Zeitschrift: Helvetica Physica Acta

Band: 57 (1984)

Heft: 3

Artikel: Lower bounds for zero energy eigenfunctions of Schrödinger operators

Autor: Amrein, W.O. / Berthier, A.M. / Georgescu, V.

DOI: https://doi.org/10.5169/seals-115508

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 01.12.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Helvetica Physica Acta, Vol. **57** (1984) 301–306

0018–0238/84/030301–06\$1.50 + 0.20/0
© Birkhäuser Verlag Basel, 1984

Lower bounds for zero energy eigenfunctions of Schrödinger operators[1])

By W. O. Amrein, Département de Physique Théorique, Université de Genève, 1211 Genève 4, Switzerland,
A. M. Berthier, Département de Mathématiques, UER 47, Université de Paris VI, F-75230 Paris Cédex 05, and
V. Georgescu, Department of Fundamental Physics, Central Institute of Physics, Bucharest, Romania

(2. XII. 1983)

Abstract. Let g be a non-zero solution in $L^2(\mathbb{R}^n)$, $n \geq 2$, of $(-\Delta + V)g = 0$. If the potential V vanishes rapidly enough at infinity, then g cannot decay (in the L^2-sense) more rapidly than any power of $|x|$, i.e. $|x|^N g \notin L^2(\mathbb{R}^n)$ for some finite N.

1. Introduction

A non-relativistic quantum mechanical particle moving on a line in a potential V cannot be bound at zero energy if V is such that

$$\int_{-\infty}^{+\infty} (1+|x|)\,|V(x)|\,dx < \infty.$$

In other words the equation $-\psi'' + V\psi = 0$ has no non-zero solutions that are square-integrable over the real line $\mathbb{R}$. If $\mathbb{R}$ is replaced by $(0, \infty)$ for example, the same is true; more precisely, if $\int_0^\infty r|V(r)|\,dr < \infty$, there are no zero energy bound states in the $l = 0$ partial wave subspace of a three-dimensional quantum mechanical system in the spherically symmetric potential $V(r)$ (see e.g. [1], Chapters XVII.1 and II.1 respectively).

In the latter case one may however have zero energy bound states in the higher order partial wave subspaces ($l \geqq 1$), even if V has finite range (see [2], footnote on page 80 for a square well, [1] or [3], Remark 11.17(c) and Problem 11.11 for more general cases). The intuitive reason for this is roughly as follows: if $l > 0$, then the effective potential is $V(r) + l(l+1)r^{-2}$ which, at large r, is roughly $l(l+1)r^{-2}$ under the above assumptions on V; hence, if the particle has zero energy, it sees a wall of infinite extension of the form cr^{-2} ($c > 0$) which can produce a bound state[2]) (no tunnelling is possible).

---

[1]) Research supported in part by the Swiss National Science Foundation and by C.N.R.S. (L.A. 213).

[2]) Notice that $\int_1^\infty r \cdot cr^{-2}\,dr = \infty$, so that the centrifugal part of the effective potential does not satisfy the condition needed for proving the non-existence of zero energy bound states.

The zero energy bound state eigenfunctions in the l-th partial wave subspace of $L^2(\mathbb{R}^3)$ are known to behave like r^{-l-1} as $r \to \infty$. This is strikingly different from the exponential decay of eigenfunctions belonging to strictly negative eigenvalues: if $\lambda < 0$, $(-\Delta + V)\psi = \lambda\psi$ and $\psi \in L^2(\mathbb{R}^3)$ and if V decays sufficiently rapidly, then $\|e^{\kappa r}\psi\|_{L^2} < \infty$ for each $\kappa < |\lambda|^{1/2}$. The purpose of our paper is to prove quite generally (i.e. in $n \geq 2$ space dimensions and without assuming spherical symmetry) that zero energy bound states are weakly localized in the sense indicated above: if V satisfies suitable decay conditions and if $\psi \in L^2(\mathbb{R}^n)$ is such that $(-\Delta + V)\psi = 0$, then there is a number $N < \infty$ such that $\|(1+|x|)^N\psi\|_{L^2} = \infty$, i.e. ψ cannot decay faster (in the L^2-sense) than some negative power of $|x|$. This follows from a more general result which we state and prove in the form of a theorem in Section 3. The proof makes heavy use of an inequality involving the Laplacean that we established in a previous paper [4].

2. Notation and preliminary results

We use the following notation: the symbol x is used for vectors in $\mathbb{R}^n$, $n \geq 2$. We set $r = |x|$, $\partial_j = \partial/\partial x_j$ $(j = 1, \ldots, n)$, $\nabla \equiv \text{grad} = (\partial_1, \ldots, \partial_n)$, $\partial_r = \sum_{j=1}^n x_j r^{-1} \partial_j$ and $\Delta = \sum_{j=1}^n \partial_j^2$. We shall refer to the operator $(1-\Delta)^{-1}$ acting on functions defined on $\mathbb{R}^n$; it is given as the convolution operator by the Green's function of the negative Laplacean (one of the Bessel potentials in the terminology of [5]).

For $0 \leq a < b \leq \infty$ we set $\Omega(a, b) = \{x \in \mathbb{R}^n \mid a < |x| < b\}$. Notice that $\Omega(0, \infty) = \mathbb{R}^n \setminus \{0\}$. The derivatives of locally integrable functions are understood to be in the sense of distributions. For $1 \leq q \leq \infty$, $k \geq 0$ and integer, $a \geq 0$ and $\Omega \equiv \Omega(a, \infty)$, $L^q(\Omega)$ denotes the Banach space of q-summable functions on Ω and $H^{k,q}(\Omega)$ the Sobolev space consisting of all $f \in L^q(\Omega)$ such that $\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f \in L^q(\Omega)$ for all n-tuples $(\alpha_1, \ldots, \alpha_n)$ of non-negative integers with $\sum_{j=1}^n \alpha_j \leq k$. We put

$$\|f\|_{H^{k,q}(\Omega)} = \sum_{\alpha_1 + \cdots + \alpha_n \leq k} \|\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f\|_{L^q(\Omega)}. \tag{1}$$

If $q = 2$, we use the simpler notation $H^k(\Omega) \equiv H^{k,2}(\Omega)$. Finally we write $\|\cdot\|_{L^q}$ for the norm in $L^q(\mathbb{R}^n)$ and $\|\cdot\|_{H^{k,q}}$ for that in $H^{k,q}(\mathbb{R}^n)$, and we denote by $H_c^{k,q}(\mathbb{R}^n \setminus \{0\})$ the set of functions $f \in H^{k,q}(\mathbb{R}^n)$ that have compact support in $\mathbb{R}^n \setminus \{0\}$.

The proof of our theorem is based on the Sobolev imbedding theorem and on the following known results that we announce as Propositions 1, 2 and 3.

Proposition 1. *If* $1 < q < \infty$, *then* $(1-\Delta)^{-1}$ *defines a bounded invertible operator from* $L^q(\mathbb{R}^n)$ *onto* $H^{2,q}(\mathbb{R}^n)$. *In particular, if* $f, \Delta f \in L^q(\mathbb{R}^n)$, *then*[3] $f \in H^{2,q}(\mathbb{R}^n)$ (see [5], Theorem V.3).

Proposition 2. *Let* $n \geq 2$, $p \in (n/2, \infty]$ *with* $p \geq n-2$. *Set* $\mu = 2 - n/p$. *Let* q *and* s *satisfy*

$$1 \leq q \leq 2 \leq s < \infty, \quad \frac{1}{p} = \frac{1}{q} - \frac{1}{s}. \tag{2}$$

[3]) Write $f = (1-\Delta)^{-1}(f - \Delta f)$.

Let $\Gamma_{ns}=\{k+n-3/2-n/s \mid k=1,2,3,\dots\}$. *Then there is a finite constant* C, *depending only on* n, p *and* s, *such that*

$$\|r^{\nu}f\|_{L^s}\leq C\,\|r^{\nu+\mu}\,\Delta f\|_{L^q} \tag{3}$$

for all $\nu\in\Gamma_{ns}$ *and all* $f\in H_c^{2,q}(\mathbb{R}^n\setminus\{0\})$. *If* $p=\infty$, *the inequality* (3) *holds with* C *replaced by* $2\nu^{-1}$. (See [4], Theorem 1 and proof of Theorem 2.)

Proposition 3. *Let* $R>0$ *and* $\Omega=\Omega(R,\infty)$, *and let* $\alpha\geq 0$. *Then* $f\in H^{k,q}(\Omega)\Rightarrow r^{-\alpha}f\in H^{k,q}(\Omega)$.

Proof. Clearly multiplication by $r^{-\alpha}$ defines a bounded operator in $L^q(\Omega)$, since $R>0$ and $\alpha\geq 0$. This proves the assertion for $k=0$. Next notice that

$$\partial_j r^{-\alpha}f=r^{-\alpha}\,\partial_j f-\alpha x_j r^{-\alpha-2}f. \tag{4}$$

Hence $f\in H^{1,q}(\Omega)\Rightarrow r^{-\alpha}f\in H^{1,q}(\Omega)$. The proof for $k>1$ is similar. ■

3. Lower bounds for zero energy eigenfunctions

We now state and prove our principal result.

Theorem. *Let* $n\geq 2$, $R_0\in[0,\infty)$ *and set* $\Omega_0=\Omega(R_0,\infty)$. *Let* $V:\Omega_0\to\mathbb{C}$ *and assume that there is a number* $p\in[1,\infty]$ *such that* $p>n/2$ *and* $p\geq n-2$ *and such that* $r^{2-n/p}V\in L^p(\Omega_0)$. *Suppose* $g\in H^1(\Omega_0)$ *is such that* Δg *is a function and*

$$|(\Delta g)(x)|\leq|V(x)|\,|g(x)|\quad\text{a.e. on }\Omega_0. \tag{5}$$

Then, if $r^{\tau}g\in L^2(\Omega_0)$ *for each* $\tau<\infty$, *one must have* $g=0$ (*in the* L^2-*sense*).

Remark. (a) If $p=\infty$, the condition on the function V means that $|x|^2\,|V(x)|\leq\text{const}<\infty$, i.e. $V(x)$ should decay at least as rapidly as $|x|^{-2}$ for $|x|\to\infty$. If $p<\infty$, the condition on V means that

$$\int_{\Omega_0}|r^2V(x)|^p\frac{d^nx}{r^n}<\infty,$$

i.e. $r^2V(x)$ must tend to zero in an L^p-sense as $|x|\to\infty$. Of course local singularities of V are allowed, and for $n=2,3,4$ the result is very natural.

(b) Let $V:\mathbb{R}^n\to\mathbb{R}$ be such that $(1+r)^{2-n/p}V\in L^p(\mathbb{R}^n)$ for some $p\in(n/2,\infty]$ with $p\geq n-2$. Then $H=-\Delta+V$ is self-adjoint in $L^2(\mathbb{R}^n)$ on the domain $\{f\in H^1(\mathbb{R}^n)\mid Hf\in L^2(\mathbb{R}^n)\}$. If zero is an eigenvalue of H, then any associated eigenvector g has the following property: there is a number $N<\infty$ such that $\|r^Nf\|_{L^2}=\infty$. (To see this, it suffices to notice that an eigenvector g corresponding to the eigenvalue zero satisfies (5) with the equality sign.)

Proof. (i) We first fix s and q satisfying the hypotheses of Propositions 1 and 2. It suffices to choose the number s; q is then defined by $q^{-1}=p^{-1}+s^{-1}$.

If $p>2$, we take $s=2$. If $p\leq 2$ (which is possible only for $n=2,3$), we define s by $s^{-1}=3/4-(2p)^{-1}-(2n)^{-1}$. The assumptions made on p imply that $s\in[3,\infty)$ in the second case and that $1<q\leq\min\{2,p\}$ in both cases.

We set $\mu=2-n/p$ and choose a number $R\in(R_0,\infty)$ as follows. If $p=\infty$, we take $R=R_0+1$; if $p<\infty$, we let $C=C(n,p,s)$ be the constant appearing in Proposition 2 and take R so large that $C\|r^\mu V\|_{L^p(\Omega(R,\infty))}<\frac{1}{2}$, which is possible by the hypothesis made on V. We set $\Omega=\Omega(R,\infty)$ and $\lambda=\|r^\mu V\|_{L^p(\Omega)}$.

The Sobolev imbedding theorem ([6], Theorem 5.4 and Corollary 5.16) implies that, if $p>n/2$ and q and s are as above, one has the following imbeddings: $H^1(\Omega)\subset L^s(\Omega)$ and $H^{2,q}(\Omega)\subset L^s(\Omega)$; here $X\subset Y$ means that each $\xi\in X$ is also an element of Y and that there is a constant $\kappa=\kappa_{XY}$ such that $\|\xi\|_Y\leq\kappa\,\|\xi\|_X$ for each $\xi\in X$.

(ii) Let $\eta\in C^\infty(\mathbb{R}^n)$ be such that $0\leq\eta\leq 1$, $\eta(x)=0$ if $|x|\leq R$ and $\eta(x)=1$ if $|x|\geq R+1$. Assume that g satisfies all the hypotheses stated in the theorem and set $g_0=\eta g$. We shall show that $r^\tau g_0$, $r^\tau\,\Delta g_0$ and each component of $r^\tau\nabla g_0$ belong to $L^q(\mathbb{R}^n)$ for each $\tau\in\mathbb{R}$.

The first assertion follows from the Hölder inequality and the hypothesis that $r^\tau g\in L^2(\Omega_0)$ for all τ: if $m\in(2,\infty]$ is defined by $m^{-1}=q^{-1}-\frac{1}{2}$, then

$$\|r^\tau g_c\|_{L^q}\leq\|r^\tau g\|_{L^q(\Omega)}\leq\|r^{-n}\|_{L^m(\Omega)}\,\|r^{\tau+n}g\|_{L^2(\Omega)}<\infty.$$

Next we observe that

$$r^\tau\,\Delta g_0=\eta r^\tau\,\Delta g+2r^\tau(\nabla\eta)\cdot\nabla g+r^\tau(\Delta\eta)g. \tag{6}$$

Since g and the components of ∇g are in $L^2(\Omega)$ and $\nabla\eta,\Delta\eta$ have compact support, the last two terms on the r.h.s. of (6) are in $L^q(\mathbb{R}^n)$ (remember that $q\leq 2$). We denote by β_τ the sum of their L^q-norms and then have by the Hölder inequality:

$$\|r^\tau\,\Delta g_0\|_{L^q}\leq\|r^\tau\,\Delta g\|_{L^q(\Omega)}+\beta_\tau\leq\|r^\mu V\|_{L^p(\Omega)}\,\|r^{\tau-\mu}g\|_{L^s(\Omega)}+\beta_\tau. \tag{7}$$

In view of the last statement in (i), this leads to the following two inequalities, in which λ is the number defined in part (i) of the proof and κ_s, κ_{qs} are finite constants depending on the values of the subscript(s):

$$\|r^\tau\,\Delta g_0\|_{L^q}\leq\lambda\kappa_s\,\|r^{\tau-\mu}g\|_{H^1(\Omega)}+\beta_\tau, \tag{8}$$

$$\begin{aligned}\|r^\tau\,\Delta g_0\|_{L^q}&\leq\lambda\kappa_{qs}\,\|r^{\tau-\mu}g_0\|_{H^{2,q}}+\lambda\,\|r^{\tau-\mu}(1-\eta)g\|_{L^s(\Omega)}+\beta_\tau\\&\leq\lambda\kappa_{qs}\,\|r^{\tau-\mu}g_0\|_{H^{2,q}}+\lambda\gamma_\tau\kappa_s\,\|g\|_{H^1(\Omega)}+\beta_\tau,\end{aligned} \tag{9}$$

where $\gamma_\tau=\|r^{\tau-\mu}(1-\eta)\|_{L^\infty(\Omega)}<\infty$.

Since $g\in H^1(\Omega)$, the inequality (8) and Proposition 3 imply that $r^\tau\,\Delta g_0\in L^q(\mathbb{R}^n)$ for $\tau\leq\mu$; in particular $\Delta g_0\in L^q(\mathbb{R}^n)$. By Proposition 1, we then have $g_0\in H^{2,q}(\mathbb{R}^n)$.

Next we notice the identity

$$\Delta r^\tau g_0=r^\tau\,\Delta g_0+2\tau\,\partial_r(r^{\tau-1}g_0)+(n\tau-\tau^2)r^{\tau-2}g_0. \tag{10}$$

Since $\|\partial_r f\|_{L^q}\leq\|f\|_{H^{1,q}}\leq\|f\|_{H^{2,q}}$, (10) leads to

$$\|\Delta r^\tau g_c\|_{L^q}\leq\|r^\tau\,\Delta g_0\|_{L^q}+2|\tau|\,\|r^{\tau-1}g_0\|_{H^{2,q}}+(n\,|\tau|+\tau^2)\,\|r^{\tau-2}g_0\|_{L^q}. \tag{11}$$

Hence, if $\tau\leq\tau_0\equiv\min\{\mu,1\}$, we have $\Delta r^\tau g_0\in L^q(\mathbb{R}^n)$. Together with Proposition 1, this implies that $r^\tau g_0\in H^{2,q}(\mathbb{R}^n)$ for $\tau\leq\tau_0$.

This last inclusion may now be combined with the inequality (9) to deduce that $r^\tau\,\Delta g_0\in L^q(\mathbb{R}^n)$ for $\tau\leq\tau_0+\mu$, and (11) then implies that $\Delta r^\tau g_0\in L^q(\mathbb{R}^n)$ if $\tau\leq 2\tau_0$. Hence, by Proposition 1, $r^\tau g_0\in H^{2,q}(\mathbb{R}^n)$ for $\tau\leq 2\tau_0$. By iterating this procedure one obtains that $\Delta r^\tau g_0\in L^q(\mathbb{R}^n)$ and $r^\tau g_0\in H^{2,q}(\mathbb{R}^n)$ for all $\tau\in\mathbb{R}$.

Finally we have for each $\tau \in \mathbb{R}$ (see (4)):

$$\|r^{\tau}\,\partial_j g_0\|_{L^q} \le \|\partial_j r^{\tau} g_0\|_{L^q} + |\tau|\,\|r^{\tau-1} g_0\|_{L^q}$$
$$\le \|r^{\tau} g_0\|_{H^{2,q}} + |\tau|\,\|r^{\tau-1} g_0\|_{L^q} < \infty.$$

(iii) We now show that $g(x)=0$ for $|x|>R+1$. For this, we let $\theta \in C_0^\infty(\mathbb{R}^n)$ be such that $\theta(x)=1$ if $|x|\le 1$ and $\theta(x)=0$ if $|x|\ge 2$. For $a>0$ we define θ_a by $\theta_a(x)=\theta(x/a)$, and we set $\delta' = \||\nabla\theta|\|_{L^\infty}$, $\delta''=\|\Delta\theta\|_{L^\infty}$. We observe that

$$|(\nabla\theta_a)(x)| \le \frac{\delta'}{a}, \qquad |(\Delta\theta_a)(x)| \le \frac{\delta''}{a^2} \quad \forall x \in \mathbb{R}^n. \tag{12}$$

The identity

$$\Delta\theta_a g_0 = \theta_a\,\Delta g_0 + 2(\nabla\theta_a)\cdot\nabla g_0 + (\Delta\theta_a) g_0 \tag{13}$$

and a similar identity for $\partial_j\theta_a g_0$ imply that $\theta_a g_0 \in H_c^{2,q}(\mathbb{R}^n\setminus\{0\})$. By setting $f=\theta_a g_0$ in (3) and using (13) and (12) one finds that, for $\nu\in\Gamma_{ns}$:

$$\|r^{\nu}\theta_a g_0\|_{L^s} \le C\,\|r^{\nu+\mu}\theta_a\,\Delta g_0\|_{L^q} + \frac{2\delta'}{a}\,C\,\|r^{\nu+\mu}\,\nabla g_0\|_{L^q} + \frac{\delta''}{a^2}\,C\,\|r^{\nu+\mu} g_0\|_{L^q}. \tag{14}$$

Remembering that $r^{\rho}\,\Delta g_0$, $r^{\rho}\,\nabla g_0$ and $r^{\rho} g_0$ are in $L^q(\mathbb{R}^n)$ for each $\rho\in\mathbb{R}$, one may take the limit $a\to\infty$ in (14) (by using for instance the dominated convergence theorem) to obtain the inequality

$$\|r^{\nu} g_0\|_{L^s} \le C\,\|r^{\nu+\mu}\,\Delta g_0\|_{L^q}, \qquad \nu\in\Gamma_{ns}. \tag{15}$$

The r.h.s. of (15) may be majorized by using the inequality (7), with $\tau=\nu+\mu$. We note that β_τ satisfies $\beta_\tau \le (R+1)^{\tau} c(\eta, g)$, where $c(\eta, g)$ is a finite number that does not depend on τ. We also have, as in (9), that

$$\|r^{\nu} g\|_{L^s(\Omega)} \le \|r^{\nu} g_0\|_{L^s} + (R+1)^{\nu}\,\|g\|_{L^s(\Omega)} \le \|r^{\nu} g_0\|_{L^s} + \kappa_s (R+1)^{\nu}\,\|g\|_{H^1(\Omega)}.$$

Consequently we obtain that

$$\|r^{\nu} g_0\|_{L^s} \le C\lambda\,\|r^{\nu} g_0\|_{L^s} + C\lambda\kappa_s (R+1)^{\nu}\,\|g\|_{H^1(\Omega)} + Cc(\eta, g)(R+1)^{\nu+\mu}. \tag{16}$$

If $p<\infty$, we have $C\lambda<\frac{1}{2}$, and (16) implies that, for $\nu\in\Gamma_{ns}$:

$$\|r^{\nu} g\|_{L^s(\Omega(R+1,\infty))} \le \|r^{\nu} g_0\|_{L^s(\mathbb{R}^n)} \le c_1(R, \eta, g)(R+1)^{\nu}, \tag{17}$$

where c_1 is a finite number independent of ν. If $p=\infty$, one may replace C by $2\nu^{-1}$ in (14)–(16) and obtains the validity of (17) for all $\nu\in\Gamma_{ns}\cap[4\lambda,\infty)$.

Now assume that $\|g\|_{L^2(\Omega(R+1,\infty))}\neq 0$. Then, as $\nu\to\infty$ ($\nu\in\Gamma_{ns}$), the l.h.s. of (17) grows faster than $(R+1)^{\nu}$, i.e. (17) is violated for ν large enough. Hence we must have $g=0$ on $\Omega(R+1,\infty)$.

(iv) To show that $g=0$ on $\Omega_0=\Omega(R_0,\infty)$, it suffices to notice that $q\ge 2p/(p+2)$, so that one may apply the unique continuation theorem proved in [4] (see [4], Theorem 2). ■

Additional remarks

(a) It is interesting to point out that A. Hinz recently obtained *upper* bounds for zero energy eigenfunctions that have the form of a negative power of $|x|$, see [7].

(b) One may ask to what extent our condition $\|r^{2-n/p}V\|_{L^p}<\infty$ is optimal. For $p=\infty$, it requires that $|V(x)|\le cr^{-2}$. The following example shows that one may have exponentially decreasing zero energy eigenfunctions for potentials V tending to zero at infinity but doing so more slowly than r^{-2}: if $-\Delta g+Vg=0$, then $V=\Delta g/g$. By taking g of the form $g(x)=\exp[-\varphi(r)]$, one obtains

$$V(x)=|\varphi'(r)|^2-\varphi''(r)-(n-1)r^{-1}\varphi'(r).$$

If for example φ is a smooth function that is constant near $r=0$ and equal to $r^\alpha, 0<\alpha<1$, near infinity, then $g\in L^2(\mathbb{R}^n)$, hence it is a zero energy bound state eigenfunction, and $V(x)$ decays at infinity like $r^{-2+2\alpha}$. This gives a class of smooth potentials that decay like $r^{-\beta}, 0<\beta<2$, and give rise to zero energy eigenfunctions that decrease more rapidly than any negative power of $|x|$.

Note. This paper is an elaboration of one of the results announced in [8]. After submission of the paper for publication, our attention was drawn to Ref. [9] which contains various L^2 lower bounds for eigenfunctions of Schrödinger operators, in particular a theorem of the type of that given here.

REFERENCES

[1] K. CHADAN and P. SABATIER, *Inverse Problems in Quantum Scattering Theory*, Springer, New York, 1977.
[2] L. I. SCHIFF, *Quantum Mechanics*, McGraw-Hill, New York, 1968.
[3] W. O. AMREIN, J. M. JAUCH and K. B. SINHA, *Scattering Theory in Quantum Mechanics*, Benjamin, Reading, 1977.
[4] W. O. AMREIN, A. M. BERTHIER and V. GEORGESCU, Ann. Inst. Fourier *31*, 153–168 (1981).
[5] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princton University Press, 1970.
[6] R. A. ADAMS, *Sobolev Spaces*, Academic Press, New York, 1975.
[7] A. HINZ, Math. Zeitschr. *185*, 291–304 (1984).
[8] W. O. AMREIN, A. M. BERTHIER and V. GEORGESCU, Comptes Rendus Acad. Sci. Paris I *295*, 575–578 (1982).
[9] R. FROESE, I. HERBST, M. HOFFMANN-OSTENHOF and T. HOFFMANN-OSTENHOF, Proc. Royal Soc. Edinburgh *95A*, 25–38 (1983).