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Lower bounds for zero energy eigenfunctions of Schrödinger operators¹⁾

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Abstract. Let g be a non-zero solution in $L^2(\mathbb{R}^n)$, $n \geq 2$, of $(-\Delta + V)g = 0$. If the potential V vanishes rapidly enough at infinity, then g cannot decay (in the L^2 -sense) more rapidly than any power of $|x|$, i.e. $|x|^N g \notin L^2(\mathbb{R}^n)$ for some finite N .

1. Introduction

A non-relativistic quantum mechanical particle moving on a line in a potential V cannot be bound at zero energy if V is such that

$$\int_{-\infty}^{+\infty} (1 + |x|) |V(x)| dx < \infty.$$

In other words the equation $-\psi'' + V\psi = 0$ has no non-zero solutions that are square-integrable over the real line \mathbb{R} . If \mathbb{R} is replaced by $(0, \infty)$ for example, the same is true; more precisely, if $\int_0^\infty r |V(r)| dr < \infty$, there are no zero energy bound states in the $l = 0$ partial wave subspace of a three-dimensional quantum mechanical system in the spherically symmetric potential $V(r)$ (see e.g. [1], Chapters XVII.1 and II.1 respectively).

In the latter case one may however have zero energy bound states in the higher order partial wave subspaces ($l \geq 1$), even if V has finite range (see [2], footnote on page 80 for a square well, [1] or [3], Remark 11.17(c) and Problem 11.11 for more general cases). The intuitive reason for this is roughly as follows: if $l > 0$, then the effective potential is $V(r) + l(l+1)r^{-2}$ which, at large r , is roughly $l(l+1)r^{-2}$ under the above assumptions on V ; hence, if the particle has zero energy, it sees a wall of infinite extension of the form cr^{-2} ($c > 0$) which can produce a bound state²⁾ (no tunnelling is possible).

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²⁾ Notice that $\int_1^\infty r \cdot cr^{-2} dr = \infty$, so that the centrifugal part of the effective potential does not satisfy the condition needed for proving the non-existence of zero energy bound states.

The zero energy bound state eigenfunctions in the l -th partial wave subspace of $L^2(\mathbb{R}^3)$ are known to behave like r^{-l-1} as $r \rightarrow \infty$. This is strikingly different from the exponential decay of eigenfunctions belonging to strictly negative eigenvalues: if $\lambda < 0$, $(-\Delta + V)\psi = \lambda\psi$ and $\psi \in L^2(\mathbb{R}^3)$ and if V decays sufficiently rapidly, then $\|e^{\kappa r}\psi\|_{L^2} < \infty$ for each $\kappa < |\lambda|^{1/2}$. The purpose of our paper is to prove quite generally (i.e. in $n \geq 2$ space dimensions and without assuming spherical symmetry) that zero energy bound states are weakly localized in the sense indicated above: if V satisfies suitable decay conditions and if $\psi \in L^2(\mathbb{R}^n)$ is such that $(-\Delta + V)\psi = 0$, then there is a number $N < \infty$ such that $\|(1 + |x|)^N \psi\|_{L^2} = \infty$, i.e. ψ cannot decay faster (in the L^2 -sense) than some negative power of $|x|$. This follows from a more general result which we state and prove in the form of a theorem in Section 3. The proof makes heavy use of an inequality involving the Laplacean that we established in a previous paper [4].

2. Notation and preliminary results

We use the following notation: the symbol x is used for vectors in \mathbb{R}^n , $n \geq 2$. We set $r = |x|$, $\partial_j = \partial/\partial x_j$ ($j = 1, \dots, n$), $\nabla \equiv \text{grad} = (\partial_1, \dots, \partial_n)$, $\partial_r = \sum_{j=1}^n x_j r^{-1} \partial_j$ and $\Delta = \sum_{j=1}^n \partial_j^2$. We shall refer to the operator $(1 - \Delta)^{-1}$ acting on functions defined on \mathbb{R}^n ; it is given as the convolution operator by the Green's function of the negative Laplacean (one of the Bessel potentials in the terminology of [5]).

For $0 \leq a < b \leq \infty$ we set $\Omega(a, b) = \{x \in \mathbb{R}^n \mid a < |x| < b\}$. Notice that $\Omega(0, \infty) = \mathbb{R}^n \setminus \{0\}$. The derivatives of locally integrable functions are understood to be in the sense of distributions. For $1 \leq q \leq \infty$, $k \geq 0$ and integer, $a \geq 0$ and $\Omega \equiv \Omega(a, \infty)$, $L^q(\Omega)$ denotes the Banach space of q -summable functions on Ω and $H^{k,q}(\Omega)$ the Sobolev space consisting of all $f \in L^q(\Omega)$ such that $\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f \in L^q(\Omega)$ for all n -tuples $(\alpha_1, \dots, \alpha_n)$ of non-negative integers with $\sum_{j=1}^n \alpha_j \leq k$. We put

$$\|f\|_{H^{k,q}(\Omega)} = \sum_{\alpha_1 + \dots + \alpha_n \leq k} \|\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f\|_{L^q(\Omega)}. \quad (1)$$

If $q = 2$, we use the simpler notation $H^k(\Omega) \equiv H^{k,2}(\Omega)$. Finally we write $\|\cdot\|_{L^q}$ for the norm in $L^q(\mathbb{R}^n)$ and $\|\cdot\|_{H^{k,q}}$ for that in $H^{k,q}(\mathbb{R}^n)$, and we denote by $H_c^{k,q}(\mathbb{R}^n \setminus \{0\})$ the set of functions $f \in H^{k,q}(\mathbb{R}^n)$ that have compact support in $\mathbb{R}^n \setminus \{0\}$.

The proof of our theorem is based on the Sobolev imbedding theorem and on the following known results that we announce as Propositions 1, 2 and 3.

Proposition 1. *If $1 < q < \infty$, then $(1 - \Delta)^{-1}$ defines a bounded invertible operator from $L^q(\mathbb{R}^n)$ onto $H^{2,q}(\mathbb{R}^n)$. In particular, if $f, \Delta f \in L^q(\mathbb{R}^n)$, then³⁾ $f \in H^{2,q}(\mathbb{R}^n)$ (see [5], Theorem V.3).*

Proposition 2. *Let $n \geq 2$, $p \in (n/2, \infty]$ with $p \geq n - 2$. Set $\mu = 2 - n/p$. Let q and s satisfy*

$$1 \leq q \leq 2 \leq s < \infty, \quad \frac{1}{p} = \frac{1}{q} - \frac{1}{s}. \quad (2)$$

³⁾ Write $f = (1 - \Delta)^{-1}(f - \Delta f)$.

Let $\Gamma_{ns} = \{k + n - 3/2 - n/s \mid k = 1, 2, 3, \dots\}$. Then there is a finite constant C , depending only on n, p and s , such that

$$\|r^\nu f\|_{L^s} \leq C \|r^{\nu+\mu} \Delta f\|_{L^q} \quad (3)$$

for all $\nu \in \Gamma_{ns}$ and all $f \in H_c^{2,q}(\mathbb{R}^n \setminus \{0\})$. If $p = \infty$, the inequality (3) holds with C replaced by $2\nu^{-1}$. (See [4], Theorem 1 and proof of Theorem 2.)

Proposition 3. Let $R > 0$ and $\Omega = \Omega(R, \infty)$, and let $\alpha \geq 0$. Then $f \in H^{k,q}(\Omega) \Rightarrow r^{-\alpha} f \in H^{k,q}(\Omega)$.

Proof. Clearly multiplication by $r^{-\alpha}$ defines a bounded operator in $L^q(\Omega)$, since $R > 0$ and $\alpha \geq 0$. This proves the assertion for $k = 0$. Next notice that

$$\partial_j r^{-\alpha} f = r^{-\alpha} \partial_j f - \alpha x_j r^{-\alpha-2} f. \quad (4)$$

Hence $f \in H^{1,q}(\Omega) \Rightarrow r^{-\alpha} f \in H^{1,q}(\Omega)$. The proof for $k > 1$ is similar. ■

3. Lower bounds for zero energy eigenfunctions

We now state and prove our principal result.

Theorem. Let $n \geq 2$, $R_0 \in [0, \infty)$ and set $\Omega_0 = \Omega(R_0, \infty)$. Let $V: \Omega_0 \rightarrow \mathbb{C}$ and assume that there is a number $p \in [1, \infty]$ such that $p > n/2$ and $p \geq n-2$ and such that $r^{2-n/p} V \in L^p(\Omega_0)$. Suppose $g \in H^1(\Omega_0)$ is such that Δg is a function and

$$|(\Delta g)(x)| \leq |V(x)| |g(x)| \quad \text{a.e. on } \Omega_0. \quad (5)$$

Then, if $r^\tau g \in L^2(\Omega_0)$ for each $\tau < \infty$, one must have $g = 0$ (in the L^2 -sense).

Remark. (a) If $p = \infty$, the condition on the function V means that $|x|^2 |V(x)| \leq \text{const} < \infty$, i.e. $V(x)$ should decay at least as rapidly as $|x|^{-2}$ for $|x| \rightarrow \infty$. If $p < \infty$, the condition on V means that

$$\int_{\Omega_0} |r^2 V(x)|^p \frac{d^n x}{r^n} < \infty,$$

i.e. $r^2 V(x)$ must tend to zero in an L^p -sense as $|x| \rightarrow \infty$. Of course local singularities of V are allowed, and for $n = 2, 3, 4$ the result is very natural.

(b) Let $V: \mathbb{R}^n \rightarrow \mathbb{R}$ be such that $(1+r)^{2-n/p} V \in L^p(\mathbb{R}^n)$ for some $p \in (n/2, \infty]$ with $p \geq n-2$. Then $H = -\Delta + V$ is self-adjoint in $L^2(\mathbb{R}^n)$ on the domain $\{f \in H^1(\mathbb{R}^n) \mid Hf \in L^2(\mathbb{R}^n)\}$. If zero is an eigenvalue of H , then any associated eigenvector g has the following property: there is a number $N < \infty$ such that $\|r^N f\|_{L^2} = \infty$. (To see this, it suffices to notice that an eigenvector g corresponding to the eigenvalue zero satisfies (5) with the equality sign.)

Proof. (i) We first fix s and q satisfying the hypotheses of Propositions 1 and 2. It suffices to choose the number s ; q is then defined by $q^{-1} = p^{-1} + s^{-1}$.

If $p > 2$, we take $s = 2$. If $p \leq 2$ (which is possible only for $n = 2, 3$), we define s by $s^{-1} = 3/4 - (2p)^{-1} - (2n)^{-1}$. The assumptions made on p imply that $s \in [3, \infty)$ in the second case and that $1 < q \leq \min\{2, p\}$ in both cases.

We set $\mu = 2 - n/p$ and choose a number $R \in (R_0, \infty)$ as follows. If $p = \infty$, we

take $R = R_0 + 1$; if $p < \infty$, we let $C = C(n, p, s)$ be the constant appearing in Proposition 2 and take R so large that $C \|r^\mu V\|_{L^p(\Omega(R, \infty))} < \frac{1}{2}$, which is possible by the hypothesis made on V . We set $\Omega = \Omega(R, \infty)$ and $\lambda = \|r^\mu V\|_{L^p(\Omega)}$.

The Sobolev imbedding theorem ([6], Theorem 5.4 and Corollary 5.16) implies that, if $p > n/2$ and q and s are as above, one has the following imbeddings: $H^1(\Omega) \subset L^s(\Omega)$ and $H^{2,q}(\Omega) \subset L^s(\Omega)$; here $X \subset Y$ means that each $\xi \in X$ is also an element of Y and that there is a constant $\kappa = \kappa_{XY}$ such that $\|\xi\|_Y \leq \kappa \|\xi\|_X$ for each $\xi \in X$.

(ii) Let $\eta \in C^\infty(\mathbb{R}^n)$ be such that $0 \leq \eta \leq 1$, $\eta(x) = 0$ if $|x| \leq R$ and $\eta(x) = 1$ if $|x| \geq R + 1$. Assume that g satisfies all the hypotheses stated in the theorem and set $g_0 = \eta g$. We shall show that $r^\tau g_0$, $r^\tau \Delta g_0$ and each component of $r^\tau \nabla g_0$ belong to $L^q(\mathbb{R}^n)$ for each $\tau \in \mathbb{R}$.

The first assertion follows from the Hölder inequality and the hypothesis that $r^\tau g \in L^2(\Omega_0)$ for all τ : if $m \in (2, \infty]$ is defined by $m^{-1} = q^{-1} - \frac{1}{2}$, then

$$\|r^\tau g_0\|_{L^q} \leq \|r^\tau g\|_{L^q(\Omega)} \leq \|r^{-n}\|_{L^m(\Omega)} \|r^{\tau+n} g\|_{L^2(\Omega)} < \infty.$$

Next we observe that

$$r^\tau \Delta g_0 = \eta r^\tau \Delta g + 2r^\tau (\nabla \eta) \cdot \nabla g + r^\tau (\Delta \eta) g. \quad (6)$$

Since g and the components of ∇g are in $L^2(\Omega)$ and $\nabla \eta, \Delta \eta$ have compact support, the last two terms on the r.h.s. of (6) are in $L^q(\mathbb{R}^n)$ (remember that $q \leq 2$). We denote by β_τ the sum of their L^q -norms and then have by the Hölder inequality:

$$\|r^\tau \Delta g_0\|_{L^q} \leq \|r^\tau \Delta g\|_{L^q(\Omega)} + \beta_\tau \leq \|r^\mu V\|_{L^p(\Omega)} \|r^{\tau-\mu} g\|_{L^s(\Omega)} + \beta_\tau. \quad (7)$$

In view of the last statement in (i), this leads to the following two inequalities, in which λ is the number defined in part (i) of the proof and κ_s, κ_{qs} are finite constants depending on the values of the subscript(s):

$$\|r^\tau \Delta g_0\|_{L^q} \leq \lambda \kappa_s \|r^{\tau-\mu} g\|_{H^1(\Omega)} + \beta_\tau, \quad (8)$$

$$\begin{aligned} \|r^\tau \Delta g_0\|_{L^q} &\leq \lambda \kappa_{qs} \|r^{\tau-\mu} g_0\|_{H^{2,q}} + \lambda \|r^{\tau-\mu} (1-\eta) g\|_{L^s(\Omega)} + \beta_\tau \\ &\leq \lambda \kappa_{qs} \|r^{\tau-\mu} g_0\|_{H^{2,q}} + \lambda \gamma_\tau \kappa_s \|g\|_{H^1(\Omega)} + \beta_\tau, \end{aligned} \quad (9)$$

where $\gamma_\tau = \|r^{\tau-\mu} (1-\eta)\|_{L^s(\Omega)} < \infty$.

Since $g \in H^1(\Omega)$, the inequality (8) and Proposition 3 imply that $r^\tau \Delta g_0 \in L^q(\mathbb{R}^n)$ for $\tau \leq \mu$; in particular $\Delta g_0 \in L^q(\mathbb{R}^n)$. By Proposition 1, we then have $g_0 \in H^{2,q}(\mathbb{R}^n)$.

Next we notice the identity

$$\Delta r^\tau g_0 = r^\tau \Delta g_0 + 2\tau \partial_r (r^{\tau-1} g_0) + (n\tau - \tau^2) r^{\tau-2} g_0. \quad (10)$$

Since $\|\partial_r f\|_{L^q} \leq \|f\|_{H^{1,q}} \leq \|f\|_{H^{2,q}}$, (10) leads to

$$\|\Delta r^\tau g_0\|_{L^q} \leq \|r^\tau \Delta g_0\|_{L^q} + 2|\tau| \|r^{\tau-1} g_0\|_{H^{2,q}} + (n|\tau| + \tau^2) \|r^{\tau-2} g_0\|_{L^q}. \quad (11)$$

Hence, if $\tau \leq \tau_0 \equiv \min\{\mu, 1\}$, we have $\Delta r^\tau g_0 \in L^q(\mathbb{R}^n)$. Together with Proposition 1, this implies that $r^\tau g_0 \in H^{2,q}(\mathbb{R}^n)$ for $\tau \leq \tau_0$.

This last inclusion may now be combined with the inequality (9) to deduce that $r^\tau \Delta g_0 \in L^q(\mathbb{R}^n)$ for $\tau \leq \tau_0 + \mu$, and (11) then implies that $\Delta r^\tau g_0 \in L^q(\mathbb{R}^n)$ if $\tau \leq 2\tau_0$. Hence, by Proposition 1, $r^\tau g_0 \in H^{2,q}(\mathbb{R}^n)$ for $\tau \leq 2\tau_0$. By iterating this procedure one obtains that $\Delta r^\tau g_0 \in L^q(\mathbb{R}^n)$ and $r^\tau g_0 \in H^{2,q}(\mathbb{R}^n)$ for all $\tau \in \mathbb{R}$.

Finally we have for each $\tau \in \mathbb{R}$ (see (4)):

$$\begin{aligned} \|r^\tau \partial_j g_0\|_{L^a} &\leq \|\partial_j r^\tau g_0\|_{L^a} + |\tau| \|r^{\tau-1} g_0\|_{L^a} \\ &\leq \|r^\tau g_0\|_{H^{2,a}} + |\tau| \|r^{\tau-1} g_0\|_{L^a} < \infty. \end{aligned}$$

(iii) We now show that $g(x) = 0$ for $|x| > R + 1$. For this, we let $\theta \in C_0^\infty(\mathbb{R}^n)$ be such that $\theta(x) = 1$ if $|x| \leq 1$ and $\theta(x) = 0$ if $|x| \geq 2$. For $a > 0$ we define θ_a by $\theta_a(x) = \theta(x/a)$, and we set $\delta' = \|\nabla \theta\|_{L^a}$, $\delta'' = \|\Delta \theta\|_{L^a}$. We observe that

$$|(\nabla \theta_a)(x)| \leq \frac{\delta'}{a}, \quad |(\Delta \theta_a)(x)| \leq \frac{\delta''}{a^2} \quad \forall x \in \mathbb{R}^n. \quad (12)$$

The identity

$$\Delta \theta_a g_0 = \theta_a \Delta g_0 + 2(\nabla \theta_a) \cdot \nabla g_0 + (\Delta \theta_a) g_0 \quad (13)$$

and a similar identity for $\partial_j \theta_a g_0$ imply that $\theta_a g_0 \in H_c^{2,a}(\mathbb{R}^n \setminus \{0\})$. By setting $f = \theta_a g_0$ in (3) and using (13) and (12) one finds that, for $\nu \in \Gamma_{ns}$:

$$\|r^\nu \theta_a g_0\|_{L^a} \leq C \|r^{\nu+\mu} \theta_a \Delta g_0\|_{L^a} + \frac{2\delta'}{a} C \|r^{\nu+\mu} \nabla g_0\|_{L^a} + \frac{\delta''}{a^2} C \|r^{\nu+\mu} g_0\|_{L^a}. \quad (14)$$

Remembering that $r^\rho \Delta g_0$, $r^\rho \nabla g_0$ and $r^\rho g_0$ are in $L^a(\mathbb{R}^n)$ for each $\rho \in \mathbb{R}$, one may take the limit $a \rightarrow \infty$ in (14) (by using for instance the dominated convergence theorem) to obtain the inequality

$$\|r^\nu g_0\|_{L^s} \leq C \|r^{\nu+\mu} \Delta g_0\|_{L^a}, \quad \nu \in \Gamma_{ns}. \quad (15)$$

The r.h.s. of (15) may be majorized by using the inequality (7), with $\tau = \nu + \mu$. We note that β_τ satisfies $\beta_\tau \leq (R+1)^\tau c(\eta, g)$, where $c(\eta, g)$ is a finite number that does not depend on τ . We also have, as in (9), that

$$\|r^\nu g\|_{L^s(\Omega)} \leq \|r^\nu g_0\|_{L^s} + (R+1)^\nu \|g\|_{L^s(\Omega)} \leq \|r^\nu g_0\|_{L^s} + \kappa_s (R+1)^\nu \|g\|_{H^1(\Omega)}.$$

Consequently we obtain that

$$\|r^\nu g_0\|_{L^s} \leq C\lambda \|r^\nu g_0\|_{L^s} + C\lambda \kappa_s (R+1)^\nu \|g\|_{H^1(\Omega)} + Cc(\eta, g)(R+1)^{\nu+\mu}. \quad (16)$$

If $p < \infty$, we have $C\lambda < \frac{1}{2}$, and (16) implies that, for $\nu \in \Gamma_{ns}$:

$$\|r^\nu g\|_{L^s(\Omega(R+1, \infty))} \leq \|r^\nu g_0\|_{L^s(\mathbb{R}^n)} \leq c_1(R, \eta, g)(R+1)^\nu, \quad (17)$$

where c_1 is a finite number independent of ν . If $p = \infty$, one may replace C by $2\nu^{-1}$ in (14)–(16) and obtains the validity of (17) for all $\nu \in \Gamma_{ns} \cap [4\lambda, \infty)$.

Now assume that $\|g\|_{L^2(\Omega(R+1, \infty))} \neq 0$. Then, as $\nu \rightarrow \infty$ ($\nu \in \Gamma_{ns}$), the l.h.s. of (17) grows faster than $(R+1)^\nu$, i.e. (17) is violated for ν large enough. Hence we must have $g = 0$ on $\Omega(R+1, \infty)$.

(iv) To show that $g = 0$ on $\Omega_0 = \Omega(R_0, \infty)$, it suffices to notice that $q \geq 2p/(p+2)$, so that one may apply the unique continuation theorem proved in [4] (see [4], Theorem 2). ■

Additional remarks

(a) It is interesting to point out that A. Hinz recently obtained *upper* bounds for zero energy eigenfunctions that have the form of a negative power of $|x|$, see [7].

(b) One may ask to what extent our condition $\|r^{2-n/p}V\|_{L^p} < \infty$ is optimal. For $p = \infty$, it requires that $|V(x)| \leq cr^{-2}$. The following example shows that one may have exponentially decreasing zero energy eigenfunctions for potentials V tending to zero at infinity but doing so more slowly than r^{-2} : if $-\Delta g + Vg = 0$, then $V = \Delta g/g$. By taking g of the form $g(x) = \exp[-\varphi(r)]$, one obtains

$$V(x) = |\varphi'(r)|^2 - \varphi''(r) - (n-1)r^{-1}\varphi'(r).$$

If for example φ is a smooth function that is constant near $r=0$ and equal to r^α , $0 < \alpha < 1$, near infinity, then $g \in L^2(\mathbb{R}^n)$, hence it is a zero energy bound state eigenfunction, and $V(x)$ decays at infinity like $r^{-2+2\alpha}$. This gives a class of smooth potentials that decay like $r^{-\beta}$, $0 < \beta < 2$, and give rise to zero energy eigenfunctions that decrease more rapidly than any negative power of $|x|$.

Note. This paper is an elaboration of one of the results announced in [8]. After submission of the paper for publication, our attention was drawn to Ref. [9] which contains various L^2 lower bounds for eigenfunctions of Schrödinger operators, in particular a theorem of the type of that given here.

REFERENCES

- [1] K. CHADAN and P. SABATIER, *Inverse Problems in Quantum Scattering Theory*, Springer, New York, 1977.
- [2] L. I. SCHIFF, *Quantum Mechanics*, McGraw-Hill, New York, 1968.
- [3] W. O. AMREIN, J. M. JAUCH and K. B. SINHA, *Scattering Theory in Quantum Mechanics*, Benjamin, Reading, 1977.
- [4] W. O. AMREIN, A. M. BERTHIER and V. GEORGESCU, *Ann. Inst. Fourier* 31, 153–168 (1981).
- [5] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.
- [6] R. A. ADAMS, *Sobolev Spaces*, Academic Press, New York, 1975.
- [7] A. HINZ, *Math. Zeitschr.* 185, 291–304 (1984).
- [8] W. O. AMREIN, A. M. BERTHIER and V. GEORGESCU, *Comptes Rendus Acad. Sci. Paris I* 295, 575–578 (1982).
- [9] R. FROESE, I. HERBST, M. HOFFMANN-OSTENHOF and T. HOFFMANN-OSTENHOF, *Proc. Royal Soc. Edinburgh* 95A, 25–38 (1983).