tunnel effect in field theories.

In the first part we discuss important properties, mainly topological, of magnetic monopoles. After recapitulating some properties of Dirac monopoles and fixing our notation we establish some results on the *topological quantum numbers* of monopoles. We find a constructive characterization of these numbers by using homology theory. One main result is, that under very general assumptions on the gauge group G and the unbroken subgroup H, the quantum numbers are degrees of maps defined by the asymptotic values of the Higgs field.

In the second chapter we work out an explicit construction procedure for *spherically symmetric monopoles*. Although the results are nearly the same as those of Goldhaber and Wilkinson, the derivation given here is, I believe, clearer and more elegant. Then we use this procedure for constructing two spherically symmetric monopoles for the Georgi–Glashow model. At the end of the second chapter, we derive some properties of spherically symmetric monopoles. For example, we compute the topological quantum numbers for such monopoles, give a handy rule how to derive the $J = 0$ *Dirac equation* in the field of spherically symmetric configurations, and finally solve the *Wong equation* for a specific model.

In the third chapter we study the action of the bounce solution, i.e. the minimal action which appears as exponent in the tunnel probability for the decay of the false vacuum. With the help of methods from nonlinear analysis we give a *variational characterization of the minimal action*. One of the consequences is a new proof that the bounce is spherically symmetric.

As an application of our results, we derive lower bounds for the WKB-exponent. Then we approximate the minimal action for a variety of models in different space-time dimensions. We also compare the variational results with known exact or thin-wall calculations.

In the last part we compute the *determinant*, which appears as prefactor in front of the exponential factor in the tunneling probability. In one dimension we connect this determinant with the transmission coefficient of the corresponding Schroedinger operator. We explicitly compute it for a model with an arbitrary fourth order Higgs potential.

In higher dimensions we relate the functional determinant to Jost functions of the corresponding Schroedinger operator. Here we are, of course, confronted with UV-divergences. We then compute the regularized determinants by introducing the regularized Jost functions. Finally, we use these results for calculating the regularized determinant in the *thin-wall approximation*, and are able to express it as a product of some Bessel and Hankel functions.

II. Magnetic monopoles

Magnetic monopoles were introduced in 1931 by P. Dirac [Di]. One motivation for their invention is evident from the quotation: "... The symmetry between electricity and magnetism is, however, disturbed by the fact that a single electric charge may occur on a particle, while a single magnetic pole has not been observed to occur on a particle...."

Although we are mainly interested in monopoles as solutions of the Yang–Mills–Higgs equations of a non-Abelian gauge theory, we give here a short summary of the properties of Dirac-monopoles. We do this also, because monopoles of the 't Hooft–Polyakov type look asymptotically like Dirac poles, and hence these properties are partially relevant for non-Abelian monopoles.

II.1. *Dirac monopoles*

If k^b is the *magnetic current-density* then the generalized Maxwell equations in the presence of magnetic poles are

$$dF = 4\pi/c\,{}^*K \tag{1.1}$$

$$d^*F = -4\pi/c\,{}^*J, \tag{1.2}$$

where d is the external derivative, $*$ the Hodge dual, F the field-strength 2-form $F = 1/2 \cdot F_{bc}\, dx^b \wedge dx^c$, and J, resp. K, the electric, resp. magnetic, current 1-forms $J = j_b\, dx^b$, $K = k_b\, dx^b$. In coordinates we have

$$*F^{bc}_{,c} = -4\pi/c \cdot k^b \tag{1.1'}$$

$$F^{bc}_{,c} = -4\pi/c \cdot j^b. \tag{1.2'}$$

Written in terms of codifferentials $\delta^* F = 4\pi/c \cdot K$, $\delta F = 4\pi/c \cdot J$, the conservation of electric and magnetic charges $\delta J = \delta K = 0$ is obvious. The energy-momentum tensor is the same as in ordinary electromagnetism and (using 1′, 2′) obeys

$$T^{bc}_{,c} + f^b = 0$$

with 4-force density ($j^b = (c\rho, j)$, $k^b = (c\sigma, k)$)

$$f^b = 1/cF^{bc}j_c + 1/c\ {*F^{bc}}k_c$$
$$= (1/c \cdot jE - 1/c \cdot kB, \rho E - \sigma B + 1/c \cdot j \wedge B + 1/c \cdot k \wedge E).$$

For *dyons* (particles with electric and magnetic charge) one has the generalized *Lorentz law*

$$dp^b/d\tau = 1/mc \cdot [qF^{bc} + g\ {*F^{bc}}]p_c. \tag{1.3}$$

For point particles the currents are (n denotes the nth particle)

$$j^b(x) = \sum q_n \int d\tau_n u_n^b\, \delta_4[x - z_n(\tau_n)] \tag{1.4}$$

$$k^b(x) = \sum g_n \int d\tau_n u_n^b\, \delta_4[x - z_n(\tau_n)] \tag{1.4'}$$

and the energy-momentum tensor is the usual one.

Using $**F = -F$, one finds that, if $(F, *F; J, K)$ is a solution of (1, 2), then $(F^\sim, *F^\sim; J^\sim, K^\sim)$, defined via the *duality transformation*

$$\begin{pmatrix} F^\sim \\ *F^\sim \end{pmatrix} = R(\varphi)\begin{pmatrix} F \\ *F \end{pmatrix} \qquad \begin{pmatrix} J^\sim \\ K^\sim \end{pmatrix} = R(\varphi)\begin{pmatrix} J \\ K \end{pmatrix} \tag{1.5}$$

or equivalently

$$\begin{pmatrix} B^\sim \\ E^\sim \end{pmatrix} = R(\varphi)\begin{pmatrix} B \\ E \end{pmatrix} \qquad \begin{pmatrix} q^\sim \\ g^\sim \end{pmatrix} = R(\varphi)\begin{pmatrix} q \\ g \end{pmatrix} \tag{1.6}$$

where $R(\varphi) \in SO(2)$, is also a solution of the equations (1), (2). Hence the electromagnetic field of a dyon at rest is $E = q_d x/r^3$, $B = -g_d x/r^3$. Using Lorentz's law (3) for an electrically charged particle in a dyon-field, we obtain

$$\dot p^0 = q/mc \cdot Ep \quad \text{or} \quad d(\gamma mc^2)/dt = qq_d \cdot xv/r^3 \tag{1.7}$$

$$\dot p = q/mc(Ep^0 + p \wedge B) \quad \text{or} \quad d(\gamma mv)/dt = q(q_d x - g_d v \wedge x)/r^3. \tag{1.8}$$

The *conserved energy and angular momentum* are

$$E = \gamma mc^2 + qq_d/r \tag{1.9}$$

$$J = J_{\text{orb}} + J_{\text{elm}} = x \wedge p + qg_d x/cr. \tag{1.10}$$

Because $J \cdot x/r = qg_d/c$, the particle *moves on a cone* with opening angle θ, where $\cos\theta = qg_d/|J|$.

From the equations of motion (7, 8), or the conservation laws, we obtain in the nonrelativistic limit ($A := qg_d \tan\theta/mc$)

$$d^2r/dt^2 = A^2/r^3 + qq_d/mr^2$$

$$d\varphi/dr = qg_d/(cmr^2\cos\theta)$$

$$[dr/dt]^2 = 2E/m - A^2/r^2 - 2qq_d/mr$$

From the last equation we see that if $E<0$ the particle is captured in $r_1<r<r_2$ and if $E>0$ there exists a radius of shortest approach. If the dyon is a 'nucleus' and the particle is an electron, then these turning points are

$$r_{2,1} = -Ze^2/2E \cdot (1 \pm w), \quad \text{with} \quad w := [1 + Ek^2\tan^2\theta/4E_b]^{1/2},$$

where E_b is the binding energy of a H-like ion. Here we used the famous *Dirac quantization condition* for a two-dyon system

$$q_1g_2 - q_2g_1 = k\hbar c/2 \qquad k \in Z, \tag{1.11}$$

which can be derived by demanding that the Dirac string is unobservable [Di] or by interpreting the vector potential of the dyon as a connection of a fibre bundle with base space $R^3-\{0\}$ and $U(1)$ as a typical fibre [Ya], or finally from the quantization of the angular momentum (10) [GO].

A monopole, which passes through a loop, *induces a flux change* of magnitude

$$\Delta\varphi = \int E\,ds\,dt = 4\pi g/c \tag{1.12}$$

which, using again (11), is $2k$-times the elementary flux quantum,

$$\Delta\varphi = 2k \cdot hc/2e = 4k \cdot 10^{-15}\,\text{Wb}.$$

This property could help in detecting magnetic poles.

Since about 1951 [Ma] many *experiments* were devoted to the search of such poles.

Some groups [Ma, Al, Fl, KGF, Ca, KVO] searched for Dirac monopoles in cosmic rays, others looked for monopoles which may have been caught in paramagnetic substances deep undersea, by applying strong magnetic fields (~100 kGauss) to these materials.

In other experiments [AERW, vH] samples of lunar material and from meteorites were moved through superconducting coils, hoping to find a change of the quantized coil flux.

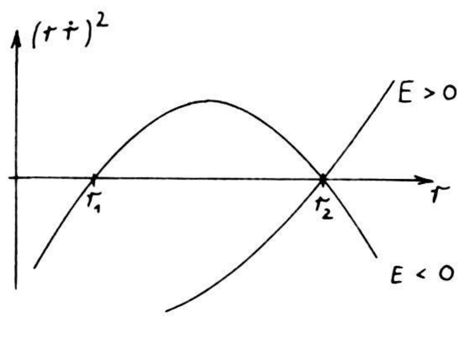

Figure 1
The allowed regions for an electron in the field of a dyon.

Searches were also made [BI, G, CGS, C] at every new higher energy accelerator, trying to detect monopoles in *production processes*, like $pN \to pNM\bar{M}$.

All these experiments gave *negative-results* and thus upper limits for the primary cosmic monopole flux, F or the cross-section for monopole-antimonopole pair creation in proton-nucleon scattering.

Typical limits are $F < 10^{-18}$ Mon/cm^2 s for monopoles with kinetic energies $T < 10^{10}$ GeV or $\sigma(pN \to pNM\bar{M}) < 10^{-43}$–$10^{-34}$ cm^2 for poles with masses $m_M <$ 1000 GeV/c^2.

II.2. 't Hooft–Polyakov type monopoles

Now we derive some properties of non-Abelian monopoles. These are regular solutions with finite energy of the (Euclidian) Yang–Mills–Higgs (YMH) equations in 3 dimensions.

First we introduce our *conventions and notations:* We denote the symmetry group of the theory with G and the Lie algebra of G with $\mathbf{G}$. Then the vector potential A is a $\mathbf{G}$-valued 1-form and the field strength F a $\mathbf{G}$-valued 2-form. Let T_p be a o.n. base of $\mathbf{G}$ with respect to an Ad-invariant scalar product. Then we have

$$A = A_b\, dx^b = A_b^p T_p\, dx^b$$
$$F = 1/2 \cdot F_{bc}^p T_p\, dx^b \wedge dx^c = dA + 1/2 \cdot [A, A] \quad \text{or}$$
$$F_{bc} = A_{c,b} - A_{b,c} + [A_b, A_c].$$

For a field φ which transforms according to a representation U of G, with induced representation U_* of $\mathbf{G}$, the *covariant derivative* is given by

$$D\varphi = d\varphi + U_*(A)\varphi \quad \text{or} \quad D_b\varphi = d\varphi/dx^b + U_*(A_b)\varphi.$$

Under a *gauge transformation*

$$\varphi'(x) = U(g(x))\varphi(x) \tag{1.13}$$
$$A'(x) = g(x)A(x)g^{-1}(x) - dg(x) \cdot g^{-1}(x) \tag{1.13'}$$

the covariant derivative and the field strength transform tensorial and hence the *action* $S = \int \{L_{YM} + L_{\text{mat}}\} d^n x$ and *Lagrangians*

$$L_{YM} = 1/2 \cdot (F, F) = 1/4 \cdot \operatorname{tr}(F_{bc}F^{bc}) \tag{1.14}$$
$$L_{\text{mat}} = L(\varphi, D\varphi) \tag{1.15}$$

are gauge invariant, if $L(\varphi, d\varphi)$ is invariant under global gauge transformations.

We always have the *Bianchi identity* $DF = 0$, and the Euler–Lagrange or *field-equations* for this action are

$$D^*F = {}^*J \quad \text{or} \quad D_c F^{bc} = j^b$$
$$+ \text{matter field-equations}, \tag{1.16}$$

where the *current* j^b is given by the variation of L_{mat} w.r.t. A,

$$j^b = \delta L_{\text{mat}}/\delta(D_b\varphi) \cdot U_*(T_p)\varphi \cdot T_p. \tag{1.17}$$

Since it is more familiar to physicists to work with *hermitian quantities* we set

$$A^{\sim} = i \cdot A \quad \text{and} \quad F^{\sim} = i \cdot F$$

and drop the $^{\sim}$ in what follows.

For a *Higgs field* the matter-Lagrangian is

$$L_{\text{mat}} = 1/2 D_b\varphi \cdot D^b\varphi + V(\varphi) \tag{1.18}$$

and the *Euclidian field equations* are $(L_p := U_*(T_p))$

$$D_i F_{ij} = j_i \quad \text{where} \quad j_{pi} = \delta L_{\text{mat}}/\delta(D_i\varphi) \cdot L_p\varphi \tag{1.19}$$

$$D_i D_i \varphi = dV/d\varphi. \tag{1.19'}$$

A *monopole* is a solution of (19) and (19′) with finite energy (3-dimensional Euclidian action).

Because we use $V(\varphi)$ for breaking the symmetry group G down to a subgroup H via the *Higgs mechanism*, V must have an absolute minimum for a nonvanishing φ (see Fig. 2). We normalize V such that V(abs. minimum) $= 0$.

If two fields φ and ψ which minimize V, i.e. $V(\varphi) = V(\psi) = 0$, can always be connected via a symmetry transformation, $\varphi = U(g)\psi$, then the *vacuum manifold* $\{\varphi \mid V(\varphi) = 0\} =: M_0$ is the coset-space

$$M_0 = G/H. \tag{1.20}$$

For (A, φ) to be a monopole solution, the φ-field must asymptotically lie in the *Higgs vacuum*

$$D\varphi = 0, \qquad \varphi \in M_0 \quad \text{for} \quad |x| \to \infty.$$

So every monopole solution defines via the map

$$\varphi^\wedge : \text{directions } S^2 \to M_0$$

$$\varphi^\wedge(x/r) = \lim_{r\to\infty} \varphi(x)$$

an element of the *second homotopy group* $\Pi_2(M_0)$. If two solutions define different elements in Π_2 then they cannot continuously be deformed into each other.

In the following we call a 3-dimensional YMH-theory *topologically nontrivial* if $\Pi_2(M_0)$ is nontrivial.

In applications the elements of $\Pi_2(M_0)$ are often characterized by the asymptotic behavior of the Higgs field. We call these integers *topological quantum numbers*. The theorem below gives a connection between these quantum numbers and flux integrals.

Hurewicz theorem [BT]. Im M_0 *is a connected and simply connected manifold*

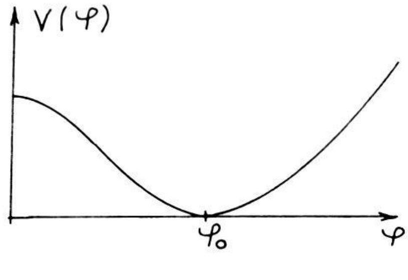

Figure 2
Plot of a typical Higgs potential.

with $\Pi_p(M_0)=0$ *for* $p=1,2,\ldots,k-1$ *and* $\Pi_k(M_0)\neq 0$, *then*

$$H_p(M_0)=0 \quad \text{for} \quad p=2,3,\ldots,k-1, \quad \text{and} \quad H_k(M_0)=\Pi_k(M_0).$$

Here $H_p(M_0)$ is the pth *homology group* i.e. p-cycles mod p-boundaries.

Using *de Rhams theorem*

$$R^p(M_0)\sim \text{Hom}\,(H_p(M_0), R),$$

where $R^p(M_0)$ is the factor space of closed exterior p-forms with respect to the subspace of exact p-forms and $\sim$ means isomorphic up to a finite group, we see that for every topologically nontrivial 3-dimensional YMH-theory the second de Rham group is nontrivial.

Because $\Pi_1(M_0)$ is generally non-Abelian whereas $H_p(M_0)$ is Abelian for all p, the Hurewicz theorem is of no help for $p=1$ (classification of strings and vortices). In this case one has [BT], however,

$$H_1=\Pi_1/[\Pi_1,\Pi_1],$$

where $[G,G]$ is the commutator subgroup of G.

In our examples M_0 will be compact (which implies that there exists a finite triangulation) and $H_p(M_0)$ has no finite torsion subgroup, hence is a finitely generated Abelian group,

$$H_p(M_0)=Z+Z+\cdots+Z. \tag{2.21}$$

Thus every p-chain c_p is of the form $c_p=\sum_r q_j\sigma_j$, where the q's are integers and the number r of terms in (21) is equal to rank$\{H_p(M_0)\}$, which is the pth *Betti number* $\beta_p(M_0)$ of M_0. The $\sigma_1,\ldots,\sigma_r$ are the generators of $H_p(M_0)$.

With the Hurewicz isomorphism we thus see that

$$\Pi_2(M_0)\sim Z+Z+\cdots+Z. \tag{2.22}$$

for every topologically nontrivial YMH-theory.

If one interprets $\varphi^\wedge(S^2)$ as a 2-chain, then

$$\varphi^\wedge(S^2)=q_1\sigma_1+\cdots+q_r\sigma_r. \tag{2.23}$$

The integers $Q[\varphi^\wedge]=\{q_1,\ldots,q_r\}$ are invariant under deformations of $\varphi^\wedge$, because two homotopic maps $M_0\to M_0$ induce the same map $H_p(M_0)\to H_p(M_0)$. Furthermore, $Q[\varphi^\wedge]$ classifies $\Pi_2(M_0)$, via the Hurewicz isomorphism, uniquely. For this reason we call $Q[\varphi^\wedge]$ the *topological charges* of $\varphi^\wedge$. Now we can state the

Lemma. *If M_0 is connected, simply connected and compact, then the topological charges $Q[\varphi^\wedge]=\{q_1,\ldots,q_r\}$, defined by*

$$\varphi^\wedge(S^2)=q_1\sigma_1+\cdots+q_r\sigma_r,$$

classify $\varphi^\wedge\in\Pi_2(M_0)$ uniquely. The number r of charges is $\beta_2(M_0)$, the second Betti number of the vacuum manifold M_0.

$\Pi_2(M_0)$ is also the set of homotopy classes of maps from the 2-tube I^2 to M_0 which send the face of I^2 to a fixed base point in M_0. The group operation on $\Pi_2(M_0)$ is defined as follows. If $\varphi^\wedge$ and $\psi^\wedge$ represent their classes, the product

$[\varphi^\wedge][\psi^\wedge]$ is the homotopy class of the map

$$\gamma^\wedge(x, y) = \begin{cases} \varphi^\wedge(2x, y) & \text{for } 0 \le x \le 1/2 \\ \psi^\wedge(2x-1, y) & \text{for } 1/2 \le x \le 1. \end{cases}$$

Using this definition it is not hard to see that in $H_2(M_0)$

$$\gamma^\wedge(S^2) = \varphi^\wedge(S^2) + \psi^\wedge(S^2), \tag{2.24}$$

and that we can interpret Q as an isomorphism $\Pi_2(M_0) \to H_2(M_0)$.

Now let us define for an arbitrary, but fixed $x \in R^r$, the homomorphism

$$Q_x : \Pi_2(M_0) \to R, \qquad \varphi^\wedge \to x \cdot Q[\varphi^\wedge] = \sum x_j q_j.$$

With de Rhams theorem we conclude that for every $x \in R^r$ there exists a closed 2-form ω_x on M_0 with

$$Q_x[\varphi^\wedge] = \sum x_j q_j = \int_{\varphi^\wedge(S^2)} \omega_x = \sum q_i \int_{\sigma_i} \omega_x,$$

and hence

$$\omega_x = \sum x_j \omega_j, \quad \text{whereby} \quad \int_{\sigma_k} \omega_j = \delta_{jk}, \qquad d\omega_j = 0. \tag{2.25}$$

So we finally obtain

$$q_j = \int_{\varphi^\wedge(S^2)} \omega_j = \int_{S^2} \varphi^{\wedge *} \omega_j. \tag{2.26}$$

If i_j denotes the imbedding of σ_j in M_0, then $i_j^*(\omega_j)$ is a closed 2-form on σ_j (pullback and exterior derivative commute!). Because $i_j(\sigma_j) = \sigma_j$, we may regard ω_j as closed 2-form on σ_j.

However, the integer

$$\deg(\varphi^\wedge, \sigma_j) = \int_{S^2} \varphi^{\wedge *} \omega_j = \sum_{p \in \varphi^{\wedge -1}(q)} [\operatorname{sgn} \det (\varphi_*^\wedge)_p] \tag{2.27}$$

is the integral formula for the *degree of the map* $\varphi^\wedge : S^2 \to \sigma_j$, if ω_j is a normalized 2-form on σ_j, as is the case [BT].

In the last expression in (27) $\varphi^\wedge$ should not be regarded as the original $\varphi^\wedge$, but as that map in the homotopy class of the original $\varphi^\wedge$ which is of the form (23). The sum if over all $p \in S^2$ which are mapped into a fixed (but arbitrary) $q \in \sigma_j$.

So we arrived at the following result: If M_0 satisfies the assumptions in the lemma, then the topological charges $Q[\varphi^\wedge] = \{q_1, \ldots, q_r\}$ of $\varphi^\wedge$ are all degrees of a map, which is homotopic to $\varphi^\wedge$. Or in other words: q_j is the number of how often $\varphi^\wedge$ winds around the jth generator σ_j of $H_2(M_0)$. This is summarized as follows:

$$\begin{gathered} \varphi^\wedge \to [\varphi^\wedge] \in \Pi_2(M_0) \leftrightarrow \varphi^\wedge(S^2) = \sum q_j \sigma_j \in H_2(M_0), \\ q_j = \deg\{[\varphi^\wedge], \sigma_j\}. \end{gathered} \tag{2.28}$$

The *determination of these topological charges*, in cases where M_0 is not simply S^2, is rather intricate. Only under very special circumstances, e.g. for $S^2 \to S^2$,

$S^3 \to S^3$, and $S^3 \to G$ [BePo], [F], [Sk] explicit formulas for the flux integrals (28) are known.

For later purposes we recall some *results from homotopy theory*, which will be relevant for us [Wh].

i) If G is a connected and compact Lie group and H a closed subgroup of G, then the long sequence

$$\cdots \to \Pi_j(H) \to \Pi_j(G) \to \Pi_j(G/H) \to \Pi_{j-1}(H) \cdots \tag{2.29}$$

is *exact*, i.e. the image of a homomorphism (indicated by an arrow) is the kernel of the next one.

ii) For two topological spaces X and Y

$$\Pi_j(X \times Y) = \Pi_j(X) + \Pi_j(Y). \tag{2.30}$$

iii) $\Pi_j(X)$ is Abelian for $j > 1$.

iv) Because every Lie group is topologically equivalent to $G_0 \times V$, where G_0 is a maximal compact subgroup and V a real vector space, and moreover $G_0 = \Pi G_j \times T^n \times C$, wherein the G_j's are simple, T^n is the n-torus and C discrete, we obtain with (30)

$$\Pi_2(\text{Lie group}) = 0. \tag{2.31}$$

Applications

a) If G *is simply connected* we find from (29) and (31) the short exact sequence

$$0 = \Pi_2(G) \to \Pi_2(G/H) \to \Pi_1(H) \to \Pi_1(G) = 0,$$

so that

$$\Pi_2(G/H) = \Pi_1(H). \tag{2.32}$$

Formula (32) is *still true if* $G = G^\sim/N^\sim$, where $G^\sim$ is the simply connected covering group of G and $N^\sim$ a discrete normal subgroup in the center of $G^\sim$ (hence $\Pi_1(G) \sim N^\sim$) and if H *is connected.* This follows because then $G^\sim/H$ is the simply connected covering space of G/H

$$0 = \Pi_1(G^\sim) \to \Pi_1(G^\sim/H) \to \Pi_0(H) = 0,$$

or $\Pi_1(G^\sim/H) = 0$. Therefore, we obtain the exact sequence

$$0 = \Pi_2(N^\sim) \to \Pi_2(G^\sim/H) \to \Pi_2(G/H) \to \Pi_1(N^\sim) = 0.$$

This means that $\Pi_2(G^\sim/H) = \Pi_2(G/H) = \Pi_1(H)$.

b) In particular, if H *is simply connected,* then

$$\Pi_2(G/H) = 0.$$

c) Let $G = G^\sim \times T^p$, $H = H^\sim \times T^q$ where $G^\sim$ is simply connected, $H^\sim$ is a subgroup of $G^\sim$ and T^q a subgroup of T^p ($T^p = p$-torus). Then

$$\Pi_2(G/H) = \Pi_1(H^\sim) \tag{2.33}$$

(33) holds because in the exact sequence

$$\begin{array}{ccccccc} 0=\Pi_2(G) & \xrightarrow{\varphi_1} & \Pi_2(G/H) & \xrightarrow{\varphi_2} & \Pi_1(H) & \xrightarrow{\varphi_3} & \Pi_1(G) \\ & & & & \| & & \| \\ & & & & \Pi_1(H^{\sim})+\underbrace{Z+\cdots+Z}_{q} & & \underbrace{Z+\cdots+Z}_{p} \end{array}$$

$\operatorname{Ker}(\varphi_3)=\Pi_1(H^{\sim})$ (since $H^{\sim}$ resp. T^q are subgroups of $G^{\sim}$ resp. T^p) and hence $\Pi_2(G/H)=\operatorname{Im}(\varphi_2)=\Pi_1(H^{\sim})$. In the last step we made use of $\operatorname{Ker}(\varphi_2)=0$.

d) Now we come to a *physically more interesting example.* Let G be of the form $G=G^{\sim}\times T^p$ with simply connected $G^{\sim}$ and $H=T^q$. Then $\operatorname{Ker}(\varphi_3)$ in

$$\begin{array}{ccccccc} 0 & \xrightarrow{\varphi_1} & \Pi_2(G/H) & \xrightarrow{\varphi_2} & \Pi_1(T^q) & \xrightarrow{\varphi_3} & \Pi_1(G^{\sim}\times T^p)\cdots \\ & & & & \| & & \| \\ & & & & \Pi_1(U(1))+\cdots+\Pi_1(U(1)) & & \Pi_1(U(1))+\cdots+\Pi_1(U(1)) \end{array}$$

is the sum of those $\Pi_1(U(1))=Z$ of $\Pi_1(H)$ which do 'not appear' in T^p. Because $\operatorname{Ker}(\varphi_2)=0$, we conclude that $\Pi_2(G/H)$ is isomorphic to $\operatorname{Im}(\varphi_2)=\operatorname{Ker}(\varphi_3)$. But the number of $U(1)$-factors of H which do not appear in T^p is the codimension of the projection of the Lie algebra t^p of T^p into t^q. Or, if $P:t^p\to t^q$ is this projection,

$$\Pi_2(G/H)=Z^r, \tag{2.34}$$

where $r=q-\dim[Pt^p]$.

Especially in the *Salam–Weinberg model* of the electro-weak interactions, where $G=SU(2)\times U(1)$ and $H=U(1)$, $r=0$, since the generator T of H is $T=Y+t^3$. So in the terminology introduced above the *Salam–Weinberg model is topologically trivial.*

Finally, we remark, that whenever G is connected and simply connected and H is connected, then

$$\Pi_1(G/H)=0 \tag{2.35}$$

$$\Pi_0(G/H)=0. \tag{2.36}$$

If a vacuum manifold is a homogeneous space, $M_0=G/H$, we conclude from this, that the Hurewicz theorem applies, and so the topological charges of a field configuration (A,φ) are degrees of the asymptotic map $\varphi^{\wedge}:S^2\to M_0$.

II.3. *Strings and domain walls*

Strings and domain walls are the counterparts of monopoles in lower dimensions.

A string is a solution of the two-dimensional Euclidian YMH-equations with finite Euclidian action (energy). Again the Higgs field must asymptotically sit in the Higgs vacuum. The asymptotic map

$$\varphi^{\wedge}:\text{directions } S^1\to M_0,$$

$$\varphi^{\wedge}(x/r)=\lim_{r\to\infty}\varphi(x)$$

defines now an element of the *first homotopy group* $\Pi_1(M_0)$.

A familiar example of a topological nontrivial theory is the *Landau–Ginzburg theory* for superconductivity, in which $G = U(1)$ and $H = e$. With the short exact sequence for the bundle $(G, H, G/H)$ one finds indeed $\Pi_1(M_0) = \Pi_1(U(1)) = Z$. The topological charge is the quantized flux, wellknown for type II superconductors below the critical temperature. It is possible [JT] to interpret this flux explicitly as degree of the map $\varphi^\wedge$.

In general, we can express the topological quantum numbers as degrees of $\varphi^\wedge$ only if Π_1 is Abelian (compare page 10).

In the 'minimal' GUT-model of Georgi and Glashow, where $SU(5)$ is broken to $SU(3)_c \times SU(2) \times U(1)_Y$, we expect no 1-dimensional defects

$$\Pi_1(M_0) = 0,$$

because $\Pi_1(SU(5)) = 0$ and $\Pi_0(SU(3) \times SU(2) \times U(1)) = 0$. On the other hand monopole solutions may exist, since $\Pi_2(M_0) = Z$.

Domain walls are 1-dimensional solutions of the Euclidian field-equations which have finite energy. The classifying group is $\Pi_0(M_0)$, so that, whenever the vacuum manifold is not connected, we expect 2-dimensional defects (domain walls) to appear.

A simple example is the *kink-solution* for the double well potential $(A, \varphi) = (0, \text{const} \cdot \tanh(mx))$. Here $G = Z_2$ and $H = 1$ and hence $\Pi_0(M_0)$ is isomorphic to Z_2 (kink and antikink).

The *cosmological implications* of the existence of these topological objects continues to be the subject of vivid discussions. The so-called *monopole problem* was even one of the main reasons (beside the flatness- and horizon-problem) which led to the development of the *inflationary universe* [Ki, Str].

II.4. *Discussion of Higgs potentials*

After the general considerations of vacuum manifolds in the previous parts we study now an explicit model. In this section we develop an efficient method for computing phase boundaries.

With the help of a lemma due to Bucella, Ruegg and Savoy, we discuss the 'phase portrait' of a Higgs potential with a cubic term, and a Higgs field sitting in the adjoint representation of $SU(N)$.

Lemma. *The* absolute extrema *of a function* $f(\alpha_1, \ldots, \alpha_N) = \sum \alpha_i^4$ *under the constraints* $\sum \alpha_i^2 = R^2$ *and* $\sum \alpha_i = \sigma$ *are never attained if* 3 *or more variables* α_i *are different. The* absolute maximum *is only attained if at least* $(N-1)$ *of the* α_i*'s are the same.*

Potential with a cubic term

The most general renormalizable potential is

$$V(\varphi) = m^2/2 \cdot \operatorname{tr} \varphi^2 + a/4 \cdot (\operatorname{tr} \varphi^2)^2 + b/2 \cdot \operatorname{tr} \varphi^4 + c/3 \cdot \operatorname{tr} \varphi^3. \tag{2.37}$$

This form has been used by Guth et al. [GT, GW] to enforce a strongly first order phase transition in the inflationary scenario. If we rescale the field, $\varphi =: c/b \cdot f$, then our potential reads

$$V = A \cdot \{\lambda/2 \cdot \operatorname{tr} f^2 + \eta/4 \cdot (\operatorname{tr} f^2)^2 + 1/2 \cdot \operatorname{tr} f^4 + 1/3 \cdot \operatorname{tr} f^3\}, \tag{2.37'}$$

where $\lambda := m^2 b/c^2$, $\eta := a/b$ and $A := c^4/b^3$ ($A > 0$ assumed).

Asymptotics

Since $V(f)$ is Ad-invariant, we can assume that f is diagonal:

$f = \operatorname{diag}(\alpha_1, \ldots, \alpha_N)$ from which

$V \sim A\{\eta/4 \cdot \|\alpha\|^4 + 1/2 \cdot \sum \alpha_i^4\}$ for $\|\alpha\| =: R \to \infty$.

Since this expression is homogeneous in α, we can restrict ourselves to S^{N-1}.

At an absolute extremum we have $\alpha_1 = \cdots = \alpha_k =: \rho_k$, $\alpha_{k+1} = \cdots = \alpha_N =: \sigma_k$. Using $R = 1$ and $\sum \alpha_i = 0$, we find

$$\rho_k = [(N-k)/Nk]^{1/2} \quad \text{and} \quad \sigma_k = [k/N(N-k)]^{1/2}. \tag{2.38}$$

The minimum is attained for $k = (N \pm 1)/2$ if N is odd and otherwise for $k = N/2$. Hence

$$\begin{aligned} V_m \sim & AR^4 \cdot [\eta/4 + 1/2N] & N \text{ even} \\ & AR^4 \cdot [\eta/4 + 1/2N \cdot (N^2+3)/(N^2-1)] & N \text{ odd} \end{aligned}$$

and we end up with the *stability condition*

$$\begin{aligned} \eta & \geq -2/N & N \text{ even} \\ & \geq -2/N \cdot (N^2+3)/(N^2-1) & N \text{ odd.} \end{aligned} \tag{2.39}$$

Phases of V

On S^{N-1} we have the constraint extremum problem for the function $g(\alpha) = A \cdot (1/2 \cdot \sum \alpha_i^4 + 1/3 \cdot \sum \alpha_i^3)$ with the constraints $R = 1$ and $\sum \alpha_i = 0$.

Using the constraints we find

$$g(\alpha) = A/2 \cdot \sum (\alpha_i + 1/6)^4 - A(216+n)/2592$$

and with $\beta_i := \alpha_i + 1/6$ the variational problem becomes

$f(\beta) = A/2 \cdot \sum \beta_i^4$ minimal with the constraints $\sum \beta_i = n/6$

and $\sum \beta_i^2 = 1 + n^2/36$.

We can again apply the lemma above and obtain $\beta_1 = \cdots = \beta_k$ and $\beta_{k+1} = \cdots = \beta_N$. In the α-variables we obtain, using the constraints,

$$\begin{aligned} \alpha_1 = \cdots = \alpha_k &= (n-k)/R \\ \alpha_{k+1} = \cdots = \alpha_N &= -k/R, \end{aligned}$$

where $R^2 = n \cdot k(n-k)$. With the definitions

$$\begin{aligned} n_k &:= [\{(n-k)/k\}^{1/2} - \{k/(n-k)\}^{1/2}]/n^{1/2} = \left(\frac{m_k}{2} - \frac{1}{n}\right)^{1/2} \\ m_k &:= 2n/[k(n-k)] - 6/n = 2 \cdot n_k^2 + 2/n, \end{aligned} \tag{2.40}$$

the potential at an absolute extremum is

$$V = A \cdot [\lambda/2 \cdot R^2 + (\eta + m_k) \cdot R^4/4 - n_k/3 \cdot R^3]. \tag{2.41}$$

The condition $dV/dR = 0$ gives

$$\begin{aligned} &R = 0 \quad \text{or} \\ &R = [n_k \pm (n_k^2 - 4h_k\lambda)^{1/2}]/2h_k, \end{aligned} \tag{2.42}$$

where we used the abbreviation $h_k := m_k + \eta$. Because $A > 0$, the upper sign is the relevant one, and with (40), (41) we see that $k < [N/2]$ at an absolute minimum.

A necessary condition for the existence of a (local) unsymmetric phase is

$$h_k\lambda < n_k^2/4. \tag{2.43}$$

From (41) we conclude that the phase diagram depends only on λ and η. At a *phase boundary* between 'phase k' and 'phase r' we have

$$g_{kr} := V_k - V_r = 0.$$

Instead of solving this often clumsy equation we can integrate the differential equation for the phase separating curve $\eta(\lambda)$

$$d\eta/d\lambda = -[dg_{kr}/d\lambda]/[dg_{kr}/d\eta]. \tag{2.44}$$

With

$$dV_k/d\lambda = AR_k^2/2 \qquad dV_k/d\eta = AR_k^4/4 \tag{2.45}$$

one obtains

$$d\eta/d\lambda = -2/[R_k^2 + R_r^2]. \tag{2.46}$$

Because $V_k(\lambda = 1/8,\ \eta = -2/n) = 0$ for all (allowed) k, the point

$$(\lambda, \eta) = (1/9, -2/n) \tag{2.47}$$

is a $([N/2]+1)$-*critical point* and serves as an *initial condition* for the differential equation (46).

At the critical point $R_k^2 = 1/9n_k^2$. Together with (46) and $R_0 = 0$ we conclude that the $k = 0$ phase is adjacent to the $k = 1$ phase, the 1-phase to the 2-phase, etc. up to $k = [N/2]$. So there are $[N/2]+1$ *stable phases.*

The $0 - k$-boundary obeys

$$\eta = -m_k + 2n_k^2/9\lambda \tag{2.48}$$

and the phase diagram looks qualitatively like indicated in Fig. 3.

For $N = 5$ (Georgi–Glashow) we find $n_1^2 = 9/20$, $n_2^2 = 1/30$, $m_1 = 13/10$ and

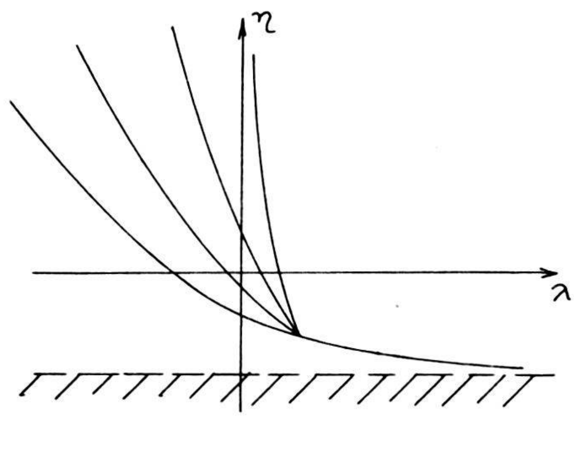

Figure 3
A qualitative sketch of the phase diagram which corresponds to the potential (37).

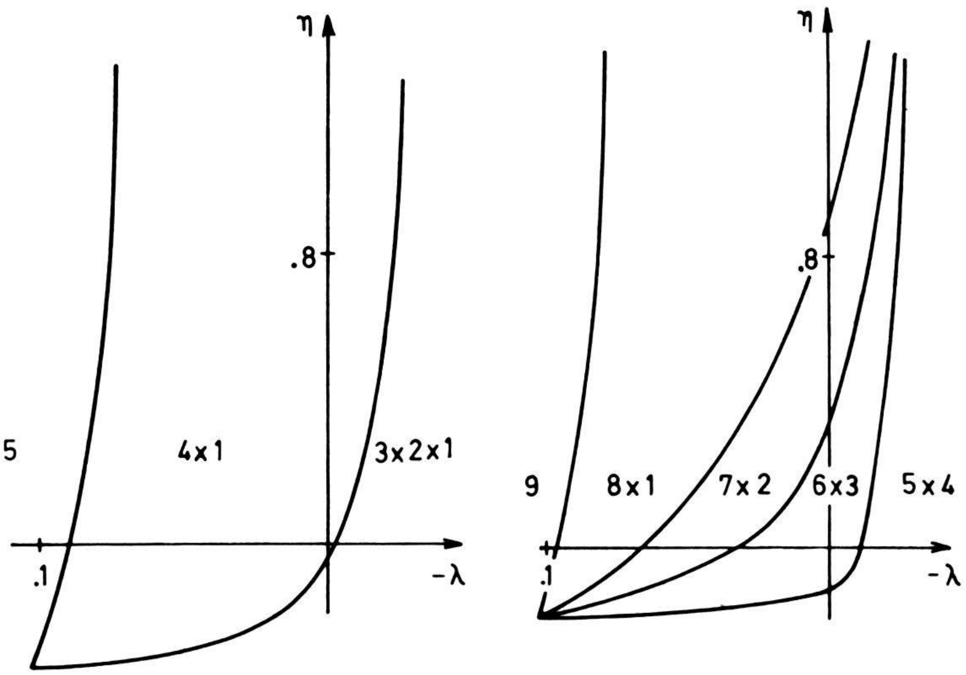

Figure 4
The different phases for the Higgs potential (37). The left hand graph shows the phase diagram for $G = SU(5)$ and the right hand graph for $G = SU(9)$.

$m_2 = 7/15$, so that

$$R_1 = 15/2sqr(5)\cdot[1 + sqr\{1 - 8\lambda/9\cdot(13 + 10\eta)\}]/[13 + 10\eta] \tag{2.49}$$

$$R_2 = 15/2sqr(30)\cdot[1 + sqr\{1 - 8\lambda\cdot(7 + 15\eta)\}]/[7 + 15\eta] \tag{2.49'}$$

With (48) the 0–1 boundary is $\eta = 1/10\lambda - 13/10$ and the 0–2 boundary $\eta = 1/135\lambda - 7/15$.

We mention also that the amplitude of the field in the phase 1 and also in the phase 2 grows linearly with λ along the 0–1 resp. 0–2 boundaries:

$$R_1(\eta(\lambda), \lambda) = 10\lambda/5^{1/2} \quad \text{resp.} \quad R_2(\eta(\lambda), \lambda) = 90\lambda/30^{1/2}.$$

For $N = 5$ the stability condition (39) means $\eta \geq -7/15$ and the critical point is at $(\lambda, \eta) = (1/9, -2/5)$. The 1–2 boundary (compare Fig. 4) was computed with (46) and (49, 49′).

III. Explicit Monopole solutions

After the general considerations in the last chapter we now try to solve the field equations (2.19), (2.19′). As usual one first imposes some symmetry properties on the solutions. But instead of calculating $D_iD_i\varphi$ or F_{ij} for a symmetric ansatz, it is more convenient to compute the Euler–Lagrange equation for the action restricted to symmetric configurations. That a critical point of the action restricted to the symmetric configurations is also a solution of the full equations (2.19 and 2.19′) is true under very general circumstances. This is a consequence of the theorem quoted below [Co], [Pa], [Str].

Let M be a smooth manifold (generally ∞-dimensional) on which a group G

acts by diffeomorphisms (a 'smooth G-manifold') and let $S: M \rightarrow R$ be a smooth G-invariant functional ($S[g\varphi] = S[\varphi]$ for all $g \in G$ and $\varphi \in M$). Then a *symmetric point of M under G* is a fixpoint under the action of G, $g\varphi = \varphi$ for all $g \in G$. We use the abbreviation SG for the set of symmetric points.

Theorem. *Let S be a G-invariant functional on a smooth G-manifold M where G is a compact Lie group. Then every critical point of S restricted to SG is also a critical point of S on M.*

This theorem is also true in other cases, e.g. if G is a group of isometries of a Riemannian manifold M and S is a C^1 functional.

In *our applications* M is a Banach space and G is compact. Therefore we indeed can use this theorem for finding solutions of the field-equations.

III.1. *Spherically symmetric monopoles*

For introducing the concept of spherical symmetry [WW, GoWi] we first need a imbedding

$$SU(2) \rightarrow G$$
$$a \rightarrow g(a)$$

from the spin group into the gauge group. Then a *spherical rotation* Γ is a simultaneous rotation in space with $R^{-1}(a)$ and gauge transformation with $g(a)$

$$\{\Gamma(a)\varphi\}(x) = U(g(a))\varphi(R^{-1}(a)x) \tag{3.1}$$
$$\{\Gamma(a)A\}(x) = g(a)R(a)A(R^{-1}(a)x)g^{-1}(a). \tag{3.2}$$

Here $U: G \rightarrow L(V)$ is the representation under which φ transforms.

So the symmetry group is the diagonal group

$$SU(2)_d := \operatorname{diag}\{SO(3)_x \times SU(2)_G\}.$$

Because of gauge and Euclidean invariance the action is clearly invariant under a spherical rotation. So we can use the theorem above with $G = SU(2)_d$, $M = \{$field configurations$\}$ and SG the set of *spherically symmetric fields*

$$SG = \{(\varphi, A) \mid \Gamma(a)\varphi = \varphi,\ \Gamma(a)A = A \text{ for all } a \in SU(2)_d\}. \tag{3.3}$$

The field A transforms as a vector and φ as a scalar field with respect to $SU(2)_d$.

Using the base $S_p = 1/2i \cdot \sigma_p$ of $SU(2)$ and setting $T_p = g_*(S_p)$ the *infinitesimal form* of the transformations (1), (2) for a rotation with angular velocity ω, $a = \exp\{-i(\omega S)\}$, of a SG-field is

$$\{\omega L + U_*(\omega T)\}\varphi = (\omega J)\varphi = 0 \tag{3.4}$$
$$\{\omega L + ad(\omega T)\}A = (\omega J)A = -i\omega \wedge A. \tag{3.5}$$

where L is the orbital angular momentum and $J = L + U_*(T)$. From (4), (5) we immediately conclude

$$U_*(x \cdot T)\varphi(x) = 0 \tag{3.6}$$
$$ix \wedge A(x) + ad(x \cdot T)A(x) = 0. \tag{3.7}$$

The 'φ-space' V is the direct sum of subspaces V_j,

$$V = \sum_{j \in Z/2} c_j V_j,$$

where every V_j is an irreducible subspace with respect to $SU(2)_G$. In every V_j we choose the base $|jm\rangle$. For the moment we assume that we have only one nontrivial irreducible representation in V, $V = V_j + V^{\perp}$, such that $SU(2)_G$ is trivially represented in $V^{\perp}$. Using the fact that φ is a scalar with respect to $SU(2)_d$, we find

$$\begin{aligned}\varphi(x) &= f^{\sim}(r) \cdot \sum_{-j \le m \le j} \langle jmj-m \mid 00\rangle Y_{jm}(x/r) \cdot \mid j-m\rangle + \sum g_q(r)\,|00\rangle_q. \\ &= f(r) \cdot \sum_{-j \le m \le j} (-1)^{j-m} Y_{jm}(x/r) \cdot \mid j-m\rangle + \sum g_q(r)\,|00\rangle_q.\end{aligned} \tag{3.8}$$

Here $\langle \cdot \cdots \mid \cdot \cdot\rangle$ are the Clebsch–Gordan coefficients, Y_{jm} the spherical harmonics and $|00\rangle_q$ a base of $V^{\perp}$ (we assumed that $SU(2)_G$ is trivially represented in $V^{\perp}$).

Clearly, *j in* (8) *must be an integer.*

Since A is a vector field w.r. to $SU(2)_d$, only the $J = 1$ sector in $L_2(S^2) \times \mathbf{G}$ contributes to A. Again we assume that there is only one nontrivial irreducible representation θ_j of $SU(2)_G$ and decompose $\mathbf{G}$ (as linear space) in $\mathbf{G}_j$ and $\mathbf{G}^{\perp}$, so that the *spherical components* $A_{\hat{M}}$ of the gauge potential A are

$$\begin{aligned}A_{\hat{M}}(x) = &\sum_{m,n} \sum_{k=-1,0,1} \langle (j+k)mjn \mid 1M\rangle f_k(r) Y_{j+k,m}(x/r) \cdot |jn\rangle \\ &+ \sum g_{Mq}(r) Y_{1M}(x/r) \cdot |00\rangle_q.\end{aligned} \tag{3.9}$$

where, as in the expansion of φ, *j must be an integer.*

From (1), (2) resp. (8), (9) we conclude that a field configuration is determined if we know the values of φ and A on a curve from the origin to infinity. One may regard these values as '*initial data*' for (φ, A).

We already used $\varphi(r) := \varphi(x = (0, 0, r))$ and $A(r) := A(x = (0, 0, r))$ as initial data.

With $F_1 = -(-)^j[3j/\{(2j+1)(2j+3)\}]^{1/2} f_1$, $F_0 = (-)^j[3/(2j+1)]^{1/2} f_0$ and $F_{-1} = -(-)^j[3(j+1)/\{(2j-1)(2j+1)\}]^{1/2} f_{-1}$ we find the following expressions for $A(r)$

$$\begin{aligned}A_{\hat{1}}(r) &= [F_1 + F_0 + F_{-1}]\,|j1\rangle \\ A_{\hat{0}}(r) &= [-\{(j+1)/j\}^{1/2} F_1 + \{j/(j+1)\}^{1/2} F_{-1}]\,|j0\rangle + \sum g_{0q} \cdot |00\rangle_q \\ A_{\hat{-1}}(r) &= [F_1 - F_0 + F_{-1}]\,|j-1\rangle.\end{aligned}$$

Now we shall prove that there exists a gauge, the so-called *spherical gauge*, in which $A_{\hat{0}}(r) = 0$. This means in other words, that the initial data of the scalar field $x \cdot A(x)$ are $\{x \cdot A(x)\}(r) = 0$. Therefore $x \cdot A(x)$ vanishes everywhere.

We prove this statement by explicitly constructing the transformation. Let $S(x)$ be the scalar with respect to J, determined by the initial data

$$S(r) = P \exp\left\{-i \int^r dr' A_{\hat{0}}(r')\right\}.$$

(P means path-ordering.) Hence $dS(r)S^{-1}(r) = -iA_{\hat{0}}(r)$. Now we perform a gauge transformation with $S(x) = gS(r)g^{-1}$, where a in $g(a)$ is a 'rotation' which rotates

e_3 into x. This is the case for the rotation with Euler angles $(\varphi, \theta, -\varphi)$,

$$a = \exp[-i\varphi S_3]\exp[-i\theta S_2]\exp[i\varphi S_3]. \tag{3.10}$$

If φ is a scalar and A a vector field, then the transformed fields

$$\varphi^{\sim}(x) = U(S(x))\varphi(x)$$
$$A^{\sim}(x) = S(x)A(x)S^{-1}(x) - i\,dS(x)S(x)^{-1}$$

are again scalar and vector fields.

Thus $A^{\sim}(x)x/r$ is a scalar field, and

$$(A^{\sim}e_3)(r) = S(r)(Ae_3)(r)S^{-1}(r) - i\,dS(r)/dr \cdot S^{-1}(r),$$

or

$$S^{-1}(r)A_0^{\sim}(r)S(r) = A_0(r) - S^{-1}(r)\,dS(r) = 0.$$

Thus we found explicitly a gauge transformation, s.t. the transformed fields are again scalar resp. vector fields. Furthermore, the *new A-field has no radial component.*

From the *reality of* A, $A = A^*$, we finally conclude that

$$\varphi(r) = (-)^j f(r) \cdot |j0\rangle + \sum g_q(r) \cdot |00\rangle_q \tag{3.11}$$

$$\begin{aligned} A_1^{\wedge}(r) &= v(r) \cdot |j1\rangle \\ A_0^{\wedge}(r) &= 0 \\ A_{-1}^{\wedge}(r) &= v^*(r) \cdot |j-1\rangle. \end{aligned} \tag{3.12}$$

The orthonormal base $|jm\rangle$ in (11) and (12) generally are not the same (e.g. if φ does not sit in the adjoint representation), as should be clear from the construction.

The *generalization* of our algorithm to the case when the homomorphism of the spin group into the gauge group allows *more than one nontrivial irreducible representation* is straightforward. In (11), (12) one simply has to sum over these irreducible representations.

The *explicit construction* of (11) and (12) simplifies enormously by observing, that $|j0\rangle$ and $|00\rangle_q$ in (11) span exactly that subspace $V^{\sim}$ of V which obeys $U_*(T_3)V^{\sim} = 0$ and the $|j1\rangle$ and $|j-1\rangle$ in (12) span exactly the subspace $\mathbf{G}^{\sim\sim} = \{ad(T_1^{\wedge})\mathbf{G}^{\sim} + ad(T_{-1}^{\wedge})\mathbf{G}^{\sim}\}$ of $\mathbf{G}$. Here $\mathbf{G}^{\sim}$, analogously to $V^{\sim}$, contains the elements of $\mathbf{G}$ which commute with T_3. In other words,

$$\{|j0\rangle, |00\rangle_q\} = \mathrm{Ker}\,[U_*(T_3)] \tag{3.13}$$

$$\{|j1\rangle, |j-1\rangle\} = ad(T_1^{\wedge})[\mathrm{Ker}\,(ad(T_3))] + ad(T_{-1}^{\wedge})[\mathrm{Ker}\,(\mathrm{ad}(\mathrm{T}_3))] \tag{3.14}$$

So we end up with the following *construction rules:*

i) Specify the imbedding $SU(2) \to G$ and find the subspaces $V^{\sim} = \mathrm{Ker}\,[U_*(T_3)]$ of V resp. $\mathbf{G}^{\sim} = \mathrm{Ker}\,[ad(T_3)]$ of $\mathbf{G}$.

ii) $\varphi(r)$ is then a r-dependent linear combination of base elements of $V^{\sim}$. If $\mathbf{G}^{\sim\sim}$ is the image of $\mathbf{G}^{\sim}$ under $ad(T_1^{\wedge})$ and $ad(T_{-1}^{\wedge})$, then $A(r)$ is an r-dependent linear combination of base elements of $\mathbf{G}^{\sim\sim}$.

iii) The configuration (A, φ) away from the positive z-axis is now given by a spherical rotation (1), resp. (2), where a is defined in (10).

What remains to be done is the *derivation of the field equations*. Here we use the theorem quoted above. We insert the general ansatz for (φ, A) into the action and consider only variations within this class of spherically symmetric functions. We know that a solution of the corresponding *ordinary differential equations* for the functions $v(r)$, $f(r)$ and $g_q(r)$ provides a solution of (2.19, 2.19′).

III.2. *Two examples*

Let us study now a specific model. We regard the $SU(5)$-*model* due to Georgi and Glashow with φ in the adjoint representation and hence $V = \mathbf{G}$. We choose the following imbedding of $SU(2)$ in G

$$g(a) = \left(\begin{array}{c|c} & \\ \hline & a \end{array}\right) \qquad T_p = \left(\begin{array}{c|c} & \\ \hline & S_p \end{array}\right) \tag{3.15}$$

Because the character of this representation is $\sum \operatorname{tr}[T_q Ad(-i\varphi T_3) T_q] = 9 + 12\cos(\varphi/2) + (1 + 2\cos(\varphi))$, we have the decomposition $\mathbf{G} = 9\mathbf{G}_0 + 6\mathbf{G}_{1/2} + \mathbf{G}_1$. Now using our rules stated above, it easy to see that

$$\varphi(r) = g_1(r)\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -3/2 & \\ & & & & 3/2 \end{pmatrix} + 5/2 \cdot f(r)\begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix} + \left(\begin{array}{c|c} g(r) & \\ \hline & \end{array}\right) \tag{3.16}$$

$$\{A^{\hat{1}}, A^{\hat{0}}, A^{\hat{-1}}\} = \left\{\left(\begin{array}{c|cc} & & \\ \hline & 0 & v \\ & 0 & 0 \end{array}\right), \left(\begin{array}{c|cc} & & \\ \hline & 0 & 0 \\ & 0 & 0 \end{array}\right), \left(\begin{array}{c|cc} & & \\ \hline & 0 & 0 \\ & v^* & 0 \end{array}\right)\right\} \tag{3.17}$$

The 3-dimensional selfadjoint matrix $g(r)$ fulfils $\operatorname{tr}\{g(r)\} = 0$.

We choose the potential (2.37) with a cubic term, discussed on page 543ff.

The action for our configuration is still clumsy and we do not write it down. But it is invariant under

$$g(x) \to U g(x) U^{-1}$$
$$v(x) \to \exp\{i\psi(x)\} v(x),$$

and has thus an $SU(3) \times U(1)$ invariance group. Because of the $SU(3)$-invariance of the action and the theorem on p. 547 we can set $g(x) = 0$. One cannot use the $U(1)$ in the same manner for a nontrivial monopole solution. However, ψ in $v = \exp\{i\psi\}k$ is a cyclic variable, therefore we can set $\psi = 0$.

If we define

$$K(x) = \{1 + irk(r)\}/\delta, \qquad F(x) = r \cdot f(r), \qquad G(x) = r \cdot g_1(r), \qquad x = c\delta r/b$$

where $\delta^2 = R^2/20$ if the broken phase is $SU(4) \times U(1)$, or $\delta^2 = 2R^2/15$ if it is $SU(3) \times SU(2) \times U(1)$ ($R^2 = \operatorname{tr} \varphi^2$ in the corresponding vacua, compare page 543ff), we end up with the following differential equations

$$\begin{aligned} x^2K'' &= F^2K + (K^2-1)K \\ x^2F'' &= 2FK^2 + b\{\lambda/\delta^2 \cdot x^2F + \eta/2 \cdot (25F^2+15G^2)F - 3x/\delta \cdot FG \\ &\quad + 27/2 \cdot G^2F + 25/2 \cdot F^3\} \\ x^2G'' &= b\{\lambda/\delta^2 \cdot x^2G + \eta/2 \cdot (25F^2+15G^2)G - x/2\delta \cdot G^2 - 5x/2\delta \cdot F^2 \\ &\quad + 7/2 \cdot G^3 + 45/2 \cdot GF^2\}. \end{aligned} \tag{3.18}$$

These equations were also derived by Steinhardt [St] who solved them numerically. A interesting property of the solution which asymptotically lies in the 4–1-vacuum ($f(r)$ and $g(r)$ approach $\delta = sqr(1/20) \cdot R$ as r approaches ∞) is, that the core is approximately in the 3–2–1 phase. That indicates that the *monopole dissociation process* might have occurred in the early universe.

As a *second example* we choose the $SO(3)$ imbedding

$$2^{1/2}T_1 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad 2^{1/2}T_2 = \begin{pmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \end{pmatrix}$$

$$T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.19}$$

The character of this representation is $ch(\varphi) = 2[2\cos(2\varphi) + 2\cos(\varphi) + 2] + 4[2\cos(\varphi) + 1] + 2$, hence $\mathbf{G} = 2\mathbf{G}_2 + 4\mathbf{G}_1 + 2\mathbf{G}_0$. Our construction rules give

$$\varphi = \begin{pmatrix} f_1 & & \\ & F^{\sim} & \\ & & f_2 \end{pmatrix} = g\begin{pmatrix} -1.5 & & \\ & I & \\ & & -1.5 \end{pmatrix} + f\begin{pmatrix} 2.5 & & \\ & 0 & \\ & & -2.5 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & F & \\ & & 0 \end{pmatrix} \tag{3.20}$$

$$A_{\hat{1}} = \begin{pmatrix} 0 & u_1 & u_2 & u_3 & 0 \\ & & & & v_1 \\ & & & & v_2 \\ & & & & v_3 \\ & & & & 0 \end{pmatrix} \tag{3.21}$$

where F is of course a 3-dimensional hermitian matrix with vanishing trace, $\mathrm{tr}\{F(r)\} = 0$.

Using the abbreviations $v = (v_1, v_2, v_3) \in C^3$ and $u = (u_1, u_2, u_3) \in C^3$, we find that the action of this configuration is invariant w.r. to

$$F(r) \to UF(r)U^{-1}$$
$$v(r) \to U_1 v(r)$$
$$u(r) \to U_2 u(r),$$

and has thus a $SU(3)\times SU(3)\times SU(3)$ invariance.

As before we may assume that $F=0$, $u=(0,0,h)$ and $v=(0,0,l)$. If we finally set

$$R^2(x)=(r^2h^2(r)-2)/\delta^2 \qquad S^2(x)=(r^2l^2(r)-2)/\delta^2$$
$$F(x)=r\cdot f(r) \qquad G(x)=r\cdot g(r)$$

where again $x:=c\delta/b\cdot r$, we obtain for our ansatz as YMH-equations:

$$\begin{aligned}
x^2S'' &= S/2\cdot(2S^2-1-R^2)+25S/4\cdot(F+G)^2\\
x^2R'' &= R/2\cdot(2R^2-1-S^2)+25R/4\cdot(F-G)^2\\
x^2F'' &= x^2R^2(F-G)/2+x^2S^2(F+G)/2+b\cdot[\lambda/\delta^2\cdot x^2F\\
&\quad +\eta(25F^2+15G^2)F/2-3xFG/\delta+27FG^2/2+25F^3/2]\\
x^2G'' &= 5x^2R^2(G-F)/6+5x^2S^2(F+G)/6+b\cdot[\lambda/\delta^2\cdot x^2G\\
&\quad +\eta(25F^2+15G^2)G/2-xG^2/2\delta-5xF^2/2\delta+7G^3/2+45GF^2/2]
\end{aligned} \tag{3.22}$$

It should be clear from our construction, that a solution of (18) provides an $SU(N)$-solution for every $N\geq 2$. For $N=2$, $G=0$ and a special choice of the 'Higgs parameters' one obtains the *'t Hooft–Polyakov monopole* [H], [P].

III.3. *Properties of spherically symmetric monopoles*

a) *A discouraging lemma*

If a large symmetry group (e.g. $SU(5)$) is broken down to $SU(3)\times SU(2)\times U(1)$, then we expect monopole solution which approach asymptotically the nonsymmetric vacuum. For the typical case, where φ belongs to the adjoint representation (Ad), the vacuum manifold M_0 is the orbit of $\varphi_0=c\cdot\mathrm{diag}\,(1,1,1,-3/2,-3/2)$. Hence one is interested in (spherically symmetric) monopole solutions, with the property that the asymptotic values $\varphi^\wedge$ of φ are in the orbit of φ_0 and fulfil the YMH-equations. These requirements are, however, frequently not compatible for a spherically symmetric configuration because of the following lemma.

Lemma. *If $\varphi_0\in SU(N)$ is regular and has an eigenvalue λ with multiplicity greater than $N/2$, then the vacuum manifold $M_0\sim G/H$ has no 3 elements X_1, X_2, X_3, which satisfy $[X_1,X_2]=ipX_3$ for some nonvanishing p.*

Proof. Let us assume that $X_j\in M_0=G/H$ and that $[X_1,X_2]=ipX_3$ holds. Then $X_j=\mathrm{Ad}\,(U_j)\varphi_0$ with $U_j\in SU(N)$, and

$$[\mathrm{Ad}\,(U_1)\varphi_0, \mathrm{Ad}\,(U_2)\varphi_0]=ip\,\mathrm{Ad}\,(U_3)\varphi_0. \tag{$*$}$$

The left hand side is also equal to $[\mathrm{Ad}\,(U_1)\varphi_0-\lambda\cdot I, \mathrm{Ad}\,(U_2)\varphi_0-\lambda\cdot I]$. Because

φ_0 has more than $N/2$ eigenvalues λ and Ad is a similarity transformation, we conclude that $\dim\{\mathrm{Ker}\,[\mathrm{Ad}\,(U_j)\varphi_0 - \lambda \cdot I]\} > N/2$ or that the dimension of the kernel of the left hand side in (*), interpreted as element of $L(N)$, is >0. On the other hand we assumed that φ_0 and hence the right hand side is regular. With this contradiction the lemma is proven.

In typical examples one can argue that the $g_q(r)$ is the expansion (8) for $\varphi(x)$ vanish. But we always have a θ_1 representation of $SU(2)_G$, because $\{T_1, T_2, T_3\}$ span an irreducible subspace V_1 (we assumed that φ is of type ad). With our lemma we conclude for example, that for $N>4$ there exists no solution of the form (8) which contains only the θ_1 representation.

Therefore, demanding $g_1 = 0$ and $g = 0$ in (16) is not consistent.

b) *$M_0 \cap S^q$ must be topologically nontrivial*

For all spherically symmetric configurations the norm of $\varphi \in V$ and the norm of A in $R^3 x$ **G** depends only on the radius r and not on the direction in R^3. This is a trivial consequence of (1), (2), and (3). Hence the asymptotic map $\varphi\hat{}: S^2 \to M_0$ is in fact a map between spheres, $\varphi\hat{}: S^2 \to S^q$.

c) *Topological quantum numbers for spherically symmetric fields*

Now we compute the topological quantum numbers for a connected and simply connected compact Lie group G and a connected H. For that we use the *isomorphism* $\alpha: H_2(M_0 = G/H) \to H_1(H)$ in the exact sequence

$$0 \to H_2(G/H) \to H_1(H) \to 0.$$

Let η be an element of $H_2(G/H)$ which is represented by the 2-chain $c_2 \in C_2(G)$ with boundary δc_2 in the cycles of H. Then $\alpha(\eta)$ is the class $[\delta c_2]$ of δc_2 in $H_1(H)$.

In our case $H_2(G/H)$ and $H_1(H)$ are both generated by a finite number of cycles. We denote the generators of $H_1(H)$ by $\sigma_1, \ldots, \sigma_r$. A connected H decomposes as $H = K \times U_1(1) \times \cdots \times U_r(1)$, with semisimple and simply connected K. Hence we may choose σ_j as a generator of $H_1(U_j(1))$.

For a spherically symmetric field we have $\varphi\hat{}(e) = U(g(a))\varphi_0$, with $a = \exp[-i\varphi T_3]\exp[i\theta T_2]$ and where φ_0 is the asymptotic value of $\varphi(r)$. From $U(g(a))\varphi_0 = \varphi_0$ for $\theta = 0$, we conclude that $g(a(\theta = 0))$ is an element of H. Thus the map

$$\varphi\hat{}: S^2\text{-Northpole} \to M_0$$
$$(\varphi, \theta) \to \varphi\hat{}(\varphi, \theta)$$

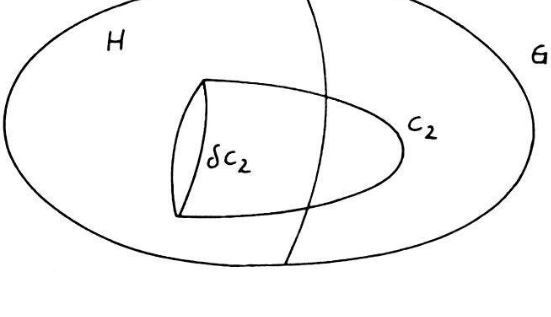

Figure 5
A 2-chain $c_2 \in C_2(G)$ and its boundary δc_2. δc_2 defines a cycle in the subgroup H of G.

defines a cycle in G/H. The boundary of this G/H-cycle is the map

$$\delta\varphi^\wedge : S^1 \to H$$
$$\varphi \to U\{\exp[-i\varphi T_3]\}.$$

Now it is straightforward how to compute the topological charges $\{q_1, \ldots, q_r\}$. Since $\delta\varphi^\wedge(\varphi) = U(k(\varphi))U(u_1(\varphi)) \cdots U(u_r(1))$, with $k(\varphi) \in K$ and $u_j(\varphi) \in U_j$, the integer q_j is the degree of the map

$$S^1 \to S^1$$
$$\varphi \to U(u_j(\varphi)).$$

But on the other hand $k(\varphi)u_1(\varphi) \cdots u_r(\varphi) = \exp[-i\varphi T_3]$, and so the q's can be determined with simple algebraic methods.

The *procedure* is the following:

In the Lie algebra $R + \cdots + R$ of the torus-subgroup T^r of H we choose the base $w_1, \ldots, w_r$ such that the w_j have the smallest possible norm under the constraint $U\{\exp[-2i\pi w_j]\} = Id$. Now we can uniquely expand

$$T_3 = q_1 w_1 + \cdots + q_r w_r + (\text{an element in Lie } (K)). \tag{3.23}$$

The expansion coefficients $q_j \in Z$ are the *topological quantum numbers*, because $\exp[-iw_j\varphi] = \sigma_j$ is the generator of $H_1\{U_j(1)\}$. (Note that the orientation has already been chosen.)

Example. The 't Hooft–Polyakov monopole

Here $G = SU(2)$, $H = U(1)$ and φ is an $SU(2)$ triplet. In our previous notation $r = 1$ and $w_1 = S_3$. On the other hand, there exists only one nontrivial homomorphism $SU(2) \to SU(2)$ so that $T_3 = S_3$ or $q_1 = 1$.

It follows that the topological quantum number of a 't Hooft–Polyakov monopole is 1.

This is an old result. It was first proven by A. Guth and E. Weinberg [GW]. Later L. O'Raifeartaigh [R] generalized this result to the case where $G = SU(2)$ and φ belongs to an arbitrary representation.

If φ is a triplet then $\varphi^\wedge$ is a map from S^2 to S^2 and the degree of (the normalized) $\varphi^\wedge$ is [JT], [Str]

$$\deg\{\varphi^\wedge\} = 1/4\pi \cdot \int (\varphi^\wedge, d\varphi^\wedge \wedge d\varphi^\wedge). \tag{3.24}$$

With the help of (8) we find $\varphi^{\wedge b}(x) = x^b f(r)$ and thus

$$N = 1/8\pi \cdot \int \varepsilon_{bcd} f^3 x^b \, dx^c \wedge dx^d.$$

It is known [JT] that $1 - |\varphi(x)| \leq \text{const} \cdot \exp[-(1-\varepsilon)m_L r]$, where $m_L = \min\{sqr(\lambda), 2\}$ and ε is an arbitrary positive constant. So we arrive at

$$N = 1/4\pi \cdot \int \varepsilon_{bcd} x^b \, dx^c \wedge dx^d = 1.$$

We emphasize that our method for computing the topological quantum numbers of spherically symmetric configurations is extremely simple and applicable to nearly all 'interesting' situations, e.g. for $G = SU(N)$ and any desired representation for φ. One only has to normalize the base $\{w_1, \ldots, w_r\}$ of the maximal Abelian ideal in the Lie algebra $\mathbf{H}$ of H and calculate $q_j = (T_3, w_j) \in Z$ for $j = 1, \ldots, r$.

d) *Dirac equation in a spherically symmetric monopole field*

From the Lagrangian of a Dirac field ψ (which transforms according to a representation V of G) in *external* YM- and Higgs fields,

$$L(\psi) = \bar{\psi}(i\gamma^b D_b - m)\psi - \psi(\Gamma, \varphi)\psi, \quad \text{where} \quad D_b = \delta/\delta x^b - iV_*(A_b), \tag{3.25}$$

one obtains the *Dirac equation*

$$i\hbar\, d\psi/dt = H\psi \quad \text{with} \quad H = \alpha_j(p_j - V_*(A_j)) + m\beta + \beta(\Gamma, \varphi) + V_*(A_0). \tag{3.26}$$

From now on we assume that (A, φ) is spherically symmetric. The Lagrangian is therefore invariant under a spherical rotation of the Dirac field,

$$(\Gamma(a)\psi)(x) = V(g(a))S(a)\psi(R(a)^{-1}x). \tag{3.27}$$

Here $V(g(a))$ is defined via the homomorphism $SU(2) \to G$ (compare page 547ff), $S(a)$ is the usual transformation of Dirac spinors with respect to $SU(2)$ (as subgroup of $SL(2, C)$) and $R(a)$ is the rotation corresponding to $a \in SU(2)$.

From the invariance property

$$L(\Gamma(a)\psi)(x) = L(\psi)(R^{-1}x)$$

we obtain the conserved angular momentum in the usual way [Str]:

The 1-parametric subgroup $a(t) = \exp[-iS \cdot \omega t]$ defines the vector field $X(x) = dR(a(t))x/dt|_{t=0} = \omega \wedge x$. Hence the Lie derivative of ψ with respect to $X(x)$ is $L_X\psi = V_*(T \cdot \omega)\psi + S_*(S \cdot \omega)\psi - (\omega \wedge x)\, D\psi$. Here we used the notation $T_b = g_*(S_b)$ from Section III.1. The conserved *Noether current* is $(J^i, \omega) = \bar{\psi}\gamma^i\omega\{V_*(T)/i + S/i - x \wedge \nabla\}\psi$.

So we obtain the following *conserved angular momentum* in the state ψ

$$(\psi, J\psi) = \int \psi^*\{L + V_*(T) + S\}\psi\, d^3x, \tag{3.28}$$

which is the sum of the orbital angular momentum $\langle L \rangle$, the spin $\langle S \rangle$ and the term $\langle V_*(T) \rangle$ from the 'internal rotations'. More precisely, we should write $\sigma_0 \circ S$ instead of S.

One can, of course, check that $J = L + S + V_*(T)$ commutes with H by using the identities (4), (5) for spherically symmetric fields. Hence the eigenstates of H are classified by their total angular momentum. Especially the state with $J = 0$ is invariant under spherical rotations, i.e.,

$$V(g(a))S(a)\psi(R^{-1}(a)x) = \psi(x). \tag{3.29}$$

Again we can use the theorem on page 547 for the $J = 0$ *sector*. The Lagrangian of a Dirac field with $J = 0$ depends only on r and is given by

$$\{L(\psi)\}(x) = \bar{\psi}(r)\{\gamma^0 d_t + \gamma^3 d_r - i\gamma^3 A(r)\}\psi(r) + \bar{\psi}(r)\{\Gamma\varphi(r)\}\psi(r), \tag{3.30}$$

where $\psi(r)$, $A(r)$ and $\varphi(r)$ are the initial data of ψ, A and φ (compare page 548).

Analogously as for scalar and vector fields one finds the following *expansion for the initial data* of the upper two components α of ψ

$$\alpha_M(r) = \sum_j \sum_{k=\pm 1/2} \langle (j+k)\, 0\, j\, M \mid 1/2\, M \rangle v_{2k}(r) \cdot |jM\rangle, \qquad M = \pm 1/2 \tag{3.31}$$

and an analogous for the lower components β.

We see that only representations θ_j with $j \in \{1/2, 3/2, 5/2, \ldots\}$ contribute in the representation space of the Dirac field.

Example. An isospin-doublet in the field of a 't Hooft–Polyakov monopole.

In the spherical gauge the 't Hooft–Polyakov monopole ($G = SU(2)$, φ in the adjoint representation) can be written in the following way [H], [P]

$$\begin{aligned} &\varphi(r) = -H(r)/r \cdot S_3 \\ &A(r) = \{-[1-K(r)]/r \cdot S_2, [1-K(r)]/r \cdot S_1, 0\}, \end{aligned} \tag{3.32}$$

where H and K are solutions of equations similar to (18) with $G = 0$.

Now we assume that $\alpha(\beta)$ is a doublet. Hence only $j = 1/2$ is possible in the expansion (31).

If $c(r) = v_{-1} - v_1/3^{1/2}$ and $d(r) = v_{-1} + v_1/3^{1/2}$ then

$$\alpha_{1/2} = (c(r), 0) \quad \text{and} \quad \alpha_{-1/2} = (0, d(r)).$$

So it is extremely simple to handle the $J = 0$ sector. We only have to insert our expression for $\psi(r)$ into (31) and vary with respect to the functions $c(r)$, $d(r)$ of α resp. $e(r)$, $f(r)$ of β to obtain the Dirac equation for the ansatz (31).

Marciano and Muzinich [MM] were able to solve the Dirac equation in the Prasad–Sommerfield field ('t Hooft–Polyakov monopole with vanishing self-coupling of the Higgs field) for $J = 0$ analytically. They required, however, that the doublet does not couple to the Higgs field. They found that in the $J = 0$ sector there is a pure charge exchange and that helicity is conserved, i.e., an 'up'-quark is scattered into a 'down'-quark. *On this level* the monopole catalyzes proton decay.

One can now ask the question, whether there is some sort of classical *Rubakov-effect* or, whether a classical Yang–Mills particle which is scattered at a monopole field suffers charge exchange.

e) *Classical limit of Yang–Mills particles in a monopole field*

S. Wong [Wo] derived the classical equations of motion for an isotopic-spin-carrying particle in an *external* YM-field. Similarly as one can extract the Lorentz force law in the classical limit of the Dirac equation in an external electromagnetic field, he derived these equations by taking the classical limit of the Dirac equation of an isotopic-spin-carrying particle in a given external YM-field.

These equations are the Euler–Lagrange equations of the action with Lagrangian

$$L = -mc[g_{bc}\dot{x}^b\dot{x}^c]^{1/2} - i\,\mathrm{tr}\,\{Ks^{-1}D_{\dot{x}}s\}, \tag{3.33}$$

where $D_{\dot{x}} = d/d\tau - ig\dot{x}^b A_b$ and the isotopic spin is $I = sKs^{-1}$. The matrix K determines the irreducible representation to which the particle belongs. Here we simplify the notation and drop the V's and V_*'s.

The equations of motion for the Lagrangian (33) are

$$dp^b/d\tau = g/mc \cdot \mathrm{tr}\,\{I \cdot F^{bc}\}p_c \tag{3.34}$$

$$D_{\dot{x}}I = dI/d\tau - ig\dot{x}^b[A_b, I] = 0 \tag{3.35}$$

or

$$\dot{p}^0 = g/mc \cdot \{p\ \mathrm{tr}\ IE\} \tag{3.36}$$

$$\dot{p} = g/mc \cdot \{p^0\ \mathrm{tr}\ IE + p \wedge \mathrm{tr}\ IB\}. \tag{3.37}$$

Equation (34) is the *generalized Lorentz law*. (35) describes the *parallel transport* of the isospin along the particle path $x^b(\tau)$. We notice as an immediate consequence of (35)

$$\frac{d}{d\tau}\mathrm{tr}\ I^2 = 0. \tag{3.38}$$

The isotopic spin of each particle thus performs a *precessional motion* [Wo].

We are especially interested in the case when the YM-field is due to magnetic monopoles. Then E vanishes and we in addition have the conservation of the kinetic energy

$$d(\gamma mc^2)/d\tau = 0. \tag{3.39}$$

Now let us specialize further and assume, that the monopole field is *spherically symmetric* and therefore angular momentum is conserved.

Again exploiting the spherical symmetry of A it is easy to see that

$$L(R(a)x, g(a)s) = L(x, s),$$

where $R(a)$ and $g(a)$ are defined as usual via the $SU(2) \to G$ homomorphism. For the *conserved angular momentum* we obtain in the nonrelativistic limit

$$J = mx \wedge \dot{x} - gx \wedge \mathrm{tr}\,\{IA\} - \mathrm{tr}\,\{IT\}. \tag{3.40}$$

Here, as before, $T = \{g_*(S_1), \dots\}$. Using $F_{jk}\dot{x}_k = (\dot{x} \wedge B)_j$ we find the following set of equations

$$m\ d^2x/dt^2 = \frac{dx}{dt} \wedge \mathrm{tr}\,(IB) \tag{3.41}$$

$$dI/dt = i[dx/dt \cdot A, I] \tag{3.42}$$

and the conservation laws

$$T = m\dot{x}^2 = \mathrm{const} \tag{3.43}$$

$$\mathrm{tr}\ I^2 = \mathrm{const} \tag{3.44}$$

$$J = mx \wedge \dot{x} - gx \wedge \mathrm{tr}\,(IA) - \mathrm{tr}\,(IT) = \mathrm{const}. \tag{3.45}$$

Now let us study the motion of a YM-particle in the field (32) of a *'t Hooft–Polyakov monopole* and neglect the interaction with the Higgs field (as was already done in the Wong equation). Here we can expand

$$I = \gamma(t)T$$

and obtain the following equations for $x(t)$ and $\gamma(t)$:

$$m\ddot{x} = -K'/r \cdot \dot{x} \wedge \gamma - (K^2 - rK' - 1)/r^4 \cdot (x, \gamma)\dot{x} \wedge x \tag{3.41'}$$

$$\dot{\gamma} = \gamma \wedge (\dot{x} \wedge x)(1-K)/r^2 = \{(\gamma, x)\dot{x} - (\gamma, \dot{x})x\}(1-K)/r^2 \tag{3.42'}$$

$$J = mx \wedge \dot{x} + (\gamma, x)x(K-1)/r^2 - \gamma K \tag{3.45'}$$

from which follows, for example, that

$$(J-\gamma)x=0. \tag{3.46}$$

Because K decays exponentially, we obtain the asymptotic equations

$$m\ddot{x}\sim\dot{x}\wedge x(x,\gamma)/r^4$$
$$\dot{\gamma}\sim\{(\gamma,x)\dot{x}-(\gamma,\dot{x})x\}/r^2$$
$$J\sim mx\wedge\dot{x}-(\gamma,x)x/r^2.$$

With the conservation of the kinetic energy we find from the first of the asymptotic equations that (d is a constant of integration)

$$r\sim[2T/m\cdot t^2+d^2]^{1/2}.$$

Using this, together with $|x\wedge\dot{x}|^2=r^2\dot{x}^2-(x\dot{x})^2=2Td^2/m$, yields

$$J^2=[(\gamma,x)/r]^2+2d^2Tm$$

or that $(\gamma, x/r)=\text{const}$. But with (46) $(\gamma, x/r)=(J, x/r)=\text{const}$, and hence the *YM-particle moves asymptotically on a cone.*

For a *vanishing total angular momentum* $J=0$ and $(J,x)=-(\gamma,x)K=0$ the equations (41′, 42′, 45′) simplify to

$$m\ddot{x}=-\dot{x}\wedge\gamma\cdot K'/r \tag{3.47}$$

$$\dot{\gamma}=0. \tag{3.48}$$

So $\gamma=\text{constant}$ and *classically there is no charge exchange* for $J=0$. The orbital motion of the YM-particle is the same as the motion of an electron in a magnetic field $B=-\gamma K'/r$. Since $(\gamma,x)=0$ and γ is constant the YM-particle moves on a plane which is perpendicular to γ.

In a *Prasad–Sommerfield* monopole field $K=Dr/\sinh(Dr)$, we can solve the equations of motion analytically. For that we introduce polar coordinates (ρ,φ) in the plane of the particle's path. With the abbreviation $A^2=D^2/2mT$ we find

$$\rho(t)=4\pi/M\cdot\operatorname{arcosh}\{(1+A^2)^{1/2}\cdot\cosh(2AT\cdot t)\}. \tag{3.49}$$

Here $M=4\pi D$ is the mass of the Prasad–Sommerfield monopole.

One should compare (49) with the solution for a Dirac pole

$$\rho(t)^2=2T/m\cdot t^2.$$

As one expects $\rho(t)$, $\varphi(t)$ approach the Dirac solution for $M\to\infty$.

f) *A remark on axisymmetric monopoles and multimonopole solutions in the Prasad–Sommerfield limit*

We have seen that *ss*-monopole solutions must have unit topological charge. Conversely, $SU(2)$-configurations with unit topological charge have to be *ss* [We]. According to these results, monopoles of charge greater than unity can only be obtained by dropping the assumption of *ss*. But then the field equations (1.19) become extremely complicated. Without the following observation [Bo] the problem would probably have remained intractable.

For $G = SU(2)$ and $\varphi \in su(2)$ we find that the Euclidian action is

$$S[A, \varphi] = \tfrac{1}{2} \|{}^*F \mp D\varphi\|^2 + \int V(\varphi)\, d^3x \pm 4\pi q \tag{3.50}$$

and hence is bounded from below by $4\pi\,|q|$. In the limit of a vanishing Higgs potential (*Prasad–Sommerfield limit*) the action attains its lower bound for fields which satisfy the Bogomolny equations

$$D\varphi = \pm B \tag{3.51}$$

Therefore, for $V = 0$, the field equations (1.19) reduce to these much simpler first order equations. The Bogomolny limit is the only case in which one can hope to describe a system of separated monopoles in static equilibrium [Man], [OPW], because only then the attractive long range force due to the φ-field may cancel the long-range force due to the A field.

The equations (3.51) have also a geometrical significance. If we identify φ with A_0 then (3.51) is the static version of the 4-dimensional self dual equations $F = \pm{}^*F$. As a result the whole progress on multimonopole solutions made in the last years is restricted to the Bogomolny limit of a vanishing self interaction of the Higgs field.

Perhaps the most fundamental development is a proof, due to Taubes [JT], that the Bogomolny equations do indeed admit static, *separated*, monopole configurations for all magnetic charges q. Furthermore, he showed that the exact solutions had to be real analytic.

The other problem, to construct *explicit solutions* with arbitrary charges and finally to prove a completeness theorem asserting that this construction generates all solutions ($4q - 1$ real parameters for $SU(2)$ [We]) has as its natural first step the construction of axisymmetric configurations.

One can define axisymmetric fields by demanding (1), (2) and (3) only for $a \in SU(2)$, which describe rotations around a fixed axis, e.g. the 3-axis. The configuration is determined by its initial values on a half plane which has the symmetry axis as its boundary. In the sense of the theorem on page 547 one has to construct again the most general axisymmetric configuration. For $G = SU(2)$ and φ in the adjoint representation this has been done by Lohe [Lo]. He remained with a 2-dimensional $SU(2)$-theory for 2 iso-triplets in a curved space. The reduced problem is still too complicated for attempting a complete solution. So Manton [Man] constructed an axially and mirror symmetric ansatz for finding solutions. Using the same model as Lohe, he was able to express all remaining fields (two doublets) in terms of *one* function, the so-called superpotential. The 5 differential equations for the 6 functions in Manton's ansatz still possess a residual $U(1)$ gauge invariance. By using this invariance Forgacs et al. [FHP] introduced 2 potentials such that 3 of these 5 equations are automatically fulfilled. The remaining 2 equations are equivalent to the celebrated *Ernst equation* of General Relativity [Er]. Then Forgacs et al. used successfully an appropriate Bäcklund transformation for generating solutions of the Ernst equation and hence axially symmetric multimonopoles in an $SU(2)$ theory.

At the same time R. Ward [Wa] constructed an axisymmetric $q = 2$ monopole solution, using twistor methods. This led then naturally to axially symmetric solutions of orbitrary charge [Pra].

But in both approaches not so much is known about the regularity of the solutions (*regularity problem*).

At first sight one should expect the axisymmetric solutions to describe both single monopoles with arbitrary charge and separated poles localized on an axis. But it was shown [Ho] that axisymmetric systems can describe only single monopoles.

Therefore one has to give up the axial symmetry to obtain the separated monopoles predicted by Taubes.

Generalizing his earlier results Ward outlined how to construct a separated 2 monopole solution using the Atiyah–Ward ansätze [Wa]. Ward's solution was immediately generalized by Corrigan and Goddard to a q-monopole solution depending on the maximal number of degrees of freedom [CG].

Independently Forgacs et al. generalized their axially symmetric monopoles to a $4q-1$ parameter family using the 'inverse scattering' method [FHP] developed for the Bogomolny equations. The advantage of their method is the obtained explicit form of the solutions.

The main drawback of both generalizations once again is that regularity is not automatically guaranteed.

Later Nahm adapted the ADHM instanton construction technique to monopoles. He replaced the $(n+1)$ dimensional quaternionic vector space for n-instanton fields by $L_2[-\frac{1}{2}, \frac{1}{2}] \otimes$ quaternionic space [Na]. The algebraic constraints in the ADHM construction are replaced by the equations

$$\frac{dT_\alpha}{dt} = \varepsilon_{\alpha\beta\gamma}[T_\beta, T_\gamma]$$

where the matrix functions $T_\alpha = T_\alpha(t)$ are antihermitian. The vector potential is then given by the zero modes of a matrix differential operator which contains essentially the T's. The question of regularity can hopefully more easily be decided within this framework.

To summarize, the approaches of Ward et al., Nahm, Forgacs et al. and, later, by Hitchin [Hi] give the complete set of $SU(2)$ monopoles in the Bogomolny limit. The generalization of the above results to groups larger than $SU(2)$ was also considered. Explicit generalization of the Forgacs et al. method was considered by these authors and by Bais. Ward generalized his own approach to $SU(3)$. The approach of Nahm makes little distinction between different groups and lends itself more readily to generalization.

As the most general $SU(2)$ q-pole configuration depends on $4q-1$ parameters of which $3q$ corresponds to positional degrees of freedom the interpretation of the remaining $q-1$ poses a problem. As it turned out these parameters have probably a deep (geometrical) significance [Hi, FPH]. For an interpretation of the parameters in some very special cases compare [OR].

For further review of magnetic monopoles we refer to [Bu, OR, Pre, Ro].

IV. The WKB-exponent in field theory

IV.1. *Generalities*

The evolution of the universe in very early epoches may have been dramatically affected by the nature of the *phase transition* which is associated to the spontaneous symmetry breakdown of the grand unified gauge symmetry. It is now widely believed that this phase transition was of strongly first order.

The 'condensation' of the metastable symmetric phase into the asymmetric equilibrium state requires the occurrence of the quantum or thermal *fluctuations* of the order-parameter (Higgs field) of a certain critical size.

The rate of birth of this critical droplets involves the theory of *homogeneous nucleation*. A satisfactory theory based on first principles is not yet available. The work of Langer, Coleman and others are, however, very promising steps toward the solution of this difficult problem.

These workers used path integral techniques for calculating the analytic continuation of the 'energy' resp. 'free energy' in the potential. If the potential is changed such that the ground state becomes unstable, then the imaginary part of the analytic continuation of the 'energy' (pole on the second sheet of the resovent, Weisskopf–Wigner pole) is the decay-width of the metastable state.

For the effective potential one uses the semiclassical (one loop) approximation and computes the decay width by summing over all multi-instanton contributions. Treating the collective coordinates in a suitable manner, one finds for the *decay rate per volume and time* of the metastable state [La, CC]

$$\Gamma/V = i\cdot[S/2\pi\hbar]^{n/2}\cdot\exp[-S/\hbar]\cdot[\det\{-\Delta+m^2\}\{-\Delta+V''(\varphi)\}^{-1}]^{1/2}. \tag{4.1}$$

Here φ is the so-called *bounce solution* of the classical Euclidian field equation, i.e. a solution of

$$-\Delta\varphi + V'(\varphi) = 0 \quad \text{with} \quad \varphi(x)\to \text{false vacuum} \quad \text{for} \quad |x|\to\infty, \tag{4.2}$$

and with minimal Euclidian action (activation energy)

$$S = S[\varphi] = \int d^n x\{1/2\cdot(\nabla\varphi,\nabla\varphi) + V(\varphi)\}. \tag{4.3}$$

In what follows we assume that the false vacuum is given by $\varphi_+ = 0$. The mass m in the determinant is given by the curvature of the potential at the false vacuum, $m^2 = V''(\varphi_+)$, and the *prime* in (1) means that one should omit the *zero mode*(*s*) of $-\Delta + V''(\varphi)$.

This nice expression for the decay rate is in general difficult to evaluate. A very important observation for applications of (1) was a theorem of Coleman, Glaser and Martin [CGM], which states that the bounce is *spherically symmetric* (*ss*) for a large class of potentials V.

With $r^2 := t^2 + x^2$ the equation (2) for a *ss*-field becomes

$$d^2\varphi/dr^2 + (n-1)/r\cdot d\varphi/dr = V'(\varphi)$$
$$\text{with boundary condition } \varphi(r)\to\varphi_+ = 0 \quad \text{for} \quad r\to\infty. \tag{4.2'}$$

The action of a spherically symmetric function is

$$S[\varphi] = 2\pi^{n/2}/\Gamma(n/2)\cdot\int dr\, r^{n-1}[1/2\cdot(d\varphi/dr)^2 + V(\varphi)] \tag{4.3'}$$

and regularity at $r = 0$ requires

$$d\varphi/dr(r=0) = 0. \tag{4.4}$$

The nature of the bounce solution becomes clearer if we interpret equation (2′) as the equation of motion of a point particle in the potential $-V$ which suffers a friction force $(n-1)/r$ (see [CC]).

In applications one chooses usually an initial condition and integrates the differential equation (2′). Depending on the resulting asymptotic value one changes the initial condition and integrates again. This procedure is repeated until the asymptotic value of φ converges to the desired one.

In this chapter we develop an *alternative method* for calculating $S[\varphi]$. We shall derive a variational principle which is very handy. It can be used to prove spherical symmetry of the bounce solution, to derive *lower bounds* on S and for computing S[bounce].

First of all we establish some preliminaries concerning the spherical rearrangement of a function and certain Sobolev inequalities which will be used later on. Then, by making use of the mountain-pass theorem (which we will prove to be applicable for a specified class of potentials), we derive upper and lower bounds for the Euclidian action of the bounce solution. Furthermore, we compare the variational results with known exact results and with the thin wall approximation.

The conclusion of these comparisons will be that the new algorithm for computing $S_{\min}$, is not only stable and handy, but leads also to fairly good quantitative results.

IV.2. *A variational characterization of S[bounce]*

In this section we prove that the solution of the field equation $\Delta\varphi = V'(\varphi)$ with minimal Euclidian action is spherically symmetric. In contrast to the proof of Coleman, Glaser, and Martin [CGM] our approach enables us in addition to derive lower and upper bounds for S[bounce]. It is also very useful for constructing approximative solutions with variational techniques.

a) *Sobolev inequalities, spherical rearrangement*

In what follows we use the Hilbert space H_1 which is the completion of $C_{0,\infty}$ with respect to the norm

$$\|\varphi\|^2 = (\varphi, \varphi) = \int [(\nabla\varphi, \nabla\varphi) + \varphi^2]\, d^n x. \tag{4.5}$$

$H_{1,r} \subset H_1$ is the subspace of all spherically symmetric functions.

For the norm in L_p we use the symbol $\|\varphi\|_p$.

The following *imbedding properties* are well known:

i) For $n \geq 3$ the inequality

$$\|\varphi\|_p \leq c_p \|\varphi\| \quad \text{holds for} \quad 2 \leq p \leq nc := 2n/(n-2); \tag{4.6}$$

ii) The imbedding $H_{1,r} \to L_p(R^n)$ is compact for $n \geq 3$, $2 < p < nc$.

For proving the main theorem we will need this compact imbedding of $H_{1,r}$ in L_p. [Although there exists a generalization of the Sobolev–Kondrachov–Rellich Theorem for unbounded domains, due to Berger and Schechter [BS], by replacing the usual L_p spaces by weighted L_p spaces, it is not clear to us, how to use this BS-generalization directly to prove a generalized Palais–Smale condition on H_1.]

We also need the notion of the *spherical rearrangement* of a nonnegative function:

The spherical rearrangement (SR) $R\varphi(r)$ of a real valued, nonnegative function φ is the spherically symmetric and with r decreasing function which fulfils the conditions (θ is the step function)

$$\int d^n x\theta[\varphi(x)-v]=\int d^n x\theta[R\varphi(r)-v] \quad \text{for all } v\geq 0,$$

or equivalently (μ denotes the Lebesgue-measure on R),

$$\mu[\varphi^{-1}(B)]=\mu[(R\varphi)^{-1}(B)] \quad \text{for all Borel sets } B \text{ in } R.$$

From the first resp. second definition one immediately derives the following equations resp. inequalities:

iii) $R(\psi^p)=(R\psi)^p$ if $p>0$

iv) $\int V(\varphi)\, d^n x=\int V(R\varphi)\, d^n x$ for a measurable function V.

With the help of an isoperimetric inequality one can show [GGMT]

v) $\|R\varphi\|\leq\|\varphi\|$, where the equal sign holds iff φ is spherically symmetric and monotone.

With the identity $\varphi(x)=\int_0^\infty dt\theta\{\varphi(x)>t\}$ one can prove

vi) $\int \varphi\cdot\psi\, d^n x\leq\int (R\varphi)(R\psi)\, d^n x.$

Until now we assumed that the functions we are dealing with, beside being nonnegative, should approach zero as r goes to infinity. One can also define the *increasing spherically symmetric rearrangement* of functions which go to infinity when r approaches infinity. One concludes

vii) $\int \varphi\cdot\psi\, d^n x\geq\int R\varphi\cdot R\psi\, d^n x$ if $R\varphi$ is the decreasing and $R\psi$ the increasing SR of φ resp. ψ. The equal sign holds iff $\psi=\varphi^{-q}$ where $q>0$ real.

If the decreasing SR of φ exists, then the increasing SR of φ^{-q} exists and

viii) $R(\varphi^{-q})=(R\varphi)^{-q}$.

For additional properties of the spherical rearrangement and proofs of the stated (in)equalities we refer to [Fa, Lu].

b) *The main theorem*

In this section we first quote the *mountain-pass theorem*, due to Rabinowitz and Ambrosetti [AmRa]. Then we conclude that the action for a certain class of potentials fulfils the condition needed for applying this theorem. As a corollary we shall see easily, that the minimal action solution is spherically symmetric.

We use the following *notation:*
The action S is the sum of the kinetic and the potential terms

$$T[\varphi]=1/2\cdot\int(\nabla\varphi,\nabla\varphi)\,d^nx$$

$$V[\varphi]=\int V(\varphi)\,d^nx=:\int\{m^2/2\cdot\varphi^2+P(\varphi)\}\,d^nx.$$

By rescaling the coordinates, the function P and the field φ

$$x^*:=m\cdot x \qquad P^*(\varphi):=1/m^2\cdot P(\varphi) \qquad \varphi^*:=m^{1-n/2}\cdot\varphi$$

one obtains (dropping the stars)

$$S[\varphi]=1/2\,\|\varphi\|^2+\int P(\varphi)\,d^nx. \tag{4.7}$$

We are interested in critical points of the functional S on the Hilbert space H_1, or equivalently, in weak solutions of

$$\Delta\varphi=\varphi+p(\varphi), \quad \text{where} \quad p(z):=dP(z)/dz.$$

If φ is a critical point of S, then $T[\varphi]$ and $V[\varphi]$ are related by the *virial theorem:*

$$(n-2)T[\varphi]+nV[\varphi]=0 \qquad S=2/n\cdot T[\varphi]=-2/(n-2)\cdot V[\varphi]. \tag{4.8}$$

Because *scaling arguments* will play a crucial role in what follows, we give the proof by this method. Let $\varphi_t:=\varphi(x/t)$. Then

$$T[\varphi_t]=t^{n-2}T[\varphi] \quad \text{and} \quad V[\varphi_t]=t^nV[\varphi] \tag{4.9}$$

Thus

$$S[\varphi_t]=t^{n-2}(T[\varphi]+t^2V[\varphi]).$$

Since φ is a critical point we have $d_tS[\varphi_t]_{t=1}=0$, from which the virial theorem follows.

For the formulation of the mountain-pass theorem we need the notion of the *Palais–Smale-condition (PS):*

Definition. *The action S fulfils* (PS) *if all sequences $\{\varphi_m\}$, for which $S[\varphi_m]$ is uniformly bounded and $S'[\varphi_m]\to 0$, have convergent subsequences.*

Mountain-pass theorem [AmRa]. *Let E be a real Banach space, and let $S\in C^1(E,R)$. Assume that S fulfils* (PS), *the normalization condition $S[0]=0$ and satisfies*

S1) *there exist $r,\alpha>0$ such that $S\geq\alpha$ on a sphere $S_r(0)$*
S2) *there exists an $e\in E-B_r(0)$ such that $S[e]\leq 0$*

Then S has a critical value $c\geq\alpha$.

Furthermore, let $\Gamma:=\{g\in C([0,1],E)\mid g(0)=0$ and $g(1)=e\}$. Then $c:=\inf_{g\in\Gamma}\max_{t\in[0,1]}S[g(t)]$ is a *critical value* of S and $c\geq\alpha$.

The mountain-pass theorem requires the (PS)-condition only *locally*, i.e. on a subset of E, on which $c-\varepsilon<S[\varphi]<c+\varepsilon$ for some $\varepsilon>0$.

Let us define the class of functions P, or equivalently of functionals S, for which we will prove the validity of the assumptions made in the mountain-pass theorem.

(PC1) There exist a $\delta > 0$ and $\gamma > 2(1+\delta)$, such that

$$\gamma P(\varphi) - \varphi p(\varphi) + \delta\varphi^2 \geq 0.$$

(PC2) There exist an $a > 0$ and $0 < d < nc - 2$, such that

$$|p(\varphi) - p(\psi)| \leq a\,|\varphi|^R\,|\varphi - \psi| + a\,|\psi|^R\,|\varphi - \psi|, \quad \text{where} \quad R := nc - d - 2.$$

(PC3) There is (at least) one φ_0, s.t. $P(\varphi_0) + 1/2 \cdot \varphi_0^2 < 0$.

(PC3) is a *necessary condition*, because for every solution of the field equation $V[\varphi] < 0$ in more than 2 dimensions. *(PC2) can be weakened* in more than 2 dimensions. The first condition is the strongest and should be weakened.

We will discuss these conditions further later in this chapter and give examples in which (PC1)–(PC3) are fulfilled.

Proof of S1. From (PC2) with $\psi = 0$ we find $|P(\varphi)| \leq a\,|\varphi|^{R+2}$. Integrating this inequality over space-time and using (6) we find

$$S[\varphi] \geq 1/2 \cdot \|\varphi\|^2\,(1 - 2a \cdot c_{R+2}^{R+2}\,\|\varphi\|^R).$$

So we conclude that $S[\varphi]$ is positive inside a ball with finite radius (0 excluded) if $nc - d - 2 > 0$.

Proof of S2. If the graph of φ is as indicated in Fig. 6 and $\tau := \max_{[0,\varphi_0]} |V(\varphi)|$, then

$$V[\varphi] = \omega_n\left\{-\varepsilon R^n/n + \int_R^{R+h} r^{n-1} V(\varphi(r))\right\} \leq \omega_n R^{n-1}\{-\varepsilon R/n + \tau \cdot h\}$$

i.e. $V[\varphi] < 0$ if R is big enough. So we can find a t such that the rescaled function $\varphi(r/t)$ has negative action.

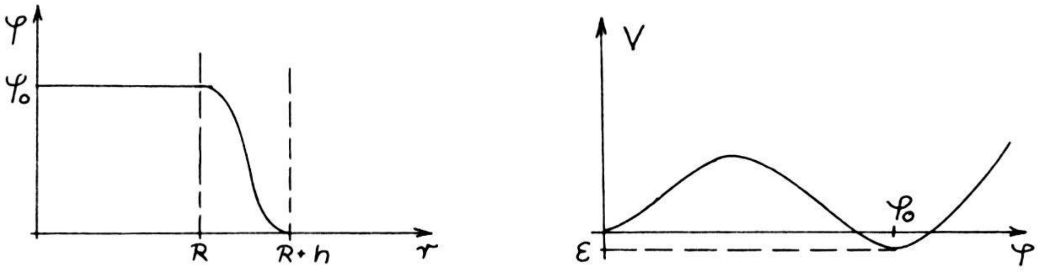

Figure 6
For fields φ, which have an extended core wherein $\varphi(r) = \varphi_0$, the potential energy is negative.

Proof of the Palais–Smale condition. First we show that $\{\varphi_m\}$ is bounded in H_1, or equivalently $\varphi_m \to \varphi$ weakly since H_1 is a reflective Hilbert space.

Using (PC1) and $\|\varphi\|_2 \leq \|\varphi\|$ we have

$$S[\varphi_m] - S'(\varphi_m)\varphi_m/\gamma = (1/2 - 1/\gamma)\,\|\varphi_m\|^2 + \left\{\gamma\int P(\varphi_m) - \int p(\varphi_m)\varphi_m\right\}\Big/\gamma$$

$$\geq (1/2 - 1/\gamma)\,\|\varphi_m\|^2 - \delta/\gamma \cdot \|\varphi_m\|_2^2 \geq [1/2 - 1/\gamma(1+\delta)] \cdot \|\varphi_m\|^2.$$

Now, if $\{\varphi_m\}$ is a sequence, with the properties required in the (PS)-condition, then

$$M - 1/\gamma \cdot o(\|\varphi_m\|) \geq \text{const} \cdot \|\varphi_m\|^2, \quad \text{where} \quad \text{const} > 0,$$

i.e., the sequence $\{\varphi_m\}$ is bounded.

We now use the isometric identification of $H_{1,r}$ and $H_{1,r}^*$ via the Riesz–Fischer theorem.

$$H_{1,r} \leftrightarrow H_{1,r}^*, \qquad \varphi + q(\varphi) \leftrightarrow S'(\varphi),$$

where

$$\int \{(\nabla q, \nabla \psi) + q\psi\} =: (q, \psi) = \int p(\varphi)\psi =: S_1(\varphi)\psi \quad \text{for all} \quad \psi \in H_{1,r}.$$

We use the following *strategy* to show that φ_m has a convergent subsequence:

First we prove that $S_1'(\varphi_m) \to S_1'(\varphi)$ in $H_{1,r}$, using the compact imbedding ii) on page 562. By Riesz-Fischer, $q(\varphi_m) \to q(\varphi)$ in $H_{1,r}$. But since $S'(\varphi_m)$, which is identified with $\varphi_m + q(\varphi_m)$, converges to zero, we conclude that φ_m must converge to φ.

Next we show $S_1'(\varphi_m) \to S_1'(\varphi)$. We use the notation $N := n-2$, $R := nc - d - 2$, $T := 4/(n-2)$, $U := 2n/(n+2)$.

Clearly

$$\|S_1'(\varphi) - S_1'(\varphi_m)\|^* = \sup_{\|\psi\|=1} \left| \int \{p(\varphi) - p(\varphi_m)\}\psi \right|$$

$$\leq \left\{ \int |p(\varphi) - p(\varphi_m)|^p \right\}^{1/p} \cdot \|\psi\|_q \qquad (1/p + 1/q = 1).$$

We divide space-time into 2 regions:

$$A := \{x \mid |\varphi(x)| \geq |\varphi_m(x)|\} \quad \text{and} \quad A^c.$$

In A $|\varphi|^R |\varphi - \varphi_m|$ dominates the same expression with φ and φ_m interchanged. So our integral is smaller than twice the integral over A of this expression plus twice the integral over A^c of the expression with φ and φ_m interchanged. Replacing A and A^c by R^n we find

$$\|\cdots\|^* \leq 2a \cdot \left\{ \int [|\varphi|^R |\varphi - \varphi_m|]^p \right\}^{1/p} \cdot \|\psi\|_q + \varphi \leftrightarrow \varphi_m.$$

Choosing $q = nc$ gives

$$\|\cdots\|^* \leq 2a \cdot \left\{ \int [|\varphi|^{T-d} |\varphi - \varphi_m|]^U \right\}^{1/U} \cdot \|\psi\|_{nc} + \varphi \leftrightarrow \varphi_m.$$

Using again the Hoelder inequality for $|\varphi|^{(T-d)U}$ and $|\varphi - \varphi_m|^U$, with $p = (n+2)/4 + e$, where we choose e such that the exponent of φ is nc, then, because $q = (n+2)/(n-2) - f$,

$$\|\cdots\|^* \leq 2ac \cdot \|\varphi\|_{nc}^T \cdot \|\varphi - \varphi_m\|_{nc-f}^{1+g} \cdot \|\psi\|,$$

where we used also the Sobolev inequality for ψ. Since $H_{1,r}$ is compactly imbedded in L_p for $2<p<nc$ and φ_m converges weakly to φ, we conclude that $\|\varphi-\varphi_m\|_{nc-f}^{1+g}\to 0$.

Consequences and remarks

a) *Spherical symmetry of the minimal action solution*

Let $\varphi=\varphi_+ + \varphi_-$, where $\varphi_+(x):=\max(\varphi(x),0)$ and analogously φ_-. Then [CGM] the scale invariant ratio $R[\varphi]=-T[\varphi]^{nc/2}/V[\varphi]$ of φ is larger than $\min\{R[\varphi_+], R[\varphi_-]\}$. Let us assume that $R[\varphi_+]<R[\varphi_-]$ and let us rescale φ_+ such that $V[\varphi_+]=V[\varphi]$, from which follows $T[\varphi_+]<T[\varphi]$. Because we assumed φ to be a solution of the field equation, we have

$$-(n-2)/n\cdot T[\varphi]/V[\varphi]=1, \quad \text{or} \quad S[\varphi]=2/n\cdot T[\varphi].$$

The maximum of $S[\varphi_{+t}]$ w.r. to t on the path $\varphi_{+t}(x)=\varphi_+(x/t)$ is assumed for $t_m^2=-(n-2)/n\cdot T[\varphi_+]/V[\varphi_+]$ and the corresponding value of S is $S_m=2T[\varphi_+]/n\cdot[(2-n)T_+/nV_+]^{n/nc}$, which is smaller or equal $2/n\cdot T[\varphi]=S[\varphi]$. The equality sign holds iff $\varphi_-=0$.

Now we spherically rearrange φ_+ and consider the path $R\varphi_+(r/t)$. Because $V[R\varphi_+]=V[\varphi_+]$, and $T[R\varphi_+]\le T[\varphi_+]$ and the equality sign holds iff φ_+ is spherically symmetric and monotonically decreasing with increasing radius, we obtain

$$\max_t S[R\varphi_{+t}]\le S[\varphi_{+tm}]=S_m\le S[\varphi],$$

i.e., there exists a critical value $\le S[\varphi]$. Thus, to every critical point φ, which is not spherically symmetric and monotone, there is another critical point with these properties and which has a lower action than φ.

b) *Weakening of condition* (PC2)

Let us assume that the potential has the shape sketched in Fig. 7. Then every solution φ^* of $\Delta\varphi=dV^*/d\varphi$ is a solution of $\Delta\varphi=dV/d\varphi$:

On $A:=\{x\mid\varphi(x)\ge\varphi_m\}$ the solution fulfils $\Delta\varphi^*=0$. If p obeys some extra conditions (e.g. Lipschitz-continuous) then the boundary of A is regular enough that we can conclude $\varphi^*=\varphi_m$ on A. In this case φ^* is also a solution of $\Delta\varphi=V'(\varphi)$, because $\varphi^*\le\varphi_m$ and for these values V and V^* agree.

Now one can use the same arguments as in a) to show, that the solution φ with minimal action fulfils $\varphi\le\varphi_m$. We simply define $\varphi_1(x)=\varphi(x)$ if $\varphi<\varphi_m$ and $\varphi_1(x)=\varphi_m$ elsewhere.

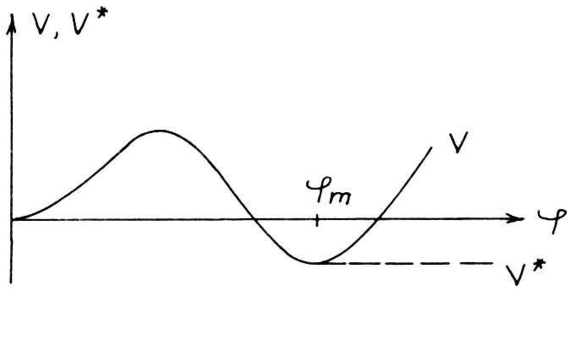

Figure 7
'Every' solution φ^* of $\Delta\varphi^*=V^{*\prime}(\varphi^*)$ is also a solution of $\Delta\varphi=V'(\varphi)$.

For $\varphi \leq \varphi_m$ one can always find a constant such that $|\varphi|^R \leq \text{const} \cdot |\varphi|^{1+d}$ for some small d. Furthermore, we saw in the proof of the (PS)-condition, that the two terms on the right hand side of (PC2) can be replaced by a sum of more terms with different R's. So (PC2) can be replaced by demanding that there exists a constant such that $|p(\varphi)| < \text{const} \cdot |\varphi|^{2+d}$ for $\varphi < \varphi_m$, if the potential has the form in Fig. 7.

c) *Remarks on* (PC1)

(PC1) for example is fulfilled for $P(\varphi) = C/m \cdot \varphi^m - A/n \cdot \varphi^n$ with $A \geq 0$, $C \geq 0$ and $2 < m < n$, because $\gamma P - p = C(\varphi/m - 1)\varphi^m - A(\gamma/n - 1)\varphi^n$ is positive if $m < \gamma < n$.

Since $|\varphi^n - \psi^n| \leq |2\varphi|^{n-1} |\varphi - \psi| + |2\psi|^{n-1} |\varphi - \psi|$ the condition (PC2) is also fulfilled if $n < nc$ (this can be weakened as we have seen). Condition (PC3) is clearly satisfied, so that our example is admissible with respect to (PC1–PC3).

Typical Higgs potentials violate, however, (PC1). For this reason we give a simple and short proof of the inf-max principle for potentials which are admissible in the sense of [CGM].

Let G be the subset

$G := \{g \mid g \in H \text{ and } V[g] < 0\}$, and let H be a subspace of H_1 depending on V.

Now we prove – using the results of Coleman, Glaser, and Martin – the

Theorem. *The minimal critical value $S_{\min}$ for an admissible potential (in the sense of* [CGM]) *is given by*

$$S_{\min} = \inf_{G} \max_{[0,1]} S[g(t)],$$

where $g(t, x) := g(x/t)$.

Proof. We first express the kinetic energy of the element g^* of $\{g(t)\}$, which has potential energy $V[g^*] = -1$, as a function of $S[g_m]$, where g_m is the element of $\{g(t)\}$ with maximal action. We find $g^* = g_m(x/t_0)$, where $t_0 = (-V[g_m])^{-1/n}$.

Using the *virial theorem*, $(n-2) \cdot T[g_m] + n \cdot V[g_m] = 0$, or

$$T[g_m] = n/2 \cdot S[g_m], \qquad V[g_m] = -(n-2)/2 \cdot S[g_m], \tag{4.10}$$

one easily finds

$$T[g^*] = nc/2 \cdot \{(n-2)/2 \cdot S[g_m]\}^{2/n} =: c(n) \cdot S[g_m]^{2/n}. \tag{4.11}$$

Next we recall the conclusion in [CGM] that there exists a minimizing sequence $\{\varphi_k\}$ for the kinetic energy with the constraint $V[\varphi_k] = -1$. This function φ_k can be chosen to be spherically symmetric and monotonically decreasing with increasing radius r. The minimal action solution φ_m is then given by rescaling the 'limit function φ^*' of the minimizing sequence of the constraint problem.

Now let us assume that $S_{\min}$ differs from the action $S[\varphi_m]$ of the CGM-solution and let $g_1(t, x) := \varphi_m(x/t)$. Because $\varphi_m \in G$ and $S[\varphi_m] = \max_t S[g_1(t)]$ we have $S_{\min} \leq S[\varphi_m]$.

If the strict inequality sign would hold, $S_{\min} < S[\varphi_m]$, then there would exist a

$g \in G$ such that $S[g_m] := \max_t S[g(t)] < S[\varphi_m]$. Hence we conclude from (11) that

$$T[g^*] = c(n) \cdot S[g_m]^{2/n} < c(n) \cdot S[\varphi_m]^{2/n} = T[\varphi^*],$$

or that φ^* cannot be the limit point of the minimizing sequence $\{\varphi_k\}$. This contradiction proves $S_{\min} = S[\varphi_m]$.

The usefulness of this theorem will be demonstrated in Section IV.3, where we will derive upper and lower bounds for the action of the bounce solution.

It must be emphasized that we have proven much more than the stated theorem. Indeed, since we showed that the (PS)-condition for a class of potentials is satisfied, we can apply the deep *deformation theorem* due to Rabinowitz and Clark. This theorem states roughly, that one can construct a pseudo gradient vector field such that the corresponding flux lowers the action of fields with actions in the interval $[c-\varepsilon, c+\varepsilon]$, if these fields are not to close to the set of critical points with critical value c.

The existence of such 'deformations' or 'homotopies' in a Banach space is very powerful for proving nice properties for solutions of the Euler–Lagrange equation of certain actions.

IV.3. *Upper and lower bounds for* $S[\textit{bounce}]$

Now we use the theorem of the preceding section to find lower and upper bounds for tunnel probabilities with variational methods. The actions we are dealing with have the form

$$S[\varphi] = \int d^n x\{1/2 \cdot (\nabla\varphi)^2 + m^2/2 \cdot \varphi^2 + P(\varphi)\}.$$

Let $\varphi(r)$ be a spherically symmetric function which depends on some parameters and σ a typical field amplitude. With a typical field amplitude we mean a point of V, to which the field tunnels. Furthermore, we set $\varphi(r) := a \cdot f(r)$, where f is a suitably chosen trial function and $y := a/\sigma$ our first dimensionless parameter. By rescaling, $x^* := m \cdot x$, we find (dropping the *'s)

$$S = \omega_n/2 \cdot (\sigma y)^2 \cdot m^{2-n} \cdot [\gamma + \alpha] \quad \text{with} \tag{4.12}$$

$$\alpha = F_2 + 2(m\sigma y)^{-2} \cdot \int dr\, r^{n-1} P(af) \tag{4.12'}$$

$$\gamma = \int dr\, r^{n-1}(df/dr)^2. \tag{4.12''}$$

Here we used the abbreviation $F_j := \int dr\, r^{n-1} \cdot f^j$. With the abbreviation $S_0 := \omega_n/n \cdot \sigma^2 m^{2-n}$, the action of $\varphi_t(r) := \varphi(r/t)$ is

$$S[\varphi_t] = t^{n-2}T[\varphi] + t^n \cdot V[\varphi] = n/2 \cdot S_0 y^2 \cdot \{t^{n-2} \cdot \gamma + t^n \cdot \alpha\}. \tag{4.13}$$

If $(2-n)\alpha > 0$ then $(n-2) \cdot S[\varphi_t]$ attains its maximum for

$$t_m = [-2\gamma/(nc\alpha)]^{1/2}, \tag{4.14}$$

where $nc := 2n/(n-2)$ is the *critical exponent* ($n/2$ and $nc/2$ are dual exponents).

The maximal (minimal for $n=1$) action is

$$S_m(y,\ldots)=S_0 y^2\gamma^{n/2}\cdot(-2/\alpha nc)^{n/nc} \tag{4.15}$$

and the minimum with respect to y is assumed if $y=:y_0$ obeys

$$(n-2)y\cdot d\alpha/dy=4\alpha. \tag{4.16}$$

Computation of α, y_0 and S_m for typical models

As a first example, we consider

$$V(\varphi)=m^2/2\cdot\varphi^2-C/3\cdot\varphi^3 \tag{4.17}$$

and the tunneling to $\sigma=3m^2/2C$ where V vanishes. Using (12′), (16) and (15) we obtain

$$\alpha=F_2-yF_3 \qquad y_0=4F_2/[(6-n)F_3] \tag{4.18}$$

$$S_m=S_0y_0^2\cdot\gamma^{n/2}\cdot[(6-n)/(nF_2)]^{n/nc} \tag{4.18′}$$

Next we consider

$$V(\varphi)=m^2/2\cdot\varphi^2-C/3\cdot\varphi^3+A/4\cdot\varphi^4. \tag{4.19}$$

V vanishes for $\sigma=2C/3A\cdot[1\pm(1-9Am^2/2C^2)^{1/2}]$. If we define

$$\mu:=A\sigma^2/2m^2, \qquad \lambda:=F_2F_4/F_3,$$

$$h=(6-n)/(4-n)\cdot(1+\mu)/4\mu\cdot[1\pm\{1-(4-n)/(6-n)^2\cdot 32\mu/(1+\mu)^2\cdot\lambda\}^{1/2}]$$

then

$$y_0=F_3/F_4\cdot h \tag{4.20}$$

and

$$-2/(nc\alpha)=(4-n)/[n\{1-(1+\mu)/2\lambda\cdot h\}]. \tag{4.20′}$$

For $\mu=1$ we remain with the *double-well potential*

$$V(\varphi)=A\varphi^2/4\cdot(\varphi-2C/3A)^2. \tag{4.19′}$$

As a final example we consider

$$V(\varphi)=A^*\cdot(\sigma^*\varphi)^2-A^*\cdot\varphi^4+B\varphi^4\cdot\ln(\varphi^2/\sigma^{*2}).$$

With the transformation $A^*=A+2B\ln(\sigma/\sigma^*)=(2A-B)(\sigma/\sigma^*)^2$ we can write

$$V(\varphi)=(2A-B)\cdot\sigma^2\varphi^2-A\cdot\varphi^4+B\cdot\varphi^4\ln(\varphi^2/\sigma^2) \tag{4.21}$$

which is of the form of a one-loop *effective potential*. Despite the fact that the first parametrization is more suitable for our purposes, we use (21) in order to have closer contact to results obtained by other authors.

Using again (12′), (16), and the notation $L:=\int dr\,r^{n-1}f^4\cdot\ln f^2$, $Q:=A/B$, we find

$$\alpha=F_2+(2Q-1)^{-1}\cdot[(L-QF_4)+F_4\ln y^2]y^2 \tag{4.22}$$

and y_0 solves the equation

$$(n-4)F_4y^2\ln y^2+\{(n-4)(L-QF_4)+(n-2)F_4\}y^2-2(2Q-1)F_2=0. \tag{4.22′}$$

This equation can be used to simplify α:

$$\alpha=(n-2)/(n-4)\cdot\{F_2-y^2/(2Q-1)\cdot F_4\}. \tag{4.23}$$

At this point one can apply the *multidimensional Newton-algorithm* for calculating $S_m(y_0,\ldots)$ in (4). However, for $n=4$ we can solve equation (22) analytically:

$$y_0^2=(2Q-1)F_2/F_4 \tag{4.24}$$

$$S_m(n=4)=\pi^2\gamma^2/(8BF_4)\cdot[Q-1-L/F_4-\ln\{(2Q-1)F_2/F_4\}]^{-1}. \tag{4.24'}$$

If we replace in (22), (23), (24), and (24') $(2Q-1)$ by Q^*, where $Q^*=A^*/B$, then we obtain the corresponding expressions or equations for the *-parametrization.

Comparison of the variational upper bounds with exact results

Now we demonstrate the *excellent agreement* of the variational results with the known exact ones. We first calculate the bounds for the actions corresponding to the potentials (17), (19), and (21) for $B=0$. For these three models the actions of the bounce solutions are, at least for some dimensions (mostly $n=1$), known exactly. We mention, without proof, that our inf-max principle is replaced in one dimension by an sup-min principle.

As a *trial function* we choose $f(r)=\exp(-r^b/2)$ and find

$$\begin{aligned} &F_2=u\cdot\Gamma(nu) && F_3=(2/3)^{nu}\cdot F_2\\ &F_4=(1/2)^{nu}\cdot F_2 && \gamma=(n-2)/4\cdot[1+(n-2)u]\cdot\Gamma((n-2)u)\end{aligned} \tag{4.25}$$

where $u:=1/b$.

1) *Cubic potential* (17)

Using (18) and (15) we find $y_0=4/(6-n)\cdot(3/2)^{nu}$ and $(m:=n-2)$

$$\begin{aligned} S_m(\cdot\cdot\cdot)=S_0[4/(6-n)]^2\cdot[m/4\cdot(3/2)^{4u}\cdot(1+mu)\Gamma(mu)]^{n/2}\\ \times[(6-n)/\{nu\Gamma(nu)\}]^{m/2},\end{aligned}$$

The minimal actions, minimal points and y_0's for $n=1,3,4$ are given in the following table:

n	y_0	u	S (min)	
1	1.008	0.5703	$00.5346\sigma^2\cdot m$	
3	3.048	0.6796	$19.5643\sigma^2/m$	(4.26)
4	6.993	0.7718	$91.9398\sigma^2/m^2$	

On the other hand, the 1-dim solution is $\varphi(r)=\sigma\cdot\cosh^{-2}(mr/2)$ and its action is $S(\text{exact})=0.5333m\sigma^2$. This exact solution is very close to the approximate one (see Fig. 8).

2) *Double-well potential* (19') *in one dimension*

If we insert (25) into (20) and (20') with $\mu=1$, we find

$$y_0=5/6\cdot(4/3)^u\cdot\{1\pm[1-24/25\cdot(9/8)^u]^{1/2}\}$$

$$2S_m=[2\pi/3]^{1/2}\cdot m/8\cdot(\sigma y_0)^2\cdot\{[2-(2/3)^u y_0][1-u]/\sin\pi u\}^{1/2}$$

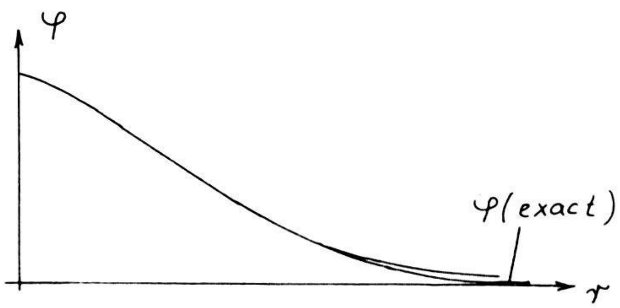

Figure 8
Plot of the exact bounce solution and the function which was obtained by the variational principle for the cubic potential (17).

For both signs $u \geq 0.34$. The action of the double-well solution is only half of the expression in (15), since this solution is only 'half a bounce'. We find

$$S_m = 0.166713 \cdot m\sigma^2 \qquad S(\text{exact}) = 0.166666 \cdot m\sigma^2. \tag{4.27}$$

The expressions for the general 4th order potential (19) are clumsy and are not given here.

3) *Model* (21) *in 3 dimensions*

For the potential (21) with $B = 0$ the exact minimal action for $n = 3$ is known [BP]. Let us compare this with the variational bound. From $B \cdot (22')$ we obtain $y^2 = 4 \cdot F_2/F_4$ and from (22) $\alpha = -F_2$. Using (15) and (25) yields

$$S_m = 2\pi\sigma^2/3m \cdot [3u\Gamma(3u)]^{-1/2} \cdot [4^u(1+u)\Gamma(u)]^{3/2}.$$

The minimum is attained for $u = 0.9136$ and the resulting S is

$$S_m = 19.27 \cdot \sigma^2/m \tag{4.28}$$

which has to be compared with $S_{\text{exact}} = 18.90 \cdot \sigma^2/m$.

Although we used a very simple one parameter (beside t, y) trial function, we see from the numbers in (26), (27) and (28), that variational results are very close to the exact ones. We thus developed a very *simple and accurate approximation scheme* for computing the minimal action, i.e. the action of the bounce solution. The usefulness and accuracy of variational methods is well known from other branches of physics (e.g. for the computation of ground state energies).

Comparison of lower bounds with exact results

Using Hoelder and Sobolev inequalities we derive in this section lower bounds for S(bounce) and compare these with the exact results in order to establish their quality.

First we list some *inequalities* which will be used throughout.

a) *Sobolev inequality:*

$$\|f\|_{nc} \leq c_n \cdot \|\nabla f\|_2 \qquad nc := 2n/(n-2), \quad \text{for} \quad n \geq 3;$$

the best value of c_n is [Fa]: $c_n^{-2} = \pi n(n-2)[\Gamma(n/2)/\Gamma(n)]^{2/n}$.

(4.29)

b) *Jensen inequality:*

$$\int dP(x) \exp[f(x)] \geq \exp\left[\int dP(x) f(x)\right].$$

Setting $\exp[f(x)] := h(x)/g(x)$ and $dP(x) := g(x)\,dx/\int g(x)\,dx$ we find $\int g(x)\,dx \cdot \ln[\int h(x)\,dx/\int g(x)\,dx] \geq \int g(x) \ln[h(x)/g(x)]$. With the definitions $g(x) := f(x)^4$ and $h(x) := f(x)^2$ we end up with

$$L := -\int dx f(x)^4 \ln f(x)^2 \leq \|f\|_4^4 \cdot \ln[\|f\|_2^2/\|f\|_4^4]. \tag{4.30}$$

We will use (30) for deriving lower bounds for S(bounce) for the potential (21).

c) *Hoelder inequality*

$$\int dx g(x) h(x) \leq \|g\|_p \cdot \|h\|_q \quad \text{if} \quad 1/p + 1/q = 1, \quad \text{or}$$
$$\|f\|_{a+b}^{a+b} \leq \|f\|_{ap}^a \cdot \|f\|_{bq}^b. \tag{4.31}$$

Setting $a = n$, $b = 4 - n$ and $p = nc/n$ for $n \leq 4$, respectively $a = 4nc/n$, $b = (n-4)nc/n$ and $p = n/nc$ for $n > 4$, we obtain

$$(n-4) \cdot \|f\|_{nc}^n \leq (n-4) \cdot \|f\|_4^4 \cdot \|f\|_2^{n-4} \quad \text{if} \quad n > 2 \tag{4.31'}$$

For $n = 1$ we use the following inequality [Fa]:

$$\|df/dx\|_2 \geq \|f\|_4^4/\|f\|_2^3. \tag{4.32}$$

With $c_1 := 1$, (29), (31′) and (32) we conclude that

$$\gamma \geq [F_4/(\omega_n F_2^2)]^{2/n} \cdot F_2/c_n^2 \quad \text{for} \quad n = 1, 2 \text{ and } 4. \tag{4.33}$$

Combining (15) and (33), we obtain the following inequality, which is valid for all interesting dimensions (i.e. 1, 3, and 4),

$$S \geq \sigma^2 m^{2-n} \beta \cdot y^2 F_4/F_2 \cdot [(2-n)F_2/n\alpha]^{n/nc} \tag{4.34}$$

where $\beta(n=1) = 1$, $\beta(n=3) = 3^{1/2}\pi^2/4$ and $\beta(n=4) = 8\pi^2/3$.

For the derivation of lower bounds for our actions it is always possible, to change α in such a way, that it only depends on $\eta := y^2 \cdot F_4/F_2$. This can be done with the help of (30) or/and (31′). The inf-max principle then tells us that a minimum of $S(\eta)$ yields a lower bound for S. Let us apply this for our previous examples.

1) For the *cubic potential* (17) we find from (34, 34′) with (18)

$$S \geq \sigma^2 m^{2-n} \beta\eta \cdot [(2-n)/\{n(1-\eta^{1/2})\}]^{n/nc}.$$

The minimum is at $\eta = [4/(6-n)]^2$ and the corresponding bound is

$$S \geq \sigma^2 m^{2-n} \beta \cdot [4/(6-n)]^2 \cdot [(6-n)/n]^{n/nc} \tag{4.35}$$

For a comparison with the bounds (26) we list numerical results

$$S(n=1) \geq 0.2862, \qquad S(n=3) \geq 7.5976, \qquad S(n=4) \geq 52.6379. \tag{4.36}$$

2) For the *double-well potential in one dimension* (19′) we find from (20′) for $\mu = 1$, $n = 1$, using (34), (31′, $n = 6$)

$$S \geq m\sigma^2 \cdot [\eta - \eta^{3/2}]. \tag{4.37}$$

Again putting in numbers gives

$$S \geq 1.4815 \cdot m\sigma^2 \tag{4.38}$$

which is 0.8889 times the exact result, given in (27).

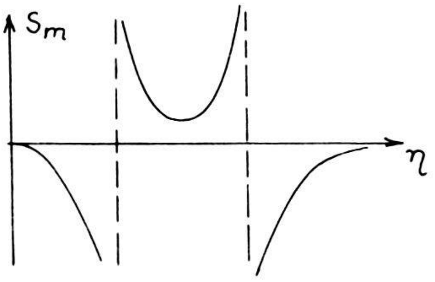

Figure 9
The dependence of the action S_m with effective potential (21) on the variational parameter η.

3) Lower bound for the action with *effective potential* (21).

Now we come to the more interesting potential (21), which plays an important role in applications (e.g. in cosmology). From (22), (30), and (34) we conclude

$$S \geq \sigma^2 m^{2-n} \beta\eta \cdot [(n-2)(2Q-1)/\{n(1-2Q+Q\eta-\eta \ln \eta)\}]^{n/nc}. \tag{4.39}$$

Because $B=(75\alpha/8)^2$ and $m^2=2B(2Q-1)$ we find in *4 dimensions*

$$S(n=4) \geq 2/3 \cdot (8\pi/75\alpha)^2 \cdot [(1-2Q)/\eta + Q - \ln \eta]^{-1}.$$

$S(\eta)$ has 2 poles, as indicated in Fig. 9. However, we know from (2), that $S>0$. So we obtain for the minimizing η: $\eta=(2Q-1)$, and thus for the corresponding lower bound

$$S(n=4) \geq 2/3 \cdot (8\pi/75\alpha)^2 \cdot [Q-1-\ln(2Q-1)]^{-1}. \tag{4.40}$$

With analogous manipulations we obtain in *3 dimensions*

$$S(n=3) \geq \mathrm{sqr}\,(2)\pi^2/75 \cdot \sigma\eta/\alpha \cdot [(1-2Q)\eta^2 + Q\eta - \eta \ln \eta]^{-1/2}. \tag{4.41}$$

In contrast to the 4-dimensional case we cannot give an analytic expression for the minimum of $S(\eta)$ and we are forced to use a numerical algorithm (Newton algorithm) for finding $\eta_{\min}(Q)$. The results for $n=3$ and $n=4$, together with the variational values are shown later in this section.

We see that the lower bounds are typically a factor 2 lower than the exact results. For the functions (25) the Jensen inequality deviates from an equality by a factor $\sim \ln 4$, independent of u. The Sobolev inequality (29) is sharp for small values of b (factor 1.03 for $b=0.5$, $n=4$) and gets worse for increasing values of b (factor 1.83 for $b=5$, $n=4$). At any rate, if Q is not too close to 1, then the bounds (40) and (41), multiplied by 2, can be used as a *first guess* for the exact actions.

Variational bounds for the action with effective potential (21)

Now we are prepared for computing the interesting bounds for the effective potential (21). From the virial theorem (8) we know, that for $n>2$ $V(\varphi)<0$. If $Q\sim 1$ this is only possible for functions φ, which have an extended core in which φ sits in the minimum of V. In such situations our trial functions (25) are not suitable and we choose different ones. We sketch the computations first for the functions (25) in 3 and 4 dimensions and then consider other trial functions.

In 4 dimensions our method is very simple. For the functions (25) we need

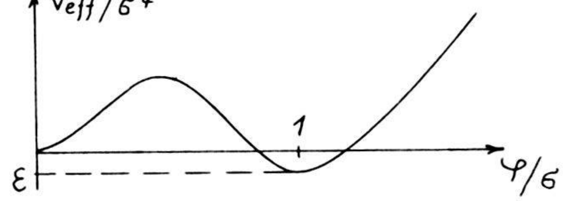

Figure 10
Depending on ε, we have to choose different trial functions.

only a 1-dimensional Newton-algorithm. Using (25) in (24′) yields

$$S_m(n=4) = (8\pi/75\alpha)^2/8 \cdot [2^{2u}(1+2u)\Gamma(2u)]^2 \cdot [4u\Gamma(4u)]^{-1} \times [Q-1+2u(1-2\ln 2) - \ln(2Q-1)]^{-1}, \tag{4.42}$$

where we used $L = -n/2 \cdot u^2 2^{-nu} \cdot \Gamma(nu)$. Now it is simple to find, for a given $0.5 \le Q < 1$, the minimal point u and minimal value S_m.

For $Q \sim 1$ the following function is more suitable as a trial function

$$\begin{aligned} f(r) &= 1 && \text{if } r \le c \\ &= \exp[(r-c)^2/2] && \text{if } r \ge c \end{aligned} \tag{4.43}$$

Using this trial function we obtain the following values for γ, L, and the F's:

$$\begin{aligned} F_2 &= c^4/4 + 3c^2/2 + 1/2 + \pi^{1/2} \cdot (c^3/2 + 3c/4) =: p(c) \\ \gamma &= \pi^{1/2} \cdot (c^3/4 + 9c/8) + 3c^2/2 + 1 =: q(c) \\ F_4 &= p(2^{1/2} \cdot c)/4 \\ L &= -q(2^{1/2} \cdot c)/8. \end{aligned} \tag{4.44}$$

Inserting (44) into (24′) results in ($d := 2^{1/2} \cdot c$)

$$S_m(n=4) = (8\pi/75\alpha)^2/2 \cdot q(c)^2/p(d) \times [Q - 1 + q(d)/2p(d) - \ln\{(8Q-4)p(c)/p(d)\}]^{-1} \tag{4.45}$$

The *core radius* is $r_B(\text{physical}) = (c+1)/mt_m$, or with (14),

$$r_B\sigma = (c+1)/\pi\sigma \cdot [p(d)S_m/\{2(2Q-1)p(c)q(c)\}]^{1/2}. \tag{4.46}$$

In this manner we have calculated upper bounds and good estimates for the exact bounce-actions for both regions, $Q \sim 0.5$ and $Q \sim 1$. One can, of course, combine the functions (25) and (43) for constructing a 4-dimensional collection of trial functions of the form $\varphi = \sigma y \cdot f(b, c, r/t)$, where

$$f(b, c, r) = 1 \quad \text{for } r \le 1, \quad \text{otherwise} \quad = \exp[-(r-c)^b/2] \tag{4.47}$$

or trial functions, which depend on even more parameter. We have done the calculations for the functions (25), (43), (47) and also computed the lower bounds (40). The results are given in Table 1.

The *3 dimensional case* is somewhat harder. As we already mentioned, we cannot find the minimum with respect to y for $n = 3$. So we have to use the 2-dimensional Newton-algorithm for finding the minimum within the family of functions (25). For the same reason as for $n = 4$ we have to use different trial functions for $Q \sim 1$. We choose again the function (43).

Table 1

Q	0.51	0.53	0.55	0.57	0.59	0.61	0.63
$S(25)$	64.22	105.33	145.77	191.02	243.91	307.32	384.72
$S(43)$	74.95	116.26	155.18	197.89	247.73	308.39	383.73
$S(47)$	64.22	104.60	144.44	188.90	240.75	302.72	378.13
$S(40)$	44.30	64.69	81.83	98.69	116.18	134.86	155.15
q	0.65	0.67	0.69	0.71	0.73	0.75	0.77
$S(25)$	480.71	601.55	756.06	956.97	1223.06	1582.91	2081.40
$S(43)$	476.59	592.88	740.98	932.85	1186.11	1527.48	1998.79
$S(47)$	471.31	588.26	737.18	929.97	1184.15	1526.19	1997.51
$S(40)$	177.52	202.45	230.54	262.51	299.29	342.09	392.55
$r_{b\sigma}$	10.27	11.22	12.22	13.30	14.51	15.88	17.47
Q	0.79	0.81	0.83	0.85	0.87	0.89	0.91
$S(25)$	2791.60	3837.84	5442.91	8032.25	12485.60	20819.50	38341.70
$S(43)$	2667.93	3650.11	5151.10	7562.73	11692.50	19384.60	35479.70
$S(47)$	2665.20	3642.95	5133.03	7520.53	11596.80	19168.33	34948.39
$S(40)$	452.90	526.31	617.46	733.50	885.99	1094.87	1397.84
$r_{B\sigma}$	19.33	21.57	24.32	27.79	32.30	37.67	46.45
Q	0.93	0.95	0.97				
$S(25)$	82087.5	226625.0					
$S(43)$	75447.9	206749.0	954116.0				
$S(47)$	74014.7	201846.7	926542.1				
$S(40)$	1875.7	2738.4	4755.9				
$r_{B\sigma}$	60.3	85.1	141.1				

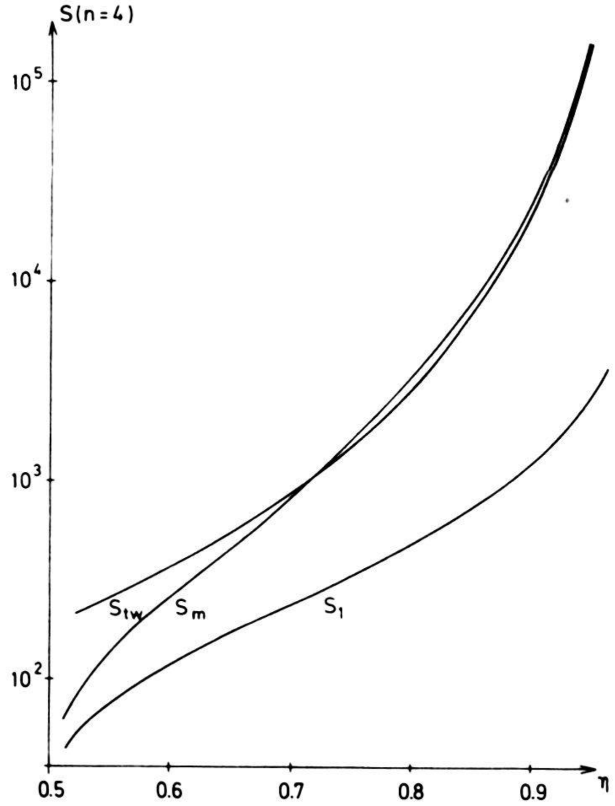

Figure 11
Plots of the lower bound S_1, upper bound S_m and thin-wall result S_{tw} for the action of the bounce solution in 4 dimensions. S_m is obtained with the variational principle in which the trial functions (47) have been used. S_1 has the same origin, but the inequalities (40) and (33) were used. S_{tw} is computed in section IV.4.

Table 2

Q	0.51	0.53	0.55	0.57	0.59	0.61
$S(25)$	0.938	2.234	3.521	4.896	6.405	8.092
$S(43)$	1.035	2.415	3.755	5.157	6.670	8.336
$S(41)$	0.428	0.976	1.483	1.991	2.515	3.062
Q	0.63	0.65	0.67	0.69	0.71	0.73
$S(25)$	10.00	12.20	14.76	17.77	21.38	25.78
$S(43)$	10.20	12.33	14.81	17.77	21.31	25.60
$S(41)$	3.64	4.25	4.91	5.61	6.38	7.21
$r_{B\sigma}$				0.04	1.03	1.13
Q	0.75	0.77	0.79	0.81	0.83	0.85
$S(25)$	31.22	38.01	46.99	58.11	75.03	98.17
$S(43)$	30.91	37.60	46.24	57.69	73.34	95.59
$S(41)$	8.12	9.13	10.26	11.54	13.00	14.71
$r_{B\sigma}$	1.24	1.36	1.51	1.69	1.91	2.19
Q	0.87	0.89	0.91	0.93		
$S(25)$	132.84	188.22	284.81			
$S(43)$	128.78	181.54	273.04	453.33		
$S(41)$	16.75	19.25	22.45	26.79		
$r_{B\sigma}$	2.55	3.05	3.77	4.26		

For the family (25) we obtain with (15), (22) and (25, $n=3$)

$$S_m(n=3)=\sigma/(6\cdot 2^{1/2})\cdot 8\pi/75\sigma\cdot y^2\cdot[(1+u)\Gamma(u)]^{3/2}$$
$$\times[\{1-2Q+y^2 8^{-u}(Q+3u/2-\ln y^2)\}3u\Gamma(3u)]^{-1/2}. \tag{4.48}$$

The results of the numerical calculations are listed in Table 2.

For the functions (43) we find

$$\begin{aligned} F_2 &= c^3/3+c+\pi^{1/2}(c^2/2+1/4)=:p(c)\\ \gamma &= \pi^{1/2}(c^2/4+3/8)+c:=q(c)\\ F_4 &= 1/8^{1/2}\cdot p(d),\\ L &= -1/32^{1/2}\cdot q(d), \end{aligned} \tag{4.49}$$

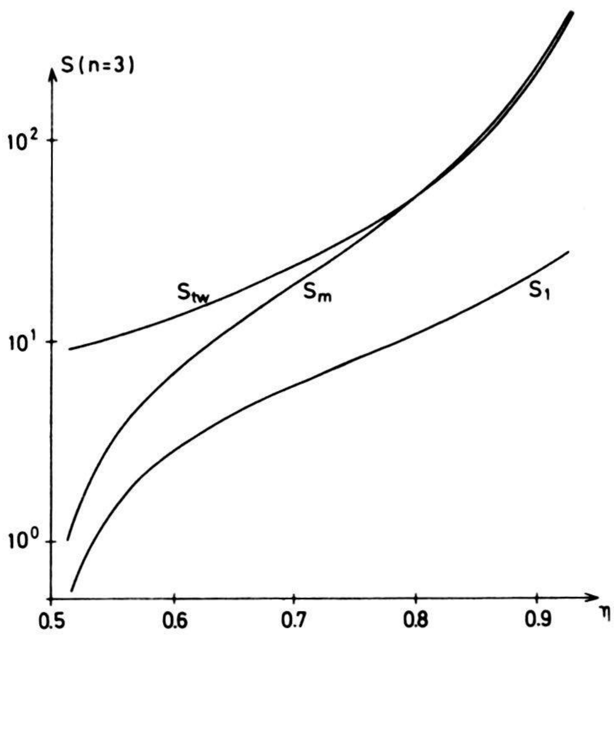

Figure 12
Plots of the lower bound S_1, upper bound S_m and thin-wall result S_{tw} for the action of the bounce solution in 3 dimensions. S_m and S_1 were computed with the help of the inf–max principle (page 55). S_{tw} is calculated in section IV.4.

where again $d := 2^{1/2}c$. For the action we find

$$S_m(n=3) = 4/54^{1/2} \cdot 8\pi\sigma/75\alpha \cdot q(c)^{3/2} \cdot y^2 \times [(1-2Q)p(c) + y^2/8^{1/2}\{Qp(d) + q(d)/2 + p(d)\ln y^2\}]^{-1/2}$$

and the *bubble radius* is $r_B\sigma = 3(1+c)S_m/\sigma \cdot [4\pi q(c)y^2]^{-1}$.

For $n=3$ we did not calculate the minimal action for the functions (47) because $S_m(47)$ is close to $S_m(25)$ for $Q \sim 0.5$ and close to $S_m(43)$ for $Q \sim 1$ as we saw in four dimensions. In addition we list the lower bounds obtained in (41).

IV.4. *Variational results versus thin-wall approximation*

The thin-wall (TW) approximation was introduced by Coleman [CC] and improved by Affleck [Af] who invented a series expansion in the difference ε between the potential energy densities of the true and false vacua. However, even for the simplest case, the double well potential the computation of the second term in the expansion is very long and tedious. For this reason it is interesting to compare variational with TW results.

Let $V = V_0 - \varepsilon \cdot V_1$ and require

$$\begin{aligned} &\text{(i)} \quad V_0(0) = V_0(\sigma) = 0, \qquad dV_0(0)/dx = dV_0(\sigma)/dx = 0 \\ &\text{(ii)} \quad V_1(\sigma) = 1, \qquad V_0(\varphi) \to V(\varphi) \quad \text{for} \quad \varepsilon \to 0. \end{aligned} \tag{4.50}$$

As in the variational calculations we split V_0 into two terms,

$$V_0(\varphi) = (m_0\sigma)^2/2 \cdot [(\varphi/\sigma)^2 + 2/(m_0\sigma)^2 \cdot P_0(\varphi)].$$

Then the action of the bounce in the thin-wall approximation is given by [CC]

$$S_{tw} = \omega_n/n \cdot [(n-1)/\varepsilon]^{n-1} \cdot S_1^n, \tag{4.51}$$

where S_1 is the one-dimensional action

$$S_1 = \int_0^1 dx \cdot x[1 + 2/(m_0\sigma x)^2 \cdot P_0(\sigma x)]^{1/2} = m_0\sigma^2 \cdot I. \tag{4.52}$$

Furthermore, we define z via $\varepsilon =: m_0^2\sigma^2 \cdot z$ and end up with

$$S_{tw} = S_0^*(n-1)^{n-1} \cdot I^n/z^{n-1}. \tag{4.53}$$

We used the star index because $S_0^* = \sigma^2 m_0^{2-n}$ depends on m_0, the mass appearing in V_0, rather than on the mass m in V.

For the *potential (21)* $\varepsilon = (1-Q)B\sigma^2$ or $z = (1-Q)/2$ and for the integral we find $I = 0.20995$, so that

$$\begin{aligned} S_{tw}(n=4) &= 54(24\pi/5)^2 \cdot I^4/(1-Q)^3 \\ S_{tw}(n=3) &= 512\pi/(50)^{1/2} \cdot I^3/(1-Q)^2. \end{aligned} \tag{4.54}$$

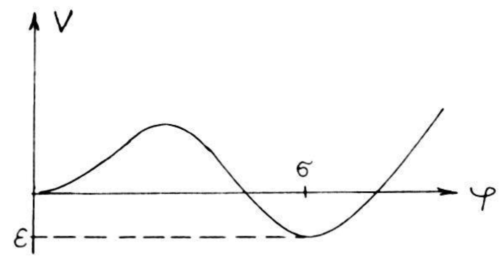

Figure 13
The thin wall parameter ε is the difference between the potential energy densities in the true resp. false vacuum.

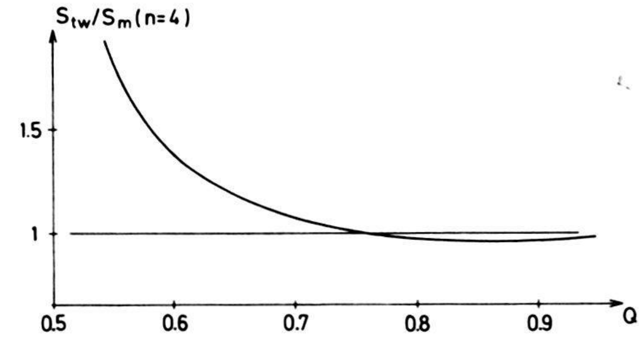

Figure 14
The ratio of the thin-wall results to the results obtained with the variational methods in 4 dimensions.

The graphs in Figs. 14, 15 show the ratio of the thin-wall results to the results obtained with the variational method.

After our lengthy discussion of the variational method for computing S[bounce] and it's application to various models we now turn to the *temperature dependent problem*.

IV.5. *Temperature-dependent effective potential*

If one computes the effective potential in the $SU(5)$ gauge theory with a quartic Higgs potential at *finite temperatures*, then, to first order in a loop expansion, the effective potential is [DJ]

$$V(T, \varphi) = V(\varphi) + V_T(\varphi) \tag{4.55}$$

where the V_T is the finite temperature contribution

$$V_T(af) = 18T^4/\pi^2 \cdot I\{(25/8)^{1/2} \cdot gy/\eta \cdot f\}.$$

We use the notation: $\eta := T/\sigma$, $y := a/\sigma$; $V(\varphi)$ is the effective potential (21) at $T=0$, and I the integral

$$\begin{aligned} I(z) &= \int_0^\infty dx x^2 \ln[1-\exp\{-(x^2+z^2)^{1/2}\}] \\ &= -z^2 \cdot \sum_n K_2(nz)/n^2. \end{aligned} \tag{4.56}$$

In [DJ] the *high temperature expansion* ($z \ll 1$) has been calculated:

$$I_4(z) = -\pi^4/45 + \pi^2 z^2/12 - \pi z^3/6 - z^4/32 \cdot \ln z^2 + 5.41/32 \cdot z^4. \tag{4.57}$$

For the computation of tunnel probabilities one usually needs $I(z)$ not only for $z \ll 1$, but also for $z \sim 1$. We now use the saddle point method for constructing an approximate representation of I, which is good for *all* values of z.

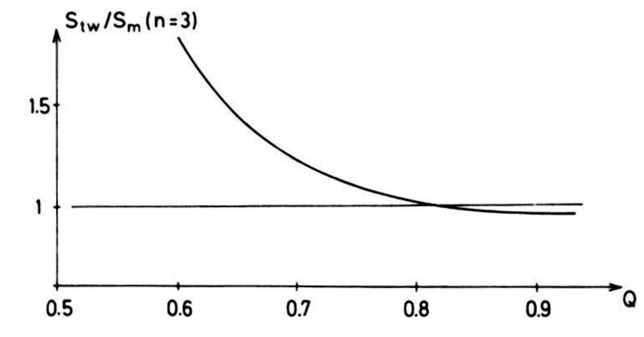

Figure 15
The ratio of the thin-wall results to the results obtained with the variational methods in 3 dimensions.

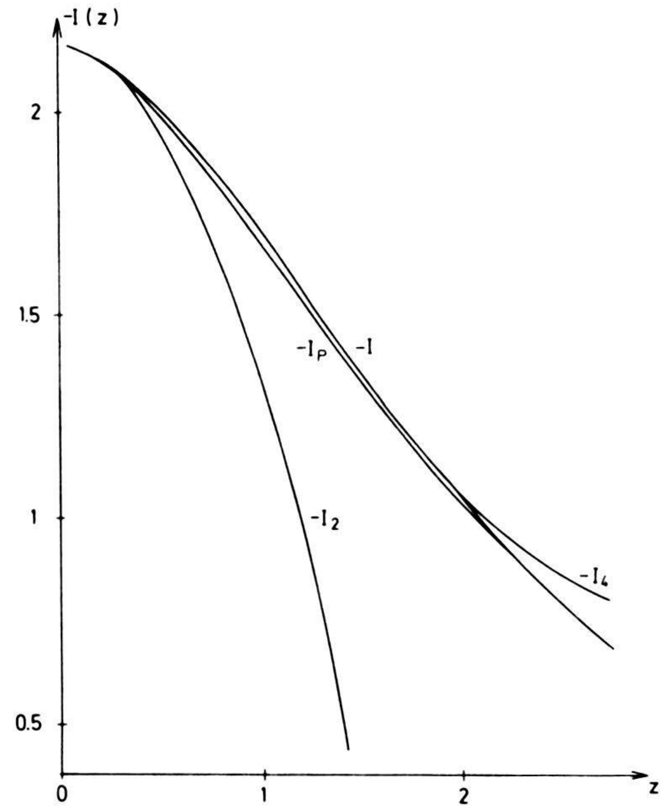

Figure 16
Different approximations for the integral $I(z)$. I_4 is the high temperature approximation (57) and I_2 the sum of the first two terms in (57). I_p is the result (58) of the saddlepoint method.

For this purpose we expand the logarithm in (56)

$$I(z) = 1/n \cdot \sum_n \int dx \exp[-f_n(x, z)], \qquad f_n(x, z) = n(x^2 + z^2)^{1/2} - 2 \ln x$$

and apply the *saddle point method* to each term. This gives

$$I(z) \sim 1/n \cdot \sum_n [2\pi\, d^2 f_n(x_m)/dx^2]^{-1/2} \cdot \exp[-f_n(x_m)].$$

Inserting the value of the saddle point x_m, we find

$$I(z) \sim -(4\pi)^{1/2}/e \cdot \sum (1+u)^2/n^4 \cdot u^{-1/2} \cdot \exp(-u),$$

where

$$u := [1 + (nz)^2]^{1/2}.$$

Instead of using this clumsy expression, which is not very accurate near the core, we retain only the first term but change the "constants 1" in such a way that $I(0) = -\pi^4/45$, $I''(0) = \pi^2/6$:

$$-I(z) \sim -I_p(z) = (4\pi)^{1/2}/e \cdot (a+u)^2/u^{1/2} \cdot \exp(-u) \qquad (4.58)$$

where $u = (b^2 + z^2)^{1/2}$, $a = (e/\pi)^{1/4}$, and $b = 0.682293823$.

Usually people only use the first 2 terms $I_2 = -\pi^4/45 + (\pi z)^2/12$ of (57) in computing S[bounce] for the effective potential (55). This treatment is, of course, only justified if $z \ll 1$, i.e., for $a^2 g/T \ll 1$.

V. The primed determinant

In this chapter we connect the generalized *Weinstein–Arenszjan determinants* $w(\lambda)$, which appear as coefficients in the expression for the tunnel probability, to some scattering data of the corresponding Schroedinger operators. In one dimension these determinants are well defined and we compute $w(\lambda)$ for a general fourth order potential.

Then we generalize the discussion to higher dimensions for spherically symmetric potentials, using the results in one dimension. Here we are clearly confronted with the problems of regularization and renormalization.

A general remark on determinants

We study determinants of the form

$$w(\lambda) := \det[(H-\lambda)(H_0-\lambda)^{-1}] \tag{5.1}$$

where $H_0 = -\Delta$ and $H = -\Delta + V''(\varphi) - m^2 = H_0 + q$ are Schroedinger operators. It's known [Ka, Ku] that, if H and H_0 are closed operators and q is of *relative trace class to H_0, then $w(\lambda)$ is a meromorphic function* in regions of the complex plane where H and H_0 have only isolated eigenvalues. The zeros and poles of $w(\lambda)$ are at points which are eigenvalues of H and H_0. The order of these zeros (poles) are the dimensions of the corresponding eigenspaces.

For dimensions $n \geq 2$ q is, however, not of relative trace class with respect to H_0, which forces us to regularize the determinants [Si].

V.1. *One-dimensional determinants*

Here we investigate $w(\lambda)$ for $H_0 = -d^2$ and $H = H_0 + q$ where q is the "massless part" of $V''(\varphi)$, i.e., $V''(\varphi) = m^2 + q$.

First we restrict ourselves to a *finite interval* $I = [-L, L]$ on which we impose *Dirichlet boundary conditions.*

It is well known that the Dirichlet spectrum of H with $q \in L_2(I)$, is real, discrete, bounded from below, and that all eigenvalues are simple.

We introduce the *fundamental solutions* φ_1 and φ_2 which fulfil

$$H\varphi = \lambda\varphi \tag{5.2}$$

with the boundary conditions

$$\varphi_1(-L, \lambda, q) = 1 \qquad \varphi_1'(-L, \lambda, q) = 0 \tag{5.2'}$$

$$\varphi_2(-L, \lambda, q) = 0 \qquad \varphi_2'(-L, \lambda, q) = 1 \tag{5.2''}$$

In terms of φ_1 and φ_2 the general solution of $-\varphi'' + q\varphi + f = \lambda\varphi$ has the representation

$$\varphi(x) = \varphi(-L)\varphi_1(x) + \varphi'(-L)\varphi_2(x) + \int_{-L}^{x} dy\{\varphi_2(x)\varphi_1(y) - \varphi_1(x)\varphi_2(y)\} \cdot f(y)$$

If we set $\varphi = \varphi_2$ and $f(y) = \varepsilon \cdot v(y)\varphi_2(y)$ we find

$$\varphi_2(x, q+\varepsilon v) = \varphi_2(x, q) + \int_{-L}^{x} dy\{\varphi_2(x)\varphi_1(y) - \varphi_1(x)\varphi_2(y)\}\varepsilon v(y)\varphi_2(y, q+\varepsilon v)$$

so that the *Frechet derivative* of $\varphi_2(x)$ with respect to $q(y)$ is

$$\varphi_{2,q} = [\varphi_2(x)\varphi_1(y)\varphi_2(y) - \varphi_1(x)\varphi_2^2(y)] \cdot \theta(x-y). \tag{5.3}$$

On the other hand, $\varphi_2(x, \lambda+\varepsilon, q) = \varphi_2(x, \lambda, q-\varepsilon)$, or

$$d\varphi_2/d\lambda = -\int dy\varphi_{2,q}. \tag{5.4}$$

With these preparations it is possible to compute the determinant $w(\lambda)$. From (1) we have

$$d(\log w)/d\lambda = \mathrm{tr}\,[R(H_0, \lambda) - R(H, \lambda)], \tag{5.5}$$

where $R(H, \lambda)$ stands for the *resolvent operator* $(H-\lambda)^{-1}$. But R is an integral operator with the *Greens function* $G_H(\lambda; x, y)$ as its kernel. Using the fundamental solutions we obtain for G_H, for Dirichlet boundary conditions,

$$\begin{aligned} G_H(\lambda; x, y) = {} & \varphi_2(x) \cdot [\varphi_1(y) - \varphi_2(y)\varphi_1(L)/\varphi_2(L)] \\ & -[\varphi_2(x)\varphi_1(y) - \varphi_1(x)\varphi_2(y)] \cdot \theta(x-y). \end{aligned} \tag{5.6}$$

For such integral operators the trace is nothing else than the integral over the diagonal elements of the Greens function [Si], and thus

$$\mathrm{tr}\,[R(H, \lambda)] = \int dx\varphi_2\varphi_1 - \varphi_1(L)/\varphi_2(L) \cdot \int dx\varphi_2^2. \tag{5.7}$$

With (3) and (4) we find

$$\mathrm{tr}[R(H, \lambda)] = -d\log\{\varphi_2(L)\}/d\lambda \tag{5.8}$$

or

$$d[\log\{w(\lambda) \cdot \varphi_2(L, \lambda, 0)/\varphi_2(L, \lambda, q)\}]/d\lambda = 0. \tag{5.8'}$$

The equations (3) and (4) tell us in addition, that $d\varphi_2(L)/d\lambda \neq 0$ for all λ, or that all poles in (8) are simple. Since the argument of the logarithm in (8′) is analytic in λ and goes to 1 as λ goes to ∞, we end up with

$$w(\lambda) = \varphi_2(L, \lambda, q)/\varphi_2(L, \lambda, 0). \tag{5.9}$$

This shows that one should interpret $\varphi_2(L, \lambda, q)$ as the generalization of the *characteristic polynom* of a linear operator on a finite dimensional vector space.

Using formula (9) we can compute the *product expansions* for certain functions. For example, if $L = 1/2$, $q(x) = \omega^2$, and $m = 0$, we obtain

$$\det[(-d^2+\omega^2)(-d^2)^{-1}] = 1/\omega \cdot \sinh(\omega).$$

On the other hand, the eigenvalues of H_0, resp. H are $\lambda_n = (n\pi)^2$, resp. $\lambda_n = (n\pi)^2 + \omega^2$, so that

$$1/\omega \cdot \sinh(\omega) = \Pi_n[1 + (\omega/n\pi)^2]. \tag{5.10}$$

Determinants on R

One can directly use the results obtained so far, in one introduces $\varphi_i^* := \exp(-mL) \cdot \varphi_i$ and let L go to infinity.

However, in order to see the connection between our determinants and scattering data more directly, we make use of the so-called *m-functions*. If $\lambda =: k^2$, they are defined by

$$\varphi_r(x, k, q) =: m_r(x, k, q) \cdot \exp(-ikx) \tag{5.11}$$

$$\varphi_l(x, k, q) =: m_l(x, k, q) \cdot \exp(+ikx), \tag{5.11'}$$

i.e. φ_r (φ_l) describes a particle moving to the right (left). The m-functions fulfil the conditions

$$m_r(x, k, q) \to 1 \qquad m_r'(x, k, q) \to 0 \quad \text{for} \quad x \to -\infty \tag{5.12}$$

$$m_l(x, k, q) \to 1 \qquad m_l'(x, k, q) \to 0 \quad \text{for} \quad x \to \infty. \tag{5.12'}$$

Carrying out the analogous calculations as on a finite interval, we find

$$w(\lambda) = \lim_{x \to \infty} m_r(x, k, q)/m_r(x, k, 0) = \lim_{x \to \infty} m_r(x, k, q).$$

Now we need the notion of the *reflection and transmission coefficients*. They are defined via the r and l-functions in the decompositions

$$\varphi_l = l_-\varphi_r + l_+\varphi_r^* \qquad \varphi_r = r_+\varphi_l + r_-\varphi_l^*. \tag{5.13}$$

$[r_-(k) = l_+(k), l_-(-k) = -r_+(k)]$. The unitary *scattering matrix* is then given by $(T := 1/r_-,\ R := r_+/r_-)$

$$S = \begin{pmatrix} T & R \\ -TR^*/T^* & T \end{pmatrix} \tag{5.14}$$

Now let us compute $\det[(-d^2 + V''(\varphi))(-d^2 + m^2)^{-1}] = w(\lambda = -m^2)$. With (13) and (11′) we obtain

$$m_r = r_+ m_l \exp(2ikx) + r_- m_l^*, \quad \text{or} \quad \lim_{x \to \infty} m_r(x, k = im, q) = r_-(k = im, q).$$

Hence we conclude

$$w(\lambda = -m^2) = r_-(k = im, q) \tag{5.15}$$

or

$$\det[(H_0 + m^2)(H + m^2)^{-1}] = T(k = im, q), \tag{5.15'}$$

i.e., we can express the functional determinant through the transmission coefficient of the corresponding Schroedinger operator.

Treatment of the zero mode $(\lambda^* := \lambda + m^2)$

Now let us assume that $\lambda^* = 0$ is an eigenvalue of $-d^2 + V''(\varphi)$, or that $\lambda = -m^2$ is an eigenvalue of H. We divide w by λ^* and use l'Hospitals rule

$$w'(-m^2) = \lim_{\lambda \to -m^2} [r_-(\lambda)]/[m^2 + \lambda] = i/2m \cdot d[r_-(im, q)]/d\lambda. \tag{5.16}$$

Thus, we end up with

$$\det{}'[(-d^2 + V''(\varphi))(-d^2 + m^2)^{-1}] = i/2m \cdot d[r_-(im, q)]/d\lambda. \tag{5.17}$$

Analogous to the formulae (3) and (4), we have [Lam]

$$d[r_-(im, q)]/d\lambda = i\int dx\varphi_2^2,$$

where φ_2 is the eigenfunction of H with eigenvalue $-m^2$ and asymptotic condition $\varphi_2(x) \to \exp(-mx)$ for $x \to \infty$.

Finally, using (17) the primed determinant becomes

$$w'(-m^2) = 1/2m \cdot \|\varphi_2\|_2^2. \tag{5.18}$$

Computation of $\|\varphi_2\|_2^2$

The zero mode is proportional to the derivative of the bounce solution $Bm^{1/2}\varphi_2 = d\varphi/dx$, or $w'(\lambda) = (mB)^{-2}T[\varphi] = 2(2mB)^{-2}S[\varphi]$. Hence we are left with computing the constant B.

From the equation of motion for φ we find

$$mx = \int_{\varphi/m^{1/2}}^{\sigma/m^{1/2}} d\psi[\psi^{-1} \cdot \{1 + 2/m\psi^2 \cdot P(\psi/sqr(m))\}^{-1/2}] =: I(\varphi) \tag{5.19}$$

Since $m^{1/2} \cdot \varphi \sim B \exp(-mx)$, this shows that

$$B = \lim_{\varphi \to 0} m^{1/2}\varphi \cdot \exp[I(\varphi)]. \tag{5.20}$$

Let us *summarize* the results obtained so far.

The primed Weinstein–Arenszjan determinant for $\lambda = -m^2$ is

$$\underline{\det'[(-d^2 + V''(\varphi))(-d^2 + m^2)^{-1}] = 1/(2m^2B^2) \cdot S[\varphi],} \tag{5.21}$$

where B is given in (20), and $S[\varphi]$ is the action of the bounce solution. For computing the tunnel probability one anyway has to calculate $S[\varphi]$. The only additional quantity that must be computed is the integral $I(\varphi)$ in (19).

Application to typical models

1) *Cubic potential*

The potential $V(\varphi) = m^2/2 \cdot \varphi^2 - C/3 \cdot \varphi^3$ was already discussed in the preceding chapter. Using the result for $S[\varphi]$ on page 571 and $B = 4\sigma m^{1/2}$, we obtain

$$w'(-m^2) = \det'[1 - 3m^2 \cosh^{-2}(mx)\{-d^2 + m^2\}^{-1}] = 1/60m^2. \tag{5.22}$$

2) *Double-well potential*

Again using the results on page 572 for the potential (4.19′) and $B = \sigma m^{1/2}$, we find

$$w'(-m^2) = \det'[1 - 3m^2/2 \cdot \{1 - \tanh^2(mx/2)\}\{-d^2 + m^2\}^{-1}] = 1/24m^2. \tag{5.23}$$

3) *Arbitrary quartic potential*

For the potential $V(\varphi) = m^2/2 \cdot \varphi^2 - C/3 \cdot \varphi^3 + A/4 \cdot \varphi^4$ on page 570 we find

$$w'(-m^2) = 9/32m \cdot (A/mzC)^2 \cdot (z-1) \cdot S \tag{5.24}$$

$z := 9Am^2/2C^2$ and

$$\underline{S = 1/72m^2z \cdot (z-1)[9/z - 6 + (z/6 - 1/6)\log\{(1 + sqr(z))/(1 - sqr(z))\}].}$$

V.2. *Determinants in higher dimensions*

In this section we first derive the angular momentum expansion of our functional determinant. For doing this it is crucial that the bounce is spherically symmetric. Analogously to the one-dimensional case, we then extract the zero modes. Finally, we have to renormalize the mass and coupling constants in the Lagrangian.

We use the *dimensional regularization*, because this seems to be the most suited regularization scheme, if one has the analytic properties of the Jost functions with respect to anglular momentum in mind.

Angular momentum expansion of $w(\lambda)$

Since the bounce solution is spherically symmetric, the potential q appearing in the Schroedinger operator in (1) depends only on r, and the eigenspaces are classified by their angular momentum j, i.e.

$$w(\lambda)=\Pi_j w_j(\lambda)^{d(j)}, \tag{5.25}$$

where the product goes from $0, 1, 2\ldots$ to ∞, and

$$w_j(\lambda)=\det\left[(H_j-\lambda)(H_j^0-\lambda)^{-1}\right] \tag{5.25'}$$

is the determinant of the radial Schroedinger operator in n dimensions, $H_j = -d^2/dr^2-(n-1)/r\cdot d/dr+j(j+n-2)/r^2+q$, and $d(j)$ the dimension of an eigenspace with angular momentum j in n dimensions: $d(j)=(j+1)(j+2)(j+3)\ldots(2j+n-2)/(n-2)!$.

Setting $\mu:=j-1+n/2$ and $\psi(r,\mu)^*:=r^{(n-1)/2}\cdot\psi(r,j)$, one sees that $\psi(r,\mu)^*$ is an eigenfunction of

$$H(\mu):=-d^2/dr^2+(\mu^2-1/4)/r^2+q, \tag{5.26}$$

iff $\psi(r,j)$ is an eigenfunction of H_j with the same eigenvalue. Therefore, the $W-A$ determinant (25) is also given by

$$w(\lambda)=\Pi_j\det\left[\{H(\mu)-\lambda\}\{H^0(\mu)-\lambda\}^{-1}\right]^d=\Pi_j w(\mu,\lambda)^d \tag{5.27}$$

$$d=2\mu\Gamma(\mu-1+n/2)\cdot\left[\Gamma(n-1)\Gamma(\mu+2-n/2)\right]^{-1}. \tag{5.27'}$$

This shows that the *dependence on n in only via the relation between* μ *and j*. Thus dimensional regularization seems to be well suited for our task. Furthermore, we are back at the 1-dimensional problem and thus we can proceed exactly in the same matter as in the preceding section.

The radial equation $H(\mu)\psi=\lambda\psi=k^2\psi$ has two kinds of solutions, the φ-solution, which is regular at $r=0$, and the Jost solution f. These have the following behaviour

$$\log[\varphi(\mu,k,r)]\sim(\mu+1/2)\log r \quad \text{for} \quad r\to 0 \tag{5.28}$$

$$\log[f(\mu,k,r)]\sim -ikr \quad \text{for} \quad r\to\infty. \tag{5.28'}$$

The relation

$$\varphi(\mu,k,r)=1/2ik\cdot\left[f(\mu,k)f(\mu,-k,r)-f(\mu,-k)f(\mu,k,r)\right] \tag{5.29}$$

defines the *Jost functions* $f(\mu, k)$, which play such a crucial role in the discussion of the scattering matrix. They are analytic in $\mathrm{Re}(\mu)>0$ and $\mathrm{Im}(k)<k_0$, whereby k_0 is nonnegative and depends on q. Using the asymptotic behaviour of the Jost functions in (29), we obtain

$$w(\mu, \lambda)=\lim_{r\to\infty} \varphi(\mu, k, r)/\varphi^0(\mu, k, r)=f(\mu, k)/f^0(\mu, k), \tag{5.30}$$

where $\mathrm{Im}(k)<0$. Introducing the *normalized Jost functions*

$$F(\mu, k):=f(\mu, k)/f^0(\mu, k), \tag{5.31}$$

we end up with

$$\underline{w(\lambda)=\Pi_j[F(j+(n-2)/2, k)]^d} \quad \text{with} \quad k^2=\lambda \text{ and } \mathrm{Im}(k)<0. \tag{5.32}$$

Here we used the relation between μ and j.

Treatment of the zero modes

Because of the Euclidian invariance of S the Schroedinger operator H has n eigenfunctions with eigenvalues $-m^2$. They are

$$\varphi_i=\mathrm{const}\cdot x_i/r\cdot d\varphi/dr \qquad i=1, \ldots, n, \tag{5.33}$$

where φ is the bounce solution and hence the *zero modes have angular momentum* $j=1$.

In order to compute the primed determinant, we only have to deal with the $j=1$ sector. Now we repeat the consideration of the preceding section. We find

$$w'(\mu=n/2, -m^2)=\lim_{\lambda\to -m^2} f(n/2, -k)\cdot[f^0(n/2, -k)(m^2+k^2)]^{-1}.$$

Using again l'Hospitals rule gives

$$w'(n/2, -m^2)=df/dk(n/2, -im)\cdot[f^0(n/2), -im)2im]^{-1}.$$

Analogously to the identity below (17), we use the formula [AR]

$$\int \psi(\mu, k, r)^2 dr=-idf(\mu, -k)/dk\cdot[f(\mu, k)]^{-1}$$

(compare with (18)), which holds if $\psi(\mu, k, r)$ is a bound state and $\psi\to\exp(-mr)$ for $r\to\infty$. Therefore,

$$w'(n/2, -m^2)=1/2m\cdot f(n/2, im)/f^0(n/2, -im)\cdot\int \psi(n/2, im, r)^2\, dr.$$

Because of (33), and the transformations $j\to\mu$, $\psi\to\psi^*$ above (26), we have

$$m^{1/2}\psi=1/B\cdot r^{(n-1)/2}\cdot d\varphi/dr. \tag{5.35}$$

With the virial theorem we find

$$\int \psi(n/2, im, r)^2\, dr=2/(B^2 m\omega_n)\cdot T[\varphi]=n/(B^2 m\omega_n)\cdot S[\varphi],$$

which results in

$$w'(n/2, -m^2) = n/(2m^2B^2\omega_n) \cdot (-1)^{(1-n)/2} \cdot S[\varphi] \cdot F(n/2, im).$$

Putting all together gives finally

$$w'(-m^2) = [n/(2m^2B^2\omega_n) \cdot (-1)^{(n-1)/2} \cdot S[\varphi] \cdot F(n/2, im)]^n \times \Pi_{j<>1}[F(j-1+n/2, -im)]^d. \tag{5.36}$$

This equation gives a *formal expression* for the primed $W-A$ determinant. It requires again the computation of the number B, which is determined by the asymptotic behaviour of the bounce. In contrast to the 1-dimensional case, we here need some information about scattering data.

Because *for* $n=1$ only the $j=0$ and $j=1$ sectors contribute, and both dimensions are equal to one, we have

$$w'(-m^2) = 1/(2mB)^2 \cdot S[\varphi] \cdot F(1/2, im) \cdot F(-1/2, -im).$$

But $F(1/2, im) \cdot F(-1/2, -im) = 2$, so we find again the formula (5.21).

In *three dimensions* (36) reads

$$w'(-m^2) = -[3/(8\pi m^2B^2) \cdot S \cdot F(3/2, im)]^3 \ \Pi_{j<>1}F(j+1/2, -im)^{2j+1},$$

and in *four dimensions* ($d=(j+1)^2$) we obtain

$$w'(-m^2) = [1/(\pi mB)^2 \cdot S \cdot F(2, im)]^4 \ \Pi_{j<>1}F(j+1, -im)^d.$$

One-loop counter terms

Now we renormalize the effective action of the *dynamical Higgs potential*

$$V(\varphi) = (2A-B)\sigma^2\varphi^2 - A\varphi^4 + B\varphi^4 \ln \varphi^2/\sigma^2$$

in order to see which parameters in the renormalized determinant are the physical ones.

The effective action S_{eff} in 1-(Higgs-boson) loop approximation is

$$S_{\text{eff}}[\varphi] = S[\varphi] + \hbar/2 \cdot \operatorname{tr} \ln [\{-\Delta + V''(\varphi)\}\{-\Delta + m^2\}^{-1}] + S^{\text{ct}}[\varphi]. \tag{5.37}$$

If $q := V''(\varphi) - m^2$, we have

$$\operatorname{tr} \ln [\,..\,] = \operatorname{tr} \ln [1 + q(-\Delta + m^2)^{-1}] =: \operatorname{tr} \ln (1+A)$$
$$= -\sum_1 \frac{(-)^j}{j} \cdot \operatorname{tr} A^j. \tag{5.38}$$

The kernel of the integral operator A^j is

$$G_j(x_1, x_j) = \int .. \int dx_2 \, .. \, dx_{j-1} G(x_1, x_2) \, .. \, G(x_{j-1}, x_j) \tag{5.39}$$

with

$$G(x, y) = q(x) \cdot C(m, x-y).$$

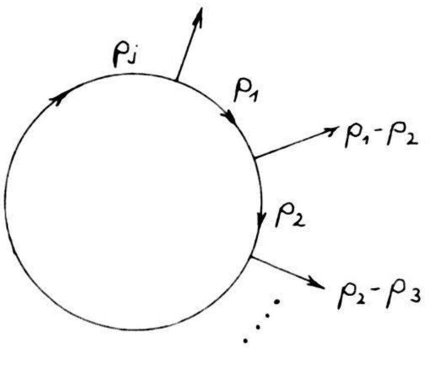

Figure 17
The Feynman diagram which corresponds to tr A^j.

Here $C(m, z) = (2\pi)^{-n} \int \exp(ipz)(p^2+m^2)^{-1} d^n p$ is the Greens function of A and hence

$$\operatorname{tr} A^j = \int \Pi_i d^n x_i \Pi_i G(x_i, x_{i+1})$$
$$= (2\pi)^{-nj/2} \int \Pi d^n p \Pi q\hat{\ }(p_i - p_{i+1})(p_i + m^2)^{-1}.$$

We used $q(x) = (2\pi)^{-n/2} \int \exp(ipx) q\hat{\ }(p) d^n p$ and $x_1 = x_{j+1}$, resp. $p_1 = p_{j+1}$.

The corresponding *Feynman diagram* is shown in Fig. 17 and the *degree of divergence* is

$$d(\Gamma(A^j)) = n - 2j,$$

i.e., tr A is quadratic and tr A^2 logarithmic divergent in 4 dimensions.

The *effective potential* is the effective action for a constant field φ divided by Vol (space-time). By substituting $y_i = x_i - x_{i+1}$ for $i = 1, \ldots, j-1$ and $y_j = x_1 + \cdots + x_j$ we find for a constant q

$$\operatorname{tr} A^j = q^j (2\pi)^{-n} \operatorname{Vol} \int d^n p (p^2 + m^2)^{-j} \tag{5.40}$$

Using the well known formula in *dimensional regularization*

$$\int d^n p (p^2+m^2)^{-j} = (\pi m^2)^{n/2} m^{-2j} \cdot \Gamma(j - n/2)/\Gamma(j)$$

we are led to

$$\operatorname{tr} \ln(1+A) = (1/4\pi)^{n/2} \cdot m^{n-4} \operatorname{Vol} \cdot [m^2 q \Gamma(1-n/2) - q^2/2 \cdot \Gamma(2-n/2)] + \operatorname{tr} \ln(1+A)_{\text{reg}}. \tag{5.41}$$

The first 2 terms come from tr A and tr A^2; $(1+A)_{\text{reg}}$ is the regularized operator in the sense of Simon [Si], i.e.

$$\operatorname{tr} \ln(1+A)_{\text{reg}} = -(1/4\pi)^{n/2} \cdot m^n \operatorname{Vol} \cdot \sum_3^\infty (-)^j \cdot (q/m^2)^j [j(j-1)(j-2)]^{-1}.$$

It's not difficult to see that

$$\operatorname{tr} \ln(1+A)_{\text{reg}} = (m^2/4\pi)^{n/2}/m^4 \cdot \operatorname{Vol} \cdot [-m^2 q/2 - 3q^2/4 + 1/2 \cdot (V'')^2 \ln V''/m^2]. \tag{5.42}$$

For the moment we set $m_0^2 = 2(2A - B)\sigma^2$ and $\lambda_0/4 = -A$, or

$$V(\varphi) = m_0^2/2 \cdot \varphi^2 + \lambda_0/4 \cdot \varphi^4 + \hbar \cdot V_1$$

where $\hbar \cdot V_1$ is the one-loop contribution of the gauge bosons. In computing the contribution (42) of the Higgs boson loops, we only have to take into account the first 2 terms in $V(\varphi)$, because $\hbar V_1$ gives a $\hbar^2$ contribution to the expression (42).

We obtain ($m_0^2 := m^2 + \delta m^2$, $\lambda_0 := \lambda + \delta\lambda$)

$$\begin{aligned} V_{\mathrm{eff}} = {} & m^2/2 \cdot \varphi^2[1 + \delta m^2/m^2 - 3\lambda/16\pi^2 \cdot \Gamma(2 - n/2) - 3\lambda/32\pi^2] \\ & + \lambda/4 \cdot \varphi^4[1 + \delta\lambda/\lambda - 9\lambda/16\pi^2 \cdot \Gamma(2 - n/2) - 27\lambda/32\pi^2] \\ & + \hbar \cdot V_1 + (V'')^2/64\pi^2 \cdot \ln V''/m^2, \end{aligned}$$

and with

$$\delta m^2 = 3\lambda (m/4\pi)^2 \cdot \Gamma(2 - n/2) \qquad \delta\lambda = (3\lambda/4\pi)^2 \cdot \Gamma(2 - n/2)$$

we finally obtain

$$\begin{aligned} V_{\mathrm{eff}} = {} & m^2/2 \cdot \varphi^2(1 - 3\lambda/32\pi^2) + \lambda/4 \cdot \varphi^4(1 - 27\lambda/32\pi^2) \\ & + \hbar \cdot V_1 + 1/64\pi^2 \cdot (m^2 + 3\lambda\varphi^2)^2 \ln[(m^2 + 3\lambda\varphi^2)/m^2] + \text{finite}. \end{aligned}$$

The described renormalization of determinants is known in literature [Si]. It amounts in dropping the terms in the expansion (38) which are not finite:

$$\det(1 + A)_{\mathrm{reg}} = \det R(A, [n/2] \quad \text{where} \quad R(A, m) = (1 + A) \exp\left(\sum_1^m (-1)^j/j \cdot A^j\right).$$

V.3. *Angular momentum expansion of the regularized determinant*

With the preceding remarks it is now easy to regularize the determinant (32), resp. the primed determinant (36):

In 2 and 3 dimensions one has to drop the linear, and in *4 dimensions* the linear and quadratic terms in q in the power expansion of $\ln\{w'(\lambda, q)\}$. From [AR] we take the following expansion of F in powers of q

$$\begin{aligned} F(\mu, k) &= 1 + \sum_1 (-)^n/n! \cdot \int \cdots \int \Pi\, dr_j \cdot K_n(r_1, \ldots, r_n) q(r_1) \cdots q(r_n) \\ &= 1 + \sum (-)^n/n! \cdot K_n(q, \ldots, q), \end{aligned} \tag{5.45}$$

where

$$K_n(r_1, \ldots, r_n) = \det \begin{pmatrix} K(r_1, r_1) & \cdots & K(r_1, r_n) \\ \vdots & & \vdots \\ K(r_n, r_1) & \cdots & K(r_n, r_n) \end{pmatrix} \tag{5.45'}$$

$$\begin{aligned} & K(r_i, r_j) = i\pi/2 \cdot (r_i r_j)^{1/2} \cdot H(\mu, kr_>) J(\mu, kr_<) \\ & r_> = (r_i + r_j)/2 + |r_j - r_i|/2 \\ & r_< = (r_i + r_j)/2 - |r_j - r_i|/2. \end{aligned}$$

We used the notation $H(\mu, z)$ for the second Hankel function of μth order and $J(\mu, z)$ for the Bessel function of the first kind of μth order [AS].

So we obtain

$$\ln[F(\mu, k)] = -K_1(q) + 1/2 \cdot K_2(q, q) - 1/2 \cdot [K_1(q)]^2 + 0(q^3),$$

with the K_n defined in (45) and (45′).

Now it is natural to define the *regularized Jost functions*

$$\underline{F_{\text{reg}}(\mu, k) = F(\mu, k) \cdot \exp\{K(\mu, q)\},} \tag{5.46}$$

with

$$K(\mu, q) = K_1(q) \qquad \text{for} \quad n = 2 \text{ or } n = 3 \tag{5.46'}$$

$$K(\mu, q) = K_1(q) - 1/2 \cdot [K_2(q, q) - \{K_1(q)\}^2] \quad \text{for} \quad n = 4. \tag{5.46''}$$

Conclusion

The angular momentum expansion of the regularized determinant is given in (32) resp. (36) if one uses the regularized Jost functions instead of the usual ones.

V.4. *Thin wall approximation for the functional determinant*

As an application of our results, as well as illustrating the first term in an asymptotic expansion of the determinant in the TW-parameter (compare page 578f), we consider the *square well potential*

$$q(r) = q \cdot \theta(R - r). \tag{5.47}$$

For $r < R$ and $r > R$ we have the free case, hence with $k' := (m^2 + q)^{1/2}$,

$$\varphi(\mu, k, r) = \varphi_0(\mu, k', r) \quad \text{for} \quad r < R$$
$$f(\mu, k, r) = f_0(\mu, k, r) \quad \text{for} \quad r > R,$$

and because of the following equation, valid for $r > R$ (we drop μ),

$$\varphi(k, r) = 1/2ik \cdot \{f(k) f_0(-k, r) - f(-k) f_0(k, r)\}$$

the Jost functions are determined by the *matching-conditions*:

$$\varphi(k, R) = \varphi_0(k', R)$$
$$\varphi'(k, R) = \varphi_0'(k', R).$$

Using the explicit expressions for φ_0 and f_0 [AR], and the abbreviations $x := kR$, $y := k'R$, we obtain

$$\begin{aligned} F(\mu, k) &= -i\pi/2 \cdot (x/y)^\mu \cdot \{xJ(\mu, y)H(\mu+1, x) - yJ(\mu+1, y)H(\mu, x)\} \\ &= i\pi/2 \cdot (x/y)^\mu \cdot \{xJ(\mu, y)H'(\mu, x) - yJ'(\mu, y)H(\mu, x)\}. \end{aligned} \tag{5.48}$$

Setting $1 + z = (y/x)^2$, and using the multiplication theorem for Bessel functions [AS], the normalized Jost function is

$$\begin{aligned} F(\mu, k) &= 1 + (x/y)^\mu z \cdot \int dx' x' I(\mu, (1+z)^{1/2} x') K(\mu, x') \\ &= 1 - z \cdot \sum (-z/2)^k / k! \cdot \int dx' x'^{k+1} I(\mu + k, x') K(\mu, x')\, dx'. \end{aligned} \tag{5.48'}$$

Here the integral extends from $x'=0$ to $x'=mR$. Comparing this expansion with (46′, 46″), we obtain for the integral operators K_n of the square-well potential ($z := 1/m^2$):

$$K_n(q, \ldots, q) = 2n \cdot (z/2)^n \cdot \int dx' x'^n I(\mu + n - 1, x') K(\mu, x'). \tag{5.49}$$

After a straightforward calculation we found the following *expansion in* $1/\mu$ ($Q := qR^2$):

$$F(\mu, -imR) = 1 + Q/4\mu + Q^2/32\mu^2 + (Q^3/24 - Q(mR)^2 - Q^2/2)/16\mu^3$$
$$+ (Q^4/24 - 2Q^3 - 4Q^2(mR)^2 + 8Q(mR)^2)/256\mu^4 + 0(1/\mu^5)$$

$$K_1(q) = -Q/4\mu + Q(mR)^2/16\mu^3 - Q(mR)^2/32\mu^4 + \cdots$$
$$K_2(q) = Q^2/16\mu^2 - Q^2/16\mu^3 - Q^2(mR)^2/32\mu^4 + \cdots.$$

From this we obtain in *2 or 3 dimensions*

$$F_{\text{reg}}(\mu, -imR) = \exp[-Q^2/32\mu^3 + 0(1/\mu^4)]. \tag{5.50}$$

Thus we find in *2 dimensions* ((f) means a finite expression)

$$w_{\text{reg}}(-im) = (f) \cdot \Pi_j \exp[-Q^2/16j^3] = (f) \cdot \exp[-zeta(3)Q^2/16] \tag{5.51}$$

and for $n = 3$ we obtain

$$w_{\text{reg}}(-im) = (f) \cdot \Pi_j \exp[-Q^2/(4j+2)^2] = (f) \cdot \exp[-(\pi Q)^2/32]. \tag{5.52}$$

In *4 dimensions* the expansion of $K(\mu, q)$, defined in (46′), is

$$K(\mu, q) = [-1/\mu + (mR)^2/4\mu^3 + (mR)^2/8\mu^4] \cdot Q/4 + Q^2/32\mu^2,$$

and hence we obtain with (46)

$$F_{\text{reg}}(\mu, -imR) = \exp[(Q/4\mu)^4 + 0(1/\mu^5)]. \tag{5.53}$$

For the regularized determinant we find in 4 dimensions

$$w_{\text{reg}}(-im) = (f) \cdot \Pi_j \exp[(Q/4)^4/(j+1)^2] = (f) \cdot \exp[(Q/4)^4 \cdot \pi^2/6]. \tag{5.54}$$

Inspecting the expressions (51, 52, 54), we can see a posteriori, that our angular momentum expansion of the regularized determinant *converges*, as expected by construction.

For obtaining more quantitative answers, one has to use the computer to calculate the Jost functions (45) and the regularizing exponent $K(\mu, q)$ in (46″).

Summary

In this chapter we first computed explicitly functional determinants in one dimension of the kind

$$w(\lambda) = \det[(H - \lambda)(H_0 - \lambda)^{-1}],$$

where $H_0 = -\Delta$ and $H = -\Delta + q(x)$, with an arbitrary fourth-order potential $q(x)$.

In (15) and (15′) these determinants were connected to the transmission coefficient of the corresponding Schroedinger operators.

If $q(x)$ is due to the fluctuations around a classical solution, then $H+m^2$ has inevitably n zero modes. In one dimension we were able to compute the determinants with the zero mode omitted explicitly. There remained only elementary integrals over $[V(\varphi)]^{1/2}$, resp. $[V''(\varphi)]^{1/2}$, for calculating $S[\varphi]$ and B in (21).

In *higher dimensions* we first gave the angular momentum expansion in arbitrary (complex) dimensions n (27, 32). Again we worked out the "primed determinant" by omitting the zero modes (36). Knowledge of the asymptotics of the bounce solution determines B. We remained with the computation of the Jost functions for all angular momenta and for momentum $k=-im$.

At this point, we clearly also had to renormalize the resulting formal expressions. The study of the renormalization of the effective potential enabled us to give a prescription for renormalizing the determinants:

The *renormalized determinants* are given by the same expressions as the unrenormalized ones if the Jost functions are replaced by the regularized ones (46).

Next we explicitly computed these regularized Jost functions for a '*thin wall potential*' which was simulated by the square well potential $q\cdot\theta(R-r)$. The renormalized determinant in this case is given by some combination of Bessel and Hankel functions (48, 49).

Finally we explicitly proved the convergence of the angular momentum expansion, by looking at the 1/(angular momentum)-behaviour of the regularized Jost functions. We did not 'work out' this infinite product for obtaining a numerical answer. With our results this is, however, only a question of computer time. Nevertheless, it would be valuable to study, for example, the dependence of the renormalized determinants on Q, where $Q:=mR^2$.

Our asymptotic expansion indicates that $w(-im)$ may depend exponentially on some power (depending on the dimension) of Q, i.e., may have a dramatic effect on the tunnel probability (in the TW approximation $S[\varphi]$ is of order const $\cdot R^n$).

There are indications [ASW], that these infrared problems are cured in supersymmetric theories.

V.5. *An application: computation of partition functions*

Using Laplace transformation techniques, it is not hard to see, that the difference of the partition functions with the Hamiltonians H, resp. H_0, is

$$Z(\beta)-Z_0(\beta)=i/2\pi\cdot\int dz\,\exp(-\beta z)\cdot\mathrm{tr}\,[R(z)-R_0(z)], \tag{5.55}$$

where C is a path which encloses both the spectrum of H and H_0 counterclockwise. (Compare the analogous result for the unitary time-evolution operator $U(t)=\exp[-itH]$ in Quantum Mechanics.) We used the difference of the partition functions for justifying the following manipulations.

Using (5), we can write

$$Z(\beta)-Z_0(\beta)=1/2\pi i\cdot\int dz\,\exp(-\beta x)\cdot d\ln w(z)/dz. \tag{5.56}$$

If both, Z and Z_0, exist then the partition function for a Hamiltonian with

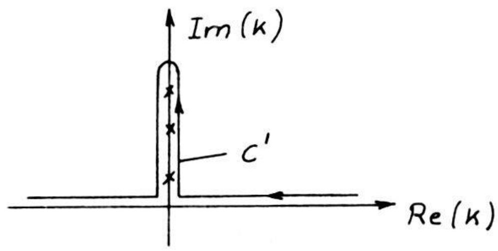

Figure 18
The path of integration C' in the complex k-plane.

Dirichlet boundary conditions on $[0, L]$ is

$$Z(\beta) = 1/2\pi i \cdot \int dz \exp(-\beta z) d \ln \varphi_2(L, z)/dz \tag{5.57}$$

where $\varphi_2(L, z)$ is defined in (2) (here we use z instead of λ for the energy). Setting $z = k^2$, the transformed path of integration C' looks as sketched in Fig. 18. If spec $(H) \subset R^+$ then we only have to integrate along $R + i\varepsilon$ (compare Fig. 18).

In this variables the integral reads

$$Z(\beta) = 1/2\pi i \cdot \int_{C'} dk \exp(-\beta k^2) d \ln \varphi_2(L, k^2)/dk. \tag{5.57'}$$

Formula (57) is obvious if one remembers that $\varphi_2(L, z)$ is the characteristic 'polynom' of $H - z$ and is also analytic in z. Since the spectrum is discrete and all eigenvalues are simple, (57) yields $Z(\beta) = \sum_k \exp[-\beta E_k]$.

A nice feature of formula (56) is that we can immediately generalize it from finite intervals to R. With (15) we conclude

$$Z(\beta) - Z_0(\beta) = 1/2\pi i \cdot \int dk \exp(-\beta k^2) d \ln r_-(k, q)/dk. \tag{5.58}$$

Examples. Reflectionless soliton potential

These potentials play an important role in the inverse scattering-theory. They are given by

$$V_n(x) = n(n+1)m^2 \cdot \cosh^{-2}(mx) \tag{5.59}$$

and have n bound states $k_j = i \cdot jm$ for $j = 1, \ldots, n$. The reflection coefficient R vanishes.

The piece of (58) around the zeros of r_- in the upper k-plane gives the expected discrete part $\sum_j \exp[-\beta E_j]$ and the integral along $R + i\varepsilon$ gives the contribution

$$\Delta Z_{con}(\beta) = -\sum_j \operatorname{erfc}\{mj(\beta)^{1/2}\} \cdot \exp(j^2 m^2 \beta).$$

So we end up with

$$Z(\beta) - Z_0(\beta) = \sum_j \operatorname{erf}\{jm(\beta)^{1/2}\} \cdot \exp(j^2 m^2 \beta). \tag{5.60}$$

Let us now assume that $Z-Z_0$ and that Z_0 exist and $Z-Z_0 \ll Z_0$. Then

$$-\beta(F-F_0)=\ln Z/Z_0=\ln[1+(Z-Z_0)/Z_0]$$
$$=-\sum_k (-)^k/k\cdot(Z-Z_0)^k/Z_0^k \sim (Z-Z_0)/Z_0$$

and the difference of the free energies is again expressible in terms of functional determinants. For example, we may think of H_0 as a Hamiltonian with quadratic or quartic potential and of q as a local disturbance.

Now let H_0, and H have ground states ψ_0, resp. ψ with energies E_0, resp. E, which are separated from the rest of the spectra.

Then

$$F-F_0 \sim -[\exp(-\beta E)-\exp(-\beta E_0)]/[\beta\exp(-\beta E_0)] \sim E-E_0,$$

if $\beta(E-E_0)\ll 1$.

Now we change the potential parameters in such a way, that ψ ceases to be the ground state of H and becomes a *metastable state.* We expect that E goes into a Weisskopf–Wigner pole on the second sheet of the resolvent i.e. $E\to E-i\Gamma/2$. Although ΔF is still real, the analytic continuation of it is approximatively (?)

$$\Delta F \sim E-E_0-i\Gamma/2.$$

Hence the *decay width* of the metastable state ψ is given by

$$\underline{\Gamma \sim 2\cdot \mathrm{im}\,(F).}$$

Acknowledgements

To my advisor, Prof. Norbert Straumann, I offer my sincerest thanks. Many of the ideas in this thesis originated with him. He has provided an inspiring model both as a research physicist and as a teacher. In addition, I am thankful for all the support and encouragement that he has given me over the years. To him I owe a special dept of gratitude for reading the draft and suggesting improvements on the style of presentation.

I have benefitted from conversations with the members of the institute for theoretical physics of the University of Zurich. Especially I would like to thank M. Camenzind, B. Kay, R. Durrer, M. Schweizer and J. Vigfusson for their contribution to this thesis.

To Prof. L. O'Raifeartaigh from the Institute for Advanced Studies in Dublin (DIAS) I am indebted for very interesting discussions, especially on various aspects of vacuum decay.

I have also benefitted from conversations with Prof. G. Boerner from the MPI in Munich on gravitational effects for the bubble-dynamics.

Most of my knowledge about the fermion–monopole system I got from Prof. C. Schmid (ETH Zurich).

The drawings were made by my brother B. Wipf, thanks to him too.

I would also like to thank my wife, Ingrid, for her understanding and patience during may career as graduate student.

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