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# Lower bounds for zero energy eigenfunctions of Schrödinger operators<sup>1)</sup>

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*Abstract.* Let  $g$  be a non-zero solution in  $L^2(\mathbb{R}^n)$ ,  $n \geq 2$ , of  $(-\Delta + V)g = 0$ . If the potential  $V$  vanishes rapidly enough at infinity, then  $g$  cannot decay (in the  $L^2$ -sense) more rapidly than any power of  $|x|$ , i.e.  $|x|^N g \notin L^2(\mathbb{R}^n)$  for some finite  $N$ .

## 1. Introduction

A non-relativistic quantum mechanical particle moving on a line in a potential  $V$  cannot be bound at zero energy if  $V$  is such that

$$\int_{-\infty}^{+\infty} (1 + |x|) |V(x)| dx < \infty.$$

In other words the equation  $-\psi'' + V\psi = 0$  has no non-zero solutions that are square-integrable over the real line  $\mathbb{R}$ . If  $\mathbb{R}$  is replaced by  $(0, \infty)$  for example, the same is true; more precisely, if  $\int_0^\infty r |V(r)| dr < \infty$ , there are no zero energy bound states in the  $l = 0$  partial wave subspace of a three-dimensional quantum mechanical system in the spherically symmetric potential  $V(r)$  (see e.g. [1], Chapters XVII.1 and II.1 respectively).

In the latter case one may however have zero energy bound states in the higher order partial wave subspaces ( $l \geq 1$ ), even if  $V$  has finite range (see [2], footnote on page 80 for a square well, [1] or [3], Remark 11.17(c) and Problem 11.11 for more general cases). The intuitive reason for this is roughly as follows: if  $l > 0$ , then the effective potential is  $V(r) + l(l+1)r^{-2}$  which, at large  $r$ , is roughly  $l(l+1)r^{-2}$  under the above assumptions on  $V$ ; hence, if the particle has zero energy, it sees a wall of infinite extension of the form  $cr^{-2}$  ( $c > 0$ ) which can produce a bound state<sup>2)</sup> (no tunnelling is possible).

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<sup>2)</sup> Notice that  $\int_1^\infty r \cdot cr^{-2} dr = \infty$ , so that the centrifugal part of the effective potential does not satisfy the condition needed for proving the non-existence of zero energy bound states.

The zero energy bound state eigenfunctions in the  $l$ -th partial wave subspace of  $L^2(\mathbb{R}^3)$  are known to behave like  $r^{-l-1}$  as  $r \rightarrow \infty$ . This is strikingly different from the exponential decay of eigenfunctions belonging to strictly negative eigenvalues: if  $\lambda < 0$ ,  $(-\Delta + V)\psi = \lambda\psi$  and  $\psi \in L^2(\mathbb{R}^3)$  and if  $V$  decays sufficiently rapidly, then  $\|e^{\kappa r}\psi\|_{L^2} < \infty$  for each  $\kappa < |\lambda|^{1/2}$ . The purpose of our paper is to prove quite generally (i.e. in  $n \geq 2$  space dimensions and without assuming spherical symmetry) that zero energy bound states are weakly localized in the sense indicated above: if  $V$  satisfies suitable decay conditions and if  $\psi \in L^2(\mathbb{R}^n)$  is such that  $(-\Delta + V)\psi = 0$ , then there is a number  $N < \infty$  such that  $\|(1+|x|)^N\psi\|_{L^2} = \infty$ , i.e.  $\psi$  cannot decay faster (in the  $L^2$ -sense) than some negative power of  $|x|$ . This follows from a more general result which we state and prove in the form of a theorem in Section 3. The proof makes heavy use of an inequality involving the Laplacean that we established in a previous paper [4].

## 2. Notation and preliminary results

We use the following notation: the symbol  $x$  is used for vectors in  $\mathbb{R}^n$ ,  $n \geq 2$ . We set  $r = |x|$ ,  $\partial_j = \partial/\partial x_j$  ( $j = 1, \dots, n$ ),  $\nabla \equiv \text{grad} = (\partial_1, \dots, \partial_n)$ ,  $\partial_r = \sum_{j=1}^n x_j r^{-1} \partial_j$  and  $\Delta = \sum_{j=1}^n \partial_j^2$ . We shall refer to the operator  $(1 - \Delta)^{-1}$  acting on functions defined on  $\mathbb{R}^n$ ; it is given as the convolution operator by the Green's function of the negative Laplacean (one of the Bessel potentials in the terminology of [5]).

For  $0 \leq a < b \leq \infty$  we set  $\Omega(a, b) = \{x \in \mathbb{R}^n \mid a < |x| < b\}$ . Notice that  $\Omega(0, \infty) = \mathbb{R}^n \setminus \{0\}$ . The derivatives of locally integrable functions are understood to be in the sense of distributions. For  $1 \leq q \leq \infty$ ,  $k \geq 0$  and integer,  $a \geq 0$  and  $\Omega \equiv \Omega(a, \infty)$ ,  $L^q(\Omega)$  denotes the Banach space of  $q$ -summable functions on  $\Omega$  and  $H^{k,q}(\Omega)$  the Sobolev space consisting of all  $f \in L^q(\Omega)$  such that  $\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f \in L^q(\Omega)$  for all  $n$ -tuples  $(\alpha_1, \dots, \alpha_n)$  of non-negative integers with  $\sum_{j=1}^n \alpha_j \leq k$ . We put

$$\|f\|_{H^{k,q}(\Omega)} = \sum_{\alpha_1 + \cdots + \alpha_n \leq k} \|\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f\|_{L^q(\Omega)}. \quad (1)$$

If  $q = 2$ , we use the simpler notation  $H^k(\Omega) \equiv H^{k,2}(\Omega)$ . Finally we write  $\|\cdot\|_{L^q}$  for the norm in  $L^q(\mathbb{R}^n)$  and  $\|\cdot\|_{H^{k,q}}$  for that in  $H^{k,q}(\mathbb{R}^n)$ , and we denote by  $H_c^{k,q}(\mathbb{R}^n \setminus \{0\})$  the set of functions  $f \in H^{k,q}(\mathbb{R}^n)$  that have compact support in  $\mathbb{R}^n \setminus \{0\}$ .

The proof of our theorem is based on the Sobolev imbedding theorem and on the following known results that we announce as Propositions 1, 2 and 3.

**Proposition 1.** *If  $1 < q < \infty$ , then  $(1 - \Delta)^{-1}$  defines a bounded invertible operator from  $L^q(\mathbb{R}^n)$  onto  $H^{2,q}(\mathbb{R}^n)$ . In particular, if  $f, \Delta f \in L^q(\mathbb{R}^n)$ , then<sup>3)</sup>  $f \in H^{2,q}(\mathbb{R}^n)$  (see [5], Theorem V.3).*

**Proposition 2.** *Let  $n \geq 2$ ,  $p \in (n/2, \infty]$  with  $p \geq n - 2$ . Set  $\mu = 2 - n/p$ . Let  $q$  and  $s$  satisfy*

$$1 \leq q \leq 2 \leq s < \infty, \quad \frac{1}{p} = \frac{1}{q} - \frac{1}{s}. \quad (2)$$

<sup>3)</sup> Write  $f = (1 - \Delta)^{-1}(f - \Delta f)$ .

Let  $\Gamma_{ns} = \{k + n - 3/2 - n/s \mid k = 1, 2, 3, \dots\}$ . Then there is a finite constant  $C$ , depending only on  $n, p$  and  $s$ , such that

$$\|r^\nu f\|_{L^s} \leq C \|r^{\nu+\mu} \Delta f\|_{L^q} \quad (3)$$

for all  $\nu \in \Gamma_{ns}$  and all  $f \in H_c^{2,q}(\mathbb{R}^n \setminus \{0\})$ . If  $p = \infty$ , the inequality (3) holds with  $C$  replaced by  $2\nu^{-1}$ . (See [4], Theorem 1 and proof of Theorem 2.)

**Proposition 3.** Let  $R > 0$  and  $\Omega = \Omega(R, \infty)$ , and let  $\alpha \geq 0$ . Then  $f \in H^{k,q}(\Omega) \Rightarrow r^{-\alpha} f \in H^{k,q}(\Omega)$ .

*Proof.* Clearly multiplication by  $r^{-\alpha}$  defines a bounded operator in  $L^q(\Omega)$ , since  $R > 0$  and  $\alpha \geq 0$ . This proves the assertion for  $k = 0$ . Next notice that

$$\partial_j r^{-\alpha} f = r^{-\alpha} \partial_j f - \alpha x_j r^{-\alpha-2} f. \quad (4)$$

Hence  $f \in H^{1,q}(\Omega) \Rightarrow r^{-\alpha} f \in H^{1,q}(\Omega)$ . The proof for  $k > 1$  is similar. ■

### 3. Lower bounds for zero energy eigenfunctions

We now state and prove our principal result.

**Theorem.** Let  $n \geq 2$ ,  $R_0 \in [0, \infty)$  and set  $\Omega_0 = \Omega(R_0, \infty)$ . Let  $V: \Omega_0 \rightarrow \mathbb{C}$  and assume that there is a number  $p \in [1, \infty]$  such that  $p > n/2$  and  $p \geq n-2$  and such that  $r^{2-n/p} V \in L^p(\Omega_0)$ . Suppose  $g \in H^1(\Omega_0)$  is such that  $\Delta g$  is a function and

$$|(\Delta g)(x)| \leq |V(x)| |g(x)| \quad \text{a.e. on } \Omega_0. \quad (5)$$

Then, if  $r^\tau g \in L^2(\Omega_0)$  for each  $\tau < \infty$ , one must have  $g = 0$  (in the  $L^2$ -sense).

*Remark.* (a) If  $p = \infty$ , the condition on the function  $V$  means that  $|x|^2 |V(x)| \leq \text{const} < \infty$ , i.e.  $V(x)$  should decay at least as rapidly as  $|x|^{-2}$  for  $|x| \rightarrow \infty$ . If  $p < \infty$ , the condition on  $V$  means that

$$\int_{\Omega_0} |r^2 V(x)|^p \frac{d^n x}{r^n} < \infty,$$

i.e.  $r^2 V(x)$  must tend to zero in an  $L^p$ -sense as  $|x| \rightarrow \infty$ . Of course local singularities of  $V$  are allowed, and for  $n = 2, 3, 4$  the result is very natural.

(b) Let  $V: \mathbb{R}^n \rightarrow \mathbb{R}$  be such that  $(1+r)^{2-n/p} V \in L^p(\mathbb{R}^n)$  for some  $p \in (n/2, \infty]$  with  $p \geq n-2$ . Then  $H = -\Delta + V$  is self-adjoint in  $L^2(\mathbb{R}^n)$  on the domain  $\{f \in H^1(\mathbb{R}^n) \mid Hf \in L^2(\mathbb{R}^n)\}$ . If zero is an eigenvalue of  $H$ , then any associated eigenvector  $g$  has the following property: there is a number  $N < \infty$  such that  $\|r^N f\|_{L^2} = \infty$ . (To see this, it suffices to notice that an eigenvector  $g$  corresponding to the eigenvalue zero satisfies (5) with the equality sign.)

*Proof.* (i) We first fix  $s$  and  $q$  satisfying the hypotheses of Propositions 1 and 2. It suffices to choose the number  $s$ ;  $q$  is then defined by  $q^{-1} = p^{-1} + s^{-1}$ .

If  $p > 2$ , we take  $s = 2$ . If  $p \leq 2$  (which is possible only for  $n = 2, 3$ ), we define  $s$  by  $s^{-1} = 3/4 - (2p)^{-1} - (2n)^{-1}$ . The assumptions made on  $p$  imply that  $s \in [3, \infty)$  in the second case and that  $1 < q \leq \min\{2, p\}$  in both cases.

We set  $\mu = 2 - n/p$  and choose a number  $R \in (R_0, \infty)$  as follows. If  $p = \infty$ , we

take  $R = R_0 + 1$ ; if  $p < \infty$ , we let  $C = C(n, p, s)$  be the constant appearing in Proposition 2 and take  $R$  so large that  $C \|r^\mu V\|_{L^p(\Omega(R, \infty))} < \frac{1}{2}$ , which is possible by the hypothesis made on  $V$ . We set  $\Omega = \Omega(R, \infty)$  and  $\lambda = \|r^\mu V\|_{L^p(\Omega)}$ .

The Sobolev imbedding theorem ([6], Theorem 5.4 and Corollary 5.16) implies that, if  $p > n/2$  and  $q$  and  $s$  are as above, one has the following imbeddings:  $H^1(\Omega) \subset L^s(\Omega)$  and  $H^{2,q}(\Omega) \subset L^s(\Omega)$ ; here  $X \subset Y$  means that each  $\xi \in X$  is also an element of  $Y$  and that there is a constant  $\kappa = \kappa_{XY}$  such that  $\|\xi\|_Y \leq \kappa \|\xi\|_X$  for each  $\xi \in X$ .

(ii) Let  $\eta \in C^\infty(\mathbb{R}^n)$  be such that  $0 \leq \eta \leq 1$ ,  $\eta(x) = 0$  if  $|x| \leq R$  and  $\eta(x) = 1$  if  $|x| \geq R + 1$ . Assume that  $g$  satisfies all the hypotheses stated in the theorem and set  $g_0 = \eta g$ . We shall show that  $r^\tau g_0$ ,  $r^\tau \Delta g_0$  and each component of  $r^\tau \nabla g_0$  belong to  $L^q(\mathbb{R}^n)$  for each  $\tau \in \mathbb{R}$ .

The first assertion follows from the Hölder inequality and the hypothesis that  $r^\tau g \in L^2(\Omega_0)$  for all  $\tau$ : if  $m \in (2, \infty]$  is defined by  $m^{-1} = q^{-1} - \frac{1}{2}$ , then

$$\|r^\tau g_c\|_{L^q} \leq \|r^\tau g\|_{L^q(\Omega)} \leq \|r^{-n}\|_{L^m(\Omega)} \|r^{\tau+n} g\|_{L^2(\Omega)} < \infty.$$

Next we observe that

$$r^\tau \Delta g_0 = \eta r^\tau \Delta g + 2r^\tau (\nabla \eta) \cdot \nabla g + r^\tau (\Delta \eta) g. \quad (6)$$

Since  $g$  and the components of  $\nabla g$  are in  $L^2(\Omega)$  and  $\nabla \eta, \Delta \eta$  have compact support, the last two terms on the r.h.s. of (6) are in  $L^q(\mathbb{R}^n)$  (remember that  $q \leq 2$ ). We denote by  $\beta_\tau$  the sum of their  $L^q$ -norms and then have by the Hölder inequality:

$$\|r^\tau \Delta g_0\|_{L^q} \leq \|r^\tau \Delta g\|_{L^q(\Omega)} + \beta_\tau \leq \|r^\mu V\|_{L^p(\Omega)} \|r^{\tau-\mu} g\|_{L^s(\Omega)} + \beta_\tau. \quad (7)$$

In view of the last statement in (i), this leads to the following two inequalities, in which  $\lambda$  is the number defined in part (i) of the proof and  $\kappa_s, \kappa_{qs}$  are finite constants depending on the values of the subscript(s):

$$\|r^\tau \Delta g_0\|_{L^q} \leq \lambda \kappa_s \|r^{\tau-\mu} g\|_{H^1(\Omega)} + \beta_\tau, \quad (8)$$

$$\begin{aligned} \|r^\tau \Delta g_0\|_{L^q} &\leq \lambda \kappa_{qs} \|r^{\tau-\mu} g_0\|_{H^{2,q}} + \lambda \|r^{\tau-\mu} (1-\eta) g\|_{L^s(\Omega)} + \beta_\tau \\ &\leq \lambda \kappa_{qs} \|r^{\tau-\mu} g_0\|_{H^{2,q}} + \lambda \gamma_\tau \kappa_s \|g\|_{H^1(\Omega)} + \beta_\tau, \end{aligned} \quad (9)$$

where  $\gamma_\tau = \|r^{\tau-\mu} (1-\eta)\|_{L^\infty(\Omega)} < \infty$ .

Since  $g \in H^1(\Omega)$ , the inequality (8) and Proposition 3 imply that  $r^\tau \Delta g_0 \in L^q(\mathbb{R}^n)$  for  $\tau \leq \mu$ ; in particular  $\Delta g_0 \in L^q(\mathbb{R}^n)$ . By Proposition 1, we then have  $g_0 \in H^{2,q}(\mathbb{R}^n)$ .

Next we notice the identity

$$\Delta r^\tau g_0 = r^\tau \Delta g_0 + 2\tau \partial_r (r^{\tau-1} g_0) + (n\tau - \tau^2) r^{\tau-2} g_0. \quad (10)$$

Since  $\|\partial_r f\|_{L^q} \leq \|f\|_{H^{1,q}} \leq \|f\|_{H^{2,q}}$ , (10) leads to

$$\|\Delta r^\tau g_c\|_{L^q} \leq \|r^\tau \Delta g_0\|_{L^q} + 2|\tau| \|r^{\tau-1} g_0\|_{H^{2,q}} + (n|\tau| + \tau^2) \|r^{\tau-2} g_0\|_{L^q}. \quad (11)$$

Hence, if  $\tau \leq \tau_0 \equiv \min\{\mu, 1\}$ , we have  $\Delta r^\tau g_0 \in L^q(\mathbb{R}^n)$ . Together with Proposition 1, this implies that  $r^\tau g_0 \in H^{2,q}(\mathbb{R}^n)$  for  $\tau \leq \tau_0$ .

This last inclusion may now be combined with the inequality (9) to deduce that  $r^\tau \Delta g_0 \in L^q(\mathbb{R}^n)$  for  $\tau \leq \tau_0 + \mu$ , and (11) then implies that  $\Delta r^\tau g_0 \in L^q(\mathbb{R}^n)$  if  $\tau \leq 2\tau_0$ . Hence, by Proposition 1,  $r^\tau g_0 \in H^{2,q}(\mathbb{R}^n)$  for  $\tau \leq 2\tau_0$ . By iterating this procedure one obtains that  $\Delta r^\tau g_0 \in L^q(\mathbb{R}^n)$  and  $r^\tau g_0 \in H^{2,q}(\mathbb{R}^n)$  for all  $\tau \in \mathbb{R}$ .

Finally we have for each  $\tau \in \mathbb{R}$  (see (4)):

$$\begin{aligned} \|r^\tau \partial_j g_0\|_{L^a} &\leq \|\partial_j r^\tau g_0\|_{L^a} + |\tau| \|r^{\tau-1} g_0\|_{L^a} \\ &\leq \|r^\tau g_0\|_{H^{2,a}} + |\tau| \|r^{\tau-1} g_0\|_{L^a} < \infty. \end{aligned}$$

(iii) We now show that  $g(x) = 0$  for  $|x| > R + 1$ . For this, we let  $\theta \in C_0^\infty(\mathbb{R}^n)$  be such that  $\theta(x) = 1$  if  $|x| \leq 1$  and  $\theta(x) = 0$  if  $|x| \geq 2$ . For  $a > 0$  we define  $\theta_a$  by  $\theta_a(x) = \theta(x/a)$ , and we set  $\delta' = \|\nabla \theta\|_{L^\infty}$ ,  $\delta'' = \|\Delta \theta\|_{L^\infty}$ . We observe that

$$|(\nabla \theta_a)(x)| \leq \frac{\delta'}{a}, \quad |(\Delta \theta_a)(x)| \leq \frac{\delta''}{a^2} \quad \forall x \in \mathbb{R}^n. \quad (12)$$

The identity

$$\Delta \theta_a g_0 = \theta_a \Delta g_0 + 2(\nabla \theta_a) \cdot \nabla g_0 + (\Delta \theta_a) g_0 \quad (13)$$

and a similar identity for  $\partial_j \theta_a g_0$  imply that  $\theta_a g_0 \in H_c^{2,a}(\mathbb{R}^n \setminus \{0\})$ . By setting  $f = \theta_a g_0$  in (3) and using (13) and (12) one finds that, for  $\nu \in \Gamma_{ns}$ :

$$\|r^\nu \theta_a g_0\|_{L^s} \leq C \|r^{\nu+\mu} \theta_a \Delta g_0\|_{L^a} + \frac{2\delta'}{a} C \|r^{\nu+\mu} \nabla g_0\|_{L^a} + \frac{\delta''}{a^2} C \|r^{\nu+\mu} g_0\|_{L^a}. \quad (14)$$

Remembering that  $r^\rho \Delta g_0$ ,  $r^\rho \nabla g_0$  and  $r^\rho g_0$  are in  $L^a(\mathbb{R}^n)$  for each  $\rho \in \mathbb{R}$ , one may take the limit  $a \rightarrow \infty$  in (14) (by using for instance the dominated convergence theorem) to obtain the inequality

$$\|r^\nu g_0\|_{L^s} \leq C \|r^{\nu+\mu} \Delta g_0\|_{L^a}, \quad \nu \in \Gamma_{ns}. \quad (15)$$

The r.h.s. of (15) may be majorized by using the inequality (7), with  $\tau = \nu + \mu$ . We note that  $\beta_\tau$  satisfies  $\beta_\tau \leq (R+1)^\tau c(\eta, g)$ , where  $c(\eta, g)$  is a finite number that does not depend on  $\tau$ . We also have, as in (9), that

$$\|r^\nu g\|_{L^s(\Omega)} \leq \|r^\nu g_0\|_{L^s} + (R+1)^\nu \|g\|_{L^s(\Omega)} \leq \|r^\nu g_0\|_{L^s} + \kappa_s (R+1)^\nu \|g\|_{H^1(\Omega)}.$$

Consequently we obtain that

$$\|r^\nu g_0\|_{L^s} \leq C \lambda \|r^\nu g_0\|_{L^s} + C \lambda \kappa_s (R+1)^\nu \|g\|_{H^1(\Omega)} + C c(\eta, g) (R+1)^{\nu+\mu}. \quad (16)$$

If  $p < \infty$ , we have  $C \lambda < \frac{1}{2}$ , and (16) implies that, for  $\nu \in \Gamma_{ns}$ :

$$\|r^\nu g\|_{L^s(\Omega(R+1, \infty))} \leq \|r^\nu g_0\|_{L^s(\mathbb{R}^n)} \leq c_1(R, \eta, g) (R+1)^\nu, \quad (17)$$

where  $c_1$  is a finite number independent of  $\nu$ . If  $p = \infty$ , one may replace  $C$  by  $2\nu^{-1}$  in (14)–(16) and obtains the validity of (17) for all  $\nu \in \Gamma_{ns} \cap [4\lambda, \infty)$ .

Now assume that  $\|g\|_{L^2(\Omega(R+1, \infty))} \neq 0$ . Then, as  $\nu \rightarrow \infty$  ( $\nu \in \Gamma_{ns}$ ), the l.h.s. of (17) grows faster than  $(R+1)^\nu$ , i.e. (17) is violated for  $\nu$  large enough. Hence we must have  $g = 0$  on  $\Omega(R+1, \infty)$ .

(iv) To show that  $g = 0$  on  $\Omega_0 = \Omega(R_0, \infty)$ , it suffices to notice that  $q \geq 2p/(p+2)$ , so that one may apply the unique continuation theorem proved in [4] (see [4], Theorem 2). ■

### Additional remarks

(a) It is interesting to point out that A. Hinz recently obtained *upper* bounds for zero energy eigenfunctions that have the form of a negative power of  $|x|$ , see [7].

(b) One may ask to what extent our condition  $\|r^{2-n/p}V\|_{L^p} < \infty$  is optimal. For  $p = \infty$ , it requires that  $|V(x)| \leq cr^{-2}$ . The following example shows that one may have exponentially decreasing zero energy eigenfunctions for potentials  $V$  tending to zero at infinity but doing so more slowly than  $r^{-2}$ : if  $-\Delta g + Vg = 0$ , then  $V = \Delta g/g$ . By taking  $g$  of the form  $g(x) = \exp[-\varphi(r)]$ , one obtains

$$V(x) = |\varphi'(r)|^2 - \varphi''(r) - (n-1)r^{-1}\varphi'(r).$$

If for example  $\varphi$  is a smooth function that is constant near  $r=0$  and equal to  $r^\alpha$ ,  $0 < \alpha < 1$ , near infinity, then  $g \in L^2(\mathbb{R}^n)$ , hence it is a zero energy bound state eigenfunction, and  $V(x)$  decays at infinity like  $r^{-2+2\alpha}$ . This gives a class of smooth potentials that decay like  $r^{-\beta}$ ,  $0 < \beta < 2$ , and give rise to zero energy eigenfunctions that decrease more rapidly than any negative power of  $|x|$ .

*Note.* This paper is an elaboration of one of the results announced in [8]. After submission of the paper for publication, our attention was drawn to Ref. [9] which contains various  $L^2$  lower bounds for eigenfunctions of Schrödinger operators, in particular a theorem of the type of that given here.

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